Article

Formal Calculation of Q-Binomial

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Keywords: Formal Calculation; q-nested sum; q-binomial; q-analysis; q-calculus

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Formal Calculation of q-Binomial

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Abstract

This article offers formulas for computing various q-binomial nested sums, give three forms of results, reveals the three forms of q-binomial and their interrelationships. It is a powerful tool for q-analysis, which can prove and generalize many classic conclusions in a simple way. This article also utilized it to obtain a large number of new results, including formulas for q-Eulerian numbers and polynomials. By taking the limit of q to 1, it can calculate general nested sums and analyze binomial coefficients.

Keywords: formal calculation; q-nested sum; q-binomial; q-analysis; q-calculus

MSC: 05A30

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1. Calculation Formula

q-binomial: $M, N, K \in \mathbb{N}$, $\begin{bmatrix} N \\ M \end{bmatrix}_q = \frac{(q^N-1)(q^{N-1}-1)...(q^{N-M+1}-1)}{(q^M-1)(q^{M-1}-1)...(q^1-1)}$, $q \neq 0,\ 1$, abbreviated as G_M^N, $[N]_q = G_1^N$.

$(a;q)_n = \begin{cases} (1-a)(1-aq)...(1-aq^{n-1}), n>0 \\ 1, n=0 \end{cases}$. $G_{g_1, g_2 \cdots g_p}^M = \frac{(q;q)_M}{\prod_{i=1}^p (q;q)_{g_i}}$, $\sum g_i = M$.

δ_{ij} is Kronecker delta, $i = j, \delta_{ij} = 1; i \neq j, \delta_{ij} = 0$. The following relationship holds:

$$G_0^N = 1, G_M^N = G_{N-M}^N; G_M^N = 0,\ M > N, M < 0. \quad (1)$$

$$G_M^N = q^M G_M^{N-1} + G_{M-1}^{N-1} = G_M^{N-1} + q^{N-M} G_{M-1}^{N-1}. \quad (2)$$

$$\sum_{n=0}^{N-1} q^n G_M^{n+K} = q^{M-K} G_{M+1}^{N+K},\ M \geq K \geq 0. \quad (3)$$

$$G_K^M = \sum_{w \in \Omega(0^{M-K}, 1^K)} q^{inv(w)},\ \text{inv(.) denotes the inversion statistic. [1]} \quad (4)$$

Lemma 1.

$$\begin{aligned} &\sum_{n=0}^{N-1} q^n [n]_q G_M^{n+K},\ M \geq K \geq 0 \\ &= q^{2(M-K)+1} G_1^{M+1} G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+1}^{N+K}\ (1) \\ &= q^{M-2K-1} G_1^{M+1} G_{M+2}^{N+K+1} + q^{M-K} (G_1^{M-K} - q^{-K-1} G_1^{M+1}) G_{M+1}^{N+K}\ (2) \\ &= (q^{2(M-K)+1} G_1^{M+1} - q^{2M-K+2} G_1^{M-K}) G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+2}^{N+K+1}\ (3). \end{aligned}$$

Proof.

$$
\begin{aligned}
&\sum q^n[n]_q G_M^{n+K} = \sum q^n \frac{q^n - q^{n-M+K}}{q-1} G_M^{n+K} + \sum q^n \frac{q^{n-M+K}-1}{q-1} G_M^{n+K} \\
&= \sum q^n (q^{n-M+K}-1)\frac{q^{M-K}-1}{q-1} G_M^{n+K} + \sum q^n \frac{q^{M-K}-1}{q-1} G_M^{n+K} + \sum q^n \frac{q^{n-M+K}-1}{q-1} G_M^{n+K} \\
&= \sum q^n (q^{M+1}-1) G_1^{M-K} G_{M+1}^{n+K} + \sum q^n G_1^{M-K} G_M^{n+K} + \sum q^n G_1^{M+1} G_{M+1}^{n+K} \\
&= (q^{M-K}-1) G_1^{M+1} G_{M+2}^{N+K} q^{M+1-K} + G_1^{M-K} G_{M+1}^{N+K} q^{M-K} + G_1^{M+1} G_{M+2}^{N+K} q^{M+1-K} \\
&= q^{2(M-K)+1} G_1^{M+1} G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+1}^{N+K} \ (1) \\
&= q^{2(M-K)+1} G_1^{M+1} q^{-M-2} (G_{M+2}^{N+K+1} - G_{M+1}^{N+K}) + q^{M-K} G_1^{M-K} G_{M+1}^{N+K} \\
&= q^{M-2K-1} G_1^{M+1} G_{M+2}^{N+K+1} + q^{M-K} (G_1^{M-K} - q^{-K-1} G_1^{M+1}) G_{M+1}^{N+K} \ (2) \\
&= q^{2(M-K)+1} G_1^{M+1} G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} (G_{M+2}^{N+K+1} - q^{M+2} G_{M+2}^{N+K}) \\
&= (q^{2(M-K)+1} G_1^{M+1} - q^{2M-K+2} G_1^{M-K}) G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+2}^{N+K+1} \ (3).
\end{aligned}
$$

□

Definition 1. *Recursively define* ∇_q^p, $p \in \mathbb{Z}$; $SUM_q(N) = SUM_q(N, PS, PT)$, $K_i, D_i \in \mathbb{C}$, $T_i \in \mathbb{N}$.

$\nabla_q^0 f(n) = f(n), \sum_{n=0}^{N-1} q^n \nabla_q^1 f(n+1) = f(N)$, $\sum_{n=0}^{N-1} q^n f(n+1) = \nabla_q^{-1} f(N), \nabla_q^1 = \nabla_q$.
$SUM_q(N, [K_1 : D_1], [T_1 = 1]) = \sum_{n=0}^{N-1} q^n (K_1 + [n]_q D_1)$.
$SUM_q(N, [K_1 : D_1, K_2 : D_2], [T_1, T_2 = T_1 + 2 - p]) = \sum_{n=0}^{N-1} q^n (K_2 + [n]_q D_2) \nabla^p SUM_q(n+1, [K_1 : D_1], [T_1])$.

Abbreviations: $[K_1 : D, K_2 : D ... K_M : D] = [K_1, K_2 ... K_M] : D, [K_1, K_2 ... K_M] : 1 = [K_1, K_2 ... K_M]$.
In this paper, the default $PS = [K_1 : D_1, K_2 : D_2 ... K_M : D_M], PT = [T_1, T_2 ... T_M], T_i < T_{i+1}$.

$SUM_q(N, PS, [1, 2 ... M]) = \sum_{n=0}^{N-1} \prod_{i=1}^{M} q^n (K_i + [n]_q D_i)$.
$SUM_q(N, PS, [1, 3 ... 2M-1]) = \sum_{n_M=0}^{N-1} q^{n_M} (K_M + [n_M]_q D_M) ... \sum_{n_2=0}^{n_3} q^{n_2} (K_2 + [n_2]_q D_2) \sum_{n_1=0}^{n_2} q^{n_1} (K_1 + [n_1]_q D_1)$.
$SUM_q(N, PS, [1, 2, 4]) = \sum_{n_3=0}^{N-1} q^{n_3} (K_3 + [n_3]_q D_3) \sum_{n=0}^{n_3} q^n (K_1 + [n]_q D_1)(K_2 + [n]_q D_2)$.
$SUM_q(N, PS, [1, 3, 4]) = \sum_{n_3=0}^{N-1} q^{n_3} (K_3 + [n_3]_q D_3)(K_2 + [n_3]_q D_2) \sum_{n=0}^{n_3} q^n (K_1 + [n]_q D_1)$.
$SUM_q(N, PS, [1, 4]) = \sum_{n_3=0}^{N-1} q^{n_3} (K_2 + [n_3]_q D_2) \sum_{n_2=0}^{n_3} q^{n_2} \sum_{n_1=0}^{n_2} q^{n_1} (K_1 + [n_1]_q D_1)$.

Use $\mathbb{K}, \mathbb{T}$ to represent the set $\{K_i\}, \{T_i\}$. $(K_1 + T_1)(K_2 + T_2) ... (K_M + T_M) = \sum \prod_{i=1}^{M} X_i, X_i = T_i \ or \ K_i$

Definition 2. $X(T)$=*Number of* $\{X_1, X_2 ... X_M\} \in \mathbb{T}$.
X_{T-1}=*Number of* $\{X_1, X_2 ... X_{i-1}\} \in \mathbb{T}$, X_{K-1}=*Number of* $\{X_1, X_2 ... X_{i-1}\} \in \mathbb{K}$.
X_T=*Number of* $\{X_1, X_2 ... X_i\} \in \mathbb{T}$, X_K=*Number of* $\{X_1, X_2 ... X_i\} \in \mathbb{K}$.

$X_{T-1} + X_{K-1} = i - 1$, $X_T + X_K = i$. Use the auxiliary form and each X_i cannot be swapped:

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Theorem 1. $H = T_M - M, SUM_q(N, PS, PT) =$

$$Form_1 \rightarrow \sum_{g=0}^{M} H_1^q(g) G_{H+1+g}^{N+H},\ B_i = \begin{cases} q^{1+(T_i-T_{i-1})X_{T-1}} G_1^{T_i-X_{K-1}} D_i,\ X_i{=}T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}} (K_i{+}G_1^{X_{T-1}} D_i), X_i{=}K_i \end{cases},$$

$$Form_2 \rightarrow \sum_{g=0}^{M} H_2^q(g) G_{H+1+g}^{N+H+g},\ B_i = \begin{cases} q^{-(T_i-X_{K-1})} G_1^{T_i-X_{K-1}} D_i,\ X_i{=}T_i \\ K_i{-}q^{-(T_i-X_{K-1})} G_1^{T_i-X_{K-1}} D_i,\ X_i{=}K_i \end{cases},$$

$$Form_3 \rightarrow \sum_{g=0}^{M} H_3^q(g) G_{T_M+1}^{N+T_M-g},\ B_i = \begin{cases} q^{1+(T_i-T_{i-1}-1)X_{T-1}} \{(q^{X_{T-1}} G_1^{T_i} - q^{T_i} G_1^{X_{T-1}}) D_i - K_i q^{T_i}\},\ X_i{=}T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}} (K_i{+}G_1^{X_{T-1}} D_i),\ X_i{=}K_i \end{cases}.$$

$$H_i^q(g) = H_i^q(g, PS, PT) = H_i^q(g, M),\ \textit{is defined as} \sum_{X(T)=g} \prod_{i=1}^{M} B_i.$$

Proof.

$SUM_q(1,[K_1:D_1],[1])=\sum_{n=0}^{N-1}q^n(K_1+[n]_qD_1)=\sum_{n=0}^{N-1}q^nG_1^nD_1+\sum_{n=0}^{N-1}q^nK_1$
$=q^1D_1G_2^N+K_1G_1^N=q^{-1}D_1G_2^{N+1}+(K_1-q^{-1}D_1)G_1^N=(q^1D_1-K_1q^2)G_2^N+K_1G_2^{N+1}.$
It's holds when M=1, Assume that M holds.
$PS1=[PS,K_{M+1}:D_{M+1}],PT1=[PT,T_{M+1}=T_M+2-p],X=T_M-M+1-p=T_{M+1}-(M+1).$
$f(n+1)=\sum A_gG_{H+1+g}^{n+H+1}\to\nabla_q^pf(n+1)=\sum A_gG_{H+1+g-p}^{n+H+1-p}q^{-pg}.$
$SUM_q(N,PS1,PT1)$
$=\sum_{n=0}^{N-1}q^n(K_{M+1}+[n]_qD_{M+1})\sum_{g=0}^{M}H_1^q(g)G_{X+g}^{n+X}q^{-pg}$
$=\sum_{g=0}^{M}(K_{M+1}q^g+q^gG_1^gD_{M+1})H_1^q(g)G_{X+1+g}^{N+X}q^{-pg}+\sum_{g=0}^{M}q^{2g+1}G_1^{X+g+1}D_{M+1}H_1^q(g)G_{X+2+g}^{N+X}q^{-pg}$
$=\sum_{g=0}^{M}(K_{M+1}+G_1^gD_{M+1})H_1^q(g)G_{T_{M+1}-(M+1)+1+g}^{N+T_{M+1}-(M+1)}q^{(1-p)g}+\sum_{g=0}^{M}q^{(2-p)g+1}G_1^{T_{M+1}-(M-g)}D_{M+1}H_1^q(g)G_{X+2+g}^{N+X}$
$=\sum_{g=0}^{M+1}H_1^q(g,PS1,PT1)G_{T_{M+1}-(M+1)+1+g}^{N+T_{M+1}-(M+1)}\to Form_1.$

$f(n+1)=\sum A_gG_{H+1+g}^{n+H+1+g}\to\nabla_q^pf(n+1)=\sum A_gG_{H+1+g-p}^{n+H+1+g-p}.$
$SUM_q(N,PS1,PT1)$
$=\sum_{n=0}^{N-1}q^n(K_{M+1}+[n]_qD_{M+1})\nabla_q^pSUM_q(n+1)=\sum_{n=0}^{N-1}q^n(K_{M+1}+[n]_qD_{M+1})\sum_{g=0}^{M}H_2^q(g)G_{X+g}^{n+X+g}$
$=\sum_{g=0}^{M}(K_{M+1}-q^{-X-g-1}G_1^{X+g+1}D_{M+1})H_2^q(g)G_{X+1+g}^{N+X+g}+\sum_{g=0}^{M}q^{-X-g-1}G_1^{X+g+1}D_{M+1}H_2^q(g)G_{X+2+g}^{N+X+g+1}$
$=\sum_{g=0}^{M}(K_{M+1}-q^{-(T_{M+1}-(M-g))}G_1^{T_{M+1}-(M-g)}D_{M+1})H_2^q(g)G_{T_{M+1}-(M+1)+1+g}^{N+T_{M+1}-(M+1)+g}$
$+\sum_{g=0}^{M}q^{-(T_{M+1}-(M-g))}G_1^{T_{M+1}-(M-g)}D_{M+1}H_2^q(g)G_{T_{M+1}-(M+1)+1+(g+1)}^{N+T_{M+1}-(M+1)+(g+1)}$
$=\sum_{g=0}^{M+1}H_2^q(g,PS1,PT1)G_{T_{M+1}-(M+1)+1+g}^{N+T_{M+1}-(M+1)+g}\to Form_2.$

$f(n+1)=\sum A_gG_{T_M+1}^{n+1+T_M-g}\to\nabla_q^pf(n+1)=\sum A_gG_{T_M+1-p}^{n+1+T_M-g-p}q^{-pg}.$
$SUM_q(N,PS1,PT1)$
$=\sum_{n=0}^{N-1}q^n(K_{M+1}+[n]_qD_{M+1})\sum_{g=0}^{M}H_3^q(g)G_{T_M+1-p}^{n+1+T_M-p-g}q^{-pg}$
$=\sum_{g=0}^{M}K_{M+1}q^gH_3^q(g)G_{T_M+2-p}^{N+1+T_M-p-g}q^{-pg}+\sum_{g=0}^{M}q^gG_1^gD_{M+1}H_3^q(g)G_{T_M+3-p}^{N+T_M-p+2-g}q^{-pg}$
$+\sum_{g=0}^{M}(q^{2g+1}G_1^{T_M+2-p}-q^{T_M+3-p+g}G_1^g)D_{M+1}H_3^q(g)G_{T_M+3-p}^{N+T_M-p+1-g}q^{-pg}$
$=\sum_{g=0}^{M}(K_{M+1}q^g+q^gG_1^gD_{M+1})H_3^q(g)G_{T_M+3-p}^{N+T_M-p+2-g}q^{-pg}$
$+\sum_{g=0}^{M}((q^{2g+1}G_1^{T_M+2-p}-q^{T_M+3-p+g}G_1^g)D_{M+1}-q^{T_M+3-p}K_{M+1}q^g)H_3^q(g)G_{T_M+3-p}^{N+T_M-p+1-g}q^{-pg}$
$=\sum_{g=0}^{M}(K_{M+1}+G_1^gD_{M+1})H_3^q(g)G_{T_{M+1}+1}^{N+T_{M+1}-g}q^{(1-p)g}$
$+\sum_{g=0}^{M}((q^gG_1^{T_{M+1}}-q^{T_{M+1}}G_1^g)D_{M+1}-q^{T_{M+1}}K_{M+1})H_3^q(g)G_{T_{M+1}+1}^{N+T_{M+1}-(g+1)}q^{1+(1-p)g}$
$=\sum_{g=0}^{M+1}H_3^q(g,PS1,PT1)G_{T_{M+1}+1}^{N+T_{M+1}-g}\to Form_3.$

$\square$

$$\sum_{n=0}^{N-1}q^n\prod_{i=1}^{M}(K_i+D_iq^n)=SUM(N,[K_i+D_i:D_i(q-1)],[1,2...M])=\sum_{g=0}^{M}H_1^q(g)G_{1+g}^N.$$

$B_i=\begin{cases}q^{1+X_T-1}G_1^{i-X_K-1}D_i(q-1)=q^{X_T}G_1^{1+X_T-1}D_i(q-1)=q^{X_T}(q^{X_T}-1)D_i,X_i=T_i\\K_i+D_i+G_1^{X_T-1}D_i(q-1)=K_i+D_i+G_1^{X_T}D_i(q-1)=K_i+q^{X_T}D_i,X_i=K_i\end{cases}$. Following a similar form, induction proves:

Theorem 2. $\sum_{n=0}^{N-1}\prod_{i=1}^{M}(K_i+D_iq^n)=\sum_{g=1}^{M}(\sum\prod B_i)G_g^N+N\prod K_i,\ B_i=\begin{cases}D_i,X_{T-1}=0;q^{X_{T-1}}(q^{X_{T-1}}-1)D_i,X_{T-1}>0,X_i=T_i\\K_i,X_{T-1}=0;K_i+q^{X_{T-1}-1}D_i,X_{T-1}>0,X_i=K_i\end{cases}.$

$(K+D)^M=D\sum_{g=0}^{M-1}K^g(K+D)^{M-1-g}+K^M.$

$(K+qD)^M=\sum_{n=0}^{1}\prod(K+q^nD)-\sum_{n=0}^{0}\prod(K+q^nD)=$

$=q(q-1)D^2\sum_{a+b+c=M-2,a,b,c\geq 0}K^a(K+D)^b(K+qD)^c+qD\sum_{g=0}^{M-1}K^g(K+D)^{M-1-g}+K^M.$

$PS=[K+D,K+D...K+D]:(q-1)D,PT=[1,2...M].$

$B_i=\begin{cases}q^{X_T}(q^{X_T}-1)D,X_i=T_i\\K+Dq^{X_T},\ X_i=K_i\end{cases},H_1^q(0)=(K+D)^M,H_1^q(1)=q^1(q^1-1)D\sum_{a+b=M-1,a,b\geq 0}(K+D)^a(K+qD)^b.$

$SUM_q(2)-SUM_q(1)=H_1^q(1)G_2^2+H_1^q(0)G_1^2-H_1^q(0)=H_1^q(1)+qH_1^q(0)=q(K+qD)^M\rightarrow$

$(K+qD)^M=(q-1)D\sum_{g=0}^{M-1}(K+D)^g(K+qD)^{M-1-g}+(K+D)^M.$

Definition 3. $\lim_{q\rightarrow 1}H^q(g)=H(g),\ \lim_{q\rightarrow 1}SUM_q(N)=SUM(N).$

$\lim_{q\rightarrow 1}[n]_q=n,\ \lim_{q\rightarrow 1}G_M^N=\binom{N}{M}$, which yields the nested summation formula for K_i+nD_i.

