Article

Not peer-reviewed version

A Note on the Bateman-Horn Conjecture

Huan Xiao *

Posted Date: 9 May 2025

doi: 10.20944/preprints202505.0651.v1

Keywords: Golomb's method; Bateman-Horn conjecture doi:10.20944/preprints202505.0651.v1

Article

A Note on the Bateman-Horn Conjecture

Huan Xiao

School of Artificial Intelligence, Zhuhai City Polytechnic, Zhuhai, China; xiaogo66@outlook.com

Abstract: The Bateman-Horn conjecture is a quantitative form of Schinzel's hypothesis (H) on prime values in polynomials. We investigate the Bateman-Horn conjecture is true by Golomb's method.

Keywords: Golomb's method; Bateman-Horn conjecture

---

1. Introduction

1.1. The Bateman-Horn Conjecture

Let p denote a prime number. In a paper with Sierpiński [12], Schinzel proposed the following conjecture, which is known as Schinzel's hypothesis (H).

Conjecture 1. *Let $k \geq 1$ and let $f_1(x), \cdots, f_k(x) \in \mathbb{Z}[x]$ be irreducible polynomials with positive leading coefficients. Assume that there does not exist any integer $n > 1$ dividing all the products $f_1(m) \cdots f_k(m)$ for every integer m. Then there are infinitely many natural numbers n such that $f_1(n), \cdots, f_k(n)$ are all primes.*

The Schinzel hypothesis (H) was recently studied on the average by Skorobogatov and Sofos [13] and they proved that the Schinzel hypothesis (H) is true for 100% of polynomials of arbitrary degree. For more results see their paper and references therein.

The Bateman-Horn conjecture is a quantitative form of Schinzel's conjecture. Let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be irreducible polynomials of degree $h_1, \cdots, h_k$ and with positive leading coefficients. We write $f = f_1 f_2 \cdots f_k$. Assume that there does not exist a prime number p that divides $f(n)$ for every positive integer n. Let

$$\pi_f(x) = \#\{n \leq x : f_1(n), \cdots, f_k(n) \text{ are all prime}\}.$$

The Bateman-Horn conjecture is the following.

Conjecture 2 ([3]). *Let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be as above, then as $x \to \infty$,*

$$\pi_f(x) \sim \frac{c(f)}{h_1 h_2 \cdots h_k} \cdot \frac{x}{\log^k x} \tag{1}$$

where

$$c(f) = \prod_p \frac{1 - N_f(p)/p}{(1 - 1/p)^k} \tag{2}$$

and $N_f(p)$ is the number of solutions of the congruence $f(n) \equiv 0(\bmod p)$.

Remark 1. *Let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be as in Conjecture 2, then Bateman and Horn proved in [3] that $c(f)$ converges and is positive.*

For an excellent survey and relevant historical literature on the Bateman-Horn conjecture we refer to the recent expository article [1].

It is clear the Bateman–Horn conjecture includes many special cases. For a single linear polynomial, it is Dirichlet's theorem on primes in arithmetic progressions, which in turn contains the prime number theorem ($f(x) = x$) as a special case. For $k \geq 2$ or for non-linear polynomials the conjecture is open.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 May 2025 doi:10.20944/preprints202505.0651.v1

The simplest case of non-linear polynomials is the case $f(x) = x(x+2)$ of the twin prime conjecture. The Bateman–Horn conjecture also includes the Hardy-Littlewood prime tuples conjecture [7]. Indeed Hardy and Littlewood [7] also proposed many other conjectures in their *Partitio Numerorum III*, many of which are special cases of the Bateman-Horn conjecture.

The conjectures of Hardy and Littlewood and the Bateman-Horn conjecture were all based on probabilistic heuristic arguments.

The purpose of this paper is to show that Schinzel's conjecture is true.

