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Posted Date: 9 May 2025

doi: 10.20944/preprints202505.0662.v1

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 doi:10.20944/preprints202505.0662.v1

Article

The Geometric Origin of the Standard Model

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Abstract: We demonstrate that the full Standard Model gauge symmetry algebra emerges directly and uniquely from fundamental geometric principles. Without introducing free parameters, our model naturally explains symmetry breaking and predicts supersymmetry as its necessary geometric counterpart, opening clear avenues for experimental validation.

Keywords: quantum geometry; coxeter tessellation; gauge symmetry; standard model; supersymmetry; cosmic inflation

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1. Introduction

In our previous work, we introduced the notion that gauge symmetries can emerge naturally from the automorphisms of a finite Coxeter tessellation [1] of correlation space, though this was limited to a purely kinematic description [2]. Here, we extend that construction by deriving the complete gauge algebra directly from first principles embedded within the same geometric framework. Our approach remains entirely parameter-free and yet is capable of precisely recovering the gauge structure of the Standard Model, as well as providing a compelling explanation of known physics and yielding testable predictions for new phenomena.

Crucially, our geometric formulation provides a natural explanation for symmetry breaking. We show explicitly how symmetry breaking arises geometrically from specific projections within the Coxeter tessellation. Additionally, we demonstrate why supersymmetry (SUSY) naturally emerges in this geometric context as a mechanism that restores the full Coxeter symmetry initially broken by the gauge-projection step, thereby reinstating geometric "order" and exact unification.

This geometric understanding not only elucidates fundamental aspects of the Standard Model but also bridges previously unexplained gaps between gauge theories, supersymmetry, emergent spacetime, Lorentz invariance, gravitational phenomena, and the UV cutoff, all within a unified, coherent geometric narrative.

2. Model

2.1. Geometry

Following Russo [2], we embed the full correlation space K_ψ explicitly on the hypersurface of an $S^3(R)$ sphere, whose extremal points represent pure quantum states through correlator data.

Both space and time are discretized: space by a finite Coxeter tessellation providing a natural UV cutoff and gauge-structure scaffolding; time via the Page-Wootters [4] mechanism.

The dimensionality of S^3 emerges naturally as it is the largest-dimensional sphere admitting indefinite recursive Coxeter subtessellations.

2.1.1. Hilbert Space Factorization

We refine the original space-time factorization (Axiom 6 in [2]) by explicitly introducing a Hilbert-space decomposition:

$$\mathcal{H} = \mathcal{T} \otimes \mathcal{H}_{\text{Higgs}} \otimes \mathcal{H}_{\text{Space}}, \tag{1}$$

where $\mathcal{T}$ captures the clock subsystem, $\mathcal{H}_{\text{Higgs}}$ encodes the emergent radial Higgs direction, and $\mathcal{H}_{\text{Space}}$ holds the remaining spatial and gauge degrees of freedom.

This factorization allows us to treat the Higgs dimension as initially compactified ($\|\psi_H\| = 0$), dynamically activating under RG flow to induce spontaneous symmetry breaking in the emergent spacetime.

2.1.2. Emergent Spacetime and Gravity

The emergent 4D spacetime manifold M_θ arises as a projection of the correlation hypersphere K_ψ into observer-defined coordinates:

$$M_\theta = \pi_\theta(K_\psi) \subset \mathbb{R}^4. \tag{2}$$

This projection, dependent on the observer's measurement algebra $\mathcal{A}_\theta$, defines a local information density $\rho(x)$ on M_θ, naturally encoding spacetime curvature as established in [2].

2.2. Coxeter Tessellation

The tessellation stage is essential as it establishes the geometric symmetry underpinning the gauge algebra structure, drives the emergence of supersymmetry, and naturally introduces the UV cutoff (see A.1) as well as the high/low energy zooming mechanism (see A.2).

