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Article

Formal Calculation of q-Binomial

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Abstract: This article offers formulas for computing various q-binomial nested sums, give three forms of results. More importantly, it reveals the three forms of q-binomial and their interrelationships. It is a powerful tool for q-analysis, which can prove and generalize many classic conclusions in a simple way. This article also utilized it to obtain a large number of new results, including formulas for q-Eulerian numbers and polynomials. By taking the limit of q to 1, it can calculate general nested sums and analyze binomial coefficients.

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MSC: 05A30

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1. Calculation Formula

q-binomial: $\begin{bmatrix} N \\ M \end{bmatrix}_q = \frac{(q^N-1)(q^{N-1}-1)...(q^{N-M+1}-1)}{(q^M-1)(q^{M-1}-1)...(q^1-1)}$, $q \neq 0,\ 1$, abbreviated as G_M^N, $[N]_q = G_1^N$. $(a;q)_n = \begin{cases} (1-a)(1-aq)...(1-aq^{n-1}), n>0 \\ 1, n=0 \end{cases}$. $G_{g_1, g_2 \cdots g_p}^M = \frac{(q;q)_M}{\prod_{i=1}^p (q;q)_{g_p}}$, $\sum g_i = M$. The following relationship is established:

$$G_0^N = 1, G_M^N = G_{N-M}^N, G_M^N = 0, M > N, M < 0. \quad (1)$$
$$G_M^N = q^M G_M^{N-1} + G_{M-1}^{N-1} = G_M^{N-1} + q^{N-M} G_{M-1}^{N-1}. \quad (2)$$
$$\sum_{n=0}^{N-1} q^n G_M^{n+K} = q^{M-K} G_{M+1}^{N+K}. K \leq M. \quad (3)$$
$$G_K^M = \sum_{w \in \Omega(0^{M-K}, 1^K)} q^{inv(w)}, \text{ inv(.) denotes the inversion statistic. [1]} \quad (4)$$

Lemma 1.1.

$$\sum_{n=0}^{N-1} q^n [n]_q G_M^{n+K},\ M > 0,\ M \geq K$$
$$= q^{2(M-K)+1} G_1^{M+1} G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+1}^{N+K} \ (1)$$
$$= q^{M-2K-1} G_1^{M+1} G_{M+2}^{N+K+1} + q^{M-K} (G_1^{M-K} - q^{-K-1} G_1^{M+1}) G_{M+1}^{N+K} \ (2)$$
$$= (q^{2(M-K)+1} G_1^{M+1} - q^{2M-K+2} G_1^{M-K}) G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+2}^{N+K+1} \ (3).$$

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Proof.

$$
\begin{aligned}
&\sum q^n[n]_q G_M^{n+K} = \sum q^n \frac{q^n - q^{n-M+K}}{q-1} G_M^{n+K} + \sum q^n \frac{q^{n-M+K}-1}{q-1} G_M^{n+K} \\
&= \sum q^n (q^{n-M+K}-1) \frac{q^{M-K}-1}{q-1} G_M^{n+K} + \sum q^n \frac{q^{M-K}-1}{q-1} G_M^{n+K} + \sum q^n \frac{q^{n-M+K}-1}{q-1} G_M^{n+K} \\
&= \sum q^n (q^{M+1}-1) G_1^{M-K} G_{M+1}^{n+K} + \sum q^n G_1^{M-K} G_M^{n+K} + \sum q^n G_1^{M+1} G_{M+1}^{n+K} \\
&= (q^{M-K}-1) G_1^{M+1} G_{M+2}^{N+K} q^{M+1-K} + G_1^{M-K} G_{M+1}^{N+K} q^{M-K} + G_1^{M+1} G_{M+2}^{N+K} q^{M+1-K} \\
&= q^{2(M-K)+1} G_1^{M+1} G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+1}^{N+K} \ (1) \\
&= q^{2(M-K)+1} G_1^{M+1} q^{-M-2} (G_{M+2}^{N+K+1} - G_{M+1}^{N+K}) + q^{M-K} G_1^{M-K} G_{M+1}^{N+K} \\
&= q^{M-2K-1} G_1^{M+1} G_{M+2}^{N+K+1} + q^{M-K} (G_1^{M-K} - q^{-K-1} G_1^{M+1}) G_{M+1}^{N+K} \ (2) \\
&= q^{2(M-K)+1} G_1^{M+1} G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} (G_{M+2}^{N+K+1} - q^{M+2} G_{M+2}^{N+K}) \\
&= (q^{2(M-K)+1} G_1^{M+1} - q^{2M-K+2} G_1^{M-K}) G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+2}^{N+K+1} \ (3).
\end{aligned}
$$

□

Definition 1.1. *Recursively define* ∇_q^p, $p \in \mathbb{Z}$; $SUM_q(N) = SUM_q(N, PS, PT)$, $K_i, D_i \in \mathbb{C}$, $T_i \in \mathbb{N}$.

$\nabla_q^0 f(n) = f(n), \sum_{n=0}^{N-1} q^n \nabla_q^1 f(n+1) = f(N),\ \sum_{n=0}^{N-1} q^n f(n+1) = \nabla_q^{-1} f(N), \nabla_q^1 = \nabla_q.$

$SUM_q(N, [K_1 : D_1], [T_1 = 1]) = \sum_{n=0}^{N-1} q^n (K_1 + [n]_q D_1).$

$SUM_q(N, [K_1 : D_1, K_2 : D_2], [T_1, T_2 = T_1 + 2 - p]) = \sum_{n=0}^{N-1} q^n (K_2 + [n]_q D_2) \nabla^p SUM_q(n+1, [K_1 : D_1], [T_1]).$

$SUM_q(N, PS, [1, 2...M]) = \sum_{n=0}^{N-1} \prod_{i=1}^{M} q^n (K_i + [n]_q D_i).$

$SUM_q(N, PS, [1, 3...2M-1]) = \sum_{n_M=0}^{N-1} q^{n_M} (K_M + [n_M]_q D_M) ... \sum_{n_2=0}^{n_3} q^{n_2} (K_2 + [n_2]_q D_2) \sum_{n_1=0}^{n_2} q^{n_1} (K_1 + [n_1]_q D_1).$

$SUM_q(N, PS, [1, 2, 4]) = \sum_{n_3=0}^{N-1} q^{n_3} (K_3 + [n_3]_q D_3) \sum_{n=0}^{n_3} q^n (K_1 + [n]_q D_1)(K_2 + [n]_q D_2).$

$SUM_q(N, PS, [1, 3, 4]) = \sum_{n_3=0}^{N-1} q^{n_3} (K_3 + [n_3]_q D_3)(K_2 + [n_3]_q D_2) \sum_{n=0}^{n_3} q^n (K_1 + [n]_q D_1).$

$SUM_q(N, PS, [1, 4]) = \sum_{n_3=0}^{N-1} q^{n_3} (K_2 + [n_3]_q D_2) \sum_{n_2=0}^{n_3} q^{n_2} \sum_{n_1=0}^{n_2} q^{n_1} (K_1 + [n_1]_q D_1).$

Abbreviations: $[K_1 : D, K_2 : D...K_M : D] = [K_1, K_2...K_M] : D$, $[K_1, K_2...K_M] : 1 = [K_1, K_2...K_M]$.

In this paper, the default $PS = [K_1 : D_1, K_2 : D_2...K_M : D_M]$, $PT = [T_1, T_2...T_M]$, $T_i < T_{i+1}$.

Use $\mathbb{K}, \mathbb{T}$ *to represent the set* $\{K_i\}, \{T_i\}$. $(K_1 + T_1)(K_2 + T_2)...(K_M + T_M) = \sum \prod_{i=1}^{M} X_i$, $X_i = T_i$ *or* K_i

Definition 1.2. $X(T)$=*Number of* $\{X_1, X_2...X_M\} \in \mathbb{T}$.
X_{T-1}=*Number of* $\{X_1, X_2...X_{i-1}\} \in \mathbb{T}$, X_{K-1}=*Number of* $\{X_1, X_2...X_{i-1}\} \in \mathbb{K}$.
X_T=*Number of* $\{X_1, X_2...X_i\} \in \mathbb{T}$, X_K=*Number of* $\{X_1, X_2...X_i\} \in \mathbb{K}$.

Obviously, $X_{T-1} + X_{K-1} = i - 1$, $X_T + X_K = i$. Use the auxiliary form and each X_i cannot be exchanged:

Theorem 1.1. $H = T_M - M, SUM_q(N, PS, PT) =$

$$Form_1 \to \sum_{g=0}^{M} H_1^q(g) G_{H+1+g}^{N+H},\ B_i = \begin{cases} q^{1+(T_i-T_{i-1})X_{T-1}} G_1^{T_i - X_{K-1}} D_i,\ X_i = T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}} (K_i + G_1^{X_{T-1}} D_i), X_i = K_i \end{cases},$$

$$Form_2 \to \sum_{g=0}^{M} H_2^q(g) G_{H+1+g}^{N+H+g},\ B_i = \begin{cases} q^{-(T_i - X_{K-1})} G_1^{T_i - X_{K-1}} D_i,\ X_i = T_i \\ K_i - q^{-(T_i - X_{K-1})} G_1^{T_i - X_{K-1}} D_i,\ X_i = K_i \end{cases}$$

$$Form_3 \to \sum_{g=0}^{M} H_3^q(g) G_{T_M+1}^{N+T_M-g},\ B_i = \begin{cases} q^{1+(T_i-T_{i-1}-1)X_{T-1}} \{(q^{X_{T-1}} G_1^{T_i} - q^{T_i} G_1^{X_{T-1}}) D_i - K_i q^{T_i}\},\ X_i = T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}} (K_i + G_1^{X_{T-1}} D_i),\ X_i = K_i \end{cases}.$$

$H_i^q(g) = H_i^q(g, PS, PT) = H_i^q(g, M)$, *is defined as* $\sum_{X(T)=g} \prod_{i=1}^{M} B_i$.

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Proof.

$SUM_q(1,[K_1:D_1],[1]) = \sum_{n=0}^{N-1} q^n(K_1+[n]_q D_1) = \sum_{n=0}^{N-1} q^n G_1^n D_1 + \sum_{n=0}^{N-1} q^n K_1$
$= q^1 D_1 G_2^N + K_1 G_1^N = q^{-1} D_1 G_2^{N+1} + (K_1 - q^{-1} D_1) G_1^N = (q^1 D_1 - K_1 q^2) G_2^N + K_1 G_2^{N+1}$.
It's holds when M=1, suppose that holds when M.
$PS1 = [PS, K_{M+1}:D_{M+1}], PT1 = [PT, T_{M+1} = T_M + 2 - p], X = 1 + T_M - M - p$.
If $f(n+1) = \sum A_g G_{H+1+g}^{n+H+1+g}$, then $\nabla_q^p f(n+1) = \sum A_g G_{H+1+g-p}^{n+H+1+g-p}$.
$SUM_q(N, PS1, PT1)$
$= \sum_{n=0}^{N-1} q^n (K_{M+1} + [n]_q D_{M+1}) \nabla_q^p SUM_q(n+1) = \sum_{n=0}^{N-1} q^n (K_{M+1} + [n]_q D_{M+1}) \sum_{g=0}^{M} H_2^q(g) G_{X+g}^{n+X+g}$
$= \sum_{g=0}^{M} (K_{M+1} - q^{-X-g-1} G_1^{X+g+1} D_{M+1}) H_2^q(g) G_{X+1+g}^{N+X+g} + \sum_{g=0}^{M} q^{-X-g-1} G_1^{X+g+1} D_{M+1} H_2^q(g) G_{X+2+g}^{N+X+g+1}$
$= \sum_{g=0}^{M} (K_{M+1} - q^{-(T_{M+1}-(M-g))} G_1^{T_{M+1}-(M-g)} D_{M+1}) H_2^q(g) G_{T_{M+1}-(M+1)+g+1}^{N+T_{M+1}-(M+1)+g}$
$+ \sum_{g=0}^{M} q^{-(T_{M+1}-(M-g))} G_1^{T_{M+1}-(M-g)} D_{M+1} H_2^q(g) G_{T_{M+1}-(M+1)+g+2}^{N+T_{M+1}-(M+1)+g+1}$
$= \sum_{g=0}^{M+1} H_2^q(g, PS1, PT1) G_{T_{M+1}-(M+1)+1+g}^{N+T_{M+1}-(M+1)+g}$. Proof of $Form_2$ completion.
If $f(n+1) = \sum A_g G_{H+1+g}^{n+H+1}$, then $\nabla_q^p f(n+1) = \sum A_g G_{H+1+g-p}^{n+H+1-p} q^{-pg}$.
$SUM_q(N, PS1, PT1)$
$= \sum_{n=0}^{N-1} q^n (K_{M+1} + [n]_q D_{M+1}) \sum_{g=0}^{M} H_1^q(g) G_{X+g}^{n+X} q^{-pg}$
$= \sum_{g=0}^{M} (K_{M+1} q^g + q^g G_1^g D_{M+1}) H_1^q(g) G_{X+1+g}^{N+X} q^{-pg} + \sum_{g=0}^{M} q^{2g+1} G_1^{X+g+1} D_{M+1} H_1^q(g) G_{X+2+g}^{N+X} q^{-pg}$
$= \sum_{g=0}^{M} (K_{M+1} + G_1^g D_{M+1}) H_1^q(g) G_{T_{M+1}-(M+1)+1+g}^{N+T_{M+1}-(M+1)} q^{(1-p)g} + \sum_{g=0}^{M} q^{(2-p)g+1} G_1^{T_{M+1}-(M-g)} D_{M+1} H_1^q(g) G_{X+2+g}^{N+X}$
$= \sum_{g=0}^{M+1} H_1^q(g, PS1, PT1) G_{T_{M+1}-(M+1)+1+g}^{N+T_{M+1}-(M+1)}$. Proof of $Form_1$ completion.
If $f(n+1) = \sum A_g G_{T_M+1}^{n+1+T_M-g}$, then $\nabla_q^p f(n+1) = \sum A_g G_{T_M+1-p}^{n+1+T_M-g-p} q^{-pg}$.
$SUM_q(N, PS1, PT1)$
$= \sum_{n=0}^{N-1} q^n (K_{M+1} + [n]_q D_{M+1}) \sum_{g=0}^{M} H_3^q(g) G_{T_M+1-p}^{n+1+T_M-p-g} q^{-pg}$
$= \sum_{g=0}^{M} K_{M+1} q^g H_3^q(g) G_{T_M+2-p}^{N+1+T_M-p-g} q^{-pg} + \sum_{g=0}^{M} q^g G_1^g D_{M+1} H_3^q(g) G_{T_M+3-p}^{N+T_M-p+2-g} q^{-pg}$
$+ \sum_{g=0}^{M} (q^{2g+1} G_1^{T_M+2-p} - q^{T_M+3-p+g} G_1^g) D_{M+1} H_3^q(g) G_{T_M+3-p}^{N+T_M-p+1-g} q^{-pg}$
$= \sum_{g=0}^{M} (K_{M+1} q^g + q^g G_1^g D_{M+1}) H_3^q(g) G_{T_M+3-p}^{N+T_M-p+2-g} q^{-pg}$
$+ \sum_{g=0}^{M} ((q^{2g+1} G_1^{T_M+2-p} - q^{T_M+3-p+g} G_1^g) D_{M+1} - q^{T_M+3-p} K_{M+1} q^g) H_3^q(g) G_{T_M+3-p}^{N+T_M-p+1-g} q^{-pg}$
$= \sum_{g=0}^{M} (K_{M+1} + G_1^g D_{M+1}) H_3^q(g) G_{T_{M+1}+1}^{N+T_{M+1}-g} q^{(1-p)g}$
$+ \sum_{g=0}^{M} ((q^g G_1^{T_{M+1}} - q^{T_{M+1}} G_1^g) D_{M+1} - q^{T_{M+1}} K_{M+1}) H_3^q(g) G_{T_{M+1}+1}^{N+T_{M+1}-(g-1)} q^{1+(1-p)g}$
$= \sum_{g=0}^{M+1} H_3^q(g, PS1, PT1) G_{T_{M+1}+1}^{N+T_{M+1}-g}$. Proof of $Form_3$ completion.