2. Property

Definition 4. $[n]_{q-}=q^{-n}G_1^n,[n!]_{q-}=[n]_{q-}...[2]_{q-}[1]_{q-},[0!]_{q-}=1,[n]_{q+}=q^nG_1^n$, *similarly defining* $[n!]_{q+}$.

Theorem 3.
(1). $\nabla_qSUM_q(n+1,PS,[1,2...M])=\prod_{i=1}^{M}(K_i+[n]_qD_i).$
(2). At $SUM_q(N,[...PS...],[...T,T+1...T+M-1...]),K_i:D_i$ *can swap orders.*
(3). $\nabla_q^pSUM_q(N)=\sum_gH_1^q(g)G_{X+1+g}^{N+X}q^{-gp}=\sum_gH_2^q(g)G_{X+1+g}^{N+X+g}=\sum_gH_3^q(g)G_{X+M+1}^{N+X+M-g}q^{-gp},X=T_M-M-p.$
(4). $SUM_q(N,[[L_1]_{q-},[L_2]_{q-}...[L_Q]_{q-},PS1],[L_1,L_2...L_Q,PT1])=\prod[L_i]_{q-}SUM_q(N,PS1,PT1).$ T_1 *can great than 1.*
(5). $SUM_q(N,[[T_1]_{q-},[T_2]_{q-}...[T_M]_{q-}],[T_1,T_2...T_M])=\prod[T_i]_{q-}G_{T_M+1}^{N+T_M}.$
(6) At $H^q(g),\sum_{X_i\in\mathbb{K}}\prod q^{X_T}=G_{M-g}^M=G_g^M.$

Proof.
Definition of $SUM_q\rightarrow(1)(2),\ \sum_{n=0}^{N-1}q^nG_M^{n+K}=q^{M-K}G_{M+1}^{N+K}\rightarrow(3)$, which has been used for the proof of [1].
At (4), $H_2^q(g<Q)=0,\ H_2^q(g\geq Q)=\prod[L_i]_{q-}H_2^q(g-Q,PS1,PT1)\rightarrow(4)(5)$. (6) is $G_K^M=\sum_{w\in\Omega(0^{M-K},1^K)}q^{inv(w)}$. □

$[x]_{q-}+[n]_q=q^{-x}[n+x]_q,PT=[2i-1],(5)\rightarrow\sum_{n_M=0}^{N-1}...\sum_{n_1=0}^{n_2}q^{\sum n_i}\prod[n_i+2i-1]_q=[1]_q[3]_q...[2M-1]_qG_{2M}^{N+2M-1}.$

Theorem 4. $PT=[1,2...M]$,
(1). $H_1^q(g)=q^{g(g+1)}\sum_kH_2^q(k)G_g^k.$
(2). $H_1^q(g)=\sum_kH_3^q(k)G_{M-g}^{M-k}q^{(g+1)(g-k)}.$
(3). $H_2^q(g)=\sum_k(-1)^{k+g}G_g^kq^{\frac{g(g+1)-k(k+3)}{2}-kg}H_1^q(k).$
(4). $H_3^q(g)=\sum_k(-1)^{k+g}G_{M-g}^{M-k}q^{\frac{g(g+3)-k(k+3)}{2}}H_1^q(k).$

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Proof.

When M=1, verify directly. Assume that M holds. $D_i \neq 0$, changing $K_i : D_i$ to $K_i / D_i : 1$ does not affect the result .
$PS1 = [PS, K_{M+1}], PT1 = [PT, T_{M+1}]$.
$H_1^q(g, M+1) = H_1^q(g-1) q^{1+X_T} G_1^{1+X_T} + H_1^q(g)(K_{M+1} + G_1^{X_T})$
$= q^{1+X_T} G_1^{1+X_T} q^{g(g-1)} \sum_{k=0}^{M} H_2^q(k) G_{g-1}^k + (K_{M+1} + G_1^{X_T}) q^{g(g+1)} \sum_{k=0}^{M} H_2^q(k) G_g^k$
$= q^g G_1^g q^{g(g-1)} \sum_{k=0}^{M} H_2^q(k)(G_g^{k+1} - q^g G_g^k) + (K_{M+1} + G_1^g) q^{g(g+1)} \sum_{k=0}^{M} H_2^q(k) G_g^k$
$= q^{g(g+1)} \sum_{k=0}^{M} H_2^q(k) q^{-g} G_1^g G_g^{k+1} + K_{M+1} q^{g(g+1)} \sum_{k=0}^{M} H_2^q(k) G_g^k.\ (1*)$
$q^{g(g+1)} \sum_{k=0}^{M+1} H_2^q(k, M+1) G_g^k = q^{g(g+1)} \sum_{k=0}^{M+1} (H_2^q(k-1) q^{-k} G_1^k + H_2^q(k)(K_{M+1} - q^{-(k+1)} G_1^{k+1})) G_g^k$
$= q^{g(g+1)} \sum_{k=1}^{M+1} (H_2^q(k-1) q^{-k} G_1^k G_g^k + q^{g(g+1)} \sum_{k=0}^{M} H_2^q(k)(K_{M+1} - q^{-(k+1)} G_1^{k+1})) G_g^k$
$= q^{g(g+1)} \sum_{k=0}^{M} (H_2^q(k) q^{-(k+1)} G_1^{k+1} G_g^{k+1} + q^{g(g+1)} \sum_{k=0}^{M} H_2^q(k)(K_{M+1} - q^{-(k+1)} G_1^{k+1})) G_g^k.\ (1**)$
$(1*) - (1**) = q^{g(g+1)} \sum_{k=0}^{M} H_2^q(k) \{q^{-g} G_1^g G_g^{k+1} - q^{-(k+1)} G_1^{k+1} G_g^{k+1} + q^{-(k+1)} G_1^{k+1} G_g^k\}.\ \{...\} = 0 \rightarrow (1).$

$H_1^q(g, M+1) = q^g G_1^g \sum_{k=0}^{M} H_3^q(k) G_{M-g+1}^{M-k} q^{g(g-1-k)} + (K_{M+1} + G_1^g) \sum_{k=0}^{M} H_3^q(k) G_{M-g}^{M-k} q^{(g+1)(g-k)}.\ (2*)$
$\sum_{k=0}^{M+1} H_3^q(k, M+1) G_{M+1-g}^{M+1-k} q^{(g+1)(g-k)}\ (2**)$
$= \sum_{k=0}^{M} q(q^k G_1^{M+1} - q^{M+1} G_1^k - K_{M+1} q^{M+1}) H_3^q(k) G_{M-g+1}^{M-k} q^{(g+1)(g-k-1)} + \sum_{k=0}^{M} (K_{M+1} + G_1^k) H_3^q(k) G_{M+1-g}^{M+1-k} q^{(g+1)(g-k}$
Items containing $K_{M+1} : K_{M+1} G_{M+1-g}^{M+1-k} q^{(g+1)(g-k)} - K_{M+1} q^{M+2} G_{M+1-g}^{M-k} q^{(g+1)(g-k-1)} = K_{M+1} G_{M-g}^{M-k} q^{(g+1)(g-k)}$.
Items does not contain K_{M+1} in (2*) $= q^g G_1^g G_{M+1-g}^{M-k} q^{g(g-1-k)} + G_1^g G_{M-g}^{M-k} q^{(g+1)(g-k)}$.
Divide by $q^g q^{g(g-1-k)} (q-1)^{-1} = (q^g - 1) G_{M+1-g}^{M-k} + (q^g - 1) G_{M-g}^{M-k} q^{g-k} = (q^g - 1) G_{M+1-g}^{M+1-k}$.
Items does not contain K_{M+1} in (2**) $= \dfrac{q^{M+2} - q^{k+1}}{q-1} G_{M-g+1}^{M-k} q^{(g+1)(g-k-1)} + G_1^k G_{M+1-g}^{M+1-k} q^{(g+1)(g-k)}$.
Divide by $q^g q^{g(g-1-k)} (q-1)^{-1} = (q^{M+1-k} - 1) G_{M+1-g}^{M-k} + (q^g - q^{g-k}) G_{M+1-g}^{M+1-k} = (q^g - 1) G_{M+1-g}^{M+1-k}$.
$(2*) = (2**) \rightarrow (2).$

$H_2^q(g, M+1) = H_2^q(g-1) q^{-g} G_1^g + H_2^q(g)(K_{M+1} - q^{-(g+1)} G_1^{g+1})\ (3*)$
$= q^{-g} G_1^g \sum_{k=0}^{M} (-1)^{k+g-1} H_1^q(k) G_{g-1}^k q^{\frac{g(g-1)-k(k+3)}{2} - k(g-1)} + (K_{M+1} - q^{-(1+g)} G_1^{g+1}) \sum_{k=0}^{M} (-1)^{k+g} H_1^q(k) G_g^k q^{\frac{g(g+1)-k(k+3)}{2} -}$
$\sum_{k=0}^{M+1} (-1)^{k+g} H_1^q(k, M+1) G_g^k q^{\frac{g(g+1)-k(k+3)}{2} - kg}\ (3**)$
$= \sum_{k=0}^{M+1} (-1)^{k+g} (q^k G_1^k H_1^q(k-1) + (K_{M+1} + G_1^k) H_1^q(k)) G_g^k q^{\frac{g(g+1)-k(k+3)}{2} - kg}$
$= \sum_{k=0}^{M} (-1)^{k+g-1} q^{1+k} G_1^{1+k} H_1^q(k) G_g^{1+k} q^{\frac{g(g+1)-(k+1)(k+4)}{2} - (k+1)g} + \sum_{k=0}^{M} (-1)^{k+g} (K_{M+1} + G_1^k) H_1^q(k) G_g^k q^{\frac{g(g+1)-k(k+3)}{2} - kg}.$
Items containing K_{M+1} in (3*) and (3**) $= (-1)^{k+g} K_{M+1} G_g^k q^{\frac{g(g+1)-k(k+3)}{2} - kg}$.
Items does not contain K_{M+1} in (3*):
$= q^{-g} G_1^g (-1)^{k+g-1} G_{g-1}^k q^{\frac{g(g-1)-k(k+3)}{2} - k(g-1)} - q^{-(1+g)} G_1^{1+g} (-1)^{k+g} G_g^k q^{\frac{g(g+1)-k(k+3)}{2} - kg}.$
Divide by $q^{-(1+g)} (-1)^{k+g-1} q^{\frac{g(g+1)-k(k+3)}{2} - kg} (q-1)^{-1}$
$= (q^g - 1) G_{g-1}^k q^{k-g+1} + (q^{1+g} - 1) G_g^k = q^{k+1} G_{g-1}^k - q^{k-g+1} G_{g-1}^k + q^{1+g} G_g^k - G_g^k.\ (A*)$
Items does not contain K_{M+1} in (3**):
$= (-1)^{k+g-1} q^{1+k} G_1^{1+k} G_g^{1+k} q^{\frac{g(g+1)-(k+1)(k+4)}{2} - (k+1)g} + (-1)^{k+g} G_1^k G_g^k q^{\frac{g(g+1)-k(k+3)}{2} - kg}.$

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Divide by $q^{-(1+g)}(-1)^{k+g-1}q^{\frac{g(g+1)-k(k+3)}{2}-kg}(q-1)^{-1}$
$= (q^{1+k}-1)(q^g G_g^k + G_{g-1}^k) - q^{1+g}(q^k-1)G_g^k = q^{1+k}G_{g-1}^k - q^g G_g^k - G_{g-1}^k + q^{1+g}G_g^k.\ (A**)$
$(A*) - (A**) = -q^{k-g+1}G_{g-1}^k - G_g^k - (-q^g G_g^k - G_{g-1}^k) = -G_g^{k+1} + G_g^{k+1} = 0.\ (3*) = (3**) \rightarrow (3).$

$H_3^q(g, M+1) = (q^g G_1^{M+1} - q^{M+2}G_1^{g-1} - K_{M+1}q^{M+2})H_3^q(g-1) + (K_{M+1} + G_1^g)H_3^q(g)$
$= (q^g G_1^{M+1} - q^{M+2}G_1^{g-1} - K_{M+1}q^{M+2})\sum_{k=0}^{M}(-1)^{k+g-1}G_{M-g+1}^{M-k}q^{\frac{(g-1)(g+2)-k(k+3)}{2}}H_1^q(k)$
$+ (K_{M+1} + G_1^g)\sum_{k=0}^{M}(-1)^{k+g}G_{M-g}^{M-k}q^{\frac{g(g+3)-k(k+3)}{2}}H_1^q(k).\ (4*)$
$\sum_{k=0}^{M+1}(-1)^{k+g}H_1^q(k, M+1)G_{M+1-g}^{M+1-k}q^{\frac{g(g+3)-k(k+3)}{2}}\ (4**)$
$= \sum_{k=0}^{M+1}(-1)^{k+g}(q^k G_1^k H_1^q(k-1) + (K_{M+1} + G_1^k)H_1^q(k))G_{M+1-g}^{M+1-k}q^{\frac{g(g+3)-k(k+3)}{2}}$
$= \sum_{k=0}^{M}(-1)^{k+g-1}q^{1+k}G_1^{1+k}H_1^q(k)G_{M+1-g}^{M-k}q^{\frac{g(g+3)-(k+1)(k+4)}{2}} + \sum_{k=0}^{M}(-1)^{k+g}(K_{M+1} + G_1^k)H_1^q(k)G_{M+1-g}^{M+1-k}q^{\frac{g(g+3)-k(k+3)}{2}}.$
Items containing K_{M+1} in (4*):
$= -K_{M+1}q^{M+2}(-1)^{k+g-1}G_{M-g+1}^{M-k}q^{\frac{(g-1)(g+2)-k(k+3)}{2}} + (-1)^{k+g}K_{M+1}G_{M-g}^{M-k}q^{\frac{g(g+3)-k(k+3)}{2}}.$
Divide by $K_{M+1}(-1)^{k+g}q^{\frac{g(g+3)-k(k+3)}{2}} = q^{M+1-g}G_{M-g+1}^{M-k} + G_{M-g}^{M-k} = G_{M-g+1}^{M-k+1}$=that in (4**) divided by (...).
Items does not contain K_{M+1} in (4*):
$= (q^g G_1^{M+1} - q^{M+2}G_1^{g-1})(-1)^{k+g-1}G_{M-g+1}^{M-k}q^{\frac{(g-1)(g+2)-k(k+3)}{2}} + G_1^g(-1)^{k+g}G_{M-g}^{M-k}q^{\frac{g(g+3)-k(k+3)}{2}}.$
Divide by $(-1)^{k+g-1}q^{\frac{g(g+3)-k(k+3)}{2}}(q-1)^{-1}$
$= [(q^g G_1^{M+1} - q^{M+2}G_1^{g-1})G_{M-g+1}^{M-k}q^{-g-1} - G_1^g G_{M-g}^{M-k}](q-1)^{-1} = (q^{M-g-1} - q^{-1})G_{M-g+1}^{M-k} - (q^g - 1)G_{M-g}^{M-k}.\ (B*)$
Items does not contain K_{M+1} in (4**):
$= (-1)^{k+g-1}q^{1+k}G_1^{1+k}G_{M+1-g}^{M-k}q^{\frac{g(g+3)-(k+1)(k+4)}{2}} + (-1)^{k+g}G_1^k G_{M+1-g}^{M+1-k}q^{\frac{g(g+3)-k(k+3)}{2}}.$
Divide by $(-1)^{k+g-1}q^{\frac{g(g+3)-k(k+3)}{2}}(q-1)^{-1}$
$= (q^{1+k}G_1^{1+k}G_{M+1-g}^{M-k}q^{-k-2} - G_1^k G_{M+1-g}^{M+1-k})(q-1)^{-1} = (q^k - q^{-1})G_{M+1-g}^{M-k} - (q^k - 1)G_{M+1-g}^{M+1-k}.\ (B**)$
$(B*) - (B**) = (q^{M-g-1} - 1)G_{M-g+1}^{M-k} - (q^{g-k} - 1)G_{M-g}^{M-k} = 0.\ (4*) = (4**) \rightarrow (4).$