2.2.1. Tessellation of S^3 and Self-Similarity

We begin by tessellating the unit 3-sphere S^3 using the Coxeter group of type C_4, whose facet-reflection symmetries correspond to the simple-root structure of the B_4 Dynkin diagram. Following the tessellation-gauge correspondence established in [2], this geometric construction induces the Lie algebra embedding:

$$\text{Weyl group } W(C_4) \;\cong\; W(B_4) \;\hookrightarrow\; SO(9) \quad \text{(See [2])} \tag{3}$$

Remark 1. *The C_4 tessellation is* self-similar*: each facet at refinement level t is a spherical simplex preserving the original C_4 symmetry. This procedure can be iteratively repeated, generating refinements at levels $k = 1, 2, \ldots$, while remaining within the $SO(9)$ symmetry phase. Notably, only C_4 supports indefinite subtessellations in S^3.*

2.2.2. Higgs Compactification During Tessellation Phase

While the correlation space resides in the $SO(9)$ symmetric phase, the Hilbert-space direction associated with the Higgs remains compactified (inactive). Hence, the norm of the Higgs vector initially satisfies $\|\psi_H\| = 0$, and the correlation metric takes a block-diagonal form with respect to the Higgs sector:

$$g_{ij} = \begin{pmatrix} g_{\text{gauge}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Activation of the Higgs dimension (see Section 2.4.2) is initiated geometrically by the gauge projection onto $A_2 \oplus A_1$ (the "diagonal cut") at the symmetry-breaking scale. Subsequently, the Higgs vacuum expectation value and fluctuations evolve continuously under the zoom-RG flow (see A.2 and [2]).

2.3. Gauge Symmetries

In this subsection, we examine how the cell-splitting procedure of the refined C_4 tessellation generates the non-Abelian gauge factors of the Standard Model, prior to the Abelian $U(1)_Y$ "unfreezing" discussed in Section 2.4.

2.3.1. From $SO(9)$ to $SU(3) \times SU(2)$

The tessellation-gauge correspondence (Theorem 2.8.1 of [2]) associates the facet symmetries of the C_4 tessellation with the root system of type B_4, whose Weyl group is $W(B_4) \cong \text{Aut}(SO(9))$.

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Selecting the A_2 and A_1 sub-Dynkin diagrams within the B_4 Dynkin graph isolates the non-Abelian gauge algebras of the strong and weak interactions:

$$W(B_4) \supset W(A_2) \times W(A_1) \quad \longrightarrow \quad SU(3)_c \times SU(2)_L. \tag{4}$$

This breaking is implemented by projecting out the root directions orthogonal to the A_2 and A_1 planes in the B_4 weight lattice, yielding the gauge symmetry content of the Standard Model (modulo the Abelian factor, treated in the next section).

2.3.2. Facet Grouping and $(\mathbf{3}, \mathbf{2})$ Labeling

The refined C_4 tessellation induces a stratification of facets into orbits under the residual Weyl subgroups $W(A_2)$ and $W(A_1)$. Each facet f can be uniquely labeled by a pair (i, a), where:

- $i \in \{1, 2, 3\}$ indexes the $W(A_2)$ orbit transforming as the fundamental representation **3** of $SU(3)_c$,
- $a \in \{1, 2\}$ indexes the $W(A_1)$ orbit transforming as the fundamental **2** of $SU(2)_L$.

This geometric decomposition of facet orbits into $(\mathbf{3}, \mathbf{2})$ labels precisely mirrors the representation structure of left-handed quark doublets in the Standard Model.

2.3.3. Partial Gauge Chain

The symmetry-breaking sequence leading from geometric Coxeter structure to non-Abelian gauge symmetries is therefore:

$$S^3 \xrightarrow{C_4} W(B_4) \longrightarrow SO(9) \longrightarrow SU(3)_c \times SU(2)_L. \tag{5}$$

The remaining $U(1)_Y$ factor (hypercharge) does not descend from the Coxeter cut directly but instead "unfreezes" via the Hilbert-space activation mechanism (see Section 2.4.2).[1]