Let $\Lambda(n)$ be the von Mangoldt function and set

$$\psi_f(x) = \sum_{n\le x} \Lambda(f_1(n))\cdots\Lambda(f_k(n)). \tag{3}$$

By partial summation we have as in [2,4]

Proposition 3. *The Bateman-Horn conjecture 2 is equivalent to*

$$\psi_f(x) = c(f)x + o(x). \tag{4}$$

An important property of $\psi_f(x)$ is the following.

Proposition 4 ([4], Theorem 1)**.** *We have* $\psi_f(x) = O(x)$.

1.2. Main Result

We now state the main result in this paper.

Theorem 5. *The Bateman-Horn conjecture 2 is true.*

We will prove this in two ways, both of which are based on Golomb's method.

For the case of twin primes, that is for $f(x) = x(x+2)$, the Bateman-Horn conjecture predicts that

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x}\Lambda(n)\Lambda(n+2) = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

This implies

$$\liminf_{n\to\infty}(p_{n+1}-p_n) = 2$$

where p_n is the nth prime. The previous results on bounded prime gaps was made by Zhang [14] by using the sieve method, who proved that

$$\liminf_{n\to\infty}(p_{n+1}-p_n) < 7\cdot 10^7.$$

Later on Maynard [10], Tao and the Polymath Project [11] reduced the bound of Zhang to the following

$$\liminf_{n\to\infty}(p_{n+1}-p_n) \le 246.$$

2. Golomb's Method

For positive integers a, b by (a,b) we mean the greatest common divisor of a and b. Let $\mu(n)$ be the Möbius function and $\omega(n)$ the number of distinct prime divisors of n.

We investigate the Bateman-Horn conjecture by Golomb's method. There are mainly three papers on the Golomb method. The first is Golomb's thesis [5] (see also [6]), the second is Conrad [4] and the third is Hindry and Rivoal [9]. The papers [5] and [9] are by using the power series, while Conrad [4] is by using the Dirichlet series. Since one can turn a power series to a Dirichlet series and vice versa, the two approaches are equivalent. In this paper we shall use the power series.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 May 2025 doi:10.20944/preprints202505.0651.v1

The key of the Golomb method is the following identity of Golomb.

Lemma 1 ([5,6]). *Let $a_i > 1$ and $(a_i, a_j) = 1$ for $i \neq j$ and let $A = a_1 a_2 \cdots a_k$. Then we have*

$$\Lambda(a_1)\Lambda(a_2)\cdots\Lambda(a_k) = \frac{(-1)^k}{k!}\sum_{d|A}\mu(d)\log^k d. \tag{5}$$

Golomb used this identity to study the twin prime conjecture by a way analogous to Wiener's proof of the prime number theorem. That is, let $n > 2$ be even, then by (5) we have

$$2\Lambda(n-1)\Lambda(n+1) = \sum_{d|(n^2-1)}\mu(d)\log^2 d. \tag{6}$$

Let

$$G(z) = 2\sum_{n\geq 1}\Lambda(n-1)\Lambda(n+1)z^n,$$

then we have [5,6]

$$(1-z)G(z) = \sum_{\substack{d=1\\(d,2)=1}}^{\infty}\frac{(1-z)\mu(d)\log^2 d}{1-z^{2d}}\sum_{i=1}^{2^{\omega(d)}} z^{a_i} \tag{7}$$

where the a_i are the $2^{\omega(d)}$ even roots of the congruence $a^2 \equiv 1(\bmod d)$ between 0 and $2d$. If the termwise limit could be justified, then we would have

$$\lim_{z\to 1^-}(1-z)G(z) = \sum_{\substack{d=1\\(d,2)=1}}^{\infty}\frac{\mu(d)\log^2 d\cdot 2^{\omega(d)}}{2d} = 4\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right), \tag{8}$$

which would lead to a proof of the Bateman-Horn conjecture for the case of twin primes.