□

Definition 5. $H^q(g, \sum T) = H^q(g, \sum T, PS, PT) = H^q(g, \sum T, M) = \sum\prod_{X_i \in \mathbb{T}} B_i, H^q(g, \sum K) = \sum\prod_{X_i \in \mathbb{K}} B_i$

Definition 6. $F_0^{\mathbb{K}} = E_0^{\mathbb{K}} = E_0^{N,q} = E_0^{N,q-} = 1,$

$F_g^{\mathbb{K}} = \sum_{1 \le \lambda_1 < ... < \lambda_g \le M}\prod_{i=1}^{g} K_{\lambda_i}, F_g^N = F_g^{\{1,2...N\}}.\ E_g^{\mathbb{K}} = \sum_{1 \le \lambda_1 \le ... \le \lambda_g \le M}\prod_{i=1}^{g} K_{\lambda_i}, E_g^N = E_g^{\{1,2...N\}}.$
$E_g^{N,q} = \sum_{1 \le \lambda_1 \le ... \le \lambda_g \le N}\prod_{i=1}^{g}[\lambda_i]_q = S_2^q(N+g, N).\ E_g^{N,q-} = \sum_{1 \le \lambda_1 \le ... \le \lambda_g \le N}\prod_{i=1}^{g}[\lambda_i]_{q-} = S_2^{q-}(N+g, N).$

Theorem 5. $PT = [T, T+1...T+M-1], D_i = 1, H_1^q(g, \sum K) = F_{M-g}^{\mathbb{K}}E_0^{g,q} + F_{M-g-1}^{\mathbb{K}}E_1^{g,q} + ... + F_0^{\mathbb{K}}E_{M-g}^{g,q}.$

Proof.

$$\begin{aligned}
&PS1 = [PS, K_{M+1}], PT1 = [PT, T_{M+1}], \mathbb{L} = \{K_1, K_2 ... K_M, K_{M+1}\}, \\
&H_1^q(g, \sum K, M+1) = H_1^q(g-1, \sum K) + H_1^q(g, \sum K)(K_{M+1} + G_1^g). \ F_A^{\mathbb{L}} \text{ has three sources.} \\
&= F_A^{\mathbb{K}} E_{M-(g-1)-A}^{g-1,q} + F_{A-1}^{\mathbb{K}} E_{M-g-(A-1)}^{g,q} K_{M+1} + F_A^{\mathbb{K}} E_{M-g-A}^{g,q} G_1^g \\
&= F_A^{\mathbb{K}} (E_{M+1-g-A}^{g-1,q} + E_{M-g-A}^{g,q} G_1^g) + F_{A-1}^{\mathbb{K}} E_{M+1-g-A}^{g,q} K_{M+1} \\
&= F_A^{\mathbb{K}} E_{M+1-g-A}^{g,q} + F_{A-1}^{\mathbb{K}} E_{M+1-g-A}^{g,q} K_{M+1} = F_A^{\mathbb{L}} E_{M+1-g-A}^{g,q}.
\end{aligned}$$

□

In this article, $\sum_{g=0}^{M} a_g G_{1+p+g}^{X} = \sum_{g=0}^{M} b_g G_{1+p+g}^{X+g} = \sum_{g=0}^{M} c_g G_{1+p+M}^{X+M-g}, 1+p \geq 0, a_g^* = a_g q^{-pg}, c_g^* = c_g q^{-pg}$.
$D_i = 1, PT = [1, 2 ... M]$, $\nabla_q^{-p} SUM_q(X-p) = \sum_{g=0}^{M} H_1^q(g) q^{pg} G_{1+p+g}^{X}, H_1^q(M) = [M!]_{q+}$.
We can choose $\mathbb{K}$ such $H_1^q(g < M) p^{pg}$ can take any value, $\sum_{g=0}^{M} a_g G_{1+p+g}^{X}$ can be converted to $\frac{a_M q^{-pM}}{[M!]_{q+}} \nabla_q^{-p} SUM_q(X-p)$.
then $a_g^* = c \times H_1^q(g), b_g = c \times H_2^q(g), c_g^* = c \times H_3^q(g)$. c is a constant. Similarly, for any PT, $SUM_q(N)$ can be converted into constant $\times SUM_q(N, PS1, [T_M - M + 1 ... T_M - 1, T_M])$. From [4], [3(3)]:

Theorem 6.
$a_g^* = q^{g(g+1)} \sum_{k=g}^{M} b_k G_g^k = \sum_{k=0}^{g} G_{M-g}^{M-k} q^{(g+1)(g-k)} c_k^*$.
$b_g = (-1)^g q^{\frac{g(g+1)}{2}} \sum_{k=g}^{M} (-1)^k G_g^k q^{\frac{-k(k+3)}{2} - kg} a_k^*$. $c_g^* = (-1)^g q^{\frac{g(g+3)}{2}} \sum_{k=0}^{g} (-1)^k G_{M-g}^{M-k} q^{\frac{-k(k+3)}{2}} a_k^*$.
If $\sum_{g=0}^{M} a_g G_{1+p+g}^{X}$ *can be converted into* $\sum_{g=0}^{M-R} (...) G_{1+p+R+g}^{X+R}, 0 < R \leq M$*, then*
$\sum_k (-1)^k G_g^k q^{\frac{-k(k+3)}{2} - kg} a_k^* = 0, 0 \leq g < R, \sum_k (-1)^k G_{M-g}^{M-k} q^{\frac{-k(k+3)}{2}} a_k^* = 0, 0 \leq M - g < R.$

The latter part refers to the necessary and sufficient conditions for merging, which correspond to $b_g = c_{M-g}^* = 0$.
$SUM_q(N, [[T]_{-q}, PS], [T, PT]) = [T]_{-q} SUM_q(N)$. By utilizing this, we can extend $SUM_q(N)$. As long as the Y of G_Y^X is greater than -1, $T_M - M \geq -1$, then $T_i \geq T_{i+1}$ is also allowed. For Example:
$SUM_q(N, [K], [2]) = q^1 G_1^2 G_3^{N+1} + K G_2^{N+1} = q^4 G_1^2 G_3^N + (q^1 G_1^2 + K q^2) G_2^N + ... = q^7 G_1^2 G_3^{N-1} + \{q^4 G_1^2 + q^3 G_1^2 + K q^4\} G_2^{N-1} + ...$
$= \frac{SUM_q(N, [[1]_{q-}, [1]_{q-}, K], [1,1,2])}{[1]_{q-} [1]_{q-}} = \frac{SUM_q(N, [[1]_{q-}, [2]_{q-}, K], [1,2,2])}{[1]_{q-} [2]_{q-}} = \frac{SUM_q(N, [[2]_{q-}, [1]_{q-}, K], [2,1,2])}{[2]_{q-} [1]_{q-}}$.
$H_1^q(2, [...], [2,1,2]) = q^1 G_1^2 q^{-1+1} G_1^1 [K + G_1^2] + q^1 G_1^2 [q^{-2}(q^{-1}+1)] q^{(2-1)+1} G_1^{2-1} + [q^{-2} G_1^2] q^1 G_1^0 q^2 G_1^1 = q^{-1} G_1^1 q^{-2} G_1^2 \{...\}$.
$H_1^q(g, [...], [2,1,2]) = H_1^q(g, [...], [1,2,2])$. This way we can expand $Form_1$.

3. Application

Proposition 1.
(1). $\sum_{0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_M \leq N} q^{\sum \lambda_i} = G_M^{N+M} = \sum_g (-1)^g q^{Mg + \frac{g(g+1)}{2}} G_g^M G_{2M}^{N+2M-g}$.
(2). $\sum_{A \leq \lambda_1 < \lambda_2 < ... < \lambda_M \leq B} q^{\sum \lambda_i} = q^{\binom{M}{2} + AM} G_M^{B-A+1}$, $A, B \in \mathbb{Z}$, $\sum_{1 \leq \lambda_1 < ... < \lambda_g \leq M} q^{\sum_{i=1}^{M} \lambda_i} = q^{\binom{g+1}{2}} G_g^M$.
(3). $\prod_{i=1}^{M} (a + q^{A+i} z) = \sum_g q^{\binom{g}{2} + (A+1)g} G_g^M z^g a^{M-g}$.
(4). $\sum_g (-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_K^{N-g} = (q^M; q^{-1})_{M-K}, N \geq M \geq K \geq 0.$
(5). $\sum_g (-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_{M+1}^{N-g} = \sum_{i=0}^{N-M-1} q^{(M+1)i}, N \geq M+1, M > 0.$
(6). $\begin{bmatrix} N+M \\ M \end{bmatrix}_{q^2} = \sum_g (-1)^g q^{g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2} G_{2M}^{N+2M-g}$.

Proof.

$PS = [1,1...1] : 0, PT = [1,3...2M-1], H_1^q(g>0) = 0, H_1^q(0) = 1, SUM(N+1) = G_M^{N+M}$ → pre-equation of (1).
$\sum_{A\le\lambda_1<...<\lambda_M\le B} q^{\Sigma\lambda_i} = q^{A+(A+1)+...+(A+M-1)} \sum_{0\le\lambda_1\le...\le\lambda_M\le B-A-M+1} q^{\Sigma\lambda_i} \to (2)$.
At $H_3^q(g), B_i = \begin{cases} q^{X_T}(-q^{2i-1}), X_i=T_i \\ q^{X_T}=q^{i-X_K}, X_i=K_i \end{cases}$, Extract q^i form B_i, after extraction, $B_i = \begin{cases} q^{X_T}(-q^{i-1}), X_i=T_i \\ q^{-X_K}, X_i=K_i \end{cases}$,
$H_3^q(g) = q^{\binom{M+1}{2}-\binom{M-g+1}{2}}(-1)^g q^{\binom{g+1}{2}-g} \sum_{1\le\lambda_1<...<\lambda_g\le M} q^{\Sigma\lambda_i} = (-1)^g q^{\binom{M+1}{2}-\binom{M-g+1}{2}+\binom{g}{2}+\binom{g+1}{2}} G_g^M$.
Using [4(4)] is easier: $H_3^q(g) = q^{g(M-1)}(-1)^g G_{M-g}^M q^{\frac{g(g+3)}{2}}$. $Form_3 \to$ post-equation of (1).
Comparing the coefficients of z^g on both sides of (3), combined with (2), proves that (3).
$A=-1$, it's Rothe's q-Binomial Theorem $\prod_{i=1}^M (a+q^{i-1}z) = \sum_g q^{\binom{g}{2}} G_g^M z^g a^{M-g}$.

(1) and $\nabla_q^K \to G_{M-K}^{N-K} = \sum_g (-1)^g q^{(M-K)g+\frac{g(g+1)}{2}} G_g^M G_{2M-K}^{N+M-K-g}, N \ge M \ge K \ge 0$.
$K=M, \sum_g (-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_M^{N-g} = 1$
$= \sum_g (-1)^g q^{\frac{g(g+1)}{2}} G_g^M (q^M G_M^{N-g-1} + G_{M-1}^{N-g-1}) = q^M + \sum_g (-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_{M-1}^{N-g-1}$
$\to \sum_g (-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_{M-1}^{N-g-1} = 1 - q^M$. Continuing the same process → (4).
Using the same recursive and inductive methods can obtain (5).

$SUM_q(N+1, [1,1...1] : q-1, [1,3...2M-1]) = \sum_{0\le\lambda_1\le...\le\lambda_M\le N} q^{2\Sigma\lambda_i} = \begin{bmatrix} N+M \\ M \end{bmatrix}_{q^2}$.
$At\ H_1^q(g), B_i = \begin{cases} q^{1+(T_i-T_{i-1})X_{T-1}} G_1^{T_i-X_{K-1}} D_i = q^{1+2X_{T-1}}(q^{2i-1-X_{K-1}}-1) = q^{2X_{T-1}}(q^{i+X_{T-1}}-1),\ X_i=T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}}(K_i+G_1^{X_{T-1}} D_i) = q^{2X_{T-1}} = q^{2X_T}, X_i=K_i \end{cases}$,
Unable to derive a concise expression for $H_1^q(g)$, the same applies to $H_2^q(g)$.
$At\ H_3^q(g), B_i = \begin{cases} q^{1+X_{T-1}}\{(q^{X_{T-1}} G_1^{T_i} - q^{T_i} G_1^{X_{T-1}}) D_i - K_i q^{T_i}\} = -q^{1+2X_{T-1}} = -q^{-1+2X_T}, X_i=T_i \\ q^{X_{T-1}}(K_i+G_1^{X_{T-1}} D_i) = q^{X_{T-1}}(1+G_1^{X_{T-1}}(q-1)) = q^{2X_{T-1}} = q^{2X_T}, X_i=K_i \end{cases}$.
$H_3^q(g) = (-1)^g q^{-g+2(\Sigma_{i=1}^g i)} \sum_{X_i\in\mathbb{K}} \prod q^{2X_T} = (-1)^g q^{g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2}$.

□

(4) is unrelated to N; it is an effect of ∇_q^K, just as the difference table of a polynomial series will have a row of constants.

This derivation: $\sum_g (-1)^g q^{\frac{g(g+1)}{2}-pg} G_g^M G_{K-p}^{N-g-p} = 0, N \geq M > K \geq 0, 0 < p \leq K$.

(5) $\rightarrow q^{(N-M-1)} \nabla_q^1 (\sum_g (-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_{M+1}^{N-g}) = q^{(M+1)(N-M-1)} \rightarrow \sum_g (-1)^g q^{\frac{g(g-1)}{2}} G_g^M G_M^{N-g} = q^{M(N-M)}, N \geq M$.

$\sum_{A \leq \lambda_1 < \lambda_2 < \ldots < \lambda_M \leq B} q^{\sum \lambda_i D} = q^{\binom{M}{2} D + AMD} \begin{bmatrix} B-A+1 \\ M \end{bmatrix}_{q^D}$.

$\prod_{i=1}^{A} (1 + q^{C+Di} z) = \sum_{g=0}^{A} q^{gC} \sum_{1 \leq \lambda_1 \leq \ldots \leq \lambda g \leq A} q^{\sum \lambda_i D} z^g = \sum_{g=0}^{A} q^{\binom{g}{2} D + gD + gC} \begin{bmatrix} A \\ g \end{bmatrix}_{q^D} z^g$.

$\prod_{i=1}^{A} (a + q^{(2i-1)D} z) = \sum_{g=0}^{A} q^{g^2 D} \begin{bmatrix} A \\ g \end{bmatrix}_{q^{2D}} z^g a^{A-g}, \prod_{i=1}^{B} (a^{-1} + q^{(2i-1)D} z^{-1}) = \sum_{g=0}^{B} q^{g^2 D} \begin{bmatrix} B \\ g \end{bmatrix}_{q^{2D}} z^{-g} a^{-(B-g)}$.

$\prod_{i=1}^{A} (a + q^{(2i-1)D} z) \prod_{i=1}^{B} (a^{-1} + q^{(2i-1)D} z^{-1}) = \sum_{k=-B}^{A} z^k f(k)$.

$f(k) = \sum_{i=0}^{A+B} \begin{bmatrix} A \\ i \end{bmatrix}_{q^{2D}} \begin{bmatrix} B \\ i-k \end{bmatrix}_{q^{2D}} q^{i^2 D} q^{(i-k)^2 D} a^{A-i} a^{-(B-(i-k))}$

$= q^{k^2 D} a^{A-B-k} \sum_{i=0}^{A+B} \begin{bmatrix} A \\ i \end{bmatrix}_{q^{2D}} \begin{bmatrix} B \\ i-k \end{bmatrix}_{q^{2D}} q^{2D(i(i-k))} = q^{k^2 D} a^{A-B-k} \begin{bmatrix} A+B \\ A-k \end{bmatrix}_{q^{2D}}$.

The last step used q-Vandermorde identity. $a = D = 1,$ it's MacMahon's q-binomial theorem [2] p.74.

$|q|, |x| < 1, \dfrac{1}{(x;q)_{M+1}} = (1 + x + x^2 + \ldots)(1 + xq + x^2 q^2 + \ldots)(1 + xq^2 + + x^2 q^4 + \ldots)\ldots(1 + xq^M + + x^2 q^{2M} + \ldots)$.

When we multiply this out, the coefficient of x^2 will be $\sum q^a q^b, 0 \leq a \leq b \leq M$, and so forth.

(1) $\rightarrow \dfrac{1}{(x;q)_{M+1}} = \sum_{k=0}^{\infty} G_k^{M+k} x^k, \dfrac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty} \dfrac{x^k}{(q;q)_k}$, it's Euler and Cauchy's identity [2] pp 121-123.

(3) $\rightarrow (-x,q)_M = \sum_{g=0}^{M} q^{\binom{g}{2}} G_g^M x^g \rightarrow (-x,q)_\infty = \sum_{k=0}^{\infty} \dfrac{q^{\binom{k}{2}} x^k}{(q;q)_k}$, it's Euler's identity [2] p.129.