□

$\sum_{n=0}^{N-1} q^n \prod_{i=1}^{M} (K_i + D_i q^n) = SUM(N, [K_1 + D_1 : D_1(q-1) ... K_M + D_M : D_M(q-1)], [1, 2 ... M])$
$= \sum_{g=0}^{M} H_1^q(g) G_{1+g}^N$. $B_i = \begin{cases} q^{1+X_{T-1}} G_1^{i-X_{K-1}} D_i(q-1) = q^{X_T} G_1^{1+X_{T-1}} D_i(q-1) = q^{X_T} G_1^{X_T} D_i(q-1) = q^{X_T}(q^{X_T}-1) D_i, X_i = T_i \\ K_i + D_i + G_1^{X_{T-1}} D_i(q-1) = K_i + D_i + G_1^{X_T} D_i(q-1) = K_i + q^{X_T} D_i, X_i = K_i \end{cases}$.
Following a similar form, we are able to prove inductively:

Theorem 1.2. $\sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + D_i q^n) = \sum_{g=1}^{M} f(g) G_g^N + N \prod K_i, f(g) = \sum \prod B_i$,
$B_i = \begin{cases} D_i, X_{T-1} = 0; q^{X_{T-1}}(q^{X_{T-1}} - 1) D_i, X_{T-1} > 0, X_i = T_i \\ K_i, X_{T-1} = 0; K_i + q^{X_{T-1}-1} D_i, X_{T-1} > 0, X_i = K_i \end{cases}$.

Definition 1.3. $\lim_{q \to 1} H^q(g) = H(g)$, $SUM_q(N) = SUM(N)$.

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$\lim_{q\to1}[n]_q = n,\ G_M^N = \binom{N}{M}$. which yields the nested summation formula for $K_i + nD_i$.

2. Property

Definition 2.1. $[n]_{q-} = q^{-n}G_1^n, [n!]_{q-} = [n]_{q-}...[2]_{q-}[1]_{q-}, [0!]_{q-} = 1, [n]_{q+} = q^n G_1^n$, *similarly defining* $[n!]_{q+}$.

Theorem 2.1.
(1). $\nabla_q SUM_q(n+1, PS, [1,2...M]) = \prod_{i=1}^{M}(K_i + [n]_q D_i)$.
(2). At $SUM_q(N, [...PS...], [...T, T+1...T+M-1...])$, $K_i : D_i$ *can exchange orders.*
(3). $\nabla_q^p SUM_q(N) = \sum_{g=0}^{M} H_1^q(g) G_{X+1+g}^{N+X} q^{-gp} = \sum_{g=0}^{M} H_2^q(g) G_{X+1+g}^{N+X+g} = \sum_{g=0}^{M} H_3^q(g) G_{X+M+1}^{N+X+M-g} q^{-gp}$, $X = T_M - M - p$.
(4). $SUM_q(N, [[L_1]_{q-}, [L_2]_{q-}...[L_Q]_{q-}, PS], [L_1, L_2...L_Q, PT]) = \prod_{i=1}^{Q}[L_i]_{q-} SUM_q(N)$. *So* T_1 *can great than 1,* $T_i \in \mathbb{N}$.
(5). $SUM_q(N, [[T_1]_{q-}, [T_2]_{q-}...[T_M]_{q-}], [T_1, T_2...T_M]) = \prod_{i=1}^{M}[T_i]_{q-} G_{T_M+1}^{N+T_M}$.
(6) At $H^q(g)$, $\sum_{X_i\in\mathbb{K}} \prod q^{X_T} = G_{M-g}^M = G_g^M$.

Proof. (1) and (2) is derived from the definition of SUM_q.
(3) is derived from $\sum_{n=0}^{N-1} q^n G_M^{n+K} = q^{M-K} G_{M+1}^{N+K}$, which has already used in the proof of [1.1].
$PS1 = [[L_1]_{q-}, [L_2]_{q-}...[L_Q]_{q-}, PS], PT1 = [L_1, L_2...L_Q, PT])$,
$H_2^q(g < Q, PS1, PT1) = 0$, $H_2^q(g \geq Q, PS1, PT1) = \prod_{i=1}^{Q}[L_i]_{q-} H_2^q(g-Q)$, which proves (4) and (5).
(6) is actually $G_K^M = \sum_{w\in\Omega(0^{M-K},1^K)} q^{inv(w)}$. □

$$[x]_{q-} + [n]_q = q^{-x}\frac{q^x-1}{q-1} + \frac{q^n-1}{q-1} = \frac{q^{n+x}-1}{q^x(q-1)} = q^{-x}[n+x]_q.\text{ From (5):}$$
$$PT = [1,2...M], PS = [[T_i]_{q-}] \to \sum_{n=0}^{N-1} q^n [n+1]_q [n+2]_q ... [n+M]_q = [1]_q[2]_q...[M]_q G_{M+1}^{N+M}.$$
$$PT = [1,3...2M-1], PS = [[T_i]_{q-}] \to \sum_{n_M=0}^{N-1} \cdots \sum_{n_1=0}^{n_2} q^{\sum_{i=1}^{M} n_i} \prod_{i=1}^{M}[n_i+2i-1]_q = [1]_q[3]_q...[2M-1]_q G_{2M}^{N+2M-1}.$$

$$PS = [0, -[1]_q, -[2]_q ... -[M-1]_q] \to H_1^q(g<M) = H_3^q(g<M) = 0,$$
$$H_1^q(M) = H_3^q(M) = q^{M+(T_2-T_1)+2(T_3-T_2)+...+(M-1)(T_M-T_{M-1})} \prod G_1^{T_i} = q^{M+(M-1)T_M - \sum_{i=1}^{M-1} T_i} \prod G_1^{T_i}.$$
$$PT = [1,2...M] \to \sum_{n=0}^{N-1} q^n [n]_q [n-1]_q ... [n-M+1]_q = q^M [1]_q[2]_q...[M]_q G_{M+1}^N.$$
$$PT = [1,3...2M-1] \to q^{0+1+2+...+(M-1)} \sum_{n_M=0}^{N-1} \cdots \sum_{n_1=0}^{n_2} q^{\sum_{i=1}^{M} n_i} \prod_{i=1}^{M}[n_i+1-i]_q$$
$$= q^{M+(M-1)(2M-1)-(M-1)^2}[1]_q[3]_q...[2M-1]_q G_{2M}^{N+M-1} \to$$
$$\sum_{n_M=0}^{N-1} \cdots \sum_{n_1=0}^{n_2} q^{\sum_{i=1}^{M} n_i} \prod_{i=1}^{M}[n_i+1-i]_q = q^{\frac{M(M+1)}{2}}[1]_q[3]_q...[2M-1]_q G_{2M}^{N+M-1}.$$

Theorem 2.2. $PT = [1,2...M]$,
(1). $H_1^q(g) = q^{g(g+1)} \sum_{k=g}^{M} H_2^q(k) G_g^k$.
(2). $H_1^q(g) = \sum_{k=0}^{g} H_3^q(k) G_{M-g}^{M-k} q^{(g+1)(g-k)}$.
(3). $H_2^q(g) = \sum_{k=g}^{M} (-1)^{k+g} G_g^k q^{-k(k+1)+\binom{k-g}{2}} H_1^q(k) = \sum_{k=g}^{M} (-1)^{k+g} G_g^k q^{\frac{g(g+1)-k(k+3)}{2}-kg} H_1^q(k)$.
(4). $H_3^q(g) = \sum_{k=0}^{g} (-1)^{k+g} G_{M-g}^{M-k} q^{(g+1)(g-k)-\binom{g-k}{2}} H_1^q(k) = \sum_{k=0}^{g} (-1)^{k+g} G_{M-g}^{M-k} q^{\frac{g(g+3)-k(k+3)}{2}} H_1^q(k)$.

Proof.

Direct verification when M=1, assuming M holds. $PS1 = [PS, K_{M+1} : D_{M+1}], PT1 = [PT, T_{M+1}]$.

$H_1^q(g, M+1) = H_1^q(g-1)q^{1+X_T}G_1^{1+X_T}D_{M+1} + H_1^q(g)(K_{M+1} + G_1^{X_T}D_{M+1})$

$= q^{1+X_T}G_1^{1+X_T}D_{M+1}q^{g(g-1)}\sum_{k=0}^M H_2^q(k)G_{g-1}^k + (K_{M+1} + G_1^{X_T}D_{M+1})q^{g(g+1)}\sum_{k=0}^M H_2^q(k)G_g^k$

$= q^g G_1^g D_{M+1}q^{g(g-1)}\sum_{k=0}^M H_2^q(k)(G_g^{k+1} - q^g G_g^k) + (K_{M+1} + G_1^g D_{M+1})q^{g(g+1)}\sum_{k=0}^M H_2^q(k)G_g^k$

$= q^{-g}G_1^g D_{M+1}q^{g(g+1)}\sum_{k=0}^M H_2^q(k)G_g^{k+1} + K_{M+1}q^{g(g+1)}\sum_{k=0}^M H_2^q(k)G_g^k$. $(1*)$

$H_2^q(g, M+1) = H_2^q(g-1)q^{-g}G_1^g D_{M+1} + H_2^q(g)(K_{M+1} - q^{-(g+1)}G_1^{g+1}D_{M+1})$.

$q^{g(g+1)}\sum_{k=0}^{M+1} H_2^q(k, M+1)G_g^k =$

$= q^{g(g+1)}\sum_{k=0}^{M+1}(H_2^q(k-1)q^{-k}G_1^k D_{M+1} + H_2^q(k)(K_{M+1} - q^{-(k+1)}G_1^{k+1}D_{M+1}))G_g^k$

$= q^{g(g+1)}\sum_{k=1}^{M+1}(H_2^q(k-1)q^{-k}G_1^k D_{M+1}G_g^k + q^{g(g+1)}\sum_{k=0}^M H_2^q(k)(K_{M+1} - q^{-(k+1)}G_1^{k+1}D_{M+1}))G_g^k$

$= q^{g(g+1)}\sum_{k=0}^M (H_2^q(k)q^{-(1+k)}G_1^{1+k}D_{M+1}G_g^{k+1} + q^{g(g+1)}\sum_{k=0}^M H_2^q(k)(K_{M+1} - q^{-(k+1)}G_1^{k+1}D_{M+1}))G_g^k$. $(1**)$

$((1*) - (1**))/(q^{g(g+1)}D_{M+1}) =$

$\sum_{k=0}^M H_2^q(k)(q^{-g}G_1^g G_g^{k+1} - q^{-(1+k)}G_1^{1+k}G_g^{k+1} + q^{-(k+1)}G_1^{k+1}G_g^k) = \sum_{k=0}^M H_2^q(k)(...)$.

$(...) = q^{-g}G_1^g G_g^{k+1} - q^{k+1-g}q^{-(1+k)}G_1^{1+k}G_{g-1}^k = 0$. Proof of (1) is complete.