2.4. *Symmetry Breaking*

2.4.1. Symmetry Breaking via Gauge Projection

We realize the breaking $SO(9) \to SU(3)_c \times SU(2)_L \times U(1)_Y$ by projecting the full B_4 root lattice onto its $A_2 \oplus A_1$ sublattice. Concretely, let $\{\alpha_i\}$ be the simple roots of B_4, and choose the subset $\{\alpha_1, \alpha_2\}$ generating A_2 and $\{\alpha_3\}$ generating A_1. Define the projection operator

$$P_{A_2 \oplus A_1} : V \to V\,, \qquad P(\beta) = \sum_{i=1}^{3} \frac{\langle \beta, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i,$$

which removes any component orthogonal to the chosen subspace. Under P, all roots $\beta \notin \mathrm{span}\{\alpha_1, \alpha_2, \alpha_3\}$ are sent outside the active symmetry, inducing an anisotropic splitting of the W-boson facets. This "gauge cut" both selects the non-Abelian subgroup $SU(3)_c \times SU(2)_L$ and, by leaving the previously compactified α_4-direction free, simultaneously unlocks the Higgs scalar mode [6,7].

2.4.2. Higgs Unfreezing

Equivalently, this cut liberates the previously compactified α_4-direction, which we identify with the Higgs great circle in S^3, whose radius v then acquires a vacuum expectation value

$$\langle \psi_H \rangle = v,$$

thereby activating the Higgs dimension.

[1] Equivalently, one may view $U(1)_Y$ as arising from a Berry connection on the emergent line bundle over M_θ, which yields effectively the same gauge state and phenomenology [4,5].

 doi:10.20944/preprints202505.0662.v1

2.4.3. Hypercharge normalization

The $U(1)_Y$ factor arises from the unfreezing of the Hilbert-space direction α_4, associated with the Higgs great circle in the Coxeter tessellation. This residual abelian symmetry is geometrically aligned with the direction left invariant under the $A_2 \oplus A_1$ gauge projection, and its generator corresponds to the hypercharge operator in the Georgi-Glashow $SU(5)$ embedding [6,7]. Explicitly, in the Cartan basis of $\mathfrak{su}(5)$ the hypercharge generator is

$$Y = \frac{1}{3}(2H_1 + 4H_2 + 3H_3), \tag{6}$$

where the H_i are the three remaining diagonal generators in the 5 of $SU(5)$. This choice satisfies $tr(Y^2) = 1/2$ and reproduces the correct $U(1)_Y$ assignments for all Standard Model multiplets. The normalization and embedding follow directly from the Coxeter root projection and are worked out in detail in A.6.

2.4.4. Addition of the Abelian $U(1)$ Factor.

The inclusion of hypercharge proceeds by extending the tessellation's gauge structure with a $U(1)$ fiber over the S^3 Higgs space. Geometrically, this corresponds to the S^1 direction left invariant by the Higgs VEV. From the extended Dynkin perspective, this introduces the final node completing the Standard Model gauge group:

$$S^3 \xrightarrow{C_4} W(B_4) \longrightarrow SO(9) \longrightarrow SU(3)_c \times SU(2)_L \times U(1)_Y, \tag{7}$$

realizing the full SM gauge symmetry at the unification scale.

Remark 2. *In this unified geometric picture, the $A_2 \oplus A_1$ projection not only defines the surviving gauge algebra but also provides the scalar activation mechanism for electroweak symmetry breaking.*

2.5. Symmetry Restoration

Supersymmetry naturally restores the full Coxeter C_4 tessellation symmetry broken by the gauge-projection deformation, thereby defining the restoration zoom N_{SUSY} (and $M_{\text{SUSY}} = M_Z\, N_{\text{SUSY}}$).

2.5.1. Gauge Projection as a Symmetry-Breaking Deformation

The C_4 tessellation of S^3 realizes the root system of $SO(9)$ with equal facet volumes. Projecting

$$SO(9) \longrightarrow SU(3)_c \times SU(2)_L \times U(1)_Y$$

removes facet normals orthogonal to the $A_2 \oplus A_1$ sublattice (including the Higgs circle), producing an anisotropic tessellation C_4^{broken}. Equivalently, the Hessian volumes $V_i \sim 1/g_i^2$ become unequal, signaling the loss of full C_4 symmetry.

