Article

Not peer-reviewed version

On the Possibility of the Majorana Nature of Neutrino

Vadim Monakhov *

Posted Date: 8 May 2025

doi: 10.20944/preprints202505.0486.v1

Keywords: neutrino mass; Majorana neutrino; Majorana mass; Majorana spinor; Majorana fermion; charge conjugation; CP conjugation; CPT conjugation; CP symmetry; CPT symmetry

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 May 2025 doi:10.20944/preprints202505.0486.v1

Article

On the Possibility of the Majorana Nature of Neutrino

Vadim Monakhov

Saint Petersburg State University, Saint-Petersburg 198504, Russia; v.v.monahov@spbu.ru

Abstract: The smallness of the neutrino mass is usually explained by the "seesaw" mechanism, in which the neutrino must be a Majorana fermion that coincides with its charge conjugate. Six nonequivalent concepts of charge conjugation are discussed in the literature, which correspond to six possible types of Majorana and Majorana-like fermions. These are charge conjugation according to Pauli; according to Majorana and Kramers; according to Schwinger; QFT conjugation according to Kramers; CP conjugation; CPTS conjugation according to Kayser and Goldhaber. We have proved that there can be only two variants of charge conjugation that would potentially allow the neutrino to be a Majorana-like fermion: unitary CP-conjugation and anti-unitary CPTS-conjugation proposed by Kayser and Goldhaber. We have obtained explicit formulas for these operators.

Keywords: neutrino mass; Majorana neutrino; Majorana mass; Majorana spinor; Majorana fermion; charge conjugation; CP conjugation; CPT conjugation; CP symmetry; CPT symmetry

---

1. Introduction

One of the most popular explanations for the smallness of the neutrino mass is the "seesaw" mechanism [1–3]. With this mechanism, the neutrino must be a Majorana spinor (fermion). A Majorana spinor is usually understood as a particle with spin 1/2, which coincides with the charge conjugate when using Majorana and Kramers conjugation [4,5]. However, there are several nonequivalent definitions of the charge conjugation operator C in the literature. Each such definition of the operator C must correspond to its own type of Majorana-like spinors, which are also called Majorana, since most authors ignore this non-equivalence. Although the properties of such conjugation operators and spinors self-adjoint with respect to them are significantly different.

We will call Majorana spinors solutions of the Dirac equation self-adjoint with respect to charge conjugation according to Majorana and Kramers. The remaining types of charge-self-adjoint spinors will be called Majorana-like.

The idea of charge conjugation in the framework of the theory of "holes" in the Dirac Sea for solutions of the Dirac equation

$$\gamma^{\mu}(i\partial_{\mu} - qA_{\mu})\psi = m\psi\,, \tag{1}$$

was proposed by Pauli [6], although he did not call it charge conjugation. In (1) γ^{μ} are Dirac matrices, $\mu = 0,1,2,3$; q is the electric charge of the particle; A_{μ} is the vector potential of the electromagnetic field. We will use the metric $g^{\mu\nu} = diag(1,-1,-1,-1)$, with $(\gamma^0)^2 = 1$.

The idea of a Majorana spinor as a truly neutral particle, coinciding with its antiparticle, was proposed by Majorana [4], who used a representation of the Dirac gamma matrices in which they are purely imaginary. In this case, charge conjugation is equivalent to complex conjugation. Kramers [5] introduced the term "charge conjugation" for solutions of the Dirac equation, generalizing a version of the anti-unitary (containing complex conjugation) Majorana charge conjugation to other representations of the Dirac gamma matrices. We will call the corresponding anti-unitary operator

$$C_{\text{M-K}} = \eta_c \mathrm{C}(\gamma^0)^*(.)^* \tag{2}$$

 doi:10.20944/preprints202505.0486.v1

the Majorana-Kramers charge conjugation operator, where $\eta_c = e^{i\varphi}$ is an arbitrary phase factor, C is the so-called charged conjugation matrix, $*$ is the complex conjugation symbol, and $(.)^*$ is the complex conjugation operator.

The wave function of the spinor which is charge conjugate to ψ is

$$C_{\text{M-K}}\psi = \eta_c \mathrm{C}(\gamma^0)^* \psi^* \,. \tag{3}$$

We can rewrite formula (3) as

$$C_{\text{M-K}}\psi = \eta_c \mathrm{C}\bar{\psi}^T = \eta_c \mathrm{C}(\gamma^0)^* \psi^{+T}, \tag{4}$$

where $\bar{\psi} = (\gamma^0\psi)^+ = \psi^+(\gamma^0)^+$ is Dirac conjugate spinor, and ^{T} denotes a transposition.

Taking into account formula (4), we can rewrite formula (2) as

$$C_{\text{M-K}} = \eta_c \mathrm{C}\,\overline{(.)}^{\,T} = \eta_c \mathrm{C}(\gamma^0)^*(.)^+(.)^T \tag{5}$$

In this case $(.)^T(.)^+ = (.)^*$, and $(C_{\text{M-K}})^2 = 1$.

The most general definition of a Majorana (Majorana-like) spinor ψ_{M} is as self-adjoint spinor (up to a phase factor η_c) with respect to the charge conjugation operator C

$$C\psi_{\text{M}}(x) = \eta_c \psi_{\text{M}}(x) \,. \tag{6}$$

If the operator C is unitary, then we can redefine it so that the conditions $\eta_c = 1$ and $C^2 = 1$ are satisfied.

Two Majorana or Majorana-like spinors ψ_{M1} and ψ_{M2} can be constructed from the Dirac spinor ψ under the condition $C^2 = 1$ as follows:

$$\begin{aligned} \psi_{\text{M1}} &= C\psi_{\text{M1}} = \frac{\psi + C\psi}{\sqrt{2}}\,, \\ \psi_{\text{M2}} &= C\psi_{\text{M2}} = \frac{-i(\psi - C\psi)}{\sqrt{2}}\,. \end{aligned} \tag{7}$$

The Dirac spinor ψ can be represented as a superposition of two independent solutions ψ_{M1} and ψ_{M2}

$$\psi = \frac{1}{\sqrt{2}}(\psi_{\text{M1}} + i\psi_{\text{M2}}) \,. \tag{8}$$

If the charge conjugation operator $C = C_{\text{M-K}}$ is used in (7), then ψ_{M1} and ψ_{M2} correspond to Majorana spinors. If C is another charge conjugation operator, ψ_{M1} and ψ_{M2} correspond to Majorana-like spinors.