$C \geq 2, M > 1, X = C(M-1) - M + 2,$ $At\ H_3^q(g, [1, 1\ldots1] : q-1, [1, C+1\ldots C(M-1)+1]), B_i = \begin{cases} -q^{CX_{T-1}+1}, X_i = T_i \\ q^{CX_T}, X_i = K_i \end{cases}$,

$f_i = \begin{cases} 1, i \not\equiv 1 \pmod{C-1} \\ 2, i \equiv 1 \pmod{C-1} \end{cases}, \sum_{0 \leq \lambda_1 \leq \ldots \leq \lambda_X \leq N-1} q^{\sum f_i \lambda_i} = \sum_g (-1)^g q^{g + C\frac{g(g-1)}{2}} \begin{bmatrix} M \\ g \end{bmatrix}_{q^C} G_{C(M-1)+2}^{N+C(M-1)+1-g}$.

Proposition 2. *$A \in \mathbb{Z}$,*
(1). $G_{M+1}^{N+M} = \sum_g q^{(g+1)g} G_g^M G_{1+g}^N; G_{M+1+p}^{N+M+p} = \sum_{g=0}^{M} q^{(1+p+g)g} G_g^M G_{1+p+g}^{N+p} = \sum_{g=0}^{M+p} q^{(g+1)g} G_g^{M+p} G_{1+g}^N, 1 + p \geq 0$.
(2). $\sum_k (-1)^{k+g} G_k^M G_g^k q^{\frac{k(k-1)}{2} - gk + Ak} z^k = q^{-\frac{g(g+1)}{2} + Ag} G_g^M (zq^A; q)_{M-g} z^g$. Generalized Rothe's q-Binomial Theorem.
(3). $\sum_k (-1)^k G_k^M G_{M-g}^{M-k} q^{\frac{k(k-1)}{2} + Ak} z^k = G_g^M (zq^A; q)_g$.

Proof.

$PS1=[[1]_{q-},[2]_{q-}...[M]_{q-}], PS=[[M]_{q-},[M-1]_{q-}...[1]_{q-}], PT=[1,2...M],$
$SUM_q(N,PS1,PT)=[M!]_{q-}G_{M+1}^{N+M}=\sum_g H_1^q(g)G_{1+g}^N=SUM_q(N).$
$In\ H_1^q(g), X_i\in\mathbb{K},\ B_i=[M+1-i]_{q-}+G_1^{X_{T-1}}=q^{-(M-i+1)}G_1^{M+1-i+X_{T-1}}.$
$H_1^q(g,\sum K)=G_1^M G_1^{M-1}...G_1^{g+1}q^{-(M+1)(M-g)}\sum_{1\le\lambda_1<...<\lambda_{M-g}\le M}q^{\sum\lambda_i}=G_1^M...G_1^{g+1}q^{\binom{M-g+1}{2}-(M+1)(M-g)}G_{M-g}^M.$
$\frac{H_1^q(g)}{[M!]_{q-}}=\frac{q^{\sum_{i=1}^g i}[g!]_{q+}H_1^q(g,\sum K)}{[M!]_{q-}}=q^{\binom{M-g+1}{2}-(M+1)(M-g)+\sum_{i=1}^M i+\sum_{i=1}^g i}G_{M-g}^M=q^{g(g+1)}G_g^M$, this and [3(3)] yields (1).

For (2)(3), when g=0 or M, verify directly[1(3)]. Using induction to prove.

Let $f(g)=q^{-\frac{g(g+1)}{2}+Ag}, L(k)=q^{\frac{k(k-1)}{2}-gk+Ak}$.

$$\begin{aligned}
&\sum_k(-1)^{k+g}(G_k^{M-1}+q^{M-k}G_{k-1}^{M-1})G_g^k q^{\frac{k(k-1)}{2}-gk+Ak}z^k\\
&=f(g)G_g^{M-1}(zq^A;q)_{M-1-g}z^g+\sum_k(-1)^{k+g}G_{k-1}^{M-1}(G_g^{k-1}+q^{k-g}G_{g-1}^{k-1})L(k)q^{M-k}z^k\\
&=f(g)G_g^{M-1}(zq^A;q)_{M-1-g}z^g+\sum_k(-1)^{k+1+g}G_k^{M-1}(G_g^k+q^{k+1-g}G_{g-1}^k)L(k+1)q^{M-(k+1)}z^{k+1}\\
&=f(g)G_g^{M-1}(zq^A;q)_{M-1-g}z^g(1-zq^Aq^{M-g-1})+q^{M+A-2g}z\sum_k(-1)^{k+g-1}G_k^{M-1}G_{g-1}^k q^{\frac{k(k-1)}{2}-(g-1)k+Ak}z^k\\
&=f(g)G_g^{M-1}(zq^A;q)_{M-g}z^g+q^{M+A-2g}zf(g-1)G_{g-1}^{M-1}(zq^A;q)_{M-g}z^{g-1}\\
&=f(g)(zq^A;q)_{M-g}z^g(G_g^{M-1}+q^{M-g}G_{g-1}^{M-1})\to(2).
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^M(-1)^k(q^kG_k^{M-1}+G_{k-1}^{M-1})G_{M-g}^{M-k}q^{\frac{k(k-1)}{2}+Ak}z^k\\
&=\sum_{k=0}^M(-1)^kG_k^{M-1}G_{M-g}^{M-k}q^{\frac{k(k-1)}{2}+(A+1)k}z^k+\sum_{k=0}^M(-1)^kG_{k-1}^{M-1}G_{M-1-(g-1)}^{M-1-(k-1)}q^{\frac{k(k-1)}{2}+Ak}z^k.\\
&=\sum_{k=0}^M(-1)^kG_k^{M-1}(G_{M-1-(g-1)}^{M-1-k}+q^{g-k}G_{M-1-g}^{M-1-k})q^{\frac{k(k-1)}{2}+(A+1)k}z^k-z\sum_{k=0}^M(-1)^kG_k^{M-1}G_{M-1-(g-1)}^{M-1-k}q^{\frac{k(k+1)}{2}+A(k+1)}z^k\\
&=G_{g-1}^{M-1}(zq^{1+A};q)_{g-1}-zq^AG_{g-1}^{M-1}(zq^{1+A};q)_{g-1}+q^gG_g^{M-1}(zq^A;q)_g=G_{g-1}^{M-1}(zq^A;q)_g+q^gG_g^{M-1}(zq^A;q)_g\to(3).
\end{aligned}$$

□

$A=0,z=1\to(2)(3)=0.$ (1) and [6] arrived at this conclusion, and (2)(3) was inspired by them.
$(2)\to g+A<M,\sum_k(-1)^kG_g^kG_k^Mq^{\binom{k}{2}-(g+A)k}=0;\ g<M,\sum_k(-1)^kG_g^kG_k^Mq^{\binom{k}{2}-(M-1)k}=0.$
$(3)\to g>A,\sum_k(-1)^kG_{M-g}^{M-k}G_k^Mq^{\binom{k}{2}-Ak}=0;\ g>0,\sum_k(-1)^kG_{M-g}^{M-k}G_k^Mq^{\binom{k}{2}-(g-1)k}=0.$
$(2)\to\sum_kG_k^MG_g^kq^{\frac{k(k-1)}{2}+Ak}z^k=q^{\frac{g(g-1)}{2}+Ag}G_g^M(-zq^{g+A};q)_{M-g}z^g.$
$A=1,z=1,$ (3) is a special case of [1(5)]: $N=M,K=M-g.$

Proposition 3.

(1). $\sum_k(-1)^{k+g}G_k^MG_g^k=\begin{cases}G_g^M(q;q^2)_{\frac{M-g}{2}},M+g\ is\ even\\0,M+g\ is\ odd\end{cases};\sum_k(-1)^{k+g}G_k^MG_{M-g}^{M-k}=\begin{cases}G_g^M(q;q^2)_{\frac{g}{2}},g\ is\ even\\0,g\ is\ odd\end{cases}.$

(2). $\sum_k(-1)^{k+g}G_k^MG_g^kq^k=\begin{cases}q^gG_g^M(q;q^2)_{\frac{M-g}{2}},M+g\ is\ even\\q^gG_g^M(q;q^2)_{\frac{M-g+1}{2}},M+g\ is\ odd\end{cases}.$

(3). $\sum_k(-1)^{k+g}G_k^MG_g^kq^{-k}=\begin{cases}(-1)^{M-g}q^{-M}G_g^M(q;q^2)_{\frac{M-g}{2}},M+g\ is\ even\\(-1)^{M-g}q^{-M}G_g^M(q;q^2)_{\frac{M-g+1}{2}},M+g\ is\ odd\end{cases}.$

Proof.

M+g is odd, $(-1)^{k+g}G_k^M G_g^k = \dfrac{(-1)^{k+g}(q;q)_M}{(q;q)_g(q;q)_{k-g}(q;q)_{M-k}} = -(-1)^{g+(M-k+g)}G_{M-k+g}^M G_g^{M-k+g} \to Sum = 0.$

M+g is even, using induction. Sum $= \sum_k (-1)^{k+g}G_k^{M-1}G_g^k + \sum_k (-1)^{k+g}G_{k-1}^{M-1}G_g^k q^{M-k}$
$= 0 + \sum_k (-1)^{k+g}G_{k-1}^{M-1}G_g^k q^{M-k} = \sum_k (-1)^{k+g}G_{k-1}^{M-1}G_g^{k-1}q^{M-k} + \sum_k (-1)^{k+g}G_{k-1}^{M-1}G_{g-1}^{k-1}q^{M-g}$
$= \sum_k (-1)^{k+g-1}G_{k-1}^{M-1}G_g^{k-1}(1-q^{M-k}) + \sum_k (-1)^{k+g}G_{k-1}^{M-1}G_{g-1}^{k-1}q^{M-g}$
$= \sum_k (-1)^{k+(g+1)}G_k^{M-1}G_{g+1}^k(1-q^{g+1}) + \sum_k (-1)^{(k-1)+(g-1)}G_{k-1}^{M-1}G_{g-1}^{k-1}q^{M-g}$
$= G_{g+1}^{M-1}(q;q^2)_{\frac{M-1-(g+1)}{2}}(1-q^{g+1}) + q^{M-g}G_{g-1}^{M-1}(q;q^2)_{\frac{M-1-(g-1)}{2}}$
$= G_g^{M-1}(q;q^2)_{\frac{M-g-2}{2}}(1-q^{M-g-1}) + q^{M-g}G_{g-1}^{M-1}(q;q^2)_{\frac{M-g}{2}} = (G_g^{M-1} + q^{M-g}G_{g-1}^{M-1})(q;q^2)_{\frac{M-g}{2}}.$
$g = 0$, it's Gauss's q-Binomial theorems [2] p.61. Replace k, g by $M-k, M-g$ to obtain the second equation of (1).

$\sum_k (-1)^{k+g}G_k^M G_g^k(1-q^k) = \sum_k (-1)^{k+g}G_{k-1}^{M-1}G_g^k(1-q^M) = -\sum_k (-1)^{k+g}G_k^{M-1}G_g^{k+1}(1-q^M)$
$= -q^g(1-q^M)\sum_k (-1)^{k+g}G_k^{M-1}G_g^k + (1-q^M)\sum_k (-1)^{k+g-1}G_k^{M-1}G_{g-1}^k.$
M+g is odd $= -q^g(1-q^M)G_g^{M-1}(q;q^2)_{\frac{M-1-g}{2}} = -q^g(1-q^{M-g})G_g^M(q;q^2)_{\frac{M-1-g}{2}} = -q^g G_g^M(q;q^2)_{\frac{M-g+1}{2}}.$
M+g is even $= (1-q^M)G_{g-1}^{M-1}(q;q^2)_{\frac{M-g}{2}} = (1-q^g)G_g^M(q;q^2)_{\frac{M-g}{2}}$. (1) → (2). Similarly, (3) can be obtained.

□

Another Gauss's identity: $\sum_k \begin{bmatrix} M \\ k \end{bmatrix}_{q^2} q^k = (-q;q)_M$ [2] p.65. Inspired by the above form:

Proposition 4. $\sum_k \begin{bmatrix} M \\ k \end{bmatrix}_{q^2} \begin{bmatrix} k \\ g \end{bmatrix}_{q^2} q^k = q^g \begin{bmatrix} M \\ g \end{bmatrix}_{q^2} (-q;q)_{M-g}$. $\sum_k \begin{bmatrix} M \\ k \end{bmatrix}_{q^2} \begin{bmatrix} k \\ g \end{bmatrix}_{q^2} q^{-k} = q^{-M} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2} (-q;q)_{M-g}$.

Proof.

$\sum_k \begin{bmatrix} M \\ k \end{bmatrix}_{q^2} \begin{bmatrix} k \\ g \end{bmatrix}_{q^2} q^k = \sum_k \begin{bmatrix} M-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} k \\ g \end{bmatrix}_{q^2} q^k + \sum_k \begin{bmatrix} M-1 \\ k-1 \end{bmatrix}_{q^2} \begin{bmatrix} k \\ g \end{bmatrix}_{q^2} q^{2M-2k+k}$
$= q^g \begin{bmatrix} M-1 \\ g \end{bmatrix}_{q^2} (-q;q)_{M-1-g} + \{\sum_k \begin{bmatrix} M-1 \\ k-1 \end{bmatrix}_{q^2} \begin{bmatrix} k-1 \\ g \end{bmatrix}_{q^2} q^{2M-k}\} + q^{2M-2k+(2k-2g)} \sum_k \begin{bmatrix} M-1 \\ k-1 \end{bmatrix}_{q^2} \begin{bmatrix} k-1 \\ g-1 \end{bmatrix}_{q^2} q^k$
$= q^g \begin{bmatrix} M-1 \\ g \end{bmatrix}_{q^2} (-q;q)_{M-1-g} + q^{2M-2g+g} \begin{bmatrix} M-1 \\ g-1 \end{bmatrix}_{q^2} (-q;q)_{M-g} + \{\sum_k \begin{bmatrix} M-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} k \\ g \end{bmatrix}_{q^2} q^{2M-k-1}\}.$
$\{...\} = \sum_k \begin{bmatrix} M-1 \\ M-1-k, g, k-g \end{bmatrix}_{q^2} q^{2M-k-1}, k := M-1-(k-g) \to \sum_k \begin{bmatrix} M-1 \\ M-1-k, g, k-g \end{bmatrix}_{q^2} q^{M-g+k} = q^M \begin{bmatrix} M-1 \\ g \end{bmatrix}_{q^2} (-q;q)_{M-1-g}.$
$\sum_k \begin{bmatrix} M \\ k \end{bmatrix}_{q^2} \begin{bmatrix} k \\ g \end{bmatrix}_{q^2} q^k = q^g(-q;q)_{M-g}(\begin{bmatrix} M-1 \\ g \end{bmatrix}_{q^2} + q^{2(M-g)} \begin{bmatrix} M-1 \\ g-1 \end{bmatrix}_{q^2})$. Similarly, the latter equation can be obtained.

□

Proposition 5.
(1). $H_1^q(g, [[T_1]_{q-}, [T_2]_{q-}...[T_M]_{q-}], [T_1, T_2...T_M]) = \prod_{i=1}^M [T_i]_{q-} q^{(g+1+p)g} G_g^M, p = T_M - M.$
(2). $H_1^q(g, [[K]_{q-}, [K+1]_{q-}...[K+M-1]_{q-}], [T, T+1...T+M-1]) = q^{\binom{g+1}{2}} \prod_{i=1}^g [T-1+i]_q \prod_{i=1}^{M-g} [K+M-i]_{q-} G_g^M.$
(3). $\mathbb{K} = \{[K]_{q-}, [K+1]_{q-}...[K+M-1]_{q-}\}, F_{M-g}^{\mathbb{K}} E_0^{g,q} + F_{M-g-1}^{\mathbb{K}} E_1^{g,q} + ... + F_0^{\mathbb{K}} E_{M-g}^{g,q} = \prod_{i=1}^{M-g} [K+M-i]_{q-} G_g^M.$
(4). $H = T_M - M, H_1^q(g, [[T]_{q-}, PS], [T, PT]) = [T]_{q-}(H_1^q(g) + q^{H+g} H_1^q(g-1)).$

Proof.

$\sum_g H_1^q(g)G_{1+p+g}^{N+p} = \prod [T_i]_{q-} G_{T_M+1}^{N+T_M} = \prod [T_i]_{q-} G_{M+p+1}^{N+M+p}$, [2(1)] yields (1), [6] can also reach the conclusion.
At (2), K_i can swap order, let $PS = [[K+M-1]_{q-}...[K+1]_{q-}, [K]_{q-}], PT = PT1$.
$B_i\ of\ H_1^q(g) = \begin{cases} q^{X_T} G_1^{T+X_{T-1}}, X_i = T_i \\ q^{-(K+M)} q^i G_1^{K+M-i+X_T}, X_i = K_i \end{cases}$. $H_1^q(g) = q^{\binom{g+1}{2}} \prod_{i=1}^{g} [T-1+i]_q H_1^q(g, \sum K)$.
$H_1^q(g, \sum K) = \prod_{i=1}^{M-g} [K+M-i]_q q^{-(K+M)(M-g)} q^{\binom{M-g+1}{2}} G_g^M$ [1(2)], this and [5] yields (2)(3).
$SUM_q(N, [[T]_{q-}, PS], [T, PT]) = \sum_{g=0}^{M+1} H_1^q(g, [[T]_{q-}, PS], [T, PT]) G_{H+g}^{N+H-1} = [T]_{q-} \sum_{g=0}^{M} H_1^q(g) G_{1+H+g}^{N+H}$
$= [T]_{q-} \sum_{g=1}^{M+1} H_1^q(g-1) q^{H+g} G_{H+g}^{N+H-1} + [T]_{q-} \sum_{g=0}^{M} H_1^q(g) G_{H+g}^{N+H-1} \to (4)$.