We now turn to the general Bateman-Horn conjecture. The identity (5) requires that $(a_i, a_j) = 1$ for $i \neq j$, for this Hindry and Rivoal [9] introduced the hypothesis F: for all integers $n \geq 1$, $(f_i(n), f_j(n)) = 1$ for $1 \leq i \neq j \leq k$. They proved the following result.

Lemma 2 ([9], Théorème 3). *Let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be as in Conjecture 2. If the Bateman-Horn conjecture 2 is true for all $f_1, \cdots, f_k$ that satisfies the hypothesis F, then it is true for all $f_1, \cdots, f_k \in \mathbb{Z}[x]$ that as in Conjecture 2.*

Note that Conrad [4] also addressed this coprime issue, see [4, Lemma 3, Theorem 4]. Henceforth in the following we let $f_1, \cdots, f_k \in \mathbb{Z}[x]$ be as in Conjecture 2 that satisfies the hypothesis F. Remember that $f = f_1 \cdots f_k$.

Now by (5) we have

$$\Lambda(f_1(n))\Lambda(f_2(n))\cdots\Lambda(f_k(n)) = \frac{(-1)^k}{k!}\sum_{d|f(n)}\mu(d)\log^k d. \tag{9}$$

Consider the absolutely convergent series for $|z| < 1$:

$$G_f(z) = (-1)^k k!\sum_{n\geq 1}\Lambda(f_1(n))\Lambda(f_2(n))\cdots\Lambda(f_k(n))z^n, \tag{10}$$

then as in Hindry and Rivoal [9] we have

$$G_f(z) = \sum_{d\geq 1}\frac{\mu(d)\log^k d}{1-z^d}\sum_{\substack{n=1\\d|f(n)}}^{d} z^n. \tag{11}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 May 2025 doi:10.20944/preprints202505.0651.v1

If the termwise limit could be justified, then we would have

$$\lim_{z\to 1^-}(1-z)G_f(z) = \sum_{d\geq 1}\mu(d)\log^k d\frac{N_f(d)}{d} \tag{12}$$

where $N_f(d)$ is the number of solutions of the congruence $f(n)\equiv 0(\mathrm{mod}\, d)$. Furthermore Conrad proved the following result, in a slightly different but equivalent form.

Proposition 6 ([4], Theorem 7)**.** *The right side of* (12) *converges and*

$$\sum_{d\geq 1}\mu(d)\log^k d\frac{N_f(d)}{d} = c(f) > 0$$

where $c(f)$ *is as in* (2)*.*

Thus if the termwise limit could be justified, the Bateman-Horn conjecture would be proved. The termwise limit can be done for the case $f(x)=x$ of the prime number theorem. Hindry and Rivoal [9] also proved Dirichlet's theorem on primes in arithmetic progressions by using Golomb's method and thus the termwise limit can be done for linear polynomials. However for nonlinear polynomials the termwise limit is elusive.

Therefore to prove the Bateman-Horn conjecture we will not consider potential applications of the termwise limit, instead we shall analyse $G_f(z)$ directly. The first way is to do series expansion of (11) and the second is to compute the limit (12) directly by definition of limit of functions.

3. First Proof of Theorem 5: Series Expansion

Recall the series we considered from (11):

$$G_f(z) = \sum_{d\geq 1}\frac{\mu(d)\log^k d}{1-z^d}\sum_{\substack{n=1\\ d|f(n)}}^{d} z^n = \sum_{d=1}^{\infty}\mu(d)\log^k d\sum_{\substack{n=1\\ d|f(n)}}^{d}\frac{z^n}{1-z^d}. \tag{13}$$

We expand $G_f(z)$ at $z=1$. We have

$$\sum_{\substack{n=1\\ d|f(n)}}^{d}\frac{z^n}{1-z^d} = \sum_{\substack{n=1\\ d|f(n)}}^{d}\Big(\frac{1}{d(1-z)} + \frac{d-1-2n}{2d} + (1-z)\frac{6n^2-6dn+d^2-1}{12d} \tag{14}$$

$$+ (1-z)^2 g_2(n,d) + (1-z)^3 g_3(n,d) + \cdots\Big) \tag{15}$$

where

$$g_1(n,d) := \frac{6n^2-6dn+d^2-1}{12d} \tag{16}$$

and $g_2(n,d), g_3(n,d),\cdots$ are all fractions in n and d which can be computed by *Mathematica*.