Kramers [5] introduced a unitary charge conjugation operator in the framework of quantum field theory (QFT), which in the spinor field operators exchanges the particle creation (annihilation) operators and the antiparticle creation (annihilation) operators. We will denote it as C_{QFT}.

Schwinger [7] introduced in the framework of QFT the anti-unitary charge conjugation operator C_{Schw}, the formula for its action on ψ looks the same as formula (4). However, unlike $C_{\text{M-K}}$, ^{T} in it denotes matrix transposition. Here $(.)^T(.)^+ = (.)^T_{QFT}(.)^*$, where $(.)^T_{QFT}$ is a transposition operator of spinor creation and annihilation operators, which does not affect the matrix. As a result, C_{Schw} transforms QFT operators of the spinor field correctly, unlike the operator $C_{\text{M-K}}$. In the literature, $C_{\text{M-K}}$ and C_{Schw} are usually mistakenly not distinguished, and for justification of the alleged unitarity of these anti-unitary operators they are identified with the unitary operator C_{QFT} [8–10]. In this paper, we eliminate this confusion.

Nieves [11] and Kayser and Goldhaber [12] suggested that under the conditions of C and P symmetries violation, the Majorana neutrino is an eigenstate of the operator CP.

 doi:10.20944/preprints202505.0486.v1

Kayser [13] and Nieves [11] noted that under C, P, and CP symmetries violations, a neutrino can be a true neutral (Majorana) spinor only if the CPT operator transforms a neutrino with a spin projection s into an identical neutrino with a spin projection $-s$. After which Kayser and Goldhaber [12] and then Mohapatra and Pal [14] suggested that the neutrino "is an eigenstate of the CPT operator", meaning by it a $CPTS$ conjugation, with an additional operator S that changes the sign of the spin projection at rest.

Thus, six nonequivalent concepts of charge conjugation are described in the literature, and they correspond to six possible types of Majorana and Majorana-like spinors. In the literature, different versions of charge conjugation are often confused. In particular, many authors consider anti-unitary operators to be unitary because the results of their action on the field operator $\psi(x)$ for non-chiral fields coincide, or confuse different types of conjugation operators because their formulas, due to insufficiently correct notations, at first glance coincide. In this paper, we analyze the properties of known versions of charge conjugation operators and the possibility that the neutrino is a Majorana-like fermion of the corresponding type.

2. Charge Conjugation According to Majorana-Kramers and According to Pauli

Majorana [4] proposed to use a representation of gamma matrices in which they are imaginary. Such representations are now called Majorana representations. In the Majorana representation, if a solution $\psi_{\rm M1}$ of the Dirac equation (1) with $q = 1$ is real at one time, it will remain real at all times. The same applies to the imaginary solution $i\psi_{\rm M2}$. Therefore, the solution ψ of the Dirac equation can always be represented as a superposition of two independent solutions $\psi_{\rm M1}$ and $\psi_{\rm M2}$, where $\psi_{\rm M1}$ and $\psi_{\rm M2}$ are real. Moreover, since in the Majorana representation all Lorentz transformation operators (rotations and boosts) are real, the Lorentz transformations preserve the realness (imaginaryness) of these solutions.

The charge conjugation operator $C_{\rm M}$ in the Majorana representation is the complex conjugation operator [4]

$$C_{\rm M} = (.)^* \,, \tag{9}$$

$$\psi^c = \psi^* = C_{\rm M}\psi \,. \tag{10}$$

The solutions $\psi_{\rm M1}$ and $\psi_{\rm M2}$ are self-adjoint with respect to Majorana charge conjugation

$$\psi^c_{\rm M1} = \psi^*_{\rm M1} = \psi_{\rm M1} \,, \tag{11}$$

$$\psi^c_{\rm M2} = \psi^*_{\rm M2} = \psi_{\rm M2} \,. \tag{12}$$

Majorana believed that these solutions describe a truly neutral particle with spin 1/2, coinciding with its antiparticle, for which there is no need to use the theory of "holes" in the Dirac Sea. Later, such solutions were called Majorana spinors.

Kramers [5] generalized the concept of Majorana charge conjugation to other representations of gamma matrices. Therefore, we will call this type of conjugation Majorana-Kramers charge conjugation. In the Dirac representation and the Weyl (chiral) representation, the following formulas for Majorana-Kramers charge conjugation are valid

$$C_{\text{M-K}} = \eta_c i\gamma^2(.)^* = \eta_c(.)^c \,, \tag{13}$$

$$\psi^c = (.)^c\psi = i\gamma^2\psi^* \,, \tag{14}$$

$$C_{\text{M-K}}\psi = \eta_c\psi^c = \eta_c i\gamma^2\psi^* \,. \tag{15}$$

In (13)-(15) we included an additional phase factor i, ensuring that the factor $i\gamma^2$ is real. At the same time, in (2)-(5), (13) and (15) we factored out the phase factor η_c separately, without including it in the formula for the charge conjugation matrix, so that formula (14) would correspond to the most widely used (up to a sign) version of the Majorana-Kramers charge conjugation formula in the literature.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 May 2025 doi:10.20944/preprints202505.0486.v1

Despite the fact that the Dirac spinor is decomposed into the sum (8) of two Majorana spinors, which are solutions of the Dirac equation, such solutions cannot correspond to physical particles. We have already noted previously the insoluble problems in such attempts [15,16]. We will show that in quantum field theory $C_{\text{M-K}}$ is not a charge conjugation operator at all.

The Dirac spinor field operator is given by the expression [5,8–10]

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E(p)}} (b_\alpha(p) u_\alpha(p) e^{-ip_\mu x^\mu} + d_\alpha^+(p) v_\alpha(p) e^{ip_\mu x^\mu}), \tag{16}$$

where $u_\alpha(p)$ and $v_\alpha(p)$ are four basic 4-spinor columns, $\alpha = 1, 2$; $b_\alpha(p)$ is annihilation operator of the particle with momentum p and spin projection at rest $s = +1/2$ for $\alpha = 1$ and $s = -1/2$ for $\alpha = 2$; $d_\alpha^+(p)$ is corresponding antiparticle creation operator.