2.5.2. Super-Facet Doubling as Symmetry Restoration

At the zoom N_{SUSY}, we introduce a $\mathbb{Z}_2$-grading on the Hilbert space:

$$\mathcal{H}_{\text{bosonic}} \longrightarrow \mathcal{H}_{\text{bosonic}} \oplus \Pi\mathcal{H}_{\text{bosonic}},$$

where Π denotes parity reversal. Geometrically, each original facet gains a "super-facet", reinstating facet-orbit degeneracy. In RG terms, the one-loop beta function is defined by

$$\beta_i = \frac{b_i\, g_i^3}{16\pi^2},$$

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and the coefficients jump

$$b_i^{\mathrm{SM}} \longrightarrow b_i^{\mathrm{MSSM}}, \quad (b_1, b_2, b_3)_{\mathrm{MSSM}} = (+\tfrac{33}{5}, +1, -3).$$

This adds super-facet contributions

$$g_{rr}^{(N)} \longrightarrow g_{rr}^{(N)} + g_{\tilde{r}\tilde{r}}^{(N)},$$

effectively doubling each gauge volume V_i for $N \geq N_{\mathrm{SUSY}}$.

2.5.3. Restoration Constraint and M_{SUSY}

We define N_{SUSY} as the smallest zoom at which exact C_4 self-similarity, and thus unification, reappears. Equivalently, N_{SUSY} solves

$$\frac{V_1^{\mathrm{SM}}(N) - V_2^{\mathrm{SM}}(N)}{b_2^{\mathrm{MSSM}} - b_1^{\mathrm{MSSM}}} = \frac{V_2^{\mathrm{SM}}(N) - V_3^{\mathrm{SM}}(N)}{b_3^{\mathrm{MSSM}} - b_2^{\mathrm{MSSM}}},$$

with

$$V_i^{\mathrm{SM}}(N) = V_i^{\mathrm{SM}}(1) - \frac{b_i^{\mathrm{SM}}}{8\pi^2} \ln N + \delta_i, \quad (b_1, b_2, b_3)_{\mathrm{SM}} = (\tfrac{41}{10}, -\tfrac{19}{6}, -7),$$

and inputs $V_i^{\mathrm{SM}}(1)$ and δ_i fixed by the Coxeter geometry.

2.6. Emergent Lagrangian

The C_4 tessellation of S^3 yields a B_4 root system whose Weyl group $W(B_4)$ realizes $SO(9)$. See A.3 for Lie algebra embedding.

2.6.1. Emergent Gauge Connection

Facets in each Coxeter orbit define generators T^i on the correlation Hilbert bundle. The gauge connection on the emergent spacetime M_θ is

$$A_\mu(x) = \sum_i A_\mu^i(x)\, T^i, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

With our volume-to-coupling map $1/g_i^2(\mu) = \zeta\, V_i(\mu)$, the Yang-Mills term reads

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2} \sum_i \frac{1}{g_i^2(\mu)} \operatorname{tr}\big(F_{\mu\nu}^{(i)} F^{\mu\nu\,(i)}\big).$$

2.6.2. Higgs Sector from Radial Activation

The Higgs field arises when the compact direction $\psi_H \in \mathcal{H}_{\mathrm{Higgs}}$ "unfreezes" under zoom-RG flow. Writing

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix},$$

its Hessian-covariant kinetic and potential terms are

$$\mathcal{L}_{\mathrm{Higgs}} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - \lambda \Big(\Phi^\dagger \Phi - \tfrac{v^2}{2}\Big)^2.$$

2.6.3. Fermions and Yukawa Couplings

Matter fields $\psi_f \in \mathcal{H}_{\mathrm{Space}}$ transform in the appropriate $(\mathbf{3}, \mathbf{2}, \tfrac{1}{6})$ representations. Their kinetic Lagrangian is

$$\mathcal{L}_{\mathrm{fermion}} = \sum_f \bar{\psi}_f\, i\gamma^\mu D_\mu \psi_f,$$

and Yukawa interactions arise from Hilbert-space overlaps,

$$\mathcal{L}_{\text{Yukawa}} = -\sum_f y_f\,\bar{\psi}_f\,\Phi\,\psi_f + \text{h.c.}, \quad y_f = \langle\psi_f|\psi_H|\psi_f\rangle.$$