□

$For\ H_1(g), (1) = \prod T_i \binom{M}{g}, (2) = \prod_{i=1}^{g} (T-1+i) \prod_{i=1}^{M-g} (K+M-i) \binom{M}{g}$.

$$\frac{(1-q)^{A+M}}{(q;q)_A (q;q)_M} \nabla_q SUM(N, [[1]_{q-}, [2]_{q-}...[A]_{q-}, [B+1]_{q-}, [B+2]_{q-}...[B+M]_{q-}], [1, 2...A, A+1, A+2...A+M])$$
$$= G_A^{n+A} G_M^{n+M+B} q^{-(...)} = \frac{(1-q)^{A+M}}{(q;q)_A (q;q)_M} [A!]_{q-} \nabla_q SUM(N, [[B+1]_{q-}...[B+M]_{q-}], [A+1...A+M]).$$

So we can obtain the formula for $G_A^{n+A} G_M^{n+M+B}$.

$$SUM_q(N, [[1]_{q-}, [2]_{q-}...[M-1]_{q-}, 1:q-1, 1:q-1...1:q-1], [1, 2...M-1, M+1, M+2...M+K-1])$$
$$= [(M-1)!]_{q-} SUM_q(N, [1, 1...1]:q-1, [M+1, M+2...M+K-1]) = [(M-1)!]_{q-} \sum_{n=0}^{N-1} q^{Kn} G_{(M-1)+1}^{(n+1)+(M-1)}.$$

So we can calculate $\sum_{n=0}^{N-1} q^{Kn} G_M^{n+M}, K > 0$.

$PT1 = [1, 2...M], PT2 = [T+1, T+2...T+M]$,
$\nabla_q SUM(n+1, PS, PT1) = \prod_i (K_i + [n]_q D_i), \nabla_q SUM(n+1, PS, PT2) = \prod_i (K_i + [n]_q D_i) G_T^{n+T}$,
At $H_1^q(g)$, B_i of $PT1 = \begin{cases} q^{X_T} G_1^{X_T} D_i,\ X_i = T_i \\ K_i + G_1^{X_T} D_i, X_i = K_i \end{cases}$, B_i of $PT2 = \begin{cases} q^{X_T} G_1^{T+X_T} D_i,\ X_i = T_i \\ K_i + G_1^{X_T} D_i, X_i = K_i \end{cases}$.
$H_1^q(g, PS, PT2) = H_1^q(g, PS, PT1) \dfrac{G_1^{T+1}...G_1^{T+g}}{G_1^1...G_1^g} = H_1^q(g, PS, PT1) G_g^{T+g} \to$
If $\sum_{g=0}^{M} a_g G_g^n = \prod_{i=1}^{M} (K_i + [n]_q D_i)$ then $\sum_{g=0}^{M} a_g G_g^{T+g} G_{T+g}^{n+T} = \prod_{i=1}^{M} (K_i + [n]_q D_i) G_T^{n+T}$.

(1) is difficult to derive from its definition. Promote it, $\sum Diff$ below is equivalent to X_{T-1}, X_{K-1}.

Definition 7. *Set $\mathbb{T}$ come from p Source: $S_1, S_2...S_p$.*
$Diff(S_x, S_x) = 0, Diff(S_x, S_y) = -Diff(S_y, S_x) = 1,\ x > y.\ Diff(T_i, T_j) = Diff(S_x, S_y),\ T_i \in S_x, T_j \in S_y$.

Proposition 6. $\sum_{g_1+...+g_p=M, g_i=|S_i|} \prod_{i=1}^{M} G_1^{T_i+\sum_{j<i} Diff(T_j,T_i)} q^{\sum_{j<i, Diff(T_j,T_i)=-1} 1} = G_{g_1,g_2...g_p}^{M} \prod_{i=1}^{M} G_1^{T_i},\ T_i \geq i$.

Proof.

Record the sum as $W_q(g_1, g_2, ...g_p, PT)$.
$W(1, 1, [T_1, T_2]) = G_1^{T_1} G_1^{T_2+1} + G_1^{T_1} G_1^{T_2-1} q = G_1^{T_1} G_1^{T_2} G_1^2$, it's holds. Assume that $W_q(g_1, g_2, PT)$ holds,
$W_q(g_1, g_2 + 1, [PT, T_{M+1}]) = T_{M+1} \in Source_1 + T_{M+1} \in Source_2$
$= W_q(g_1, g_2, PT) G_1^{T_{M+1}+g_1} + W_q(g_1 - 1, g_2 + 1, PT) G_1^{T_{M+1}-(g_2+1)} q^{g_2+1}$.
$= (\prod_i G_1^{T_i}) G_{g_1,g_2}^M G_1^{T_{M+1}+g_1} + (\prod_i G_1^{T_i}) G_{g_1-1,g_2+1}^M G_1^{T_{M+1}-(g_2+1)} q^{g_2+1}$.
Just need to prove: $G_{g_1}^M G_1^{T_{M+1}+g_1} + G_{g_1-1}^M G_1^{T_{M+1}-(M-g_1+1)} q^{M-g_1+1} = G_1^{T_{M+1}} G_{g_1}^{M+1}$.
$(Right\ side) \times (q^{M-g_1+1} - 1)/G_{g_1}^M = (q^{T_{M+1}-1} + ... + q + 1)(q^{M+1} - 1) = (1)$.
$(Left\ side) \times (q^{M-g_1+1} - 1)/G_{g_1}^M = (q^{M-g_1+1} - 1) G_1^{T_{M+1}+g_1} + (q^{g_1} - 1) G_1^{T_{M+1}-(M-g_1+1)} q^{M-g_1+1}$
$= (q^{M-g_1+1} - 1)(q^{T_{M+1}+g_1-1} + ... + q + 1) + (q^{g_1} - 1)(q^{T_{M+1}-1} + ... + q^{M-g_1+2} + q^{M-g_1+1}) = (2)$.
$(1) - (2) = 0 \rightarrow$ It's holds when p=2. $W_q(g_1, g_2 + g_3, PT) = G_{g_1,g_2+g_3}^{g_1+g_2+g_3} \prod_i G_1^{T_i}$.
Every product has $g_2 + g_3$ factors come from $Source_2$, divide them to $g_2 \times Source_2 + g_3 \times Source_3$, g_1-factors are invariant, $(g_2 + g_3)$-factors are variant.
$\sum \prod(\text{variant factors}) = W_q(g_2, g_3, [X_1, X_2...X_{g_2+g_3}]) = G_{g_2,g_3}^{g_2+g_3} \prod_{i=1}^{g_2+g_3} G_1^{X_i}$.
$W_q(g_1, g_2, g_3, [T_1, T_2...T_M]) = G_{g_1,g_2+g_3}^{g_1+g_2+g_3} G_{g_2,g_3}^{g_2+g_3} \prod_i G_1^{T_i} = G_{g_1,g_2,g_3}^{g_1+g_2+g_3} \prod_i G_1^{T_i}$.

□

Definition 8. $\langle {M \atop M} \rangle^q = 0, \left\langle {M \atop g} \right\rangle^q = \sum_{\lambda_1+...+\lambda_{g+1}=M-g} \prod_{i=1}^{g+1} [i]_q^{\lambda_i} [1 + \lambda_1]_q [1 + \lambda_1 + \lambda_2]_q ... [1 + \lambda_1 + ... + \lambda_g]_q, \lambda_i \geq 0$.

Easy to obtain: $\left\langle {M \atop g} \right\rangle^q = [M - g]_q \left\langle {M-1 \atop g-1} \right\rangle^q + [g + 1]_q \left\langle {M-1 \atop g} \right\rangle^q, \left\langle {M \atop g} \right\rangle^q = \left\langle {M \atop M-g-1} \right\rangle^q$.

Proposition 7.
$[N]_q^M = \sum_{g=1}^M [g!]_{q+} S_2^q(M, g) q^{-g} G_g^N = \sum_{g=1}^M (-1)^{M-g} [g!]_{q-} S_2^{q-}(M, g) q^g G_g^{N+g-1} = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle^q q^{\binom{M-g}{2}} G_M^{N+g}$.

Proof.

$$\nabla_q^1 SUM(N,[[1]_{q-},[1]_{q-}...[1]_{q-}],[1,2...M]) = \prod(\frac{1}{q}+\frac{q^{N-1}-1}{q-1}) = q^{-M}[N]_q^M$$

$$= q^{-1}\nabla_q^1 SUM(N,[1]_{q-},[1]_{q-}...[1]_{q-}],[2,3...M]) = q^{-1}\sum_{g=0}^{M-1} H_1^q(g)G_{1+g}^{N}q^{-g}. = q^{-1}\sum_{g=0}^{M-1} H_2^q(g)G_{1+g}^{N+g}.$$

$$In\ H_1^q(g), B_i = \begin{cases} q^{1+X_T-1}G_1^{(i+1)-X_{K-1}} = q^{-1}q^{1+X_T}G_1^{1+X_T},\ X_i=T_i \\ q^{-1}+G_1^{X_T-1} = q^{-1}G_1^{1+X_T-1} = q^{-1}G_1^{1+X_T}, X_i=K_i \end{cases}.$$

$$H_1^q(g) = \frac{[(g+1)!]_{q+}}{[1]_{q+}}q^{-g}H_1^q(g,\sum K) = [(g+1)!]_{q+}q^{-g-1}q^{-(M-1-g)}E_{M-1-g}^{g+1,q}.$$

$$q^{-M}[N]_q^M = q^{-1}\sum_{g=0}^{M-1}[(g+1)!]_{q+}q^{-g-1-(M-g-1)}S_2^q(M,g+1)G_{1+g}^{N}q^{-g} \rightarrow Form_1.$$

$$In\ H_2^q(g), B_i = \begin{cases} q^{-(1+X_T)}G_1^{1+X_T},\ X_i=T_i \\ q^{-1}-q^{-(i+1-X_{K-1})}G_1^{i+1-X_{K-1}} = q^{-1}-q^{-(2+X_T)}G_1^{2+X_T} = -q^{-1}q^{-(1+X_T)}G_1^{1+X_T}, X_i=K_i \end{cases}.$$

$$H_2^q(g) = \frac{[(g+1)!]_{q-}}{[1]_{q-}}H_2^q(g,\sum K) = q[(g+1)!]_{q-}(-1)^{M-1-g}q^{-(M-1-g)}E_{M-1-g}^{g+1,q-}.$$

$$q^{-M}[N]_q^M = q^{-1}\sum_{g=0}^{M-1} q[(g+1)!]_{q-}(-1)^{M-1-g}q^{-(M-1-g)}E_{M-1-g}^{g+1,q-}G_{1+g}^{N+g} \rightarrow Form_2.$$

$$\nabla_q^1 SUM(N,[[1]_{q-},[1]_{q-}...[1]_{q-}],[1,2...M]) = \sum_{g=0}^{M} H_3^q(g)G_M^{N+M-1-g}q^{-g}.$$

$$B_i = \begin{cases} q(q^{X_T-1}G_1^i - q^iG_1^{X_T-1} - q^iq^{-1}) = q\frac{q^{i-1}-q^{X_T-1}}{q-1} = q^{X_T}G_1^{i-X_T},\ X_i=T_i \\ q^{-1}+G_1^{X_T-1} = q^{-1}G_1^{1+X_T-1} = q^{-1}G_1^{1+X_T}, X_i=K_i \end{cases}, H_3^q(g) = q^{-(M-g)}q^{\frac{g(g+1)}{2}}\left\langle {M \atop g} \right\rangle^q, H_3^q(M) = 0.$$

$$[N]_q^M = \sum_{g=0}^{M-1}\left\langle {M \atop g} \right\rangle^q q^{\frac{g(g+1)}{2}}G_M^{N+M-1-g} = \sum_{g=0}^{M-1}\left\langle {M \atop M-1-g} \right\rangle^q q^{\frac{(M-1-g)(M-g)}{2}}G_M^{N+g}.$$

□

$$N^M = \sum_{g=1}^{M} g!S_2(M,g)\binom{N}{g} = \sum_{g=1}^{M}(-1)^{M-g}g!S_2(M,g)\binom{N+g-1}{g} = \sum_{g=0}^{M-1}\left\langle {M \atop g} \right\rangle\binom{N+g}{M}.$$

$S_2(M,g)$ is Stirling number of the second kind. $S_2(M,g) = E_{M-g}^{g}$, $\left\langle {M \atop g} \right\rangle$ is Eulerian number,

$\left\langle {M \atop g} \right\rangle = \sum_{\lambda_1+\lambda_2+..+\lambda_{g+1}=M-g}\prod_{i=1}^{g+1} i^{\lambda_i}(1+\lambda_1)(1+\lambda_1+\lambda_2)...(1+\lambda_1+..+\lambda_g)$ [3].

Proposition 8.

(1). $0 \le A < M, 0 \le T, \sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_{A+T+1+g}^{N+A+T} q^{g(g+1+T)} = \sum_{k=0}^{A} G_{k+T}^{A+T} G_{M+T}^{M+T+k} G_{M+T+1+k}^{N+M+T} q^{k(k+1+T)}.$

(2). $0 \le A, B, T, 0 \le A+B < M, \sum_g G_g^M G_A^{A+T+g} G_B^g q^{\binom{g}{2}-g(A+B)}(-1)^g = 0.$

(3). $0 \le K, T, \sum_g (-1)^g G_g^M G_{M+K}^{M+K+T+g} q^{\binom{M+1-g}{2}+(M-g)K} = (-1)^M G_K^{T+M+K}.$

Proof.

$PS = [[T+1]_{q-}, [T+2]_{q-} ... [T+M]_{q-}], PT = [T+A+1, T+A+2 ... T+A+M]),$

$SUM_q(N) =$

$= \sum_{g=0}^{M} G_{A+T+1+g}^{N+A+T} [T+A+1]_q ... [T+A+g]_q \times [T+g+1]_q ... [T+M]_q q^{\binom{g+1}{2} - (T+M+1)(M-g) + \binom{M+1-g}{2}} G_g^M$[5(2)]

$= \frac{(q;q)_A (q^{T+A+1};q)_{M-A}}{(1-q)^M} \sum_{g=0}^{M} G_{A+T+1+g}^{N+A+T} q^{\binom{g+1}{2} - (T+M+1)(M-g) + \binom{M+1-g}{2}} G_A^{A+T+g} G_g^M \ (*)$

$= \prod_{i=1}^{M-A} [T+A+i]_{q-} SUM_q(N, [[T+1]_{q-} ... [T+A]_{q-}], [T+M+1 ... T+M+A]).$ [3(4)]

$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} \prod_{i=1}^{M-A} [T+A+i]_q \sum_{k=0}^{A} G_{M+T+1+k}^{N+M+T} q^{\binom{k+1}{2} - (T+A+1)(A-k) + \binom{A+1-k}{2}}$

$\times [T+M+1]_q ... [T+M+k]_q \times [T+1+k]_q ... [T+A]_q G_k^A$

$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} \sum_{k=0}^{A} G_{M+T+1+k}^{N+M+T} q^{\binom{k+1}{2} - (T+A+1)(A-k) + \binom{A+1-k}{2}} \prod_{i=1}^{M} [T+k+i]_q G_k^A. \ (**)$

Compare (*) and (**) :

$\sum_{g=0}^{M} G_{A+T+1+g}^{N+A+T} q^{\frac{g(1+g)-(M-g)(g+M+1+2T)}{2}} G_A^{A+T+g} G_g^M$

$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} \sum_{k=0}^{A} G_{M+T+1+k}^{N+M+T} q^{\frac{k(1+k)-(A-k)(k+A+1+2T)}{2}} \frac{(q^{T+k+1};q)_M}{(q^{T+A+1};q)_{M-A} (q;q)_k (q;q)_{A-k}}$

$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} \sum_{k=0}^{A} G_{M+T+1+k}^{N+M+T} q^{\frac{k(1+k)-(A-k)(k+A+1+2T)}{2}} G_{M+T}^{M+T+k} G_{k+T}^{A+T} \rightarrow (1).$

[6] and (1) $\rightarrow \sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_B^g q^{g(1+g+T)} q^{\frac{-g(g+3)}{2} - Bg - (A+T)g} (-1)^g = 0 \rightarrow (2).$

$SUM_q(N, [[T+1]_{q-}, [T+2]_{q-} ... [T+M]_{q-}], [T+K+M+1, T+K+M+2 ... T+K+2M]).$

$H_1^q(g) = q^{\binom{g+1}{2} - (T+M+1)(M-g) + \binom{M-g+1}{2}} \prod_{i=1}^{g} [T+K+M+i]_q \prod_{i=1}^{M-g} [T+M+1-i]_q G_g^M$ [5(2)].

$H_2^q(0) = (-1)^M q^{-M(T+K+M+1)} \prod_{i=1}^{M} [K+i]_q = \sum_g (-1)^g H_1^q(g) q^{\frac{-g(g+3)}{2} - (T+K+M)g}$ [6(3)].