In the Dirac representation for the case $p = 0$ it is convenient to choose

$$u_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; u_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; v_1(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \; v_2(0) = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{17}$$

The values of $u_\alpha(p)$ and $v_\alpha(p)$ for a non-zero spatial momentum p are obtained from $u_\alpha(0)$ and $v_\alpha(0)$ by the active Lorentz transformation $exp(\gamma^0\gamma^k\omega_{0k}/2)$, $k = 1,2,3$, where ω_{0k} are the angles corresponding to the boost that yields momentum p [17]

$$u_\alpha(p) = e^{\gamma^0\gamma^k\omega_{0k}/2} u_\alpha(0), \; v_\alpha(p) = e^{\gamma^0\gamma^k\omega_{0k}/2} v_\alpha(0). \tag{18}$$

For $u_\alpha(p)$ and $v_\alpha(p)$ the relations

$$\begin{aligned} (.)^c u_\alpha(p) = u_\alpha^c(p) = v_\alpha(p), \\ (.)^c v_\alpha(p) = v_\alpha^c(p) = u_\alpha(p) \end{aligned} \tag{19}$$

are satisfied [9,10,18].

The signs and numbering of $u_\alpha(0)$ and $v_\alpha(0)$ in (17) are chosen so that the formulas (19) are fulfilled.

The field operator $\psi^a(x)$ of the antiparticle is given up to a phase factor by the expression in which in the field operator (16) the annihilation operators $b_\alpha(p)$ are replaced by $d_\alpha(p)$, and the creation operators $d_\alpha^+(p)$ by $b_\alpha^+(p)$ [5,8–10]:

$$\psi^a(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E(p)}} (d_\alpha(p) u_\alpha(p) e^{-ip_\mu x^\mu} + b_\alpha^+(p) v_\alpha(p) e^{ip_\mu x^\mu}). \tag{20}$$

The action of $C_{\text{M-K}}$ on the operators $b_\alpha(p)$ and $d_\alpha(p)$ is reduced to their complex conjugation. But due to the arbitrariness in the choice of the complex phase, the creation and annihilation operators can always be specified as real [10]. Therefore, we can assume that $b_\alpha^*(p) = b_\alpha(p)$ and $d_\alpha^{+*}(p) = d_\alpha^+(p)$, due to which we can write

$$\begin{aligned} C_{\text{M-K}}\psi(x) &= \frac{\eta_c}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E(p)}} (b_\alpha^*(p) u_\alpha^c(p) e^{ip_\mu x^\mu} + d_\alpha^{+*}(p) v_\alpha^c(p) e^{-ip_\mu x^\mu}) = \\ &= \frac{\eta_c}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E(p)}} (d_\alpha^{+*}(p) u_\alpha(p) e^{-ip_\mu x^\mu} + b_\alpha^*(p) v_\alpha(p) e^{ip_\mu x^\mu}) = \\ &= \frac{\eta_c}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E(p)}} (d_\alpha^+(p) u_\alpha(p) e^{-ip_\mu x^\mu} + b_\alpha(p) v_\alpha(p) e^{ip_\mu x^\mu}). \end{aligned} \tag{21}$$

The result (21) of such conjugation is obviously not given by the field operator of the antiparticle (20). In the field operator (21) there is the operator $d_\alpha^+(p)$ of creation of the antiparticle in the positive-frequency term instead of the operator $d_\alpha(p)$ of annihilation of the antiparticle in (20), and the operator $b_\alpha(p)$ of annihilation of the particle in the negative-frequency term instead of the operator $b^+_\alpha(p)$ of creation of the particle in (20).

Consequently, the condition that the charge conjugation operator should exchange a particle to an antiparticle is not satisfied for $C_{\text{M-K}}$.

Moreover, it does not satisfy the unitarity condition required for the charge conjugation operator [8,9,18], since $C_{\text{M-K}}$ (13) is obviously anti-unitary.

In addition, the charge conjugation operator must preserve the spin projection [9,18], whereas $C_{\text{M-K}}$ reverses it.

Another reason why $C_{\text{M-K}}$ cannot play the role of the charge conjugation operator C of the CPT theorem is that it maps the left-chiral field of the particle ψ_L to the right-chiral field of the antiparticle $(\psi^c)_R$. This issue is discussed in more detail in the next section.

Thus, despite the fact that the Dirac spinor is a superposition of two Majorana spinors, the Majorana-Kramers conjugation operator does not satisfy the requirements for a charge conjugation operator. Earlier we found a number of other arguments proving that Majorana spinors cannot be physical particles [15,16].

The Pauli charge conjugation formulas coincide with the formulas (2)-(5) and (9)-(15) of the Majorana-Kramers charge conjugation, but are considered in the framework of the non-QFT theory of "holes" in the Dirac Sea. Therefore, the result of charge conjugation obtained using these formulas is intermediate and gives the wave function of a particle in the Dirac Sea with negative mass and energy. A "hole" in this sea has opposite values of mass, energy, momentum, spin projection, and electric charge and is treated as an antiparticle [17,19,20]. Pauli charge conjugation solves some of the problems inherent in Majorana-Kramers conjugation, but is a non-QFT analogue of Schwinger conjugation. Therefore, at present it is of historical interest only, and further we will consider Schwinger charge conjugation.

3. Charge Conjugation According to Schwinger

The formula for the Schwinger charge conjugation operator [7]

$$C_{\text{Schw}} = \eta_c \mathrm{C}\,\overline{(.)}^{\,T} = \eta_c \mathrm{C}(\gamma^0)^*(.)^+(.)^T \tag{22}$$

outwardly coincides with (5), which often leads to the fact that in the literature the Schwinger charge conjugation and the Majorana-Kramers charge conjugation are not distinguished. But in the Schwinger charge conjugation (22) the transpose operator $(.)^T = (.)^T_{matr}$ is a matrix operator, and in (5) it acts on both the matrices and the creation and annihilation operators.

Below we will consider gamma matrices in the Dirac and Weyl representations, since many formulas are simpler and more intuitive in them. In these representations, the formula for C_{Schw} looks like

$$C_{\text{Schw}} = \eta_c i\gamma^2 (.)^+\,(.)^T_{matr} = \eta_c i\gamma^2 (.)^*\,(.)^T_{QFT} = C_{\text{M-K}}\,(.)^T_{QFT}\,, \tag{23}$$

where $(.)^T_{QFT}$ is the transposition operator acting on the creation and annihilation operators, but does not affect the matrices.