Thus in view of (13),

$$\begin{aligned} G_f(z) &= \frac{1}{1-z}\sum_{d=1}^{\infty}\mu(d)\log^k d\frac{N_f(d)}{d} + \sum_{d=1}^{\infty}\mu(d)\log^k d\sum_{\substack{n=1\\ d|f(n)}}^{d}\frac{d-1-2n}{2d} \\ &+ (1-z)\sum_{d=1}^{\infty}\mu(d)\log^k d\sum_{\substack{n=1\\ d|f(n)}}^{d} g_1(n,d) + (1-z)^2\sum_{d=1}^{\infty}\mu(d)\log^k d\sum_{\substack{n=1\\ d|f(n)}}^{d} g_2(n,d) + \cdots \end{aligned}$$

The following is key for our purpose.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 May 2025 doi:10.20944/preprints202505.0651.v1

Lemma 3. *The series*

$$\sum_{d=1}^{\infty} \mu(d) \log^k d \frac{N_f(d)}{d},$$

$$\sum_{d=1}^{\infty} \mu(d) \log^k d \sum_{\substack{n=1 \\ d|f(n)}}^{d} \frac{d-1-2n}{2d},$$

$$\sum_{d=1}^{\infty} \mu(d) \log^k d \sum_{\substack{n=1 \\ d|f(n)}}^{d} g_1(n,d),$$

$$\vdots$$

$$\vdots$$

are convergent.

Proof. This is because $G_f(z)$ is convergent for every $-1 < z < 1$. □

Thus immediately

Corollary 1. *We have*

$$\lim_{z\to 1^-} (1-z)G_f(z) = \sum_{d=1}^{\infty} \mu(d) \log^k d \frac{N_f(d)}{d}$$

and in view of Propositions 3 and 6 the Bateman-Horn conjecture is true.

4. Second Proof of Theorem 5: Limit Computation

To prove the Bateman-Horn conjecture we need to compute the limit

$$\lim_{z\to 1^-} (1-z)G_f(z) = \lim_{z\to 1^-} (1-z) \sum_{d=1}^{\infty} \mu(d) \log^k d \sum_{\substack{n=1 \\ d|f(n)}}^{d} \frac{z^n}{1-z^d} = \lim_{z\to 1^-} \sum_{d=1}^{\infty} \mu(d) \log^k d \sum_{\substack{n=1 \\ d|f(n)}}^{d} \frac{(1-z)z^n}{1-z^d}.$$

In this section we compute this limit directly by the definition of limit of functions. Let $z = 1-\varepsilon$, then

$$(1-z)G_f(z) = \sum_{d=1}^{\infty} \mu(d) \log^k d \sum_{\substack{n=1 \\ d|f(n)}}^{d} \frac{\varepsilon(1-\varepsilon)^n}{1-(1-\varepsilon)^d}. \tag{17}$$

We need to show as $\varepsilon \to 0$,

$$\sum_{d=1}^{\infty} \mu(d) \log^k d \sum_{\substack{n=1 \\ d|f(n)}}^{d} \frac{\varepsilon(1-\varepsilon)^n}{1-(1-\varepsilon)^d} \longrightarrow \sum_{d=1}^{\infty} \mu(d) \log^k d \frac{N_f(d)}{d}. \tag{18}$$