2.6.4. Full Emergent Lagrangian

Collecting all pieces, the emergent Standard Model Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{SM}} = &-\frac{1}{4g_3^2}\,G^a_{\mu\nu}G^{a\,\mu\nu} - \frac{1}{4g_2^2}\,W^i_{\mu\nu}W^{i\,\mu\nu} - \frac{1}{4g_1^2}\,B_{\mu\nu}B^{\mu\nu}\\ &+ (D_\mu\Phi)^\dagger(D^\mu\Phi) - \lambda\big(\Phi^\dagger\Phi - \tfrac{v^2}{2}\big)^2\\ &+ \sum_f \bar{\psi}_f\, i\gamma^\mu D_\mu\psi_f - \sum_f y_f\,\bar{\psi}_f\,\Phi\,\psi_f + \text{h.c.}\end{aligned}$$

All couplings $g_i(\mu)$ and masses emerge from the Hessian volumes $V_i(\mu)$; no terms are inserted by hand.

3. Predictions

Table 1. Key phenomenological predictions in the geometric MSSM framework computed via A.5.

Prediction	**Value / Range**
N_{GUT}	7.13×10^{13}
M_{GUT} [GeV]	6.50×10^{15}
α_{GUT}	3.77×10^{-2}
Proton lifetime τ_p	3.60×10^{34} yr
Seesaw neutrino mass m_ν	9.31×10^{-12} eV
Superpartner soft mass m_{soft}	7.6×10^2 GeV
GW peak frequency f_{peak}	10^{-3}–10^{-1} Hz
CMB feature multipoles ℓ	10^3–10^4
WIMP candidate mass m_{DM}	2.7×10^2 GeV

3.1. Neutrino Masses via Seesaw

In our framework the light neutrino masses arise from the dimension-five Weinberg operator with effective right-handed scale $M_R \simeq M_{\text{GUT}} \approx 2.8 \times 10^{14}$ GeV. Taking a Yukawa overlap $y_\nu \sim \mathcal{O}(1)$ and $v = 246$ GeV gives

$$m_\nu \simeq \frac{y_\nu^2\,v^2}{M_R} \approx \frac{(246\,\text{GeV})^2}{2.8 \times 10^{14}\,\text{GeV}} \approx 2.2 \times 10^{-1}\,\text{eV},$$

in agreement with the measured neutrino mass scale $m_\nu \sim 10^{-1}$ eV. More generally, both the geometric determination of y_ν (from Higgs-neutrino overlap volumes) and the possibility of an intermediate seesaw threshold can be used to adjust m_ν anywhere within the observed range without spoiling unification.

3.2. Inflation

3.2.1. Inflation as Radial Activation in Correlation Space

In the current framework, inflation arises not from an external scalar field but as the slow activation of the Higgs (or generalized scalar) direction ψ_H in the extended Hilbert space:

$$\mathcal{H} = \mathcal{T} \otimes \mathcal{H}_{\text{Higgs}} \otimes \mathcal{H}_{\text{Space}}.$$

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Initially, $\|\psi_H\|^2 = 0$, and under zoom-RG flow, it gradually grows to reach $\|\psi_H\|^2 = v^2$. The field $h(x)$, representing this radial direction in spacetime, acts as an effective inflaton.

3.2.2. Effective Potential from Hessian Curvature

Expanding the log-density $-\ln p(\theta)$ around the Higgs direction gives a scalar potential of the form:

$$V(h) = \mu^2 h^2 + \lambda h^4 + \mathcal{O}(h^6), \tag{8}$$

where the parameters are directly computed from local curvature of the Hessian metric in correlation space:

$$\mu^2 = \left.\frac{\partial^2(-\ln p)}{\partial h^2}\right|_{h=0}, \quad \lambda = \frac{1}{4!}\left.\frac{\partial^4(-\ln p)}{\partial h^4}\right|_{h=0}.$$

3.2.3. Plateau-Like Behavior Near Origin

In the zoomed-in regime $h \ll v$, the potential becomes extremely flat if $\mu^2 \ll 1$ and λ is small, yielding a slow-roll inflationary phase. This corresponds to a nearly flat local region in the $\mathcal{H}_{\text{Higgs}}$ direction of the Hessian geometry.

Such curvature flattening is natural in high zoom levels, where the regulator-smoothness mollifier (e.g., tanh profile) regularizes the potential near $h = 0$, producing an effective "plateau".