$(-1)^M \prod_{i=1}^{M} [K+i]_q = \sum_g (-1)^g q^{\binom{M-g+1}{2} + (M-g)K} \prod_{i=1}^{g} [T+K+M+i]_q \prod_{i=1}^{M-g} [T+M+1-i]_q G_g^M.$

$(-1)^M \prod_{i=1}^{M} [K+i]_q \prod_{i=1}^{K} [T+M+i]_q = \sum_g (-1)^g q^{\binom{M-g+1}{2} + (M-g)K} \prod_{i=1}^{K+M} [T+g+i]_q G_g^M.$

$(-1)^M \prod_{i=1}^{K} [T+M+i]_q = \sum_g (-1)^g q^{\binom{M-g+1}{2} + (M-g)K} \prod_{i=1}^{K+M} [T+g+i]_q / \prod_{i=1}^{M} [K+i]_q \times G_g^M.$

$(-1)^M G_K^{T+M+K} = \sum_g (-1)^g q^{\binom{M-g+1}{2} + (M-g)K} \times G_{M+K}^{T+K+M+g} G_g^M \rightarrow (3),$ the case where $A+B \geq M$ of (2).

□

Proposition 9. $\sum_{n_M=0}^{N-1} ... \sum_{n_1=0}^{n_2} \prod_{i=1}^{M} [K-1+i+2n_i]_q q^{-\sum_{j=1}^{M} n_j} = q^{-(N-1)M} \prod_{i=1}^{M} [K+N-2+i]_q G_M^{M+N-1}.$

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Proof.

$PS = [[K]_{q-}, [K+1]_{q-} ... [K+M-1]_{q-}], PT = [1, 2 ... M].$

$H_1^q(g, \sum K) = q^{-(K+M)(M-g)+\binom{M-g+1}{2}} \prod_{i=1}^{M-g} G_1^{K+g-1+i} G_g^M.$ [5(2)]

$B_i = K_i \, of \, H_1^q(g) = q^{-(K-1+i)} G_1^{K-1+i} + G^{X_T} = q^{-(K-1)-i} G_1^{K-1+i+X_T}$. Expand by $B_i \rightarrow$

$H_1^q(g, \sum K) = \sum_{n_{M-g}=0}^{g} \cdots \sum_{n_1=0}^{n_2} q^{-(K-1)(M-g)-\sum_{j=1}^{M-g} j - \sum_{j=1}^{M-g} n_j} \prod_{i=1}^{M-g} G_1^{K-1+i+2n_i}.$

$\sum_{n_{M-g}=0}^{g} \cdots \sum_{n_1=0}^{n_2} \prod_{i=1}^{M-g} G_1^{K-1+i+2n_i} q^{g(M-g)-\sum_{j=1}^{M-g} n_j} = \prod_{i=1}^{M-g} G_1^{K+g-1+i} G_g^M.$

$\sum_{n_M=0}^{g} \cdots \sum_{n_1=0}^{n_2} \prod_{i=1}^{M} G_1^{K-1+i+2n_i} q^{gM-\sum_{j=1}^{M} n_j} = \prod_{i=1}^{M} G_1^{K+g-1+i} G_g^{M+g}.$

$\sum_{n_M=0}^{N-1} \cdots \sum_{n_1=0}^{n_2} \prod_{i=1}^{M} G_1^{K-1+i+2n_i} q^{(N-1)M-\sum_{j=1}^{M} n_j} = \prod_{i=1}^{M} G_1^{K+N-2+i} G_{N-1}^{M+N-1}.$

□

$K = 1, M = 1 \rightarrow \sum_{n=0}^{N-1} [1+2n]_q q^{-n} = q^{-(N-1)} [N]_q^2.$

4. Extensions of q-Euler Polynomials and Relationships between Three Forms

In this section, $a \neq 0, 1, q^{-1}, q^{-2} ... q^{-M}$....

Lemma 2. $\sum_{n=0}^{N-1} a^n G_M^{n+A} = -a^N \sum_{g=0}^{M} \frac{q^{(N+A-M)g} G_{M-g}^{N+A-1-g}}{(a;q)_{g+1}} + \frac{a^{M-A}}{(a;q)_{M+1}}, 0 \leq A \leq M, N > M - A.$

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Proof.

$A=M, M=0, \sum_{n=0}^{N-1} a^n = \frac{-a^N}{1-a} + \frac{1}{1-a}$, holds.

$A=M, M=1, \sum_{n=0}^{N-1} a^n G_1^{n+1} = \sum_{n=0}^{N-1} a^n (1+q+...+q^n)$

$= (1+q+...+q^{N-1}) \sum_{n=0}^{N-1} a^n - q\sum_{n=0}^{0} a^n - q^2 \sum_{n=0}^{1} a^n ... - q^{N-1} \sum_{n=0}^{N-2} a^n$

$= G_1^N \frac{1-a^N}{1-a} - \frac{q(1-a)}{1-a} - \frac{q^2(1-a^2)}{1-a} ... \frac{q^{N-1}(1-a^{N-1})}{1-a}$

$= G_1^N \frac{1-a^N}{1-a} + \frac{1+aq+...+a^{N-1}q^{N-1}}{1-a} - \frac{1+q+...+q^{N-1}}{1-a} = \frac{G_1^N - G_1^N a^N}{1-a} + \frac{1-a^N q^N}{(1-a)(1-aq)} - \frac{G_1^N}{1-a}$

$= \frac{-a^N q^N}{(1-a)(1-aq)} + \frac{-a^N G_1^N}{1-a} + \frac{1}{(1-a)(1-aq)}$, holds.

$A=M, N=1, -a^1 \sum_{g=0}^{M} \frac{q^g}{(a;q)_{g+1}} + \frac{1}{(a;q)_{M+1}} = \frac{1-aq^M}{(a;q)_{M+1}} - a^1 \sum_{g=0}^{M-1} \frac{q^g}{(a;q)_{g+1}}$

$= \frac{(1-aq^M)(1-aq^{M-1})}{(a;q)_{M+1}} - a^1 \sum_{g=0}^{M-2} \frac{q^g}{(a;q)_{g+1}} = ... = 1$, holds. Assume that N holds.

$\sum_{n=0}^{N} a^n G_M^{n+M} = \sum_{n=0}^{N} a^n (q^M G_M^{n-1+M} + G_{M-1}^{n-1+M}) = aq^M \sum_{n=0}^{N-1} a^n G_M^{n+M} + \sum_{n=0}^{N} a^n G_{M-1}^{n+M-1}$

$= -q^M a^{N+1} \sum_{g=0}^{M} \frac{q^{Ng} G_{M-g}^{N+M-1-g}}{(a;q)_{g+1}} + \frac{aq^M}{(a;q)_{M+1}} - a^{N+1} \sum_{g=0}^{M-1} \frac{q^{(N+1)g} G_{M-1-g}^{N+M-1-g}}{(a;q)_{g+1}} + \frac{1}{(a;q)_M}$

$= -a^{N+1} \sum_{g=0}^{M} \frac{q^{Ng+M} G_{M-g}^{N+M-1-g} + q^{(N+1)g} G_{M-1-g}^{N+M-1-g}}{(a;q)_{g+1}} + \frac{1}{(a;q)_{M+1}}$

$= -a^{N+1} \sum_{g=0}^{M} \frac{q^{(N+1)g} q^{M-g} G_{M-g}^{N+M-1-g} + q^{(N+1)g} G_{M-1-g}^{N+M-1-g}}{(a;q)_{g+1}} + \frac{1}{(a;q)_{M+1}}$

$= -a^{N+1} \sum_{g=0}^{M} \frac{q^{(N+1)g} G_{M-g}^{(N+1)+M-1-g}}{(a;q)_{g+1}} + \frac{1}{(a;q)_{M+1}}$. Proof of A=M completed.

$\sum_{n=0}^{N-1} a^n G_M^{n+A} = a^{M-A} \sum_{n=0}^{N-M+A-1} a^n G_M^{n+M}$, complete the remaining proofs.

□

Theorem 7. $X = T_M - M - y \geq -1, 0 \leq Y \leq 1, f(g) = (aq^{2+X+g};q)_{M-g} = (aq^{2+T_M-M-y+g};q)_{M-g}$,
(1). $\sum_g H_1^q(g) a^g q^{-yg} f(g) = \sum_g H_2^q(g) f(g) = \sum_g H_3^q(g) a^g q^{-yg}$,*define as* $A_a^q(PS,PT,y)$.
(2). $\sum_{n=0}^{N-1} a^n \nabla_q^y SUM_q(n+Y) = -a^N \sum_{k=0}^{M} \frac{q^{(N+Y-1)k} \nabla_q^{y+k} SUM_q(N+Y-1)}{(a;q)_{k+1}} + \frac{a^{1-Y} A_a^q(PS,PT,y)}{(a;q)_{T_M+2-y}}$.
(3). $|a|,|q|<1, \sum_{n=0}^{\infty} a^n \nabla_q^y SUM_q(n+Y) = \frac{a^{1-Y} A_a^q(PS,PT,y)}{(a;q)_{T_M+2-y}}$.
(4). $\nabla_q^y SUM_q(\infty) = \frac{A_q^q(PS,PT,1+y)}{(q;q)_{T_M+1-y}}, -\sum_{k=0}^{R} \frac{q^{(N-R)(k+1)} G_{R-k}^{N-1-k}}{(q;q)_{k+1}} + \frac{1}{(q;q)_{1+R}} = G_{1+R}^N$.

Proof.

$$\sum_{n=0}^{N-1} a^n \nabla_q^y SUM_q(n+Y) = \sum_{g=0}^{M} H_1^q(g) \sum_{n=0}^{N-1} a^n G_{1+X+g}^{n+Y+X} q^{-yg}$$
$$= \sum_{g=0}^{M} H_1^q(g)(-a^N \sum_{k=0}^{1+X+g} \frac{q^{(N+Y-1-g)k} q^{-yg} G_{1+X+g-k}^{N+Y+X-1-k}}{(a;q)_{k+1}} + \frac{a^{1+g-Y} q^{-yg}}{(a;q)_{2+X+g}}).$$
$$= -a^N \sum_{k=0}^{M} \frac{q^{(N+Y-1)k} \sum_{g=0}^{M} H_1^q(g) q^{-(y+k)g} G_{1+X+g-k}^{N+Y+X-1-k}}{(a;q)_{k+1}} + \frac{\sum_{g=0}^{M} H_1^q(g) a^{1+g-Y} q^{-yg} f(g)}{(a;q)_{T_M+2-y}}$$
$$= -a^N \sum_{k=0}^{M} \frac{q^{(N+Y-1)k} \nabla_q^{y+k} SUM_q(N+Y-1)}{(a;q)_{k+1}} + \frac{\sum_{g=0}^{M} H_1^q(g) a^{1+g-Y} q^{-yg} f(g)}{(a;q)_{T_M+2-y}}.$$

$$\sum_{n=0}^{N-1} a^n \nabla_q^y SUM_q(n+Y) = \sum_{g=0}^{M} H_2^q(g) \sum_{n=0}^{N-1} a^n G_{1+X+g}^{n+Y+X+g}$$
$$= \sum_{g=0}^{M} H_2^q(g)(-a^N \sum_{k=0}^{1+X+g} \frac{q^{(N+Y-1)k} G_{1+X+g-k}^{N+Y-1+X+g-k}}{(a;q)_{k+1}} + \frac{a^{1-Y}}{(a;q)_{2+X+g}}).$$
$$= -a^N \sum_{k=0}^{M} \frac{q^{(N+Y-1)k} \nabla_q^{y+k} SUM_q(N+Y-1)}{(a;q)_{k+1}} + \frac{a^{1-Y} \sum_{g=0}^{M} H_2^q(g) f(g)}{(a;q)_{T_M+2-y}}.$$

$$\sum_{n=0}^{N-1} a^n \nabla_q^y SUM_q(n+Y) = \sum_{g=0}^{M} H_3^q(g) \sum_{n=0}^{N-1} a^n G_{T_M+1-y}^{n+Y+T_M-y-g} q^{-yg}$$
$$= \sum_{g=0}^{M} H_3^q(g)(-a^N \sum_{k=0}^{T_M+1-y} \frac{q^{(N+Y-1)k} q^{-(y+k)g} G_{T_M+1-y-k}^{N+Y+T_M-y-g-1-k}}{(a;q)_{k+1}} + \frac{a^{1+g-Y} q^{-yg}}{(a;q)_{T_M+2-y}}).$$
$$= -a^N \sum_{k=0}^{M} \frac{q^{(N+Y-1)k} \nabla_q^{y+k} SUM_q(N+Y-1)}{(a;q)_{k+1}} + \frac{\sum_{g=0}^{M} H_3^q(g) a^{1+g-Y} q^{-yg}}{(a;q)_{T_M+2-y}}.$$

Three summations are identical → (1)(2). (3) is obvious. $SUM_q(N) = \sum_{n=0}^{N-1} q^n \nabla_q^1 SUM_q(n+1) \rightarrow$ (4).
Calculations show that the equations derived by $Form_2$ and $Form_3$ are the same as the latter half of (4).

□

$\sum_{n=0}^{\infty} a^n [n]_q^M = \frac{E_M(a,q)}{(a;q)_{M+1}}$, $E_M(a,q)$ is q-Eularian polynomials[2] p.332. [7] → three expressions for $E_M(a,q)$.
Eularian polynomials: $\sum_{n \geq 1} a^n n^M = \frac{a A_M(a)}{(1-a)^{M+1}}$. $A_M(a) = \sum_g H_1(g) a^g (1-a)^{M-g} = \sum_g H_2(g)(1-a)^{M-g} = \sum_g H_3(g) a^g$
$= \sum_g g! S_2(M,g) a^g (1-a)^{M-g} = \sum_g (-1)^{M-g} g! S_2(M,g)(1-a)^{M-g} = \sum_g \left\langle {M \atop g} \right\rangle a^g$.

At [6], some relationships have been obtained, and now the remaining ones can be deduced:

Theorem 8.
(1). $c_g^* = (-1)^g q^{\frac{g(g+3)}{2}} \sum_k b_k G_g^{M-k} q^{gk} = (-1)^g q^{\frac{g(g+3)}{2}+Mg} \sum_k b_{M-k} G_g^k q^{-gk}$.
(2). $b_{M-g} = (-1)^g q^{\frac{g(g-1)}{2}} \sum_k c_k^* G_g^k q^{-(M+1)k}$, $b_g = (-1)^{M-g} q^{\frac{(M-g)(M-g-1)}{2}} \sum_k c_k^* G_{M-g}^k q^{-(M+1)k}$.
(3). If $\sum_{g=0}^{M} b_g G_{1+p+g}^{X+g}$ *can be converted into* $\sum_{g=0}^{M-R} (...) G_{1+p+R+g}^{X+g}, 0 < R \leq M$, *then*
$\sum_k G_g^k b_k = 0, 0 \leq g < R, \sum_k G_g^{M-k} q^{gk} b_k = 0, 0 \leq g < R$.
(4). If $\sum_{g=0}^{M} c_g G_{1+p+M}^{X+M-g}$ *can be converted into* $\sum_{g=0}^{M-R} (...) G_{1+p+M-R}^{X+M-R-g}, 0 < R \leq M$, *then*
$\sum_k G_{M-g}^{M-k} q^{-k(g+1)} c_k^* = 0, 0 \leq M-g < R, \sum_k G_g^k q^{-(M+1)k} c_k^* = 0, 0 \leq g < R$.

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Proof.

At [7], $PT = [1, 2...M], f(g) = (1 - aq^{2+g})(1 - aq^{3+g})...(1 - aq^{M+1})$,
$\sum_g b_g f(g) = \sum_g b_g \sum_k G_k^{M-g} (-a)^k q^{\frac{g(g-1)}{2}+(2+g)k} = \sum_g c_g q^{-pg} a^g$. Compare a^g on both sides $\rightarrow$ (1).

$\sum_x c_x = \sum_x (-1)^x q^{\frac{x(x+1)}{2}+(M+p+1)x} \sum_k b_{M-k} G_x^k q^{-xk}$.
$-\dfrac{x(x+1)}{2} - (M+p+1)x + \dfrac{(x-g)(x-g-1)}{2} + gx + x = \dfrac{g(g+1)}{2} - (M+p+1)x$.
$\sum_x c_x G_g^x q^{\frac{g(g+1)}{2}-(M+p+1)x} = \sum_x (-1)^x G_g^x q^{\frac{(x-g)(x-g-1)}{2}+gx+x} \sum_k b_{M-k} G_x^k q^{-xk}$.
$= \sum_k b_{M-k} \sum_x (-1)^x G_g^x G_x^k q^{\frac{(x-g)(x-g-1)}{2}} q^{(g+1-k)x} = \sum_k b_{M-k} q^{(...)} \sum_x (-1)^x G_g^x G_x^k q^{\frac{x(x-1)}{2}-(k-1)x}.(*)$
[2(3)] $\rightarrow k > g, \sum_{x=g}^k (...) = 0 \rightarrow (*) = (-1)^g b_{M-g} q^g \rightarrow (2)$.
(3) is correspond to $a_g^* = c_g^* = 0$, (4) is correspond to $a_{M-g}^* = b_{M-g} = 0$.

□

Theorem 9. $A \in \mathbb{Z}$,
(1). $\sum_g a_g^* q^{-\frac{g(g+1)}{2}+Ag} a^{M-g} z^g = a^M \sum_g b_g (-q^{A+1}\frac{z}{a}; q)_g = \sum_g c_g^* q^{-\frac{g(g+1)}{2}+Ag} a^{M-g} z^g (-q^{A+1}\frac{z}{a}; q)_{M-g}$.
(2). $\sum_g b_g q^{Ag} a^{M-g} z^g = a^M \sum_g (-1)^g a_g^* q^{-\frac{g(g+3)}{2}} (q^{A-g+1}\frac{z}{a}; q)_g = q^{AM} z^M \sum_g c_g^* q^{-(M+1)g} (q^{-A}\frac{a}{z}; q)_g$.
(3). $\sum_g c_g^* q^{Ag} a^{M-g} z^g = \sum_g a_g^* q^{Ag} a^{M-g} z^g (q^{A+2+g}\frac{z}{a}; q)_{M-g} = a^M \sum_g b_g (q^{A+2+g}\frac{z}{a}; q)_{M-g}$.