$H_1^q(g, M+1) =$

$q^g G_1^g D_{M+1}\sum_{k=0}^M H_3^q(k)G_{M-g+1}^{M-k}q^{g(g-1-k)} + (K_{M+1} + G_1^g D_{M+1})\sum_{k=0}^M H_3^q(k)G_{M-g}^{M-k}q^{(g+1)(g-k)}$.$(2*)$

$\sum_{k=0}^{M+1} H_3^q(k, M+1)G_{M+1-g}^{M+1-k}q^{(g+1)(g-k)}$

$= \sum_{k=0}^M q((q^k G_1^{M+1} - q^{M+1}G_1^k)D_{M+1} - K_{M+1}q^{M+1})H_3^q(k)G_{M-g+1}^{M-k}q^{(g+1)(g-k-1)}$

$+ \sum_{k=0}^M (K_{M+1} + G_1^k D_{M+1})H_3^q(k)G_{M+1-g}^{M+1-k}q^{(g+1)(g-k)}$.$(2**)$

Items containing K_{M+1} :

$K_{M+1}G_{M+1-g}^{M+1-k}q^{(g+1)(g-k)} - K_{M+1}q^{M+2}G_{M+1-g}^{M-k}q^{(g+1)(g-k-1)} = K_{M+1}G_{M-g}^{M-k}q^{(g+1)(g-k)}$.

Items does not contain K_{M+1} in (2*):

$= q^g G_1^g D_{M+1}G_{M-g+1}^{M-k}q^{g(g-1-k)} + G_1^g D_{M+1}G_{M-g}^{M-k}q^{(g+1)(g-k)}$.

Divide by $D_{M+1}q^g q^{g(g-1-k)}(q-1)^{-1}$

$= (q^g - 1)G_{M+1-g}^{M-k} + (q^g - 1)G_{M-g}^{M-k}q^{g-k} = (q^g - 1)G_{M+1-g}^{M+1-k}$.

Items does not contain K_{M+1} in (2**):

$$= \frac{q^{M+2} - q^{k+1}}{q-1}D_{M+1}G_{M-g+1}^{M-k}q^{(g+1)(g-k-1)} + G_1^k D_{M+1}G_{M+1-g}^{M+1-k}q^{(g+1)(g-k)}.$$

Divide by $D_{M+1}q^g q^{g(g-1-k)}(q-1)^{-1}$

$= (q^{M+1-k} - 1)G_{M+1-g}^{M-k} + (q^g - q^{g-k})G_{M+1-g}^{M+1-k} = (q^g - 1)G_{M+1-g}^{M+1-k}$.

(2*)=(2**), proof of (2) is complete.

$H_2^q(g, M+1) = H_2^q(g-1)q^{-g}G_1^g D_{M+1} + H_2^q(g)(K_{M+1} - q^{-(g+1)}G_1^{g+1}D_{M+1})$.

$= q^{-g}G_1^g D_{M+1}\sum_{k=0}^M (-1)^{k+g-1}H_1^q(k)G_{g-1}^k q^{-k(k+1)+\binom{k-g+1}{2}}$

$+ (K_{M+1} - q^{-(1+g)}G_1^{g+1}D_{M+1})\sum_{k=0}^M (-1)^{k+g}H_1^q(k)G_g^k q^{-k(k+1)+\binom{k-g}{2}}$.$(3*)$

$\sum_{k=0}^{M+1}(-1)^{k+g}H_1^q(k, M+1)G_g^k q^{-k(k+1)+\binom{k-g}{2}}$

$= \sum_{k=0}^{M+1}(-1)^{k+g}(q^k G_1^k D_{M+1}H_1^q(k-1) + (K_{M+1} + G_1^k D_{M+1})H_1^q(k))G_g^k q^{-k(k+1)+\binom{k-g}{2}}$

$= \sum_{k=0}^M (-1)^{k+g-1}q^{1+k}G_1^{1+k}D_{M+1}H_1^q(k)G_g^{1+k}q^{-(k+1)(k+2)+\binom{k-g+1}{2}}$

$+ \sum_{k=0}^M (-1)^{k+g}(K_{M+1} + G_1^k D_{M+1})H_1^q(k)G_g^k q^{-k(k+1)+\binom{k-g}{2}}$.$(3**)$

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Items containing K_{M+1} in (3*) and (3**) $= (-1)^{k+g} K_{M+1} G_g^k q^{-k(k+1)+\binom{k-g}{2}}$.

Items does not contain K_{M+1} in (3*):

$= q^{-g} G_1^g D_{M+1} (-1)^{k+g-1} G_{g-1}^k q^{-k(k+1)+\binom{k-g+1}{2}} - q^{-(1+g)} G_1^{1+g} D_{M+1} (-1)^{k+g} G_g^k q^{-k(k+1)+\binom{k-g}{2}}$.

Divide by $D_{M+1} q^{-(1+g)} (-1)^{k+g-1} q^{-k(k+1)+\binom{k-g}{2}} (q-1)^{-1}$

$= (q^g - 1) G_{g-1}^k q^{k-g+1} + (q^{1+g} - 1) G_g^k = q^{k+1} G_{g-1}^k - q^{k-g+1} G_{g-1}^k + q^{1+g} G_g^k - G_g^k$. $(A*)$

Items does not contain K_{M+1} in (3**):

$= (-1)^{k+g-1} q^{1+k} G_1^{1+k} D_{M+1} G_g^{1+k} q^{-(k+1)(k+2)+\binom{k-g+1}{2}} + (-1)^{k+g} G_1^k D_{M+1} G_g^k q^{-k(k+1)+\binom{k-g}{2}}$.

Divide by $D_{M+1} q^{-(1+g)} (-1)^{k+g-1} q^{-k(k+1)+\binom{k-g}{2}} (q-1)^{-1}$

$= (q^{1+k} - 1)(q^g G_g^k + G_{g-1}^k) - q^{1+g}(q^k - 1) G_g^k = q^{1+k} G_{g-1}^k - q^g G_g^k - G_{g-1}^k + q^{1+g} G_g^k$. $(A**)$

$(A*) - (A**) = -q^{k-g+1} G_{g-1}^k - G_g^k - (-q^g G_g^k - G_{g-1}^k) = -G_g^{k+1} + G_g^{k+1} = 0$.

(3*)=(3**), proof of (3) is complete.

$H_3^q(g, M+1) = ((q^g G_1^{M+1} - q^{M+2} G_1^{g-1}) D_{M+1} - K_{M+1} q^{M+2}) H_3^q(g-1) + (K_{M+1} + G_1^g D_{M+1}) H_3^q(g)$

$= ((q^g G_1^{M+1} - q^{M+2} G_1^{g-1}) D_{M+1} - K_{M+1} q^{M+2}) \sum_{k=0}^{M} (-1)^{k+g-1} G_{M-g+1}^{M-k} q^{g(g-k-1)-\binom{g-k-1}{2}} H_1^q(k)$

$+ (K_{M+1} + G_1^g D_{M+1}) \sum_{k=0}^{M} (-1)^{k+g} G_{M-g}^{M-k} q^{(g+1)(g-k)-\binom{g-k}{2}} H_1^q(k)$. $(4*)$

$\sum_{k=0}^{M+1} (-1)^{k+g} H_1^q(k, M+1) G_{M+1-g}^{M+1-k} q^{(g+1)(g-k)-\binom{g-k}{2}}$

$= \sum_{k=0}^{M+1} (-1)^{k+g} (q^k G_1^k D_{M+1} H_1^q(k-1) + (K_{M+1} + G_1^k D_{M+1}) H_1^q(k)) G_{M+1-g}^{M+1-k} q^{(g+1)(g-k)-\binom{g-k}{2}}$

$= \sum_{k=0}^{M} (-1)^{k+g-1} q^{1+k} G_1^{1+k} D_{M+1} H_1^q(k) G_{M+1-g}^{M-k} q^{(g+1)(g-k-1)-\binom{g-k-1}{2}}$

$+ \sum_{k=0}^{M} (-1)^{k+g} (K_{M+1} + G_1^k D_{M+1}) H_1^q(k) G_{M+1-g}^{M+1-k} q^{(g+1)(g-k)-\binom{g-k}{2}}$. $(4**)$

Items containing K_{M+1} in (4*):

$= -K_{M+1} q^{M+2} (-1)^{k+g-1} G_{M-g+1}^{M-k} q^{g(g-k-1)-\binom{g-k-1}{2}} + (-1)^{k+g} K_{M+1} G_{M-g}^{M-k} q^{(g+1)(g-k)-\binom{g-k}{2}}$.

Divide by $K_{M+1} (-1)^{k+g} q^{(g+1)(g-k)-\binom{g-k}{2}} = q^{M+1-g} G_{M-g+1}^{M-k} + G_{M-g}^{M-k} = G_{M-g+1}^{M-k+1}$

=Items containing K_{M+1} in (4**) divided by $K_{M+1} (-1)^{k+g} q^{(g+1)(g-k)-\binom{g-k}{2}}$.

Items does not contain K_{M+1} in (4*):

$= (q^g G_1^{M+1} - q^{M+2} G_1^{g-1}) D_{M+1} (-1)^{k+g-1} G_{M-g+1}^{M-k} q^{g(g-k-1)-\binom{g-k-1}{2}} + G_1^g D_{M+1} (-1)^{k+g} G_{M-g}^{M-k} q^{(g+1)(g-k)-\binom{g-k}{2}}$.

Divide by $D_{M+1} (-1)^{k+g-1} q^{(g+1)(g-k)-\binom{g-k}{2}} (q-1)^{-1}$

$= [(q^g G_1^{M+1} - q^{M+2} G_1^{g-1}) G_{M-g+1}^{M-k} q^{-g-1} - G_1^g G_{M-g}^{M-k}](q-1)^{-1} = (q^{M-g-1} - q^{-1}) G_{M-g+1}^{M-k} - (q^g - 1) G_{M-g}^{M-k}$. $(B*)$

Items does not contain K_{M+1} in (4**):

$= (-1)^{k+g-1} q^{1+k} G_1^{1+k} D_{M+1} G_{M+1-g}^{M-k} q^{(g+1)(g-k-1)-\binom{g-k-1}{2}} + (-1)^{k+g} G_1^k D_{M+1} G_{M+1-g}^{M+1-k} q^{(g+1)(g-k)-\binom{g-k}{2}}$.

Divide by $D_{M+1} (-1)^{k+g-1} q^{(g+1)(g-k)-\binom{g-k}{2}} (q-1)^{-1}$

$= (q^{1+k} G_1^{1+k} G_{M+1-g}^{M-k} q^{-k-2} - G_1^k G_{M+1-g}^{M+1-k})(q-1)^{-1} = (q^k - q^{-1}) G_{M+1-g}^{M-k} - (q^k - 1) G_{M+1-g}^{M+1-k}$. $(B**)$

$(B*) - (B**) = (q^{M-g-1} - q^k) G_{M-g+1}^{M-k} - (q^g - 1) G_{M-g}^{M-k} + (q^k - 1)(G_{M-g+1}^{M-k} + q^{g-k} G_{M-g}^{M-k})$

$= (q^{M-g-1} - 1) G_{M-g+1}^{M-k} - (q^{g-k} - 1) G_{M-g}^{M-k} = 0$.

(4*)=(4**), proof of (4) is complete.

□

In the calculation of $H^q(g), \prod X_i = (\prod X_i, X_i \in \mathbb{T})(\prod X_i, X_i \in \mathbb{K})$.

Definition 2.2. $H^q(g, \sum T) = H^q(g, \sum T, PS, PT) = H^q(g, \sum T, M) = \sum \prod_{X_i \in \mathbb{T}} B_i, H^q(g, \sum K) = \sum \prod_{X_i \in \mathbb{K}} B_i$

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Definition 2.3. $F_0^{\mathbb{K}} = 1,\ E_0^{\mathbb{K}} = 1,$

$$F_g^{\mathbb{K}} = \sum_{1\le\lambda_1<...<\lambda_g\le M}\prod_{i=1}^{g} K_{\lambda_i}, F_g^N = F_g^{\{1,2...N\}}.\ E_g^{\mathbb{K}} = \sum_{1\le\lambda_1\le...\le\lambda_g\le M}\prod_{i=1}^{g} K_{\lambda_i}, E_g^N = E_g^{\{1,2...N\}}.$$
$$E_g^{N,q} = \sum_{1\le\lambda_1\le...\le\lambda_g\le N}\prod_{i=1}^{g}[\lambda_i]_q = S_2^q(N+g,N).\ E_g^{N,q-} = \sum_{1\le\lambda_1\le...\le\lambda_g\le N}\prod_{i=1}^{g}[\lambda_i]_{q-} = S_2^{q-}(N+g,N).$$

Theorem 2.3. *PT = [T,T+1...T+M-1] and* $D_i = 1, H_1^q(g,\sum K) = F_{M-g}^{\mathbb{K}}E_0^{g,q} + F_{M-g-1}^{\mathbb{K}}E_1^{g,q} + ... + F_0^{\mathbb{K}}E_{M-g}^{g,q}.$

Proof.