Under Schwinger charge conjugation, the field operator (16) of a particle is transformed into the expression

$$\begin{aligned} C_{\text{Schw}}\psi(x) &= \frac{\eta_c}{(2\pi)^{3/2}}\int d^3p\sqrt{\frac{m}{E(p)}}(b^+_\alpha(p)u^c_\alpha(p)e^{ip_\mu x^\mu} + d_\alpha(p)v^c_\alpha(p)e^{-ip_\mu x^\mu}) = \\ &= \frac{\eta_c}{(2\pi)^{3/2}}\int d^3p\sqrt{\frac{m}{E(p)}}(d_\alpha(p)u_\alpha(p)e^{-ip_\mu x^\mu} + b^+_\alpha(p)v_\alpha(p)e^{ip_\mu x^\mu}) = \eta_c\psi^a(x)\,, \end{aligned} \tag{24}$$

up to the phase factor η_c coinciding with the operator (20) of the antiparticle field $\psi^a(x)$.

The result (24) of such a conjugation corresponds to the fact that in the formula (16) for the non-chiral field operator the particle annihilation operators $b_\alpha(p)$ are replaced by the antiparticle annihilation operators $\eta_c d_\alpha(p)$, and the particle creation operators $d^+_\alpha(p)$ are replaced by the antiparticle creation operators $\eta_c b^+_\alpha(p)$ [5,8–10]. As we will see below, this is not the case for chiral operators. This is why $C_{\rm Schw}$ cannot be identified with the operator C_{QFT}, which performs such a replacement.

From (16) and (24) on the basis of (7) under the condition $C = C_{\rm Schw}$ one can construct Majorana-like spinors

$$\psi_{\rm M1}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E(p)}} (c_{1\alpha}(p) u_\alpha(p) e^{-ip_\mu x^\mu} + \eta_c c^+_{1\alpha}(p) v_\alpha(p) e^{ip_\mu x^\mu}),$$
$$\psi_{\rm M2}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E(p)}} (c_{2\alpha}(p) u_\alpha(p) e^{-ip_\mu x^\mu} + \eta_c c^+_{2\alpha}(p) v_\alpha(p) e^{ip_\mu x^\mu}), \tag{25}$$

where $c_{1\alpha}(p) = \frac{b_\alpha(p)+\eta_c d_\alpha(p)}{\sqrt{2}}$, $c_{2\alpha}(p) = \frac{-i(b_\alpha(p)-\eta_c d_\alpha(p))}{\sqrt{2}}$.

In this case, the Dirac spinor turns out to be a composition of two Majorana-like ones, and in the case of one independent Majorana-like spinor, one should abandon the extra degree of freedom and set $c_\alpha(p) = b_\alpha(p) = \eta_c d_\alpha(p)$.

Let us consider the standard scheme of using Schwinger charge conjugation to construct Majorana-like neutrinos within the framework of the "seesaw" mechanism [2,3,14,18,21,22]. The Majorana-like spinor $\psi_{\rm M}$ is constructed from the left-chiral state $\psi_L(p) = \frac{1-\gamma^5}{2}\psi$ using the operator C_{Schw}

$$\psi_{\rm M} = \psi_L + C_{Schw}\, \psi_L\,. \tag{26}$$

Since C_{Schw} in (23) contains a factor γ^2 that anticommutes with γ^5, the term $C_{Schw}\,\psi_L$ is right-chiral, and, as previously believed, it is possible to form the field operator of a massive Majorana-like spinor.

There are three problems in the approach described above that have not been mentioned before in the literature, two of which are related to the CPT theorem and the third to the properties of the vacuum and the creation and annihilation operators.

The first problem is the anti-unitary nature of the operator C_{Schw}.

CPT operator must be anti-unitary [10], as proved by Jost [23,24] in the framework of axiomatic algebraic quantum field theory.

The unitarity of the parity (space inversion) operator P is well known [8–10,17,19–21,25,26].

Wigner [27] proved that the time inversion operator T must be anti-unitary in order for the inversion to preserve the positive energy (positive frequency) of the solutions of the Schrodinger equation. Wigner's arguments are also applicable to QFT, with the correction that in the field operator under time inversion, positive-frequency solutions must preserve positive frequency, and negative-frequency solutions must preserve negative frequency. Based on these and other considerations, Schwinger [28,29] proved the anti-unitarity of the time inversion operator in the framework of QFT.

Therefore, for the operator CPT to be anti-unitary, the charge conjugation operator C must be unitary. However, C_{Schw} (23) is obviously anti-unitary operator.

The anti-unitarity of the Majorana-Kramers charge conjugation operator, which differs from C_{Schw} only by an unitary factor $(.)^T_{QFT}$, was noted by Messiah [19], Schweber [20] and other authors. Itzikson and Zuber [25] also pointed out the anti-linearity (anti-unitarity) of this operator, but they interpreted it within the framework of the theory of "holes" in the Dirac Sea [6], that is, they considered it as an anti-unitary Pauli charge conjugation operator. The Pauli conjugation operator is a non-QFT analogue of the Schwinger conjugation operator, so the problem described applies to it as well.

Thus, the Schwinger charge conjugation operator C_{Schw}, due to its anti-unitarity, cannot be the charge conjugation operator C of the CPT theorem and cannot be used to construct a Majorana or Majorana-like neutrino.

The second problem is that C_{Schw} changes the chirality of the spinor field operator.

The formula for the spatial inversion (parity) operator P [8–10] is

$$P = \eta_P R(-x^k)\gamma^0 \,, \tag{27}$$

where η_P is the phase factor, and $R(-x^k)$, $k = 1,2,3$ is the operator changing the coordinates x^k to $-x^k$ and the components of the potential A_k in (1) to $-A_k$.

P changes the chirality of the field, since γ^0 anticommutes with γ^5: $P\psi_L = \psi'_R$.

The formula for the time inversion operator T [8–10] is

$$T = \eta_T R(-x^0)\gamma^1\gamma^3(.)^* \,, \tag{28}$$

where η_T is the phase factor, and $R(-x^0)$ is the operator changing x^0 to $-x^0$ and A_k to $-A_k$.

T preserves the chirality of the field, since $\gamma^1\gamma^3$ commutes with γ^5: $T\psi_L = \psi'_L$.

C_{Schw} changes the chirality of the field, since $C_{\text{Schw}} = \eta_c i\gamma^2(.)^*\,(.)^T_{QFT}$ and γ^2 anticommutes with γ^5: $C_{\text{Schw}}\psi_L = \eta_c(\psi^a)_R$.