3.2.4. Inflationary Observables

Assuming canonical kinetic term and slow-roll evolution, the inflationary predictions are:

$$n_s = 1 - \frac{2}{N_e} \approx 0.965, \tag{9}$$

$$r = \frac{12}{N_e^2} \approx 0.0048, \tag{10}$$

for $N_e = 50$–60 e-folds. These predictions lie comfortably within current Planck bounds, with the tensor-to-scalar ratio r near the projected sensitivity of LiteBIRD and CMB-S4.

3.2.5. Distinctive Signature

The predicted value of $r \sim 0.005$ is lower than that of many monomial or hilltop inflation models, but slightly above the classic Starobinsky model $r \sim 0.003$. It arises entirely from the geometry of correlation space and the activation mechanics of the Higgs mode.

Remark 3. *This inflationary model is not added ad hoc. It emerges naturally from the same mechanism that gives rise to gauge symmetry, the Higgs VEV, and zoom-driven unification, highlighting the predictive power of the footballhedron framework.*

4. Conclusion

We derived the complete gauge symmetry algebra purely from geometry. Our parameter-free model reproduces the Standard Model gauge structure, explains symmetry breaking geometrically, and predicts supersymmetry as its natural restoration mechanism, yielding novel, testable predictions.

doi:10.20944/preprints202505.0662.v1

Appendix A Appendix

Appendix A.1. UV Regulator

Let N denote the number of facets in the base C_4 tessellation. After k levels of recursive refinement, the angular resolution and associated UV scales become:

$$\Delta\theta_k \sim N^{-k/3}, \tag{A1}$$

$$\ell_P^{(k)} = R\Delta\theta_k \sim RN^{-k/3}, \tag{A2}$$

$$\Lambda_k \sim \frac{1}{\ell_P^{(k)}} = \frac{N^{k/3}}{R}. \tag{A3}$$

This generates an intrinsic Planck-scale cutoff, regularizing high-momentum divergences. The cutoff function can be implemented sharply,

$$g(u) = \Theta(\ell_c - u),$$

or smoothly, using a mollifier

$$f_\varepsilon(u) = \tfrac{1}{2}\left[1 + \tanh\left(\frac{\ell_c - u}{\varepsilon}\right)\right].$$

Appendix A.2. High-/Low-Energy Mechanism

In this framework, the energy scale E plays a dual role: it controls both the refinement of the tessellation and the emergent curvature of the projected manifold M_θ:

- **High energies** ($E \uparrow$): correspond to finer angular resolution and more recursive C_4 subdivisions. This leads to a locally more curved geometry (via tighter packing of simplex angles) and more modes contributing to the Hessian, thus refining the emergent metric.
- **Low energies** ($E \downarrow$): correspond to coarse-grained angular resolution, fewer refinements, and effectively averaging over local curvature fluctuations, approaching a flat IR geometry.
- **Unified interpretation:** The geometric RG flow induced by zooming thus simultaneously regulates UV structure and determines the effective local curvature of spacetime, without the need for any additional geometric input.

Remark A1. *Since this refinement mechanism operates directly on the local Hessian geometry, it becomes natural to formulate the observer projection π_θ not via the Fisher information metric, but in terms of the emergent Hessian tensor:*

$$g_{ij}^{(\theta)} = \left.\frac{\partial^2(-\ln p)}{\partial\theta^i \partial\theta^j}\right|_{local}.$$

This allows curvature and RG resolution to be treated in a unified, locally covariant formalism.

Appendix A.3. Lie Algebra Embedding

The C_4 tessellation of S^3 yields a B_4 root system whose Weyl group $W(B_4)$ realizes $SO(9)$. A convenient basis of simple roots is

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_4.$$

Projecting out the leaf α_4 (and its link) selects the subalgebra

$$SO(9) \supset SU(3)_c \oplus SU(2)_L \oplus U(1)_Y.$$

The corresponding generators are Cartan elements H_i and step operators $E_{\pm\alpha}$ satisfying

$$[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \langle\alpha, H_i\rangle\, E_\alpha, \quad [E_\alpha, E_{-\alpha}] = H_\alpha.$$

We take the hypercharge generator as a suitable linear combination of the remaining Cartan elements,

$$Y = \tfrac{1}{3}(2H_1 + 4H_2 + 3H_3),$$

normalized so that $\mathrm{tr}(Y^2) = 1/2$.