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Proof.

$$\sum_g a_g^* q^{-g(g+1)} q^{\frac{g(g-1)}{2}+(A+1)g} a^{M-g} z^g = \sum_g \sum_k b_k G_g^k q^{\frac{g(g-1)}{2}+(A+1)g} a^{M-g} z^g$$

$$= \sum_k b_k a^{M-k} \sum_g G_g^k q^{\frac{g(g-1)}{2}+(A+1)g} a^{k-g} z^g = \sum_k b_k a^{M-k} \prod_{i=1}^{k} (a + q^{A+i} z).$$

$$\sum_g a_g^* q^{-\frac{g(g+3)}{2}+(A+1)g} a^{M-g} z^g = \sum_k c_k^* q^{-\frac{k(k+3)}{2}+(A+1)k} z^k \sum_g G_{g-k}^{M-k} q^{\binom{g-k}{2}+(A+1)(g-k)} a^{(M-k)-(g-k)} z^{g-k}$$

$$= \sum_k c_k^* q^{-\frac{k(k+1)}{2}+Ak} z^k \prod_{i=1}^{M-k} (a + q^{A+i} z) \to (1).$$

$$\sum_g b_g q^{Ag} a^{M-g} z^g = \sum_g q^{Ag} a^{M-g} z^g \sum_k (-1)^{k+g} G_g^k q^{\frac{g(g+1)-k(k+3)}{2}-kg} a_k^*$$

$$= \sum_k (-1)^k a_k^* q^{-\frac{k(k+3)}{2}} a^{M-k} \sum_g G_g^k q^{\binom{g}{2}+(A-k+1)g} a^{k-g} (-z)^g = \sum_k (-1)^k a_k^* q^{-\frac{k(k+3)}{2}} a^{M-k} \prod_{i=1}^{k} (a - q^{A-k+i} z).$$

$$\sum_g b_g q^{Ag} a^{M-g} z^g = \sum_g q^{Ag} a^{M-g} z^g (-1)^{M-g} q^{\frac{(M-g)(M-g-1)}{2}} \sum_k G_{M-g}^k q^{-(M+1)k} c_k^*$$

$$= \sum_k q^{-(M+1)k} c_k^* \sum_g a^{M-g} z^g (-1)^{M-g} G_{M-g}^k q^{\frac{(M-g)(M-g-1)}{2}+Ag}, \text{ replace g with M-g } \to$$

$$= \sum_k q^{-(M+1)k} c_k^* \sum_g z^{M-g} (-a)^g G_g^k q^{\frac{g(g-1)}{2}+A(M-g)} = q^{AM} \sum_k q^{-(M+1)k} c_k^* z^{M-k} \prod_{i=1}^{k} (z - q^{-A-1+i} a) \to (2).$$

$$\sum_g c_g^* q^{Ag} a^{M-g} z^g = \sum_g \sum_k (-1)^{k+g} a_k^* G_{g-k}^{M-k} q^{\frac{g(g+3)}{2}-\frac{k(k+3)}{2}+Ag} a^{M-g} z^g$$

$$= \sum_k (-1)^k a_k^* q^{-\frac{k(k+3)}{2}} \sum_g (-1)^g G_{g-k}^{M-k} q^{\frac{g(g+3)}{2}+Ag} a^{M-g} z^g, \text{ replace g with g+k} \to$$

$$= \sum_k (-1)^k a_k^* q^{-\frac{k(k+3)}{2}} \sum_g (-1)^{g+k} G_g^{M-k} q^{\frac{(g+k)(g+k+3)}{2}+A(g+k)} z^{g+k} a^{M-k-g}$$

$$= \sum_k a_k^* q^{Ak} z^k \sum_g G_g^{M-k} q^{\frac{g(g-1)}{2}+(A+k+2)g} (-z)^g a^{M-k-g} = \sum_k a_k^* q^{Ak} z^k \prod_{i=1}^{M-k} (a - q^{A+k+1+i} z).$$

$$\sum_g c_g^* q^{Ag} a^{M-g} z^g = \sum_k b_k \sum_g (-1)^g q^{\frac{g(g+3)}{2}} G_g^{M-k} q^{gk+Ag} a^{M-g} z^g$$

$$= \sum_k b_k a^k \sum_g (-1)^g q^{\frac{g(g-1)}{2}+(A+k+2)g} G_g^{M-k} a^{M-k-g} z^g = \sum_k b_k a^k \prod_{i=1}^{M-k} (a - q^{A+k+1+i} z) \to (3).$$

□

Combining [6], [8], [9] and $\sum a_g^* q^{pg} G_{1+p+g}^{X} = \sum b_g G_{1+p+g}^{X+g} = \sum c_g^* q^{pg} G_{1+p+M}^{X+M-g}$,
we can arbitrarily specify one set of values, calculate the other two sets,
and treat A, a, z as an independent variable or part of a_g, b_g, c_g to derive the corresponding relationship.

$a_g^{*,1} = q^{g(g+1)+Ag} G_g^M, a_g^{*,2} = a_g^{*,1} z^g$ [2(2)(3)] $\rightarrow$
$b_g^{*,1} = q^{Ag} G_g^M (q^A;q)_{M-g}, b_g^{*,2} = q^{Ag} G_g^M (zq^A;q)_{M-g} z^g, c_g^{*,1} = (-1)^g q^{\frac{g(g+3)}{2}} G_g^M (q^A;q)_g, c_g^{*,2} = (-1)^g q^{\frac{g(g+3)}{2}} G_g^M (zq^A;q)_g -$
$\sum_g q^{g(g+1+A+p)} G_g^M z^g G_{1+p+g}^N = \sum_g q^{Ag} (zq^A;q)_{M-g} G_g^M G_{1+p+g}^{N+g} = \sum_g (-1)^g q^{\frac{g(g+3)}{2}+pg} G_g^M (zq^A;q)_g G_{1+p+M}^{N+M-g}$.
$\sum_g a_g^{*,1} q^{-\frac{g(g+1)}{2}} z^g = \sum_g a_g^{*,2} q^{-\frac{g(g+1)}{2}} = (-zq^{1+A};q)_M$
$= \sum_g q^{Ag} G_g^M (q^A;q)_{M-g} (-zq;q)_g \ (**) = \sum_g q^{Ag} z^g G_g^M (zq^A;q)_{M-g} (-q;q)_g$
$= \sum_g (-1)^g q^g z^g G_g^M (q^A;q)_g (-zq;q)_{M-g} = \sum_g (-1)^g q^g G_g^M (zq^A;q)_g (-q;q)_{M-g}$.
$A = 1$, replace zq by z, $(**) \rightarrow \dfrac{(-zq;q)_\infty}{(q;q)_\infty} = \sum_g \dfrac{q^g (-z;q)_g}{(q;q)_g}$.
$A = 0, a_g^{*,2} = q^{g(g+1)} G_g^M z^g, b_g^{*,2} = G_g^M (z;q)_{M-g} z^g$, calculate $a_g^{*,2}$ using $b_g^{*,2} \rightarrow \sum_g (z;q)_{M-g} z^g G_g^M G_k^g = z^k G_k^M$.
It's equivalent to $\sum_g \dfrac{q^g}{(q;q)_{g-k}} = \dfrac{q^k}{(q;q)_{M-k}} \rightarrow \sum_g \dfrac{q^g}{(q;q)_g} = \dfrac{1}{(q;q)_M}$ $(*)$, a known formula [2].
[9(1)] $\rightarrow \sum_g a_g^{*,2} q^{-\frac{g(g+1)}{2}-g} z^{-g} x^g (-1)^g = (1-x)(1-xq)...(1-xq^{M-1}) = \sum_g b_g^{*,2} (\frac{x}{z};q)_g$
$= \sum_g G_g^M (1-z)(1-zq)...(1-zq^{M-g-1})(z-x)(z-xq)...(z-xq^{g-1})$.
Replace x by $\dfrac{a}{b}$ and z by $\dfrac{c}{b}$, then multiply through by b^M to get Jacobi's q-binomial theorem [2] p.71.
$(b-a)(b-aq)...(b-aq^{M-1}) = \sum_g G_g^M (b-c)(b-cq)...(b-cq^{M-g-1})(c-a)(c-aq)...(c-aq^{g-1})$.
$A = 0, c_g^{*,1} = \delta_{g0} \rightarrow \sum_g a_g^{*,1} q^{(r-1)g} a^g (aq^{1+r+g};q)_{M-g} = 1 \rightarrow \dfrac{\sum_g G_g^M q^{g^2+gr} a^g (aq^{1+r+g};q)_{M-g}}{(aq;q)_M} = \dfrac{1}{(aq;q)_M}$.
It's a finite form of Jacobi's Durfee square identity [2] pp.158-159: $\sum_g \dfrac{q^{g^2+gr} a^g}{(q;q)_g (aq;q)_{g+r}} = \dfrac{1}{(aq;q)_\infty}$.

$b_g = q^g, \sum_{k=0}^M G_g^k q^{k-g} = G_{g+1}^{M+1} \rightarrow a_g^* = q^{g(g+2)} G_{g+1}^{M+1}; \sum_k G_g^{M-k} q^{(g+1)k} = G_{g+1}^{M+1}$ [2] p.22 $\rightarrow c_g^* = (-1)^g q^{\frac{g(g+3)}{2}} G_{g+1}^{M+1}$.
So $\sum_g q^{g(g+2+p)} G_{g+1}^{M+1} G_{1+p+g}^N = \sum_g q^g G_{1+p+g}^{N+g} = \sum_g (-1)^g q^{\frac{g(g+3)}{2}+pg} G_{g+1}^{M+1} G_{1+p+M}^{N+M-g}$.
calculate b_g by $a_g^*, c_g^* \rightarrow \sum_k (-1)^{k+g} G_g^k q^{\frac{g(g+1)-k(k+1)}{2}-gk} G_{k+1}^{M+1} = q^{\binom{M-g}{2}} \sum_k (-1)^{M+k+g} G_{M-g}^k q^{\frac{k(k+3)}{2}-(M+1)k} G_{k+1}^{M+1} = q^g$.
calculate a_g^*, c_g^* by $b_g \rightarrow G_{g+1}^{M+1} = \sum_k (-1)^k G_{M-g}^{M-k} G_{k+1}^{M+1} q^{\frac{k(k+1)}{2}} = \sum_k (-1)^k G_{M-g}^{M-k} G_{k+1}^{M+1} q^{\frac{k(k+1)}{2}-g(k+1)}$.
[9(3)] $\rightarrow \sum_g q^g (q^{1+g};q)_{M-g} = \sum_g q^{g(g+1)} G_{g+1}^{M+1} (q^{1+g};q)_{M-g} = \sum_g (-1)^g q^{\frac{g(g+1)}{2}} G_{g+1}^{M+1} = 1 - (1;q)_{M+1} = 1$.
$\sum_g c_g^* q^{-g} z^g = z^{-1} - z^{-1} (z;q)_{M+1} = \sum_g q^g (zq^{1+g};q)_{M-g} \rightarrow 1 + \sum_{g=0}^M \dfrac{zq^g}{(z;q)_{g+1}} = \dfrac{1}{(z;q)_{M+1}}$, promotion of (*) [2] p.11
$\sum_g b_g q^{-g} = M+1 = q^{-M} \sum_g c_g^* q^{-(M+1)g} (q;q)_g = \sum_g (-1)^g G_{g+1}^{M+1} q^{\frac{(g+1)g}{2}-M(g+1)} (q;q)_g$.

If q^{-1} replaces q, the result will be $M+1$, which is Euler's identity: $\sum_{g=1}^{M} G_g^M (q;q)_{g-1} = M$ [2] p.83.

$c_g^{*,1} = (-1)^g q^{\frac{g(g+1)}{2}+Bg} G_g^M, c_g^{*,2} = c_g^{*,1} z^g [2(3)] \rightarrow a_g^{*,1} = q^{g(g+1)} G_g^M (q^{B-g};q)_g, a_g^{*,2} = q^{g(g+1)} G_g^M (q^{B-g}z;q)_g.$

$[9(1)] \rightarrow \sum_g (-1)^g q^{Ag+Bg} G_g^M z^g (-q^{A+1}z;q)_{M-g} = \sum_g q^{\frac{g(g+1)}{2}+Ag} G_g^M (q^{B-g};q)_g z^g.$

$[9(1)] \rightarrow \sum_g (-1)^g q^{Ag+Bg} G_g^M z^g (-q^{A+1};q)_{M-g} = \sum_g q^{\frac{g(g+1)}{2}+Ag} G_g^M (q^{B-g}z;q)_g.$

$[9(2)] \rightarrow q^{AM} z^M \sum_g (-1)^g q^{\frac{g(g+1)}{2}+Bg-(M+1)g} G_g^M (q^{-A}z^{-1};q)_g = \sum_g (-1)^g q^{\frac{g(g-1)}{2}} G_g^M (q^{B-g};q)_g (q^{A+1-g}z;q)_g.$

$[9(2)] \rightarrow q^{AM} \sum_g (-1)^g q^{\frac{g(g+1)}{2}+Bg-(M+1)g} G_g^M z^g (q^{-A};q)_g = \sum_g (-1)^g q^{\frac{g(g-1)}{2}} G_g^M (q^{B-g}z;q)_g (q^{A+1-g};q)_g.$

$B = 1 \rightarrow \sum_g (-1)^g q^{Ag+g} G_g^M z^g (-q^{A+1}z;q)_{M-g} = q^{AM} z^M \sum_g (-1)^g q^{\frac{g(g+1)}{2}-Mg} G_g^M (q^{-A}z^{-1};q)_g = 1.$

$A = -1; A = 0 \rightarrow \sum_g (-1)^g G_g^M z^g (-z;q)_{M-g} = z^M \sum_g (-1)^g q^{\frac{g(g+1)}{2}-Mg} G_g^M (z^{-1};q)_g = 1.$

$z = -q; z = q^{-1}, g := M-g \rightarrow \sum_g \frac{q^g}{(q;q)_g} = \frac{1}{(q;q)_M}; \sum_{g=0}^{M} \frac{(-1)^g q^{\frac{g(g-1)}{2}}}{(q;q)_g} = \frac{(-1)^M q^{\frac{M(M+1)}{2}}}{(q;q)_M}$ (4.1∗).

In the previous text, $T_i \in \mathbb{N}$, but excluding the actual meaning of $SUM_q(N)$, T_i can be any number.
$a = bq^{-T}, PS = [a, a...a] : a(q-1), PT = [T, T+1....T+M-1]$, due to T, a and b are independent.
At $H_1^q(g), B_i = \begin{cases} q^{X_T} G_1^{T+i-1-X_{K-1}} a(q-1) = q^{X_T}(bq^{X_T-1}-a),\ X_i = T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}}(K_i + G_1^{X_T-1} D_i) = aq^{X_T}, X_i = K_i \end{cases}, H_1^q(g) = G_g^M q^{\frac{g(g+1)}{2}} (b-a)(bq-a)...(bq^{g-1}-a)a^{M-g}.$
At $H_3^q(g), B_i = \begin{cases} q^1\{(q^{X_T-1} G_1^{T_i} - q^{T_i} G_1^{X_T-1}) D_i - K_i q^{T_i}\} = -aq^{X_T},\ X_i = T_i \\ (K_i + G_1^{X_T-1} D_i) = aq^{X_T},\ X_i = K_i \end{cases}, H_3^q(g) = G_g^M (-1)^g q^{\frac{g(g+1)}{2}} a^M.$
$a_g^* = H_1^q(g) q^{-(T-1)g}, c_g^* = H_3^q(g) q^{-(T-1)g}.$

$$\sum_g c_g^* (bx)^g q^{(T-1)g-(T+1)g} = \sum_g H_3^q(g)(bx)^g q^{-(T+1)g} = a^M \sum_g G_g^M (-1)^g q^{\frac{g(g-1)}{2}} (bq^{-T}x)^g = a^M (ax;q)_M$$
$$= \sum_g a_g^* (bx)^g q^{(T-1)g-(T+1)g} (bxq^g;q)_{M-g} = \sum_g H_1^q(g)(bx)^g q^{-(T+1)g} (bxq^g;q)_{M-g}$$
$$= \sum_g G_g^M q^{\frac{g(g+1)}{2}} (b-a)(bq-a)...(bq^{g-1}-a)(bq^{-T})^{M-g} (bx)^g q^{-(T+1)g} (bxq^g;q)_{M-g}.$$
$$= a^M \sum_g G_g^M q^{\frac{g(g-1)}{2}} (b-a)(bq-a)...(bq^{g-1}-a) x^g (bxq^g;q)_{M-g}.$$

That is to say: $\frac{(ax;q)_M}{(bx;q)_M} = \frac{\sum_g G_g^M q^{\frac{g(g-1)}{2}} (b-a)(bq-a)...(bq^{g-1}-a) x^g (bx \times q^g;q)_{M-g}}{(bx;q)_M}.$

This proves Cauchy's identity [2] p.260 : $\frac{(ax;q)_\infty}{(bx;q)_\infty} = \sum_{g=0}^{\infty} \frac{q^{\frac{g(g-1)}{2}} x^g (b-a)(bq-a)...(bq^{g-1}-a)}{(q;q)_g (bx;q)_g}.$

$M = \infty, |x| < 1, b_g = x^g, [2] \rightarrow a_g^* = \frac{q^{g(g+1)} x^g}{(x;q)_{g+1}}, [9(1)(2)] \rightarrow$

$$\sum_{g=0}^{\infty} \frac{q^{\frac{g(g+1)}{2}+Ag} x^g z^g}{(x;q)_{g+1}} = \sum_{g=0}^{\infty} x^g (-q^{A+1}z;q)_g \rightarrow A \geq 0, \sum_{g=0}^{\infty} (-1)^g \frac{q^{\frac{g(g-1)}{2}-Ag} x^g}{(x;q)_{g+1}} = \sum_{g=0}^{A} x^g (q^{-A};q)_g.$$

$$\sum_{g=0}^{\infty} x^g z^g q^A g = \frac{1}{1-xzq^A} = \sum_{g=0}^{\infty} \frac{(-1)^g q^{\frac{g(g-1)}{2}} x^g}{(x;q)_{g+1}} (q^{A+1-g}z;q)_g \rightarrow A \geq 0, \frac{1}{1-xq^A} = \sum_{g=0}^{A} \frac{(-1)^g q^{\frac{g(g-1)}{2}} x^g}{(x;q)_{g+1}} (q^A;q^{-1})$$

$a_0^* = \prod K_i = \sum b_g$, we can arbitrarily specify q, T_i and D_i to compute $H_2^q(g)$, or specify q and $a_g^*, g > 0$ to compute b_g, thereby obtaining an arbitrary number of expansions of $\prod K_i$. Other forms also have similar situations.