Therefore, $C_{\text{Schw}}PT$ preserves the chirality of the field: $C_{\text{Schw}}PT\psi_L = \eta_c\eta_P^*\eta_T^*(\psi^a)_L$. Hence, $C_{\text{Schw}}PT$ transforms a left-chiral neutrino participating in the electroweak interaction into a left-chiral antineutrino, which should participate in the electroweak interaction. However, this contradicts the fact that the CPT transformation, as is well known both from experiment and from the Standard Model, changes the chirality of the neutrino participating in the electroweak interaction to the opposite: the CPT operator turns the left-chiral neutrino ψ_L into the right-chiral antineutrino $(\psi^a)_R$.

Similar arguments apply to CP symmetry, which is preserved within one or two generations of fermions [21]. CP conjugation of the left-chiral neutrino field operator should lead to obtaining the field operator of a right-chiral antineutrino. But $C_{\text{Schw}}P$ preserves the chirality, so when it acts on the field operator of a left-chiral neutrino, we obtain the field operator of a left-chiral antineutrino. This violates CP symmetry.

Therefore, an attempt to use C_{Schw} as a charge conjugation operator C contradicts both experimentally established facts and the Standard Model.

Therefore, since C_{Schw} changes the chirality of the spinor, it cannot be the charge conjugation operator C of the CPT theorem.

The third problem is that C_{Schw} does not map the particle state vector to the antiparticle state vector as it should, and its action on the vacuum state does not keep the vacuum invariant.

Let us have a state vector $b^+_\alpha(p)\psi_{\text{V}}$, where ψ_{V} is the vacuum state vector. According to (23) $C_{Schw}b^+_\alpha(p)(C_{Schw})^{-1}$ takes $b^+_\alpha(p)$ to $b_\alpha(p)$, so

$$C_{Schw}b^+_\alpha(p)\psi_{\text{V}} = C_{Schw}b^+_\alpha(p)(C_{Schw})^{-1}C_{Schw}\psi_{\text{V}} = b_\alpha(p)C_{Schw}\psi_{\text{V}} \,. \tag{29}$$

It is usually believed that the vacuum is invariant under the action of the charge conjugation operator. However, if we assume that $C_{Schw}\psi_{\text{V}} = \psi_{\text{V}}$, and given that $b_\alpha(p)\psi_{\text{V}} = 0$, (29) can be rewritten as

$$C_{Schw}b^+_\alpha(p)\psi_{\text{V}} = b_\alpha(p)\psi_{\text{V}} = 0 \,. \tag{30}$$

In this case, multiplying both sides of (30) by C_{Schw} and taking into account that $(C_{Schw})^2 = 1$, we obtain

$$(C_{Schw})^2 b^+_\alpha(p)\psi_{\text{V}} = b^+_\alpha(p)\psi_{\text{V}} = 0 \,. \tag{31}$$

That is, we obtain a contradiction that the initial state vector $b^+_\alpha(p)\psi_{\text{V}}$ is identically equal to zero. Therefore, the assumption that the vacuum ψ_{V} is invariant under the action of the operator C_{Schw} is incorrect, and C_{Schw} maps the vacuum ψ_{V} to the alternative $C_{Schw}\psi_{\text{V}} = \psi_{\text{altV}}$, for which the annihilation operator $b_\alpha(p)$ plays the role of the creation operator. The presence of an alternative

 doi:10.20944/preprints202505.0486.v1

vacuum for spinors is not unusual. We have proven the existence of such alternative vacua in the theory of algebraic spinors [30,31].

As a result, C_{Schw} does not map the state vector of a particle to the state vector of an antiparticle, as it should be, but maps the state vector to a state with an alternative vacuum. In particular, it does not keep the vacuum invariant, but maps it to an alternative vacuum, which cannot correspond to the vacuum of our Universe.

Since the formula (22) contains the Hermitian conjugation of the creation and annihilation operators, one can assume the possibility that when C_{Schw} acts on the state vector, the product $b^+_\alpha(p)\psi_V$ is transformed into $\psi^+_V(b^+_\alpha(p))^+ = \psi_V b_\alpha(p)$. But even in this case, we obtain that the result corresponds to an alternative vacuum, in which $b_\alpha(p)$ acts on the vacuum from right to left and acts as a creation operator.

Therefore, the Schwinger charge conjugation operator acts on the state vectors in an unphysical way and cannot be used to construct a Majorana-like neutrino. The same applies to the Pauli charge conjugation operator.

Thus, there are three irrefutable reasons why the Schwinger and Pauli charge conjugation operators cannot be used to construct Majorana-like neutrinos.

4. Unitary Charge Conjugation Operator $C_{\mathbf{QFT}}$

Kramers in [5] introduced a charge conjugation operator that changes the particle annihilation operators $b_\alpha(p)$ to the antiparticle annihilation operators $d_\alpha(p)$ and the antiparticle creation operators $d^+_\alpha(p)$ to the particle creation operators $b^+_\alpha(p)$. This operator acts only on the creation and annihilation operators and commutes with the Dirac matrices. We use a generalization of this transformation in an application to the construction of Majorana-like spinors, taking into account possible phase factors, and denote this operator as C_{QFT}.

C_{QFT} is unitary. It is often denoted as $\mathcal{C}$, and its action on the spinor ψ is written as $\mathcal{C}\psi\mathcal{C}^{-1}$ [8–10,21,32–34].

C_{QFT} transforms the annihilation operators as

$$\begin{aligned} b_i(p) &\to \eta_b d_i(p), \\ d_i(p) &\to \eta_d b_i(p), \end{aligned} \tag{32}$$

where η_b and η_d are phase factors.

By Hermitian conjugation of (32), we obtain the transformation of the creation operators

$$\begin{aligned} b^+_i(p) &\to \eta^*_b d^+_i(p), \\ d^+_i(p) &\to \eta^*_d b^+_i(p). \end{aligned} \tag{33}$$

Under the transformations (32)-(33), the canonical anticommutation relations (CAR) of the creation and annihilation operators are preserved, since $\eta^*_b\eta_b = \eta^*_d\eta_d = 1$. Therefore, the transformations (33) are consistent with the CAR.

From (32)-(33), (16) and (20) it follows

$$\begin{aligned} C_{QFT}\psi(x) = &\frac{\eta_c}{(2\pi)^{3/2}}\int d^3p\sqrt{\frac{m}{E(p)}}(d_\alpha(p)u_\alpha(p)e^{-ip_\mu x^\mu} + b^+_\alpha(p)v_\alpha(p)e^{ip_\mu x^\mu}) = \\ &= \eta_c\psi^a(x), \end{aligned} \tag{34}$$

that is, when C_{QFT} acts on the particle field operator, up to phase, the antiparticle field operator is obtained.