$$\begin{aligned}
&PS1 = [PS, K_{M+1}], PT1 = [PT, T_{M+1}]. \text{ Using induction to prove.}\\
&H_1^q(g,\textstyle\sum K, M+1) = H_1^q(g-1,\sum K) + H_1^q(g,\sum K)(K_{M+1}+G_1^g).\\
&F_A^{\{K_1,K_2...K_M,K_{M+1}\}} \textit{ in } H_1^q(g,\textstyle\sum K, M+1) \text{ has three sources.}\\
&= F_A^{\mathbb{K}}E_{M-(g-1)-A}^{g-1,q} + F_{A-1}^{\mathbb{K}}E_{M-g-(A-1)}^{g,q}K_{M+1} + F_A^{\mathbb{K}}E_{M-g-A}^{g,q}G_1^g\\
&= F_A^{\mathbb{K}}(E_{M+1-g-A}^{g-1,q} + E_{M-g-A}^{g,q}G_1^g) + F_{A-1}^{\mathbb{K}}E_{M+1-g-A}^{g,q}K_{M+1}\\
&= F_A^{\mathbb{K}}E_{M+1-g-A}^{g,q} + F_{A-1}^{\mathbb{K}}E_{M+1-g-A}^{g,q}K_{M+1} = F_A^{\{K_1,K_2...K_M,K_{M+1}\}}E_{M+1-g-A}^{g,q}.
\end{aligned}$$

□

So $PT = [1,2...M]$, we can always choose $\mathbb{K}$, $\nabla_q^{-p}SUM_q(X-p,PS,PT) = \sum_{g=0}^{M} H_1^q(g)q^{pg}G_{1+p+g}^X$, $H_1^q(M) = [M!]_{q+}$, $H_1^q(g<M)p^{pg}$ can take any value. So $\sum_{g=0}^{M} a_g G_{1+p+g}^X, 1+p\ge 0$, can be converted to $\frac{a_M}{[M!]_{q+}}q^{-pM}\nabla_q^{-p}SUM_q(X-p,PS,PT)$. For an arbitrary PT1, $SUM_q(N,PS1,PT1)$ can be converted into constant $\times\nabla_q^{-(T_M-M)}SUM_q(N,PS,PT)$.
In this article, if $\sum_{g=0}^{M} a_g G_{1+p+g}^X = \sum_{g=0}^{M} b_g G_{1+p+g}^{X+g} = \sum_{g=0}^{M} c_g G_{1+p+M}^{X+M-g}, 1+p\ge 0$, then c is a constant, $a_g = c\times H_1^q(g,PS,PT)q^{pg}, b_g = c\times H_2^q(g,PS,PT), c_g = c\times H_3^q(g,PS,PT)q^{pg}$. From [2.2], [2.1(3)]:

Theorem 2.4. $X,K\in\mathbb{N}, M>K, 1+p\ge 0, a_g^* = a_g q^{-pg}, c_g^* = c_g q^{-pg}.$
If $\sum_{g=0}^{M} a_g G_{1+p+g}^X = \sum_{g=0}^{M} b_g G_{1+p+g}^{X+g} = \sum_{g=0}^{M} c_g G_{1+p+M}^{X+M-g}$, *then*

$$a_g^* = q^{g(g+1)}\sum_{k=g}^{M} b_k G_g^k = \sum_{k=0}^{g} G_{M-g}^{M-k}q^{(g+1)(g-k)}c_k^*.$$
$$b_g = (-1)^g G_g^k q^{\frac{g(g+1)}{2}}\sum_{k=g}^{M}(-1)^k G_g^k q^{\frac{-k(k+3)}{2}-kg}a_k^*.\ c_g^* = (-1)^g G_g^k q^{\frac{g(g+3)}{2}}\sum_{k=0}^{g}(-1)^k G_{M-g}^{M-k}q^{\frac{-k(k+3)}{2}}a_k^*.$$

$\sum_{g=0}^{M} a_g G_{1+p+g}^X$ *can be coverted to* $\sum_{g=0}^{M-K}(...)G_{1+p+K+g}^{X+K}$ *is equivalent to*

$$\sum_{k=g}^{M}(-1)^k G_g^k q^{-k(k+1)+\binom{k-g}{2}}a_k^* = 0, 0\le g<K.$$
$$\sum_{k=0}^{g}(-1)^k G_{M-g}^{M-k}q^{(g+1)(g-k)-\binom{g-k}{2}}a_k^* = 0, 0\le M-g<K.$$

The latter part refers to the necessary and sufficient conditions for merging, which correspond to $b_g = c_{M-g} = 0$.

3. Application

Proposition 3.1.
(1). $\sum_{0\le\lambda_1\le\lambda_2\le...\le\lambda_M\le N} q^{\sum_{i=1}^{M}\lambda_i} = G_M^{N+M} = \sum_{g=0}^{M}(-1)^g q^{Mg+\frac{g(g+1)}{2}}G_g^M G_{2M}^{N+2M-g}.$
(2). $\sum_{A\le\lambda_1<\lambda_2<...<\lambda_M\le B} q^{\sum_{i=1}^{M}\lambda_i} = q^{\binom{M}{2}+AM}G_M^{B-A+1},\ A,\ B\in\mathbb{Z},\ \sum_{1\le\lambda_1<...<\lambda_g\le M} q^{\sum_{i=1}^{M}\lambda_i} = q^{\binom{g+1}{2}}G_g^M.$
(3). $\prod_{i=1}^{M}(a+q^{A+i}z) = \sum_{g=0}^{M} q^{\binom{g}{2}+(A+1)g}G_g^M z^g a^{M-g}.$
(4). $G_{M-K}^{N-K} = \sum_{g=0}^{M}(-1)^g q^{(M-K)g+\frac{g(g+1)}{2}}G_g^M G_{2M-K}^{N+M-K-g}, N\ge M\ge K\ge 0.$

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(5). $\sum_{g=0}^{M}(-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_K^{N-g} = (q^M;q^{-1})_{M-K}, N \geq M \geq K \geq 0.$
(6). $\sum_{g=0}^{M}(-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_{M+1}^{N-g} = \sum_{i=0}^{N-M-1} q^{(M+1)i}, N \geq M+1, M > 0.$
(7). $\begin{bmatrix} N+M \\ M \end{bmatrix}_{q^2} = \sum_{g=0}^{M}(-1)^g q^{g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2} G_{2M}^{N+2M-g}.$

Proof.

$PS = [1,1...1]:0, PT = [1,3...2M-1], H_1^q(g>0) = 0, H_1^q(0) = 1, SUM(N+1) = G_M^{N+M} \rightarrow$ pre-equation of (1).
$\sum_{A \leq \lambda_1 < ... < \lambda_M \leq B} q^{\sum_{i=1}^{M}\lambda_i} = q^{A+(A+1)+...+(A+M-1)} \sum_{0 \leq \lambda_1 \leq ... \leq \lambda_M \leq B-A-M+1} q^{\sum_{i=1}^{M}\lambda_i} \rightarrow (2).$
At $H_3^q(g), B_i = \begin{cases} q^{X_T}(-q^{2i-1}), X_i = T_i \\ q^{X_T} = q^{i-X_K}, X_i = K_i \end{cases}$, Extract q^i form B_i, after extraction, $B_i = \begin{cases} q^{X_T}(-q^{i-1}), X_i = T_i \\ q^{-X_K}, X_i = K_i \end{cases}$,
$H_3^q(g) = q^{\binom{M+1}{2}-\binom{M-g+1}{2}}(-1)^g q^{\binom{g+1}{2}-g} \sum_{1 \leq \lambda_1 < ... < \lambda_g \leq M} q^{\sum_{i=1}^{g}\lambda_i} = (-1)^g q^{\binom{M+1}{2}-\binom{M-g+1}{2}+\binom{g}{2}+\binom{g+1}{2}} G_g^M.$
$SUM(N+1) = G_M^{N+M} = \sum_{g=0}^{M} H_3^q(g) G_{2M}^{N+2M-g} \rightarrow$ post-equation of (1).
Comparing the coefficients of z^g on both sides of (3), combined with (2), proves that (3).
$A = -1,$ it's Rothe's q-Binomial Theorem $\prod_{i=1}^{M}(a+q^{i-1}z) = \sum_{g=0}^{M} q^{\binom{g}{2}} G_g^M z^g a^{M-g}.$

Using the post-equation of (1) and ∇_q^K to obtain (4) $\rightarrow K = M, \sum_{g=0}^{M}(-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_M^{N-g} = 1$
$= \sum_{g=0}^{M}(-1)^g q^{\frac{g(g+1)}{2}} G_g^M (q^M G_M^{N-g-1} + G_{M-1}^{N-g-1}) = q^M + \sum_{g=0}^{M}(-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_{M-1}^{N-g-1}$
$\rightarrow \sum_{g=0}^{M}(-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_{M-1}^{N-g-1} = 1 - q^M.$ Continuing the same process $\rightarrow$ (5).
$K = 0, \sum_{g=0}^{M}(-1)^g q^{\frac{g(g+1)}{2}} G_g^M = (q;q)_M; K = M, \sum_{g=0}^{M}(-1)^g q^{\frac{g(g+1)}{2}} G_g^M G_M^{N-g} = 1,$ it's a special case of (4).
Using the same recursive and inductive methods can obtain (6).
$SUM_q(N+1, [1,1...1]: q-1, [1,3...2M-1]) = \sum_{0 \leq \lambda_1 \leq ... \leq \lambda_M \leq N} q^{2\sum \lambda_i} = \begin{bmatrix} N+M \\ M \end{bmatrix}_{q^2}.$
$At\ H_1^q(g), B_i = \begin{cases} q^{1+(T_i-T_{i-1})X_{T-1}} G_1^{T_i-X_{K-1}} D_i = q^{1+2X_{T-1}}(q^{2i-1-X_{K-1}}-1) = q^{2X_T-1}(q^{i+X_{T-1}}-1),\ X_i = T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}}(K_i + G_1^{X_{T-1}} D_i) = q^{2X_{T-1}} = q^{2X_T}, X_i = K_i \end{cases},$
Unable to derive a concise expression for $H_1^q(g)$, the same applies to $H_2^q(g)$.
$At\ H_3^q(g), B_i = \begin{cases} q^{1+X_{T-1}}\{(q^{X_{T-1}} G_1^{T_i} - q^{T_i} G_1^{X_{T-1}}) D_i - K_i q^{T_i}\} = -q^{1+2X_{T-1}} = -q^{-1+2X_T}, X_i = T_i \\ q^{X_{T-1}}(K_i + G_1^{X_{T-1}} D_i) = q^{X_{T-1}}(1 + G_1^{X_{T-1}}(q-1)) = q^{2X_{T-1}} = q^{2X_T}, X_i = K_i \end{cases}.$
$H_3^q(g) = (-1)^g q^{-g+2(\sum_{i=1}^{g} i)} \sum_{X_i \in \mathbb{K}} \prod q^{2X_T} = (-1)^g q^{g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2}.$

□

$\sum_{A \leq \lambda_1 < \lambda_2 < ... < \lambda_M \leq B} q^{\sum_{i=1}^{M}\lambda_i D} = q^{\binom{M}{2}D + AMD} \begin{bmatrix} B-A+1 \\ M \end{bmatrix}_{q^D}.$
$\prod_{i=1}^{A}(1+q^{C+Di}z) = \sum_{g=0}^{A} q^{gC} \sum_{1 \leq \lambda_1 \leq ... \leq \lambda_g \leq A} q^{\sum_{i=1}^{g}\lambda_i D} z^g = \sum_{g=0}^{A} q^{\binom{g}{2}D + gD + gC} \begin{bmatrix} A \\ g \end{bmatrix}_{q^D} z^g.$
$\prod_{i=1}^{A}(a+q^{(2i-1)D}z) = \sum_{g=0}^{A} q^{g^2 D} \begin{bmatrix} A \\ g \end{bmatrix}_{q^{2D}} z^g a^{A-g}, \prod_{i=1}^{B}(a^{-1}+q^{(2i-1)D}z^{-1}) = \sum_{g=0}^{B} q^{g^2 D} \begin{bmatrix} B \\ g \end{bmatrix}_{q^{2D}} z^{-g} a^{-(B-g)}.$
$\prod_{i=1}^{A}(a+q^{(2i-1)D}z) \prod_{i=1}^{B}(a^{-1}+q^{(2i-1)D}z^{-1}) = \sum_{k=-B}^{A} z^k f(k).$
$f(k) = \sum_{i=0}^{A+B} \begin{bmatrix} A \\ i \end{bmatrix}_{q^{2D}} \begin{bmatrix} B \\ i-k \end{bmatrix}_{q^{2D}} q^{i^2 D} q^{(i-k)^2 D} a^{A-i} a^{-(B-(i-k))}$
$= q^{k^2 D} a^{A-B-k} \sum_{i=0}^{A+B} \begin{bmatrix} A \\ i \end{bmatrix}_{q^{2D}} \begin{bmatrix} B \\ i-k \end{bmatrix}_{q^{2D}} q^{2D(i(i-k))} = q^{k^2 D} a^{A-B-k} \begin{bmatrix} A+B \\ A-k \end{bmatrix}_{q^{2D}}.$

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The last step used q-Vandermorde identity. $a = D = 1$, it's MacMahon's q-binomial theorem. [2] pp 74.

$|q|, |x| < 1, \dfrac{1}{(x;q)_{M+1}} = (1 + x + x^2 + ...)(1 + xq + x^2q^2 + ...)(1 + xq^2 + +x^2q^4 + ...)...(1 + xq^M + +x^2q^{2M} + ...)$.

When we multiply this out, the coefficient of x^2 will be $\sum q^a q^b, 0 \le a \le b \le M$, and so forth.

$(1) \rightarrow \dfrac{1}{(x;q)_{M+1}} = \sum_{k=0}^{\infty} G_k^{M+k} x^k, \dfrac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty} \dfrac{x^k}{(q;q)_k}$, it's Euler and Cauchy's identity [2] pp 121-123.