5. Inferences of Relationships Among the Three Forms

Simplifying the mutual expressions yields the inversion formulas.

Theorem 10. *Sum from 0 to M,*
(1). $a_g = \sum_k b_k G_g^k, b_g = \sum_k (-1)^{k+g} G_g^k q^{\frac{k(k-1)+g(g+1)}{2} - gk} a_k$.
(2). $a_g = \sum_k c_k G_{M-g}^{M-k} q^{-gk}, c_g = \sum_k (-1)^{k+g} G_{M-g}^{M-k} q^{\frac{k(k-1)+g(g+1)}{2}} a_k$
(3). $c_g = \sum_k b_k G_g^{M-k} q^{gk}, b_g = \sum_k (-1)^{M-g-k} c_k G_{M-g}^k q^{\frac{M^2+k^2+x^2-M+k+x}{2} - Mg - Mk}$.

Arbitrariness of a_g, b_g, c_g can derive the formulas of δ_{xg}.

Theorem 11. *Sum from 0 to M,* $0 \le x, g \le M$,
(1). $\sum_k (-1)^{k+x} G_g^k G_k^x q^{\frac{x(x-1)+k(k+1)}{2} - xk} = \sum_k (-1)^{k+x} G_{M-g}^{M-k} G_{M-k}^{M-x} q^{\frac{x(x-1)+k(k+1)}{2} - gk} = \delta_{xg}$.
(2). $\sum_k (-1)^{k+g} G_g^k G_k^x q^{\frac{k(k-1)+g(g+1)}{2} - gk} = \sum_k (-1)^{M-g-k} G_{M-g}^k G_k^{M-x} q^{\frac{M^2+g^2+k^2-M+g+k}{2} - Mg - Mk + xk} = \delta_{xg}$.
(3). $\sum_k (-1)^{k+g} G_{M-g}^{M-k} G_{M-k}^{M-x} q^{\frac{k(k-1)+g(g+1)}{2} - xk} = \sum_k (-1)^{M-x-k} G_g^{M-k} G_{M-k}^x q^{\frac{M^2+g^2+k^2-M+g+k}{2} - Mk - Mx + gk} = \delta_{xg}$.

Combining [6] and [8(3)(4)] , $b_g = 0 \leftrightarrow c_{M-g} = 0; a_g = 0 \leftrightarrow c_g = 0; a_{M-g} = 0 \leftrightarrow b_{M-g} = 0$. That is to say:

Theorem 12. *Sum from 0 to M,*
(1). $\sum_k (-1)^k G_g^k q^{\frac{-k(k+3)}{2} - gk} x_k = 0, 0 \le g < R \leftrightarrow \sum_k (-1)^k G_{M-g}^{M-k} q^{\frac{-k(k+3)}{2}} x_k = 0, 0 \le M - g < R$.
(2). $\sum_k G_g^k x_k = 0, 0 \le g < R \leftrightarrow \sum_k G_{M-g}^{M-k} q^{gk} x_k = 0, 0 \le M - g < R$.
(3). $\sum_k G_{M-g}^{M-k} q^{-k(g+1)} x_k = 0 \leftrightarrow \sum_k G_{M-g}^k q^{-(M+1)k} x_k = 0, 0 \le M - g < R$.

From [1], $0 \le g < R, 1 \le i \le R, \begin{cases} D_i = 1, K_i = [T_i]_{q-} \to b_g = c_{M-g} = 0 \\ D_i = 1, K_i = -[i-1]_q \to a_g = c_g = 0 \\ D_i = 0 \to a_{M-g} = b_{M-g} = 0 \end{cases}$. This is not a necessary condition. $PT = [T, T+1 ... T+M-1]$, it's a necessary and sufficient condition by adjusting $K_i : D_i$ and the order of PS.
$\sum a_g^* q^{pg} G_{1+p+g}^X = \sum b_g G_{1+p+g}^{X+g} = \sum c_g^* q^{pg} G_{1+p+M}^{X+M-g}$ can yields q-Vandermorde identity and its generalizations.

Theorem 13. *Sum from 0 to M,* $p \ge -1, 0 \le k \le M$,
(1). $\sum_g q^{g(g+1+p)} G_g^k G_{1+p+g}^N = G_{1+p+k}^{N+k}, \sum_g q^{g(g+1+p)-k(g+1+p)} G_{M-g}^{M-k} G_{1+p+g}^N = G_{1+p+M}^{N+M-k}$.
(2). $\sum_g (-1)^{k+g} q^{\frac{g(g+1)-k(k+3)}{2} - gk - pk} G_g^k G_{1+p+g}^{N+g} = G_{1+p+k}^N, \sum_g (-1)^{M-g} q^{\binom{M-g}{2} - (M+1+p)k} G_{M-g}^k G_{1+p+g}^{N+g} = G_{1+p+M}^{N+M-k}$.
(3). $\sum_g (-1)^{k+g} q^{\frac{g(g+3)-k(k+3)}{2} - pk + pg} G_{M-g}^{M-k} G_{1+p+M}^{N+M-g} = G_{1+p+k}^N, \sum_g (-1)^g q^{\frac{g(g+3)}{2} + gk + pg} G_g^{M-k} G_{1+p+M}^{N+M-g} = G_{1+p+k}^{N+k}$.

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This yields: if $\sum_g a_g\binom{X}{Y+g} = \sum_g b_g\binom{X+g}{Y+g} = \sum_g c_g\binom{X+M-g}{Y+M}$, then
$\sum(-1)^g a_g\binom{X+g}{Y+g} = \sum(-1)^g b_g\binom{X}{Y+g} = (-1)^M\sum c_{M-g}\binom{X+M-g}{Y+M}$.
$\sum(-1)^g a_g\binom{X+M-g}{Y+M} = \sum b_{M-g}\binom{X+g}{Y+g} = \sum(-1)^g c_g\binom{X}{Y+g}$.
$\sum b_g\binom{X+M-g}{Y+M} = \sum(-1)^{M-g} a_{M-g}\binom{X+g}{Y+g} = \sum(-1)^{M-g} c_{M-g}\binom{X}{Y+g}$.
$\sum c_g\binom{X+g}{Y+g} = \sum(-1)^g a_{M-g}\binom{X+M-g}{Y+M} = \sum(-1)^g b_{M-g}\binom{X}{Y+g}$.

[9] → $\sum a_g a^{M-g} z^g = \sum b_g a^{M-g}(a+z)^g = \sum c_g (a+z)^{M-g} z^g, a \neq 0$. From above, there are:
$\sum(-1)^g a_g a^{M-g}(a+z)^g = \sum(-1)^g b_g a^{M-g} z^g = (-1)^M \sum c_{M-g}(a+z)^{M-g} z^g$.
$\sum(-1)^g a_g (a+z)^{M-g} z^g = \sum b_{M-g} a^{M-g}(a+z)^g = \sum(-1)^g c_g a^{M-g} z^g$.
$\sum(-1)^{M-g} a_{M-g} a^{M-g}(a+z)^g = \sum b_g (a+z)^{M-g} z^g = \sum(-1)^{M-g} c_{M-g} a^{M-g} z^g$.
$\sum(-1)^g a_{M-g}(a+z)^{M-g} z^g = \sum(-1)^g b_{M-g} a^{M-g} z^g = \sum c_g a^{M-g}(a+z)^g$.

Theorem 14. $\sum_g b_g q^{(1+p+M)g} G_{1+p+M}^{N+p+M-g} = \sum_g (-1)^{M-g} c^*_{M-g} q^{(1+p+M)g+\frac{g(g+3)-M(M+3)}{2}} G_{1+p+g}^{N+p}$.

Proof.

$$\sum_g b_g G_{1+p+M}^{N+p+M-g} = \sum_g G_{1+p+M}^{N+p+M-g}(-1)^{M-g} q^{\frac{(M-g)(M-g-1)}{2}} \sum_k c^*_k G_{M-g}^k q^{-(M+1)k}$$
$$= \sum_k c_k q^* \sum_g G_{M-g}^k q^{-(M+1)k} G_{1+p+M}^{N+p+M-g}(-1)^{M-g} q^{\frac{(M-g)(M-g-1)}{2}}$$
$$= \sum_k c_{M-k} q^* \sum_g G_{M-g}^{M-k} q^{-(M+1)(M-k)} G_{1+p+M}^{N+p+M-g}(-1)^{M-g} q^{\frac{(M-g)(M-g-1)}{2}},\ [13(3)] \rightarrow$$
$$\sum_g b_g q^{\frac{g(g+3)}{2}+pg-\frac{(M-g)(M-g-1)}{2}} G_{1+p+M}^{N+p+M-g} = \sum_g (-1)^{M-g} c^*_{M-g} q^{-(M+1)(M-g)+pg+\frac{g(g+3)}{2}} G_{1+p+g}^{N+p}.$$

The reason why other forms could not be obtained is because q^{gk} appeared.

□

Theorem 15. *Sum from 0 to M, $0 \le y \le M$,*
(1). $(x;q)_y = \sum_g (x;q)_{M-g} x^g G_g^{M-y} q^{gy}$, $x^y (x;q)_{M-y} = \sum_g (x;q)_g (-1)^{M-g-y} G_{M-g}^y q^{\frac{(M-g)(M-g-1)}{2}-(M+1)y+\frac{y(y+3)}{2}}$.
(2). $(x;q)_y = x^M \sum_g (x^{-1}q^{1-y};q)_g (-1)^g G_{M-g}^{M-y} q^{\frac{g(g+3)}{2}-(M+1)g+(y-1)M}$, $(x;q)_y = x^M \sum_g (x^{-1}q^{1-g};q)_g (-1)^g G_{M-g}^{M-y} q^{\frac{g(g-1)}{2}+(}$
(3). $x^y (x;q)_{M-y} = \sum_g (q^{g-y}x;q)_{M-g} (-1)^{g+y} G_g^y q^{\frac{g(g+1)+y(y+1)}{2}-gy}$, $(x;q)_{M-y} = \sum_g (q^{g-y}x;q)_{M-g} x^g G_g^y q^{g(g-y-1)}$.
(4). $z^y = \sum_g (-1)^{y+g} q^{\frac{g(g+1)}{2}-y(g+1)} G_g^y (-zq;q)_g = \sum_g (-1)^{y+g} q^{g-y} z^g G_{M-g}^{M-y} (-zq;q)_{M-g}$.
(5). $z^y = \sum_g (-1)^g q^{\frac{g(g-1)}{2}} G_g^y (zq^{1-g};q)_g = z^M \sum_g (-1)^g q^{\frac{g(g+1)}{2}-Mg+gy} G_g^{M-y} (z^{-1};q)_g$.
(6). $z^y = \sum_g z^g q^{g(g+1)-y(g+1)} G_{M-g}^{M-y} (zq^{2+g};q)_{M-g} = \sum_g (-1)^{M-g} q^{\frac{(M-g)(M-g-1)}{2}-(M+1)y} G_{M-g}^y (zq^{2+g};q)_{M-g}$.

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Proof.

[9] $\rightarrow \sum_g b_g(-qz;q)_g = \sum_g c^*_g q^{-\frac{g(g+1)}{2}} z^g (-qz;q)_{M-g} = \sum_g q^{-\frac{g(g+1)}{2}} z^g (-qz;q)_{M-g} (-1)^g q^{\frac{g(g+3)}{2}} \sum_k b_k G_g^{M-k} q^{gk}$ (1∗).
$\sum_g c^*_g q^{-\frac{g(g+1)}{2}} z^g (-qz;q)_{M-g} = \sum_g (-qz;q)_g (-1)^{M-g} q^{\frac{(M-g)(M-g-1)}{2}} \sum_k c^*_k G^k_{M-g} q^{-(M+1)k}$ (2∗).
$x = -qz,\ b_g = \delta_{yg}$ and (1*); $c^*_g = \delta_{yg}$ and (2*) $\rightarrow (1)$. Similarly, (2) and (3) can be proven.

[9] $\rightarrow \sum_g a^*_g q^{-\frac{g(g+1)}{2}} z^g = \sum_g b_g(-zq;q)_g = \sum_g (-zq;q)_g q^{\frac{g(g+1)}{2}} \sum_k (-1)^{k+g} G_g^k q^{-\frac{k(k+3)}{2} - gk} a^*_g$
$= \sum_g c^*_g q^{-\frac{g(g+1)}{2}} z^g (-zq;q)_{M-g} = \sum_g q^{-\frac{g(g+1)}{2}} z^g (-zq;q)_{M-g} q^{\frac{g(g+3)}{2}} \sum_k (-1)^{k+g} G^{M-k}_{M-g} q^{-\frac{k(k+3)}{2}} a^*_k$
$a^*_g = \delta_{yg} \rightarrow (4)$. Similarly, (5) and (6) can be proven.

□

(1) or (4) $\rightarrow x^M = \sum_g (-1)^g (x;q)_g G_g^M q^{\frac{g(g+1)}{2} - gM}, x = q \rightarrow$ (4.1*).

6. An Example

$PS = [A,B,C], PT = [1,3,5], SUM_q(N) = \sum_g H_1^q(g) G_{3+g}^{N+2} = \sum_g H_2^q(g) G_{3+g}^{N+2+g} = \sum_g H_3^q(g) G_6^{N+5-g}$.
$H_1^q(0) = ABC$.
$H_1^q(1) = ABq^1G_1^3 + Aq^1G_1^2(C+1) + q^1G_1^1q^1(B+1)q^1(C+1)$.
$H_1^q(2) = q^1G_1^1q^3G_1^3q^2(C+G_1^2) + q^1G_1^1q^1(B+1)q^3G_1^4 + Aq^1G_1^2q^3G_1^4$.
$H_1^q(3) = q^1G_1^1q^3G_1^3q^5G_1^5$.
$H_2^q(0) = (A - q^{-1}G_1^1)(B - q^{-2}G_1^2)(C - q^{-3}G_1^3)$.
$H_2^q(1) = (A - q^{-1}G_1^1)(B - q^{-2}G_1^2)q^{-3}G_1^3 + (A - q^{-1}G_1^1)q^{-2}G_1^2(C - q^{-4}G_1^4) + q^{-1}G_1^1(B - q^{-3}G_1^3)(C - q^{-4}G_1^4)$.
$H_2^q(2) = q^{-1}G_1^1q^{-3}G_1^3(C - q^{-5}G_1^5) + q^{-1}G_1^1(B - q^{-3}G_1^3)q^{-4}G_1^4 + (A - q^{-1}G_1^1)q^{-2}G_1^2q^{-4}G_1^4$.
$H_2^q(3) = q^{-1}G_1^1q^{-3}G_1^3q^{-5}G_1^5$.
$H_3^q(0) = ABC$.
$H_3^q(1) = ABq^1(G_1^5 - q^5C) + Aq^2(G_1^3 - q^3B)(C+1) + q^3(1 - q^1A)(B+1)(C+1)$.
$H_3^q(2) = q^5(1 - q^1A)(q^1G_1^3 - q^3 - q^3B)(C + G_1^2) + q^4(1 - q^1A)(B+1)(q^1G_1^5 - q^5 - q^5C) + Aq^3(G_1^3 - q^3B)(q^1G_1^5 - q^5 -$
$H_3^q(3) = q^6(1 - q^1A)(q^1G_1^3 - q^3 - q^3B)(q^2G_1^5 - q^5G_1^2 - q^5C)$.
$PS = [A,B], PT = [1,3]. SUM(N) = \sum_g H_1(g)\binom{N+1}{2+g} = \sum_g H_2(g)\binom{N+1+g}{2+g} = \sum_g H_3(g)\binom{N+3-g}{4}$.
$H_1(0) = AB, H_1(1) = A \times 2 + 1 \times (B+1), H_1(2) = 1 \times 3$.
$H_2(0) = (A-1)(B-2), H_2(1) = (A-1) \times 2 + 1 \times (B-3), H_2(2) = 1 \times 3$.
$H_3(0) = AB, H_3(1) = A(3-B) + (1-A)(B+1), H_3(2) = (1-A)(2-B)$.

Conflicts of Interest: The authors declare that they have no conflict of interest.

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