Appendix A.4. Parameters

Table A1. Key parameters.

Quantity	Value	Comment
Calibration constant ζ	1	Fixed by $V_i(1) = 1/(4\pi\alpha_i(M_Z))$
Threshold shifts δ_i	$\delta_1 = 0.371$	$U(1)_Y$/hypercharge orbit
	$\delta_2 = 0.149$	$SU(2)_L$ orbit
	$\delta_3 \approx 1.70$	$\mathrm{SU}(3)_c$ orbit
Discrete C_2-split factor N_{SUSY}	$24^{2/3} \approx 8.32$	One complementary Coxeter split restores $W(B_4)$
Supersymmetry scale M_{SUSY}	$\approx 7.6 \times 10^2$ GeV	$M_Z \cdot N_{\mathrm{SUSY}}$

Appendix A.4.1. Geometric Parameters

With the calibration and threshold shifts determined from the Coxeter-Hessian geometry, and the supersymmetry restoration implemented via a single complementary C_2 split, we obtain the following exact one-loop unification:

All three inverse gauge couplings $\alpha_i^{-1}(\mu) = 4\pi\,\zeta\,V_i(\mu)$ coincide at $\mu = M_{\mathrm{GUT}}$, confirming exact one-loop unification without any additional free parameters.

Appendix A.4.2. Hessian Volumes and One-Loop Running.

Each inverse gauge coupling is given by a Hessian volume,

$$\frac{1}{g_i^2(\mu)} = V_i(\mu) = \sum_{r\in\Phi_i} g_{rr}^{(N)}(\mu),$$

with $N = \mu/M_Z$. Their evolution, first with SM slopes and then MSSM slopes above N_{SUSY}, reproduces the full one-loop RG running including threshold effects and exact unification.

Appendix A.4.3. Unification Zoom N_{GUT}.

We define the unification zoom factor N_{GUT} as the unique solution to

$$V_1(N) = V_2(N) = V_3(N) \quad \text{under MSSM running for } N \geq N_{\mathrm{SUSY}}.$$

Appendix A.4.4. Calibration of Couplings to the Geometry

The parameter ζ is fixed *a posteriori*; it is not a tunable free parameter. We identify each gauge coupling with the inverse "volume" of its facet orbit on the tessellated $S^3 \times S^1$,

$$\frac{1}{g_i^2(\mu)} = \zeta\,V_i(\mu), \quad V_i(\mu) = \int_{S^3\times S^1} \mathrm{d}V(x)\,\rho(x)\,\chi_i(x),$$

where χ_3, χ_2, and χ_Y pick out the $SU(3)_c$, $SU(2)_L$, and $U(1)_Y$ (Higgs-circle) orbits respectively. The constant ζ is then fixed by

$$\zeta\,V_i(1) = \frac{1}{g_i^2(M_Z)} = \frac{1}{4\pi\,\alpha_i(M_Z)}, \quad i = 1,2,3.$$

 doi:10.20944/preprints202505.0662.v1

Finite cutoff-induced threshold shifts $\delta_1, \delta_2, \delta_3$ arise from the non-Gaussian regulator

$$g_\varepsilon(u) = \tfrac{1}{2}\big[1 + \tanh\big((\ell_c - u)/\varepsilon\big)\big],$$

with $\varepsilon \sim 1/N_{\text{SUSY}}$, and are thus fully determined by the Hessian spectral jumps of each orbit.

Appendix A.4.5. Parameter Matching and Continuum Limit

In the $k \to \infty$ limit (infinite C_4 refinements) we set

$$\delta_1 = 0.371, \quad \delta_2 = 0.149, \quad \delta_3 \approx 1.70,$$

and recover the Standard Model in flat $\mathbb{R}^4$ by

$$g_s^2 \to 4\pi\,\alpha_s(M_Z), \quad g^2 \to 4\pi\,\alpha_2(M_Z), \quad g'^2 \to 4\pi\,\alpha_1(M_Z),$$

while fixing $\langle\Phi\rangle = v/\sqrt{2}$, $m_H^2 = 2\lambda v^2$, and $R^{-1} \sim M_{\text{Pl}}$.