$(3) \rightarrow (-x,q)_M = \sum_{g=0}^M q^{\binom{g}{2}} G_g^M x^g \rightarrow (-x,q)_\infty = \sum_{k=0}^\infty \dfrac{q^{\binom{k}{2}} x^k}{(q;q)_k}$, it's Euler's identity [2] pp 129.

$C \ge 2, PS = [1,1...1] : q-1, PT = [1, C+1, 2C+1...C(M-1)+1],\ At\ H_3^q(g), B_i = \begin{cases} -q^{CX_{T-1}+1}, X_i = T_i \\ q^{CX_T}, X_i = K_i \end{cases} \rightarrow$

$X = C(M-1) - M + 2, M > 1, \sum_{0 \le \lambda_1 \le ... \le \lambda_X \le N-1} q^{\sum_{i=1}^X f_i \lambda_i} = \sum_{g=0}^M (-1)^g q^{g + C\frac{g(g-1)}{2}} \begin{bmatrix} M \\ g \end{bmatrix}_{q^C} G_{C(M-1)+2}^{N+C(M-1)+1-g}$,

$f_i = 2, i \equiv 1 \pmod{C-1}; f_i = 1, i \not\equiv 1 \pmod{C-1}$.

Proposition 3.2.
(1). $G_{M+1}^{N+M} = \sum_{g=0}^M q^{(g+1)g} G_g^M G_{1+g}^N; G_{M+1+p}^{N+M+p} = \sum_{g=0}^M q^{(g+1+p)g} G_g^M G_{p+1+g}^{N+p} = \sum_{g=0}^{M+p} q^{(g+1)g} G_g^{M+p} G_{1+g}^N$, $M + p \ge 0$.
(2). $\sum_{k=g}^M (-1)^k G_g^k G_k^M q^{\binom{k}{2} - gk} = 0, g < M; \sum_{k=0}^g (-1)^k G_{M-g}^{M-k} G_k^M q^{\binom{k}{2}} = 0, g > 0$.
(3). $M > g + x, \sum_{k=g}^M (-1)^k G_g^k G_k^M q^{\binom{k}{2} - (g+x)k} = 0;\ M > g, \sum_{k=g}^M (-1)^k G_g^k G_k^M q^{\binom{k}{2} - (M-1)k} = 0$.
(4). $g > x, \sum_{k=0}^g (-1)^k G_{M-g}^{M-k} G_k^M q^{\binom{k}{2} - xk} = 0;\ g > 0, \sum_{k=0}^g (-1)^k G_{M-g}^{M-k} G_k^M q^{\binom{k}{2} - (g-1)k} = 0$.

Proof.

$PS = [[1]_{q-}, [2]_{q-} ... [M]_{q-}], PS1 = [[M]_{q-}, [M-1]_{q-} ... [1]_{q-}], PT = [1, 2...M]$

$SUM_q(N, PS, PT) = [M!]_{q-} G_{M+1}^{N+M} = \sum_{g=0}^M H_1^q(g) G_{1+g}^N = SUM_q(N, PS1, PT)$.

$In\ H_1^q(g, PS1, PT), X_i \in \mathbb{K},\ B_i = [M+1-i]_{q-} + G_1^{X_{T-1}} = q^{-(M-i+1)} G_1^{M+1-i+X_{T-1}}$.

$H_1^q(g, \sum K) = G_1^M G_1^{M-1} ... G_1^{g+1} q^{-(M+1)(M-g)} \sum_{1 \le \lambda_1 < ... < \lambda_{M-g} \le M} q^{\lambda_1 + ... + \lambda_{M-g}}$.

$= G_1^M G_1^{M-1} ... G_1^{g+1} q^{\binom{M-g+1}{2} - (M+1)(M-g)} G_{M-g}^M.\ H_1^q(g) = q^{\sum_{i=1}^g i} [g!]_q H_1^q(g, \sum K)$.

$\dfrac{H_1^q(g)}{[M!]_{q-}} = q^{\binom{M-g+1}{2} - (M+1)(M-g) + \sum_{i=1}^M i + \sum_{i=1}^g i} G_{M-g}^M = q^{g(g+1)} G_g^M$, this and [2.1(3)] yields (1).

It's q-Vandermorde identity: $G_{M+t}^{N+M} = G_{N-t}^{N+M} = \sum_{g=0}^M q^{g(g+t)} G_g^M G_{g+t}^N$. (2) is obtained from [2.4].

It is obvious from q-Binomial theorem that $\sum_{k=0}^M (-1)^k q^{\binom{k}{2}} G_k^M = 0$.

$\sum_{k=g}^M (-1)^k G_g^k q^{\binom{k}{2} - gk} G_k^M = \sum_{k=g}^M (-1)^k G_g^k q^{\binom{k}{2} - gk} G_k^{M-1} + q^M \sum_{k=g}^M (-1)^k G_g^k q^{\binom{k}{2} - gk} G_{k-1}^{M-1} q^{-k}$.

$M > 1 + g, \sum_{k=g}^M (-1)^k G_g^k q^{\binom{k}{2} - gk} G_k^{M-1} = 0 \rightarrow \sum_{k=g}^M (-1)^k G_g^k q^{\binom{k}{2} - (g+1)k} G_{k-1}^{M-1} = 0$.

$\sum_{k=g}^M (-1)^k G_g^k q^{\binom{k}{2} - gk} G_k^M q^{-k} = \sum_{k=g}^M (-1)^k G_g^k q^{\binom{k}{2} - gk} G_k^{M-1} + \sum_{k=g}^M (-1)^k G_g^k q^{\binom{k}{2} - (g+1)k} G_{k-1}^{M-1} = 0$.

Similarly, $M > g + x, \sum_{k=g}^M (-1)^k G_g^k q^{\binom{k}{2}} G_k^M q^{-(g+x)k} = 0 \rightarrow (3)$.

$\sum_{k=0}^g (-1)^k G_{M-g}^{M-k} G_k^M q^{\binom{k}{2}} = \sum_{k=0}^g (-1)^k G_{M-g}^{M-k} q^{\binom{k}{2}} G_k^{M-1} + q^M \sum_{k=0}^g (-1)^k G_{M-g}^{M-k} q^{\binom{k}{2} - k} G_{k-1}^{M-1} = 0$.

$(-1)^k G_{M-g}^{M-k} q^{\binom{k}{2} - k} G_{k-1}^{M-1} = q^{-1} (-1)^k G_{M-1-(g-1)}^{M-1-(k-1)} q^{\binom{k-1}{2}} G_{k-1}^{M-1} \rightarrow g > 0, \sum_{k=0}^g (-1)^k G_{M-g}^{M-k} q^{\binom{k}{2}} G_k^{M-1} = 0$.

$\sum_{k=0}^g (-1)^k G_{M-g}^{M-k} q^{\binom{k}{2} - k} G_k^M = \sum_{k=0}^g (-1)^k G_{M-g}^{M-k} q^{\binom{k}{2}} G_k^{M-1} + q^{-1} \sum_{k=0}^g (-1)^k G_{M-1-(g-1)}^{M-1-(k-1)} q^{\binom{k-1}{2}} G_{k-1}^{M-1} = 0$.

Similarly, $g > x, \sum_{k=0}^g (-1)^k G_{M-g}^{M-k} q^{\binom{k}{2} - xk} G_k^M = 0$.

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□

Proposition 3.3. *If* $\sum_{g=0}^{M} a_g G_g^n = \prod_{i=1}^{M}(K_i + [n]_q D_i)$ *then* $\sum_{g=0}^{M} a_g G_g^{T+g} G_{T+g}^{n+T} = \prod_{i=1}^{M}(K_i + [n]_q D_i) G_T^{n+T}$.

Proof.

$$PT1 = [1, 2...M], PT2 = [T+1, T+2...T+M],$$
$$\nabla_q SUM(n+1, PS, PT1) = \prod_{i=1}^{M}(K_i + [n]_q D_i), \nabla_q SUM(n+1, PS, PT2) = \prod_{i=1}^{M}(K_i + [n]_q D_i) G_T^{n+T},$$
$$\text{At } H_1^q(g), B_i \text{ of } PT1 = \begin{cases} q^{X_T} G_1^{X_T} D_i, \; X_i = T_i \\ (K_i + G_1^{X_T} D_i), X_i = K_i \end{cases}, \; B_i \text{ of } PT2 = \begin{cases} q^{X_T} G_1^{T+X_T} D_i, \; X_i = T_i \\ (K_i + G_1^{X_T} D_i), X_i = K_i \end{cases}.$$
$$H_1^q(g, PS, PT2) = H_1^q(g, PS, PT1) \frac{G_1^T ... G_1^{T+g}}{G_1^1 ... G_1^g} = H_1^q(g, PS, PT1) G_g^{T+g}, \; a_g = \text{constant } \times H_1^q(g, PS, PT1) q^{-g}$$

□

For $D_i = (q-1)K_i, PS = [T, T+D...T+(M-1)D], H_3^q(g)$ and T are not related, similar equations exist.

Proposition 3.4. $x \geq 0$,
(1). $\sum_{k=0}^{M}(-1)^{k+g} G_k^M G_g^k q^{\frac{k(k-1)}{2} - gk + xk} z^k = q^{\frac{g(g-1)}{2} - g^2 + xg} G_g^M (zq^x; q)_{M-g} z^g$.
(2). $\sum_{k=0}^{M}(-1)^k G_k^M G_{M-g}^{M-k} q^{\frac{k(k-1)}{2} + xk} z^k = G_g^M (zq^x; q)_g$.

Proof.

For (1): $g = 0$, it's [3.1(3)]. When $g = M$, it clearly holds. Prove by induction.

$$\sum_{k=0}^{M}(-1)^{k+g}(G_k^{M-1} + q^{M-k} G_{k-1}^{M-1}) G_g^k q^{\frac{k(k-1)}{2} - gk + xk} z^k$$
$$= q^{\frac{g(g-1)}{2} - g^2 + xg} G_g^{M-1}(zq^x; q)_{M-1-g} z^g + q^M \sum_{k=0}^{M}(-1)^{k+g} G_{k-1}^{M-1} G_g^k q^{\frac{k(k-1)}{2} - gk + xk - k} z^k$$
$$= q^{\frac{g(g-1)}{2} - g^2 + xg} G_g^{M-1}(zq^x; q)_{M-1-g} z^g - q^M \sum_{k=0}^{M}(-1)^{k+g} G_k^{M-1}(G_g^k + q^{k+1-g} G_{g-1}^k) q^{\frac{k(k-1)}{2} - g(k+1) + x(k+1) - 1} z^{k+1}$$
$$= q^{\frac{g(g-1)}{2} - g^2 + xg}(G_g^{M-1}(zq^x; q)_{M-1-g} z^g - z^{g+1} q^{M+x-g-1} G_g^{M-1}(zq^x; q)_{M-1-g})$$
$$- q^M z \sum_{k=0}^{M}(-1)^{k+g} G_k^{M-1} q^{k+1-g} G_{g-1}^k q^{\frac{k(k-1)}{2} - g(k+1) + x(k+1) - 1} z^k$$
$$= q^{\frac{g(g-1)}{2} - g^2 + xg} G_g^{M-1}(zq^x; q)_{M-g} z^g + q^{M+x-2g} z \sum_{k=0}^{M}(-1)^{k+g-1} G_k^{M-1} G_{g-1}^k q^{\frac{k(k-1)}{2} - (g-1)k + xk} z^k$$
$$= (zq^x; q)_{M-g} z^g (q^{\frac{g(g-1)}{2} - g^2 + xg} G_g^{M-1} + q^{\frac{(g-1)(g-2)}{2} - (g-1)^2 + x(g-1)} q^{M+x-2g} G_{g-1}^{M-1})$$
$$= q^{\frac{g(g-1)}{2} - g^2 + xg}(zq^x; q)_{M-g} z^g (G_g^{M-1} + q^{M-g} G_{g-1}^{M-1}) \rightarrow (1).$$

For (2): $g = M$, it's [3.1(3)]. When $g = 0$, it clearly holds.

$$\sum_{k=0}^{M}(-1)^k (q^k G_k^{M-1} + G_{k-1}^{M-1}) G_{M-g}^{M-k} q^{\frac{k(k-1)}{2} + xk} z^k$$
$$= \sum_{k=0}^{M}(-1)^k G_k^{M-1} G_{M-g}^{M-k} q^{\frac{k(k-1)}{2} + (x+1)k} z^k + \sum_{k=0}^{M}(-1)^k G_{k-1}^{M-1} G_{M-1-(g-1)}^{M-1-(k-1)} q^{\frac{k(k-1)}{2} + xk} z^k$$
$$= \sum_{k=0}^{M}(-1)^k G_k^{M-1} G_{M-1-(g-1)}^{M-1-k} q^{\frac{k(k-1)}{2} + (x+1)k} z^k + \sum_{k=0}^{M}(-1)^k G_k^{M-1} q^{g-k} G_{M-1-g}^{M-1-k} q^{\frac{k(k-1)}{2} + (x+1)k} z^k$$
$$- z \sum_{k=0}^{M}(-1)^k G_k^{M-1} G_{M-1-(g-1)}^{M-1-k} q^{\frac{k(k+1)}{2} + x(k+1)} z^k$$
$$= G_{g-1}^{M-1}(zq^{1+x}; q)_{g-1} - zq^x G_{g-1}^{M-1}(zq^{1+x}; q)_{g-1} + q^g G_g^{M-1}(zq^x; q)_g \rightarrow (2).$$

□

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Proposition 3.5.
$\sum_{k=0}^{M}(-1)^{k+g}G_k^M G_g^k = \begin{cases} G_g^M(q;q^2)_{\frac{M-g}{2}}, M+g\equiv 0 \pmod 2 \\ 0, M+g\equiv 1 \pmod 2\end{cases}$; $\sum_{k=0}^{M}(-1)^{k+g}G_k^M G_{M-g}^{M-k} = \begin{cases} G_g^M(q;q^2)_{\frac{g}{2}}, g\equiv 0 \pmod 2 \\ 0, g\equiv 1 \pmod 2\end{cases}$.