We expand the inner sum at $\varepsilon = 0$, then

$$\sum_{\substack{n=1 \\ d|f(n)}}^{d} \frac{\varepsilon(1-\varepsilon)^n}{1-(1-\varepsilon)^d} = \sum_{\substack{n=1 \\ d|f(n)}}^{d} \left(\frac{1}{d} + \varepsilon \frac{d-1-2n}{2d} + \varepsilon^2 g_1(n,d) + \varepsilon^3 g_2(n,d) + \varepsilon^4 g_3(n,d) + \cdots \right) \tag{19}$$

where $g_1(n,d), g_2(n,d), g_3(n,d), \cdots$ are same as those in (14). Thus

$$\begin{aligned}\sum_{d=1}^{\infty}\mu(d)\log^k d \sum_{\substack{n=1\\ d|f(n)}}^{d}\frac{\varepsilon(1-\varepsilon)^n}{1-(1-\varepsilon)^d} &= \sum_{d=1}^{\infty}\mu(d)\log^k d\frac{N_f(d)}{d} + \varepsilon\sum_{d=1}^{\infty}\mu(d)\log^k d \sum_{\substack{n=1\\ d|f(n)}}^{d}\frac{d-1-2n}{2d}\\ &+ \varepsilon^2\sum_{d=1}^{\infty}\mu(d)\log^k d \sum_{\substack{n=1\\ d|f(n)}}^{d} g_1(n,d)\\ &+ \varepsilon^3\sum_{d=1}^{\infty}\mu(d)\log^k d \sum_{\substack{n=1\\ d|f(n)}}^{d} g_2(n,d) + \cdots\end{aligned}$$

By Lemma 3 we have (18) as $\varepsilon \to 0$.

Acknowledgments: This work was started during my stay at Nagoya University. I would like to thank Professor Keith Conrad for sending me a copy of the paper [?], which was (and is) not available in the internet, also for his correspondence and encouragement. Special thanks to the staff of the library of Department of Science of Nagoya University, who kindly allowed me to use this library.

References

1. Aletheia-Zomlefer, S. L., Fukshansky, L and Garcia, S. R. The Bateman-Horn Conjecture: Heuristics, History, and Applications, Expositiones Mathematicae, **38**(2020) 430-479.
2. Baier, S., On the Bateman-Horn conjecture, J. number theory, **96**, 432-448, 2002.
3. Bateman, P. T., Horn, R. A., A heuristic asymptotic formula concerning the distribution of prime numbers, Mathematics of Computation, **16**: 363-367, 1962.
4. Conrad, K., Hardy-Littlewood constants, Mathematical properties of sequences and other combinatorial structures (Los Angeles, CA, 2002), 133-154, Kluwer Acad. Publ., Boston, MA, 2003.
5. Golomb. S.W. Problems in the distribution of the prime numbers, PhD Thesis, Harvard University, 1956.
6. Golomb. S.W. The Lambda Method in Prime Number Theory. Journal of number theory, **2**(1970), 193-198.
7. Hardy, G.H; Littlewood, J.E. Some problems in "Partitio Numerorum", III: On the expression of a number as a sum of primes. Acta Math. **44** (1923), 1-70.
8. Halberstam, H and Richert, H.E. Sieve Methods. London Mathematical Society Monographs. Vol. 4. London-New York: Academic Press. 1974.
9. Hindry, M and Rivoal, T. Le Λ-calcul de Golomb et la conjecture de Bateman-Horn. Enseign. Math. (2) **51** (2005), 265-318.
10. Maynard, J, Small gaps between primes, Annals of Mathematics, **181** (1): 383-413, 2015.
11. Polymath, D. H. J., Variants of the Selberg sieve, and bounded intervals containing many primes, Research in the Mathematical Sciences, **1**: Art. 12, 83, 2014
12. Schinzel, A.; Sierpiński, W. Sur certaines hypothèses concernant les nombres premiers. Acta Arithmetica. **4**(3): 185-208, 1958.
13. Skorobogatov, A. N and Sofos, E. Schinzel Hypothesis on average and rational points, Inventiones math., **231**(2): 673-739, 2023.
14. Zhang, Y. Bounded gaps between primes. Annals of Mathematics. **179**(3): 1121-1174, 2014.