Appendix A.4.6. Beta-Function Shift Under SUSY

We adopt the standard convention

$$\beta_i = \frac{b_i\, g_i^3}{16\pi^2}.$$

Above the symmetry-restoration scale N_{SUSY}, the one-loop coefficients jump from SM to MSSM values,

$$b_i^{\text{SM}} \longrightarrow b_i^{\text{MSSM}}, \quad (b_1, b_2, b_3)_{\text{MSSM}} = \big(+\tfrac{33}{5}, +1, -3\big).$$

Hence the volumes run as

$$V_i(N) = \begin{cases} V_i^{\text{SM}}(N), & N < N_{\text{SUSY}}, \\ V_i^{\text{SM}}(N_{\text{SUSY}}) - \dfrac{b_i^{\text{MSSM}}}{8\pi^2} \ln \dfrac{N}{N_{\text{SUSY}}}, & N \geq N_{\text{SUSY}}, \end{cases}$$

with the thresholds δ_i unchanged. This completes the geometric repair of the C_4 distortion and yields exact one-loop unification.

Appendix A.5. Piecewise RG

We evolve inverse gauge couplings in two regimes:

$$\frac{1}{\alpha_i(\mu)} = \begin{cases} \frac{1}{\alpha_i(M_Z)} - \dfrac{b_i^{\text{SM}}}{2\pi} \ln \dfrac{\mu}{M_Z}, & \mu < M_{\text{SUSY}}, \\ \Big[\frac{1}{\alpha_i(M_Z)} - \dfrac{b_i^{\text{SM}}}{2\pi} \ln N_{\text{SUSY}} + \delta_i\Big] - \dfrac{b_i^{\text{MSSM}}}{2\pi} \ln \dfrac{\mu}{M_{\text{SUSY}}}, & \mu \geq M_{\text{SUSY}}, \end{cases}$$

where $(b_1, b_2, b_3)_{\text{SM}} = (41/10, -19/6, -7)$, $(b_1, b_2, b_3)_{\text{MSSM}} = (33/5, 1, -3)$, $N_{\text{SUSY}} = 8.32$, and δ_i are the finite threshold shifts. The unification scale M_{GUT} is found by minimizing $\max_i[1/\alpha_i(\mu)] - \min_i[1/\alpha_i(\mu)]$, yielding the results in Table 1.

Appendix A.6. Hypercharge Embedding in SU(5)

In the fundamental **5** of SU(5), choose the simple coroots

$$h_i = E_{ii} - E_{i+1,i+1} = diag(0, \ldots, 0, 1, -1, 0, \ldots, 0), \quad i = 1, 2, 3, 4.$$

We seek coefficients a_i such that

$$Y = \sum_{i=1}^{4} a_i\, h_i = diag(y_1, \dots, y_5) = diag(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, +\tfrac{1}{2}, +\tfrac{1}{2}).$$

Matching diagonal entries gives

$$\begin{aligned} a_1 &= -\tfrac{1}{3}, \\ -a_1 + a_2 &= -\tfrac{1}{3}, \\ -a_2 + a_3 &= -\tfrac{1}{3}, \\ -a_3 + a_4 &= +\tfrac{1}{2}, \\ -a_4 &= +\tfrac{1}{2}, \end{aligned}$$

so

$$(a_1, a_2, a_3, a_4) = \left(-\tfrac{1}{3},\, -\tfrac{2}{3},\, -1,\, -\tfrac{1}{2}\right),$$

and indeed

$$Y = \sum_{i=1}^{4} a_i\, h_i = diag(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, +\tfrac{1}{2}, +\tfrac{1}{2}),$$

with $trY = 0$, so $Y \in \mathfrak{su}(5)$ and centralizes $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L$.

To reduce to the standard three-generator Cartan of the SM subgroup, define

$$H_1 = h_1, \quad H_2 = h_2, \quad H_3 = h_4.$$

Projecting Y onto this basis yields

$$Y = a_1 H_1 + a_2 H_2 + a_4 H_3 = -\tfrac{1}{6}(2H_1 + 4H_2 + 3H_3) \;\propto\; \tfrac{1}{3}(2H_1 + 4H_2 + 3H_3),$$

in agreement with the formula used in the main text [6,7].

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