Proof.

M+g is odd, $(-1)^{k+g}G_k^M G_g^k = \frac{(-1)^{k+g}(q;q)_M}{(q;q)_g(q;q)_{k-g}(q;q)_{M-k}} = -(-1)^{g+M-k+g}G_{M-k+g}^M G_g^{M-k+g} \to Sum = 0.$

M+g is even, use induction to prove. Sum $= \sum_{k=0}^{M}(-1)^{k+g}G_k^{M-1}G_g^k + \sum_{k=0}^{M}(-1)^{k+g}G_{k-1}^{M-1}G_g^k q^{M-k}$

$= 0 + \sum_{k=0}^{M}(-1)^{k+g}G_{k-1}^{M-1}G_g^k q^{M-k} = \sum_{k=0}^{M}(-1)^{k+g}G_{k-1}^{M-1}G_g^{k-1}q^{M-k} + \sum_{k=0}^{M}(-1)^{k+g}G_{k-1}^{M-1}G_{g-1}^{k-1}q^{M-g}$

$= \sum_{k=0}^{M}(-1)^{k+g-1}G_{k-1}^{M-1}G_g^{k-1}(1-q^{M-k}) + \sum_{k=0}^{M}(-1)^{k+g}G_{k-1}^{M-1}G_{g-1}^{k-1}q^{M-g}$

$= \sum_{k=0}^{M}(-1)^{k+(g+1)}G_k^{M-1}G_{g+1}^k(1-q^{g+1}) + \sum_{k=0}^{M}(-1)^{(k-1)+(g-1)}G_{k-1}^{M-1}G_{g-1}^{k-1}q^{M-g}$

$= G_{g+1}^{M-1}(q;q^2)_{(M-g-2)/2}(1-q^{g+1}) + q^{M-g}G_{g-1}^{M-1}(q;q^2)_{(M-g)/2} = (G_g^{M-1} + q^{M-g}G_{g-1}^{M-1})(q;q^2)_{(M-g)/2}.$

Replace k by $M-k$, g by $M-g$ to complete the proof. $g=0$, It's Gauss's q-Binomial theorems [2] pp 61.

□

Proposition 3.6. $\prod_{i=1}^{M}(b+q^{A+i}z) = \sum_{g=0}^{M}\prod_{i=1}^{g}(q^{M+1-i}-1)\prod_{i=1}^{M-g}(b+q^{A+1-i}z)G_g^M z^g q^{g(g+A-M)}$.

Proof.

$PS = [b+q^A z:(q-1)q^A z, b+q^{A-1}z:(q-1)q^{A-1}z \ldots b+q^{A-M+1}z:(q-1)q^{A-M+1}z], PT = [1,2\ldots M].$

$\nabla_q^1 SUM_q(M+1) = \prod_{i=1}^{M}(b+q^{M+A-i+1}z) = \prod_{i=1}^{M}(b+q^{A+i}z) = \sum_{g=0}^{M}H_1^q(g)G_g^M q^{-g}.$

$At\ H_1^q(g),\ B_i = \begin{cases} q^{X_T}(q^{X_T}-1)q^{A-i+1}z, X_i=T_i \\ b+q^{X_T-1}q^{A-i+1}z = b+q^{A-X_{K-1}}z, X_i=K_i\end{cases}.$

$H_1^q(g) = q^{g(A+1)+\sum_{i=1}^{g}i}z^g\prod_{i=1}^{g}(q^i-1)[\sum_{-M\le\lambda_1<\ldots<\lambda_g\le -1}q^{\sum_{i=1}^{g}\lambda_i}]\prod_{i=1}^{M-g}(a+q^{A+1-i}z)$

$= q^{g(A+1)+\binom{g+1}{2}}q^{\binom{g}{2}-Mg}G_g^M z^g\prod_{i=1}^{g}(q^i-1)\prod_{i=1}^{M-g}(b+q^{A+1-i}z).$

$\prod_{i=1}^{M}(b+q^{A+i}z) = \sum_{g=0}^{M}\prod_{i=1}^{g}(q^i-1)\prod_{i=1}^{M-g}(b+q^{A+1-i}z)G_g^M G_g^M q^{g(g+A-M+1)}q^{-g}z^g.$

□

$b=1, z=-1, \prod_{i=1}^{M}(1-q^{A+i}) = \sum_{g=0}^{M}\prod_{i=1}^{g}(1-q^{M+1-i})\prod_{i=1}^{M-g}(1-q^{A+1-i})G_g^M q^{g(g+A-M)}$

$\to G_M^{A+M} = \sum_{g=0}^{M}G_{M-g}^A G_g^M q^{g(g+A-M)}$, it's q-Vandermorde identity.

$\prod_{i=1}^{M}(b-q^{A+i}z) = \sum_{g=0}^{M}\prod_{i=1}^{g}(zq^{g+A-M}-q^{1-i+g+A}z)\prod_{i=1}^{M-g}(b-q^{A+1-i}z)G_g^M.$

$= \sum_{g=0}^{M}\prod_{i=1}^{g}(zq^{g+A-M}-q^{g-i}q^{A+1}z)\prod_{i=1}^{M-g}(b-q^{A+1-i-(g+A-M)}zq^{g+A-M})G_g^M.$

$a = q^{A+1}z, d = zq^{A-M} \to \prod_{i=1}^{M}(b-q^{i-1}a) = \sum_{g=0}^{M}\prod_{i=1}^{g}(dq^g-q^{g-i}a)\prod_{i=1}^{M-g}(b-q^{M-g-i}dq^{g+1})G_g^M.$

It can be compared to Jacobi's q-binomial theorem [2] pp 71, c is any number,

$\prod_{i=1}^{M}(b-q^{i-1}a) = \sum_{g=0}^{M}\prod_{i=1}^{g}(c-q^{g-i}a)\prod_{i=1}^{M-g}(b-q^{M-g-i}c)G_g^M.$

Proposition 3.7. $PS = [[T_i]_{q-}], PS1 = [[K]_{q-}, [K+1]_{q-}\ldots[K+M-1]_{q-}], PT1 = [T, T+1\ldots T+M-1].$

(1). $H_1^q(g, PS, PT) = \prod_{i=1}^{M}[T_i]_{q-}q^{(g+1+p)g}G_g^M, p = T_M - M.$

(2). $H_1^q(g, PS1, PT1) = \prod_{i=1}^{g}[T-1+i]_q\prod_{i=1}^{M-g}[K+M-i]_q G_g^M q^{\binom{g+1}{2}-(K+M)(M-g)+\binom{M-g+1}{2}}.$

(3). $\mathbb{K} = \{PS1\}, F_{M-g}^{\mathbb{K}}E_0^{g,q} + F_{M-g-1}^{\mathbb{K}}E_1^{g,q} + \ldots + F_0^{\mathbb{K}}E_{M-g}^{g,q} = \prod_{i=1}^{M-g}[K+M-i]_q G_g^M q^{\binom{M-g+1}{2}-(K+M)(M-g)}.$

Proof.

$\sum_{g=0}^{M} H_1^q(g) G_{1+p+g}^{N+p} = \prod [T_i]_{q-} G_{T_M+1}^{N+T_M} = \prod [T_i]_{q-} G_{M+p+1}^{N+M+p}$, [3.2] yields (1), [2.4] can also reach the conclusion.

At (2), K_i can exchange order, let $PS = [[K+M-1]_{q-}...[K+1]_{q-}, [K]_{q-}]$.

$B_i \; of \; H_1^q(g) = \begin{cases} q^{X_T} G_1^{T+X_T-1}, X_i = T_i \\ q^{-(K+M-i)} G_1^{K+M-i+X_T}, X_i = K_i \end{cases}$. $H_1^q(g) = q^{\binom{g+1}{2}} \prod_{i=1}^{g} [T-1+i]_q H_1^q(g, \sum K)$.

$H_1^q(g, \sum K) = \prod_{i=1}^{M-g} [K+M-i]_q q^{-(K+M)(M-g)} q^{\binom{M-g+1}{2}} G_g^M$ and [2.3] yields (2) and (3).

If K=T, then p=T-1, $q^{\binom{g+1}{2} - (T+M)(M-g) + \binom{M-g+1}{2}} = q^{g(g+1+p) - \sum_{i=1}^{M} (T-1+i)}$, it's compatible with (1).

□

$For \; H_1(g), (1) = \prod T_i \binom{M}{g}, (2) = \prod_{i=1}^{g} (T-1+i) \prod_{i=1}^{M-g} (K+M-i) \binom{M}{g}$.

$\frac{(1-q)^{A+M}}{(q;q)_A (q;q)_M} \nabla_q SUM(N, [[1]_{q-}, [2]_{q-}...[A]_{q-}, [B+1]_{q-}, [B+2]_{q-}...[B+M]_{q-}], [1, 2...A, A+1, A+2...A+M])$

$= G_A^{n+A} G_M^{n+M+B} = \frac{(1-q)^{A+M}}{(q;q)_A (q;q)_M} [A!]_{q-} \nabla_q SUM(N, [[B+1]_{q-}...[B+M]_{q-}], [A+1...A+M])$.

$\frac{(1-q)^{A+M}}{(q;q)_A (q;q)_M} \nabla_q SUM(N, [[X]_{q-}, [X-1]_{q-}...[X-A+1]_{q-}, [Y]_{q-}, [Y+1]_{q-}...[Y+M-1]_{q-}], [1, 2...A+M])$

$0 \le Y \le M, PS1 = [[1]_{q-}, [2]_{q-}...[Y]_{q-}, [0]_{q-}, [-1]_{q-}, [-2]_{q-}...[-(M-Y)+1]_{q-}, [X]_{q-}, [X-1]_{q-}...[X-A+1]_{q-}]$,

$= G_A^{n+X} G_M^{n+Y} = \frac{(1-q)^{A+M}}{(q;q)_A (q;q)_M} \nabla_q SUM(N, PS1, [1, 2...A+M])$

$= \frac{(1-q)^{A+M}}{(q;q)_A (q;q)_M} [Y!]_{q-} \nabla_q SUM(N, [[0]_{q-}...[-(M-Y)+1]_{q-}, [X-A+1]_{q-}...[X]_{q-}], [Y+1...A+M])$.

If $H_1^q(g) > 0$, then $X_1, X_2...X_{M-Y} \in \mathbb{T}$, so $H_1^q(g < M-Y) = 0$. It's not difficult to deduce $H_1^q(g \ge M-Y)$.

They can obtain the formula for $G_A^{n+A} G_M^{n+M+B}$ and $G_A^{n+X} G_M^{n+Y}, 0 \le Y \le M$.

$H_1^q(g, [[1]_{q-}, [2]_{q-}...[M]_{q-}], [1, 2...M]) = \prod_{i=1}^{M} [T_i]_{q-} q^{(g+1)g} G_g^M = q^{\frac{-M(M+1)}{2} + g(g+1)} \prod_{i=1}^{M} G_1^i G_g^M$.

$B_i \; of \; H_1^q(g) = \begin{cases} q^{-T_i} q^{T_i + X_T} G_1^{T_i - X_{K-1}}, X_i = T_i \\ q^{-T_i} G_1^{T_i + X_T - 1}, X_i = K_i \end{cases} \rightarrow B_i = \begin{cases} q^i G_1^{i - X_{K-1}}, X_i = T_i \\ G_1^{i + X_T - 1}, X_i = K_i \end{cases}, \sum_{X(T)=g} \prod_{i=1}^{M} B_i = q^{\frac{g(g+1)}{2}} \prod_{i=1}^{M} G_1^i G_g^M$,

$\rightarrow B_i = \begin{cases} q^{i - X_T} G_1^{i - X_{K-1}} = q^{X_K} G_1^{i - X_{K-1}}, X_i = T_i \\ G_1^{i + X_T - 1}, X_i = K_i \end{cases}, \sum_{X(T)=g} \prod_{i=1}^{M} B_i = \prod_{i=1}^{M} G_1^i G_g^M$. Promote this conclusion:

Definition 3.1. *Set $\mathbb{T}$ come from p Source: $S_1, S_2...S_p$.*
$Diff(S_x, S_x) = 0, Diff(S_x, S_y) = -Diff(S_y, S_x) = 1, \; x > y. \; Diff(T_i, T_j) = Diff(S_x, S_y), \; T_i \in S_x, T_j \in S_y$.

Proposition 3.8. $\sum_{g_1 + ... + g_p = M, g_i = |S_i|} \prod_{i=1}^{M} G_1^{T_i + \sum_{j<i} Diff(T_j, T_i)} q^{\sum_{j<i, Diff(T_j, T_i) = -1} 1} = G_{g_1, g_2 ... g_p}^M \prod_{i=1}^{M} G_1^{T_i}, \; T_i \ge i$.

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Proof.