In the case of a Majorana-like spinor, $d_i(p)$ must coincide with $b_i(p)$ up to some phase factor η

$$d_i(p) = \eta b_i(p). \tag{35}$$

 doi:10.20944/preprints202505.0486.v1

Therefore,

$$\psi_{\mathrm{M}}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E(p)}} (b_\alpha(p) u_\alpha(p) e^{-ip_\mu x^\mu} + \eta^* b_\alpha^+(p) v_\alpha(p) e^{ip_\mu x^\mu}) \,. \tag{36}$$

When C_{QFT} acts on $\psi_{\mathrm{M}}(x)$, we have in accordance with (6)

$$C_{QFT}\psi_{\mathrm{M}}(x) = \eta_c \psi_{\mathrm{M}}(x) \,. \tag{37}$$

From (36)-(37) taking into account (32), (33) and (35) it follows

$$\eta_b = \eta_b^* = \eta_c \,. \tag{38}$$

Therefore, the phase factor η_c is real, that is,

$$\eta_c = \pm 1 \,, \tag{39}$$

and the phase factor η^* in (36) remains arbitrary.

From (37) and the commutation of C_{QFT} with all Dirac gamma matrices it follows that for Majorana-like spinors (36)

$$C_{QFT} = \eta_c E \,, \tag{40}$$

where E is the 4×4 identity matrix and $\eta_c = \pm 1$.

The equality of the right-hand sides of (34) and (24) when C_{QFT} acts on $\psi(x)$ as $\mathcal{C}\psi(x)\mathcal{C}^{-1}$ is often mistakenly used to consider C_{Schw} as a unitary charge conjugation operator [8,9,18,26]. Although C_{Schw} is anti-unitary, and C_{QFT} is unitary. This equality is a consequence of the coincidence in (34) and (24) of the results of the action on $\psi(x)$ of two different operators, C_{Schw} and C_{QFT}. It arises from the symmetry of the solutions of the Dirac equation. This coincidence guarantees the correctness of the proofs of the invariance under charge conjugation of the Lagrangian and the equations for spinors, available in numerous monographs and textbooks. However, this symmetry is broken for chiral fields. For example,

$$C_{QFT}\psi_L = \frac{1-\gamma^5}{2} \frac{\eta_c}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E(p)}} (d_\alpha(p) u_\alpha(p) e^{-ip_\mu x^\mu} + b_\alpha^+(p) v_\alpha(p) e^{ip_\mu x^\mu}) \,, \tag{41}$$

$$C_{\mathrm{Schw}}\psi_L = \frac{1+\gamma^5}{2} \frac{\eta_c}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E(p)}} (d_\alpha(p) u_\alpha(p) e^{-ip_\mu x^\mu} + b_\alpha^+(p) v_\alpha(p) e^{ip_\mu x^\mu}) \,. \tag{42}$$

From the comparison of (41) and (42) we see that C_{QFT} does not change the chirality of the field, since it commutes with gamma matrices, while C_{Schw} reverses it due to the presence of the factor γ^2 in the operator (23).

It is the charge conjugation operator C_{QFT} that is the operator C of the CPT theorem. However, it cannot be used for the "seesaw" mechanism, in which the right-chiral component of a massive neutrino is obtained by charge conjugation of the left-chiral component using the operator C_{Schw}. If we try to construct a massive Majorana-like spinor ψ_{M} from the left-chiral state $\psi_{\mathrm{M}L}$ using C_{QFT}, we obtain

$$\psi_{\mathrm{M}} = \frac{1}{2}(\psi_{\mathrm{M}L} + C_{QFT}\,\psi_{\mathrm{M}L}) = \frac{1}{2}(\psi_{\mathrm{M}L} + \eta_c \psi_{\mathrm{M}L}) = \frac{1+\eta_c}{2}\psi_{\mathrm{M}L} \,. \tag{43}$$

For $\eta_c = -1$ we get $\psi_{\mathrm{M}} = 0$, and for $\eta_c = 1$ from (43) it follows $\psi_{\mathrm{M}} = \psi_{\mathrm{M}L}$, and we again obtain a left-chiral state. That is, it is impossible to obtain a right-chiral state from a left-chiral one using C_{QFT}.

Moreover, within one or two generations of fermions, the neutrino must be CP-symmetric (CP-self-adjoint). However, the C-self-adjoint left-chiral spinor (43) is not CP-symmetric, since the CP

operator changes the chirality to the opposite. As a result, to construct a massive Majorana-like neutrino in the approximation of one or two generations, it is necessary to use CP conjugation.

5. CP-Conjugation

The fact that the Majorana-Kramers charge conjugation operator $C_{\text{M-K}}$ reverses the chirality of the neutrino was apparently first noticed by Nieves [11] in 1982. He proposed that the Majorana neutrino is an eigenstate of the CP operator, but mistakenly assumed that $C_{\text{M-K}}$ is the CP operator. Kayser and Goldhaber [12] also proposed that, under the conditions of C and P symmetry violation in weak interactions, the Majorana neutrino is an eigenstate of the CP operator.

As already mentioned, in the theory of Majorana neutrinos arising from the "seesaw" mechanism, the right-chiral part of the field operator is obtained using charge conjugation of the left-chiral part [2,3,14,18,21,22]. In the case $C = C_{QFT}$, CP is unitary operator and provides chirality reversal. Therefore, it does not contradict the CPT theorem and can be used to construct the right-chiral part of a massive spinor from the left-chiral one. However, the parity operator P changes a spatial momentum p to the opposite. Therefore, if we try to construct a massive Majorana spinor ψ_{M} from a left-chiral Majorana state $\psi_{\text{M}L}(p)$ using the CP operator, this state can have a definite momentum p only if the CP operator acts on a field operator with opposite momentum

$$\psi_{\text{M}}(p) = \psi_{\text{M}L}(p) + \eta^*_{CP} CP \psi_{\text{M}L}(-p) = \psi_{\text{M}L}(p) + \psi_{\text{M}R}(p)\,, \tag{44}$$

where $C = C_{QFT}$, η_{CP} is the phase factor of the CP transformation, and the factor η^*_{CP} compensates it.