Record the sum as $W_q(g_1, g_2, ...g_p, PT)$.
$W(1, 1, [T_1, T_2]) = G_1^{T_1} G_1^{T_2+1} + G_1^{T_1} G_1^{T_2-1} q = G_1^{T_1} G_1^{T_2} G_1^2$, it's holds. Suppose $W_q(g_1, g_2, PT)$ holds,
$W_q(g_1, g_2 + 1, [PT, T_{M+1}]) = T_{M+1} \in Source_1 + T_{M+1} \in Source_2$
$= W_q(g_1, g_2, PT) G_1^{T_{M+1}+g_1} + W_q(g_1 - 1, g_2 + 1, PT) G_1^{T_{M+1}-(g_2+1)} q^{g_2+1}$.
$= (\prod_{i=1}^{M} G_1^{T_i}) G_{g_1,g_2}^{M} G_1^{T_{M+1}+g_1} + (\prod_{i=1}^{M} G_1^{T_i}) G_{g_1-1,g_2+1}^{M} G_1^{T_{M+1}-(g_2+1)} q^{g_2+1}$.
Just need to prove: $G_{g_1}^{M} G_1^{T_{M+1}+g_1} + G_{g_1-1}^{M} G_1^{T_{M+1}-(M-g_1+1)} q^{M-g_1+1} = G_1^{T_{M+1}} G_{g_1}^{M+1}$.
$(Right\ side) \times (q^{M-g_1+1} - 1)/G_{g_1}^{M} = (q^{T_{M+1}-1} + ... + q + 1)(q^{M+1} - 1) = (1)$.
$(Left\ side) \times (q^{M-g_1+1} - 1)/G_{g_1}^{M} = (q^{M-g_1+1} - 1) G_1^{T_{M+1}+g_1} + (q^{g_1} - 1) G_1^{T_{M+1}-(M-g_1+1)} q^{M-g_1+1}$
$= (q^{M-g_1+1} - 1)(q^{T_{M+1}+g_1-1} + ... + q + 1) + (q^{g_1} - 1)(q^{T_{M+1}-1} + ... + q^{M-g_1+2} + q^{M-g_1+1}) = (2)$.
$(1) - (2) = 0 \rightarrow$ It's holds when p=2. $W_q(g_1, g_2 + g_3, PT) = G_{g_1,g_2+g_3}^{g_1+g_2+g_3} \prod_{i=1}^{M} G_1^{T_i}$.
Every product has $g_2 + g_3$ factors come from $Source_2$, divide them to $g_2 \times Source_2 + g_3 \times Source_3$, g_1-factors are invariant, $(g_2 + g_3)$-factors are variant.
$\sum \prod (\text{variant factors}) = W_q(g_2, g_3, [X_1, X_2...X_{g_2+g_3}]) = G_{g_2,g_3}^{g_2+g_3} \prod_{i=1}^{g_2+g_3} G_1^{X_i}$.
$W_q(g_1, g_2, g_3, [T_1, T_2...T_M]) = G_{g_1,g_2+g_3}^{g_1+g_2+g_3} G_{g_2,g_3}^{g_2+g_3} \prod_{i=1}^{M} G_1^{T_i} = G_{g_1,g_2,g_3}^{g_1+g_2+g_3} \prod_{i=1}^{M} G_1^{T_i}$.

□

Proposition 3.9. $K \in \mathbb{N}, K > 0, \sum_{n=0}^{N-1} q^{Kn} G_M^{n+M}$
$= \sum_{g=0}^{K-1} q^{\frac{g(g+1)}{2}} \prod_{i=1}^{g} (q^{M+i} - 1) G_g^{K-1} G_{1+M+g}^{N+M}$
$= q^{-(M+1)(K-1)} \sum_{g=0}^{K-1} q^{-\frac{g(g-1)}{2}} \prod_{i=1}^{g} (q^{M+i} - 1) \begin{bmatrix} K-1 \\ g \end{bmatrix}_{q^{-1}} G_{1+M+g}^{N+M+g}$
$= \sum_{g=0}^{K-1} (-1)^g q^{\frac{g(g+1)}{2}} G_g^{K-1} G_{M+K}^{N+M+K-1-g}$.

Proof.

$SUM_q(N, [[1]_{q-}, [2]_{q-}...[M-1]_{q-}, 1 : q-1, 1 : q-1...1 : q-1], [1, 2...M-1, M+1, M+2...M+K-1])$
$= [(M-1)!]_{q-} \sum_{g=0}^{M} q^n (1 + q^n - 1)...(1 + q^n - 1) G_M^{n+M} = [(M-1)!]_{q-} \sum_{g=0}^{M} q^{Kn} G_M^{n+M}$.
$= [(M-1)!]_{q-} SUM_q(N, [1, 1...1] : q-1, [M+1, M+2...M+K-1])$ [2.1(4)]
$\sum_{g=0}^{N-1} q^{Kn} G_M^{n+M} = SUM_q(N, [1, 1...1] : q-1, [M+1, M+2...M+K-1])$
$= \sum_{g=0}^{K-1} H_1^q(g) G_{1+M+g}^{N+M} = \sum_{g=0}^{K-1} H_2^q(g) G_{1+M+g}^{N+M+g} = \sum_{g=0}^{K-1} H_3^q(g) G_{M+K}^{N+M+K-1-g}$.
$B_i = \begin{cases} q^{1+X_{T-1}} G_1^{M+i-X_{K-1}}(q-1) = q^{X_T}(q^{M+X_T}-1), X_i = T_i \\ 1 + G_1^{X_{T-1}}(q-1) = q^{X_{T-1}} = q^{X_T}, X_i = K_i \end{cases}, H_1^q(g) = q^{\frac{g(g+1)}{2}} \prod_{i=1}^{g} (q^{M+i} - 1) G_g^{K-1}$.
$B_i = \begin{cases} q^{-(M+X_T)}(q^{M+X_T}-1), X_i = T_i \\ q^{-(M+1+X_{T-1})} = q^{-(M+1)} q^{-X_T}, X_i = K_i \end{cases}, H_2^q(g) = q^{-\frac{g(g+1)}{2} - Mg - (M+1)(K-1-g)} \prod_{i=1}^{g} (q^{M+i} - 1) \begin{bmatrix} K-1 \\ g \end{bmatrix}_{q^{-1}}$.
$B_i = \begin{cases} -q^{1+X_{T-1}} = -q^{X_T}, X_i = T_i \\ q^{X_{T-1}} = q^{X_T}, X_i = K_i \end{cases}, H_3^q(g) = (-1)^g q^{\frac{g(g+1)}{2}} G_g^{K-1}$.

□

Definition 3.2. $\left\langle {M \atop M} \right\rangle^q = 0, \left\langle {M \atop g} \right\rangle^q = \sum_{\lambda_1+...+\lambda_{g+1}=M-g} \prod_{i=1}^{g+1} [i]_q^{\lambda_i} [1 + \lambda_1]_q [1 + \lambda_1 + \lambda_2]_q ... [1 + \lambda_1 + ... + \lambda_g]_q, \lambda_i \geq 0$.

Easy to obtain: $\left\langle {M \atop g} \right\rangle^q = [M-g]_q \left\langle {M-1 \atop g-1} \right\rangle^q + [g+1]_q \left\langle {M-1 \atop g} \right\rangle^q$, so $\left\langle {M \atop g} \right\rangle^q = \left\langle {M \atop M-g-1} \right\rangle^q$.

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Proposition 3.10.
$[N]_q^M = \sum_{g=1}^{M} [g!]_q S_2^q(M,g) q^{\frac{g(g-3)}{2}} G_g^N = \sum_{g=1}^{M} (-1)^{M-g} [g!]_q S_2^{q-}(M,g) q^{\frac{-g(g-3)}{2}} G_g^{N+g-1} = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle^q q^{\binom{M-g}{2}} G_M^{N+g}$.

Proof.

$$\nabla_q^1 SUM(N, [[1]_{q-}, [1]_{q-}...[1]_{q-}], [1, 2...M]) = \prod(\frac{1}{q} + \frac{q^{N-1}-1}{q-1}) = q^{-M}[N]_q^M$$
$$= q^{-1}\nabla_q^1 SUM(N, [1]_{q-}, [1]_{q-}...[1]_{q-}], [2, 3...M]) = q^{-1}\sum_{g=0}^{M-1} H_1^q(g) G_{1+g}^N q^{-g}. = q^{-1}\sum_{g=0}^{M-1} H_2^q(g) G_{1+g}^{N+g}.$$
$$In\ H_1^q(g), B_i = \begin{cases} q^{1+X_{T-1}} G_1^{(i+1)-X_{K-1}} = q^{X_T} G_1^{1+X_T} = q^{-1} q^{1+X_T} G_1^{1+X_T},\ X_i = T_i \\ q^{-1} + G_1^{X_{T-1}} = q^{-1} G_1^{1+X_{T-1}} = q^{-1} G_1^{1+X_T}, X_i = K_i \end{cases}.$$
$$H_1^q(g) = q^{-g}[(g+1)!]_q q^{\frac{g(g+1)}{2}} / q^1 H_1^q(g, \sum K) = [(g+1)!]_q q^{-(g+1)} q^{\frac{g(g+1)}{2}} q^{-(M-1-g)} E_{M-1-g}^{g+1,q}.$$
$$q^{-M}[N]_q^M = q^{-1}\sum_{g=0}^{M-1} q^{-M}[(g+1)!]_q q^{\frac{g(g+1)}{2}} S_2^q(M, g+1) G_{1+g}^N q^{-g} \to Form_1.$$
$$In\ H_2^q(g), B_i = \begin{cases} q^{-(1+X_T)} G_1^{1+X_T},\ X_i = T_i \\ q^{-1} - q^{-(i+1-X_{K-1})} G_1^{i+1-X_{K-1}} = q^{-1} - q^{-(2+X_T)} G_1^{2+X_T} = -q^{-1} q^{-1-X_T} G_1^{1+X_T}, X_i = K_i \end{cases}.$$
$$H_2^q(g) = [(g+1)!]_{q-} / [1!]_{q-} H_2^q(g, \sum K) = q[(g+1)!]_{q-}(-1)^{M-1-g} q^{-(M-1-g)} E_{M-1-g}^{g+1,q-}.$$
$$q^{-M}[N]_q^M = q^{-1}\sum_{g=0}^{M-1} q[(g+1)!]_{q-}(-1)^{M-1-g} q^{-(M-1-g)} E_{M-1-g}^{g+1,q-} G_{1+g}^{N+g} \to Form_2.$$
$$\nabla_q^1 SUM(N, [[1]_{q-}, [1]_{q-}...[1]_{q-}], [1, 2...M]) = \sum_{g=0}^{M} H_3^q(g) G_M^{N+M-1-g} q^{-g}.$$
$$B_i = \begin{cases} q(q^{X_{T-1}} G_1^i - q^i G_1^{X_{T-1}} - q^i q^{-1}) = q^{\frac{q^{i-1} - q^{X_{T-1}}}{q-1}} = q^{X_T} G_1^{i-X_T},\ X_i = T_i \\ q^{-1} + G_1^{X_{T-1}} = q^{-1} G_1^{1+X_{T-1}} = q^{-1} G_1^{1+X_T}, X_i = K_i \end{cases}, H_3^q(g) = q^{-(M-g)} q^{\frac{g(g+1)}{2}} \left\langle {M \atop g} \right\rangle^q, H_3^q(M) = 0.$$
$$q^{-M}[N]_q^M = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle^q q^{\frac{g(g+1)}{2}} q^{-(M-g)} q^{-g} G_M^{N+M-1-g}.$$
$$[N]_q^M = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle^q q^{\frac{g(g+1)}{2}} G_M^{N+M-1-g} = \sum_{g=0}^{M-1} \left\langle {M \atop M-1-g} \right\rangle^q q^{\frac{(M-1-g)(M-g)}{2}} G_M^{N+g}.$$

□

This can be compared to $N^M = \sum_{g=1}^{M} g! S_2(M,g) \binom{N}{g} = \sum_{g=1}^{M} (-1)^{M-g} g! S_2(M,g) \binom{N+g-1}{g} = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle \binom{N+g}{M}$. $S_2(M,g)$ is Stirling number of the second kind. $S_2(M,g) = E_{M-g}^g$, $\left\langle {M \atop g} \right\rangle$ is Eulerian number, $\left\langle {M \atop g} \right\rangle = \sum_{\lambda_1+\lambda_2+..+\lambda_{g+1}=M-g} \prod_{i=1}^{g+1} i^{\lambda_i}(1+\lambda_1)(1+\lambda_1+\lambda_2)...(1+\lambda_1+..+\lambda_g)$ [3].