Since there is a symmetry of the field operator of any spinor $\psi(x)$ with respect to the sign of the momentum p in the momentum expansion, it follows from (44) that

$$\psi_{\text{M}}(x) = \psi_{\text{M}L}(x) + \eta^*_{CP} CP \psi_{\text{M}L}(x) = \psi_{\text{M}L}(x) + \psi_{\text{M}R}(x)\,. \tag{45}$$

Thus, the unitary operator CP, where $C = C_{QFT}$, in the case of one or two generations of leptons can be used to construct the right-chiral components of a massive Majorana-like spinor from the left-chiral components of a Majorana-like spinor. The resulting spinor is self-adjoint both with respect to CP-conjugation (44)-(45) and with respect to charge conjugation via C_{QFT}. The question of the influence of CP symmetry breaking for such spinors in the case of three generations requires further study.

6. $CPTS$-Conjugation According to Kayser and Goldhaber

In one of the first papers that started developing the "seesaw" mechanism to explain the smallness of the neutrino mass, the formation of Majorana-like neutrinos using CPT conjugation was used [1]. However, when trying to construct a Majorana-like spinor ψ from the left-chiral state $\psi_L(p)$ in this way, a problem with a spin projection s arises, since the resulting state

$$\psi = \psi_L(p,s) + CPT\psi_L(p,s) = \psi_L(p,s) + \psi_R(p,-s) \tag{46}$$

cannot have a definite spin projection, but is a mixture of left-chiral and right-chiral states with opposite spin projections. Such constructions cannot correspond to real particles with non-zero spin.

Kayser and Goldhaber [12] proposed a solution to this problem by using, instead of the CPT operator, the $CPTS$ conjugation, in which, in addition to CPT, there is an operator S that changes the sign of the spin projection at rest. In the field operator (16), the summation is over the spin projections s_3, so at rest $S = \gamma^1\gamma^3$ is the rotation operator by an angle π in the plane γ^1, γ^3 (reflection of the spin projection along the γ^2 axis).

This definition of the S conjugation operator seems simple and clear. However, it does not follow from it how to write it explicitly for the field operator corresponding to an arbitrary momentum. But for the operator S it is enough to reproduce our reasoning from the previous section for the parity

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 May 2025 doi:10.20944/preprints202505.0486.v1

operator P, with the only difference being that not all three spatial axes and corresponding components of the momentum are reflected, but only the axes with generators γ^1 and γ^3. Therefore, when acting on the field operator, $\gamma^1\gamma^3$ must be supplemented by the inversion of the axes x^1 and x^3. As a result, the Lorentz-covariant operator S of the inversion of the spin projection s_3 should have the form

$$S = \eta_S R(-x^1, -x^3)\gamma^1\gamma^3 \,, \tag{47}$$

where η_S is the phase factor, and $R(-x^1, -x^3)$ is the operator that changes the sign of the coordinates x^1 and x^3 and A_1 and A_3 by analogy with the operator $R(-x^k)$, $k = 1, 2, 3$ in (27).

It is important to note that the operator (47), although defined as the operator of reflection of two axes, is simultaneously the operator of rotation of space by the angle π in the plane γ^1, γ^3. It is easy to check that it commutes with γ^5 and $\gamma^\mu i\partial_\mu$, and therefore leaves both the Standard Model Lagrangian and the extended Dirac equation with the corresponding Standard Model "long" derivatives [21,35] invariant.

Thus, a formula of the $CPTS$ conjugation in the Dirac and Weyl representations, taking into account (27), (28) and (47), can be written as

$$CPTS = \eta_{CPTS}\, C_{QFT} R(-x^0, -x^2)\gamma^0\, (.)^* \,, \tag{48}$$

where η_{CPTS} is the phase factor, C_{QFT} is given by formula (40), and $R(-x^0, -x^2)$ is the operator changing the signs of x^0 and x^2.

The constructed anti-unitary operator $CPTS$, where $C = C_{QFT}$, is the symmetry operator of the Standard Model Lagrangian and the Dirac equation. It can be used to construct right-chiral components of Majorana-like spinors (36) from left-chiral ones. The resulting spinor is self-adjoint with respect to charge conjugation using C_{QFT}. In the approximation of one or two generations of fermions, T-symmetry is preserved, and S-symmetry is always preserved due to the isotropy of space. Because of which, in this case, $CPTS$-conjugation is equivalent to CP-conjugation, and the $CPTS$-self-conjugated Majorana-like neutrino is also CP-self-conjugated.

7. Conclusions

Majorana spinors are solutions of the Dirac equation that are self-adjoint with respect to the Majorana-Kramers charge conjugation $C_{\text{M-K}}$ (2),(5), (13). Any Dirac spinor can be described as a superposition of two Majorana spinors. However, Majorana spinors cannot correspond to really existing particles.

The "seesaw" mechanism is usually realized in the literature by constructing right-chiral components of Majorana-like spinors from left-chiral ones by means of the Schwinger charge conjugation operator C_{Schw} (22)-(23). However, it is anti-unitary and reverses the chirality of the fermion field components, which contradicts the CPT theorem. In addition, it does not map the particle state vector to the antiparticle state vector, as it should, and its action on the vacuum does not keep the vacuum invariant. Therefore, it cannot be used to construct Majorana-like spinors.

The Pauli charge conjugation operator is a non-QFT analogue of the Schwinger conjugation operator C_{Schw}, and therefore it also cannot be used to construct Majorana-like neutrinos.

The charge conjugation operator C_{QFT}, which provides the transformations (32)-(34) and (37), for the Majorana-like spinors has the form (40). It is unitary and is the C operator of the CPT theorem. However, this operator does not change chirality and therefore is not suitable for constructing right-chiral components of a Majorana-like spinor from left-chiral components.

The unitary operator of the CP conjugation, with $C = C_{QFT}$, can be used to construct Majorana-like neutrinos (44)-(45) in the approximation of one or two fermion generations and to implement the "seesaw" mechanism.

The anti-unitary operator (48) of the $CPTS$ conjugation, with $C = C_{QFT}$, can be used to construct Majorana-like neutrinos and to implement the "seesaw" mechanism.

Thus, the charge conjugation operators according to Majorana-Kramers, Pauli and Schwinger cannot be the charge conjugation operator C of the CPT theorem and be used to construct a Majorana-like massive neutrino. There can be only two variants of conjugation that potentially allow the neutrino to be a Majorana-like fermion. These are the unitary CP conjugation (44)-(45) and the anti-unitary $CPTS$ conjugation proposed by Kayser and Goldhaber [12], the formula (48) of which we found explicitly.

Acknowledgments: The author acknowledges Saint-Petersburg State University for a research project 103821868

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

QFT	Quantum field theory
CPTS	Charge, Parity, Time, Spin

References

1. Minkowski, P. $\mu \to e\gamma$ at a rate of one out of 10^9 muon decays? *Phys. Letters B* **1977**, *67*, 421–428. https://doi.org/10.1016/0370-2693(77)90435-x
2. Mohapatra, R. N.; Senjanovic, G. Neutrino Mass and Spontaneous Parity Nonconservation. *Phys. Rev. Lett.* **1980**, *44*, 912–915. https://doi.org/10.1103/PhysRevLett.44.912
3. Granelli, A. Standard and Non-Standard Aspects of Neutrino Physics. *Universe* **2024**, *10*, 164. https://doi.org/10.3390/universe10040164
4. Majorana, E. Teoria simmetrica dell'elettrone e del positrone. *Nuovo Cimento* **1937**, *14*, 171–184. https://doi.org/10.1007/BF02961314
5. Kramers, H. A. The use of charge-conjugated wave-functions in the hole-theory of the electron. *Proc. of the Royal Academy of Amsterdam* **1937**, *40*, 814–823.
6. Pauli, W. Contributions mathématiques à la théorie des matrices de Dirac. *Annales de l'Institut Henri Poincare* **1936**, *6*, 109–136.
7. Schwinger, J. Quantum Electrodynamics. I. A Covariant Formulation. *Phys. Rev.* **1948**, *74*, 1439. https://doi.org/10.1103/PhysRev.74.1439
8. Bjorken, J. D.; Drell, S. D. *Relativistic quantum fields*; Mcgraw-Hill College: N.Y., 1965.
9. Peskin, M. E.; Schroeder, D. V. *An Introduction to Quantum Field Theory*; Addison-Wesley Publishing Company: USA, 1995.
10. Weinberg, S. *The Quantum Theory of Fields - Foundations. Vol. 1*; Cambridge University Press: Cambridge, 1995.
11. Nieves, J. F. Electromagnetic properties of Majorana neutrinos. *Phys. Rev. D* **1982** *26*, 3152. https://doi.org/10.1103/PhysRevD.26.3152
12. Kayser, B.; Goldhaber, A. S. CPT and CP properties of Majorana particles, and the consequences. *Phys. Rev. D* **1983**, *28*, 2341. https://doi.org/10.1103/PhysRevD.28.2341
13. Kayser, B. Majorana neutrinos and their electromagnetic properties. *Phys. Rev. D.* **1982**, *26*, 1662. https://doi.org/10.1103/PhysRevD.26.1662
14. Mohapatra, R. N.; Pal, P. B. *Massive neutrinos in physics and astrophysics*; World Scientific, 2004.
15. Monakhov, V. Majorana Mass Term of Majorana Spinors. *Phys. Sci. Forum* **2023**, *7*, 1. https://doi.org/10.3390/ECU2023-14016
16. Monakhov, V. The impossibility of the existence of Majorana spinors as physical particles *Space, time and fundamental interactions* **2023**, *3-4*, 211–217.
17. Bjorken, J. D.; Drell, S. D. *Relativistic quantum mechanics*; Mcgraw-Hill College: N.Y., 1964.
18. Sozzi, M. *Discrete symmetries and CP violation: From experiment to theory*; Oxford University Press: Oxford, 2008.
19. Messiah, A. *Quantum mechanics. Vol 2*; North-Holland Pub. Co.: Amsterdam, 1962.
20. Schweber, S. S. *Introduction to Relativistic Quantum Field Theory*; Harper and Row: N.Y., 1961.
21. Donoghue, J. F.; Golowich, E.; Holstein B. R. *Dynamics of the Standard Model*; Cambridge University Press: Cambridge, 2022.

 doi:10.20944/preprints202505.0486.v1

22. Mannheim, P. D. Introduction to Majorana masses. *Int. J. of Theor. Phys.* **1984**, *23*, 643–674. https://doi.org/10.1007/BF02214135
23. Jost, R. A remark on the CTP theorem. *Helv. Phys. Acta* **1957**, *30*, 409–416.
24. Jost, R. Das Pauli-Prinzip und die Lorentz-Gruppe. In *Theoretical physics in the twentieth century. A memorial volume to Wolfgang Pauli*; Fierz, M., Weisskopf, V. F., Eds.; Interscience Publishers: N.Y., 1960; pp. 107–136.
25. Itzikson, K.; Zuber, J.-B. *Quantum field theory*; McGraw-Hill: N.Y., 1980.
26. Sakurai, J. J. *Invariance Principles and Elementary Particles*; Princeton University Press: Princeton,1964.
27. Wigner, E. *Group theory and its application to the quantum mechanics of atomis spectra*; Acad. press: N.Y. and London, 1959.
28. Schwinger, J. The Theory of Quantized Fields. I. *Phys. Rev.* **1951**, *82*, 914. https://doi.org/10.1103/PhysRev.82.914
29. Schwinger, J. Spin, statistics, and the TCP theorem. *Proc. of the National Acad. of Sci.* **1958**, *44*, 223–228. https://doi.org/10.1073/pnas.44.2.223
30. Monakhov, V. V. Construction of a fermionic vacuum and the fermionic operators of creation and annihilation in the theory of algebraic spinors. *Phys. Part. Nucl.* **2017**, *48*, 836—838. https://doi.org/10.1134/s1063779617050318.
31. Monakhov, V. V. Modules over Clifford algebras as a basis for the theory of second quantization of spinors. *What Comes Beyond the Standard Models*, **2022**, *23*, 290–303.
32. Wolfenstein, L.; Ravenhall, D. G. Some Consequences of Invariance under Charge Conjugation. *Phys. Rev.* **1952**, *88*, 279. https://doi.org/10.1103/PhysRev.88.279
33. Federbush, P. G.; Grisaru, M. T. Representations of symmetry operators. *Il Nuovo Cimento* **1958**, *9*, 890–894. https://doi.org/10.1007/BF02903129
34. Mariwalla, K. H. Representation of Discrete Symmetry Operators. *J. of Math. Phys.* **1966**, *7*, 114–122. https://doi.org/10.1063/1.1704797
35. Cheng, T.-P.; Li, L.-F. *Gauge theory of elementary particle physics*; Clarendon Press: Oxford, 1984.