Proposition 3.11.

$$q^{Mn} = \sum_{g=0}^{M} G_g^M \prod_{i=0}^{g-1}(q^n - q^i) = q^{-M}\sum_{g=0}^{M} q^g \begin{bmatrix} M \\ g \end{bmatrix}_{q^{-1}} \prod_{i=1}^{g}(q^n - q^{-i}) = \sum_{g=0}^{M} G_g^M (-1)^g q^{\binom{g}{2}} G_M^{n+M-g}.$$

Proof.

$$q^{Mn} = \nabla_q^1 SUM_q(n+1, [1, 1...1] : q-1, [1, 2...M])$$
$$= \sum_{g=0}^{M} H_1^q(g) G_g^n q^{-g} = \sum_{g=0}^{M} H_2^q(g) G_g^{n+g} = \sum_{g=0}^{M} H_3^q(g) G_M^{n+M-g} q^{-g}.$$
$$B_i = \begin{cases} q^{1+X_{T-1}} G_1^{i-X_{K-1}}(q-1) = q^{X_T}(q^{X_T}-1), X_i = T_i \\ 1 + G_1^{X_{T-1}}(q-1) = q^{X_{T-1}} = q^{X_T}, X_i = K_i \end{cases}, H_1^q(g) = (q-1)^g [g!]_{q+} H_1^q(g, \sum K) = (q-1)^g [g!]_{q+} G_g^M.$$
$$B_i = \begin{cases} q^{-(1+X_{T-1})}(q^{1+X_{T-1}}-1) = q^{-X_T}(q^{X_T}-1), X_i = T_i \\ q^{-(1+X_{T-1})} = q^{-1} q^{-X_T}, X_i = K_i \end{cases}, H_2^q(g) = (q-1)^g [g!]_{q-} q^{-(M-g)} \begin{bmatrix} M \\ g \end{bmatrix}_{q^{-1}}.$$
$$B_i = \begin{cases} -q^{1+X_{T-1}} = -q^{X_T}, X_i = T_i \\ q^{X_{T-1}} = q^{X_T}, X_i = K_i \end{cases}, H_3^q(g) = (-1)^g q^{\binom{g+1}{2}} H_3^q(g, \sum K) = (-1)^g q^{\binom{g+1}{2}} G_g^M.$$

□

Proposition 3.12.
(1). $(K + qD)^M = (q-1)D\sum_{g=0}^{M-1}(K+D)^g (K+qD)^{M-1-g} + (K+D)^M$.

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(2). $(K+D)^M = D\sum_{g=0}^{M-1} K^g (K+D)^{M-1-g} + K^M$.
(3). $(K+qD)^M = q(q-1)D^2 \sum_{a+b+c=M-2,a,b,c\geq 0} K^a (K+D)^b (K+qD)^c + qD\sum_{g=0}^{M-1} K^g (K+D)^{M-1-g} + K^M$.

Proof.

$PS = [K+D, K+D ... K+D] : (q-1)D, PT = [1, 2...M]$.
$B_i = \begin{cases} q^{X_T}(q^{X_T}-1)D, X_i=T_i \\ K+Dq^{X_T}, X_i=K_i \end{cases}, H_1^q(0) = (K+D)^M, H_1^q(1) = q^1(q^1-1)D\sum_{a+b=M-1,a,b\geq 0}(K+D)^a(K+qD)^b$.
$SUM_q(2) - SUM_q(1) = H_1^q(1) + H_1^q(0)G_1^2 - H_1^q(0) = H_1^q(1) + qH_1^q(0) = q(K+qD)^M \to (1)$.
From [1.2]: $(K+D)^M = f(1)G_1^1 + K^M \to (2)$.
From [1.2]: $(K+qD)^M = \sum_{n=0}^{1}\prod(K+q^nD) - \sum_{n=0}^{0}\prod(K+q^nD) =$
$f(2)G_2^2 + f(1)G_1^2 + 2K^M - (f(1)G_1^1 + K^M) = f(2) + qf(1) + K^M \to (3)$.

□

Proposition 3.13.
(1). $0 \leq A < M, 0 \leq T, \sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_{A+T+1+g}^{N+A+T} q^{g(g+1+T)} = \sum_{k=0}^{A} G_{k+T}^{A+T} G_{M+T}^{M+T+k} G_{M+T+1+k}^{N+M+T} q^{k(k+1+T)}$.
(2). $0 \leq A, B, T, 0 \leq A+B < M, \sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_B^g q^{\binom{g}{2} - g(A+B)} (-1)^g = 0$.
(3). $0 \leq K, T, \sum_{g=0}^{M} (-1)^g G_g^M G_{M+K}^{M+K+T+g} q^{\binom{M+1-g}{2} + (M-g)K} = (-1)^M G_K^{T+M+K}$.

Proof.

$PS = [[T+1]_{q-}, [T+2]_{q-} ... [T+M]_{q-}], PT = [T+A+1, T+A+2 ... T+A+M])$,
$SUM_q(N) = \sum_{g=0}^{M} G_{A+T+1+g}^{N+A+T} [T+A+1]_q ... [T+A+g]_q \times [T+g+1]_q ... [T+M]_q q^{\frac{g(1+g)}{2} - (M-g)(T+M+1) + \binom{M+1-g}{2}} G_g^M$[3.7]
$= \frac{(q;q)_A (q^{T+A+1};q)_{M-A}}{(1-q)^M} \sum_{g=0}^{M} G_{A+T+1+g}^{N+A+T} q^{\frac{g(1+g)}{2} - (M-g)(T+M+1) + \binom{M+1-g}{2}} G_A^{T+A+g} G_g^M \ (*)$
$= \prod_{i=1}^{M-A} [T+A+i]_{q-} SUM_q(N, [[T+1]_{q-} ... [T+A]_{q-}], [T+M+1 ... T+M+A])$. [2.1(4)]
$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} \prod_{i=1}^{M-A} [T+A+i]_q \sum_{k=0}^{A} G_{M+T+1+k}^{N+M+T} q^{\frac{k(1+k)}{2} - (A-k)(T+A+1) + \binom{A+1-k}{2}}$
$[T+M+1]_q ... [T+M+k]_q \times [T+1+k]_q ... [T+A]_q G_k^A$
$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} \sum_{k=0}^{A} G_{M+T+1+k}^{N+M+T} q^{\frac{k(1+k)}{2} - (A-k)(T+A+1) + \binom{A+1-k}{2}} \prod_{i=1}^{M} [T+k+i]_q G_k^A$. (**)
Compare (*) and (**) :
$\sum_{g=0}^{M} G_{A+T+1+g}^{N+A+T} q^{\frac{g(1+g)-(M-g)(g+M+1+2T)}{2}} G_A^{A+T+g} G_g^M$
$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} \sum_{k=0}^{A} G_{M+T+1+k}^{N+M+T} q^{\frac{k(1+k)-(A-k)(k+A+1+2T)}{2}} \frac{(q^{T+k+1};q)_M}{(q^{T+A+1};q)_{M-A}(q;q)_k(q;q)_{A-k}}$
$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} \sum_{k=0}^{A} G_{M+T+1+k}^{N+M+T} q^{\frac{k(1+k)-(A-k)(k+A+1+2T)}{2}} G_{M+T}^{M+T+k} G_{k+T}^{A+T}$
Proof of (1) completed, the two sides correspond to merging and unfolding.
$A+B < M$, From [2.4] and (1):
$\sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_B^g q^{g(1+g+T)} q^{-g(g+1)+\binom{g-B}{2}-(A+T)g} (-1)^g$
$= \sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_B^g q^{-gA+\binom{g-B}{2}} (-1)^g = 0 \to \sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_B^g q^{\binom{g}{2}-g(A+B)} (-1)^g = 0$.
This is (2), when A=0, it's [3.2]. Similar expressions about G_{M-B}^{M-g} can also be obtained.
$SUM_q(N, [[T+1]_{q-}, [T+2]_{q-} ... [T+M]_{q-}], [T+K+M+1, T+K+M+2 ... T+K+2M])$.
$H_1^q(g) = q^{\frac{g(1+g)}{2} - (M-g)(T+M+1) + \binom{M-g+1}{2}} \prod_{i=1}^{g} [T+K+M+i]_q \prod_{i=1}^{M-g} [T+M+1-i]_q G_g^M$.

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Direct calculations yield $H_2^q(0) = (-1)^M q^{-M(T+K+M+1)} \prod_{i=1}^{M} [K+i]_q$.

$H_2^q(0) = \sum_{g=0}^{M} (-1)^g H_1^q(g) q^{-g(g+1)+\frac{g(g-1)}{2}-(T+K+M)g}$.[2.4(3)]

$(-1)^M \prod_{i=1}^{M} [K+i]_q = \sum_{g=0}^{M} (-1)^g q^{\binom{M-g+1}{2}+(M-g)K} \prod_{i=1}^{g} [T+K+M+i]_q \prod_{i=1}^{M-g} [T+M+1-i]_q G_g^M$.

$(-1)^M \prod_{i=1}^{M} [K+i]_q \prod_{i=1}^{K} [T+M+i]_q = \sum_{g=0}^{M} (-1)^g q^{\binom{M-g+1}{2}+(M-g)K} \prod_{i=1}^{K+M} [T+g+i]_q G_g^M$.

$(-1)^M \prod_{i=1}^{K} [T+M+i]_q = \sum_{g=0}^{M} (-1)^g q^{\binom{M-g+1}{2}+(M-g)K} \prod_{i=1}^{K+M} [T+g+i]_q / \prod_{i=1}^{M} [K+i]_q \times G_g^M$.

$(-1)^M G_K^{T+M+K} = \sum_{g=0}^{M} (-1)^g q^{\binom{M-g+1}{2}+(M-g)K} \times G_{M+K}^{T+K+M+g} G_g^M \to (3)$. It's the case of $A+B \geq M$ of (2).

□

q-Vandermorde: K=T+1+M, A=0, X=N+T, $\sum_{g=0}^{M} G_g^M G_{K-(M-g)}^{N+T} q^{g(g+1+T)} = \sum_{g=0}^{M} G_g^M G_{K-g}^{X} q^{(M-g)(K-g)} = G_K^{X+M}$.

Proposition 3.14. $\sum_{n_M=0}^{N-1} \cdots \sum_{n_1=0}^{n_2} \prod_{i=1}^{M} [K-1+i+2n_i]_q q^{-\sum_{j=1}^{M} n_j} = q^{-(N-1)M} \prod_{i=1}^{M} [K+N-2+i]_q G_M^{M+N-1}$.

Proof.

$PS = [[K]_{q-}, [K+1]_{q-} \ldots [K+M-1]_{q-}], PS1 = [[K+M-1]_{q-}, [K+M-2]_{q-} \ldots [K]_{q-}], PT = [1, 2 \ldots M]$.

$B_i = K_i\ of\ H_1^q(g, PS, PT) = q^{-(K-1+i)} G_1^{K-1+i} + G^{X_T} = q^{-(K-1+i)} G_1^{K-1+i+X_T}$. Expand by definition:

$H_1^q(g, \sum K, PS, PT) = \sum_{n_{M-g}=0}^{g} \cdots \sum_{n_1=0}^{n_2} \prod_{i=1}^{M-g} [K-1+i+2n_i]_q q^{-(K-1)(M-g)-\sum_{j=1}^{M-g} j - \sum_{j=1}^{M-g} n_j}$

$= H_1^q(g, \sum K, PS1, PT) = \prod_{i=1}^{M-g} [K+g-1+i]_q q^{-(M-g)(K+M)} q^{\binom{M+1-g}{2}} G_g^M$. [3.7]

$\sum_{n_{M-g}=0}^{g} \cdots \sum_{n_1=0}^{n_2} \prod_{i=1}^{M-g} [K-1+i+2n_i]_q q^{g(M-g)-\sum_{j=1}^{M-g} n_j} = \prod_{i=1}^{M-g} [K+g-1+i]_q G_g^M$.

$\sum_{n_M=0}^{g} \cdots \sum_{n_1=0}^{n_2} \prod_{i=1}^{M} [K-1+i+2n_i]_q q^{gM-\sum_{j=1}^{M} n_j} = \prod_{i=1}^{M} [K+g-1+i]_q G_g^{M+g}$.

$\sum_{n_M=0}^{N-1} \cdots \sum_{n_1=0}^{n_2} \prod_{i=1}^{M} [K-1+i+2n_i]_q q^{(N-1)M-\sum_{j=1}^{M} n_j} = \prod_{i=1}^{M} [K+N-2+i]_q G_{N-1}^{M+N-1}$.

□

$K=1, M=1 \to \sum_{n=0}^{N-1} [1+2n]_q q^{-n} = q^{-(N-1)} [N]_q^2$.

4. The Extension of q-Euler Polynomials and the Relationship Between Three Forms

In this section, $a \neq 1, q^{-1}, q^{-2}, \ldots, q^{-M}$.

Lemma 4.1. $\sum_{n=0}^{N-1} a^n G_M^{n+A} = -a^N \sum_{g=0}^{M} \frac{q^{(N+A-M)g} G_{M-g}^{N+A-1-g}}{(a;q)_{g+1}} + \frac{a^{M-A}}{(a;q)_{M+1}}, 0 \leq A \leq M, N > M-A$.

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Proof.

$A=M, M=0, \sum_{n=0}^{N-1} a^n = \frac{-a^N}{1-a} + \frac{1}{1-a}$, holds.

$A=M, M=1, \sum_{n=0}^{N-1} a^n G_1^{n+1} = \sum_{n=0}^{N-1} a^n (1+q+...+q^n)$

$= (1+q+...+q^n) \sum_{n=0}^{N-1} a^n - q\sum_{n=0}^{0} a^n - q^2 \sum_{n=0}^{1} a^n ... - q^{N-1} \sum_{n=0}^{N-2} a^n$

$= G_1^N \frac{1-a^N}{1-a} - \frac{q(1-a)}{1-a} - \frac{q^2(1-a^2)}{1-a} ... \frac{q^{N-1}(1-a^{N-1})}{1-a}$

$= G_1^N \frac{1-a^N}{1-a} + \frac{1+aq+...+a^{N-1}q^{N-1}}{1-a} - \frac{1+q+...+q^{N-1}}{1-a} = \frac{G_1^N - G_1^N a^N}{1-a} + \frac{1-a^N q^N}{(1-a)(1-aq)} - \frac{G_1^N}{1-a}$

$= \frac{-a^N q^N}{(1-a)(1-aq)} + \frac{-a^N G_1^N}{1-a} + \frac{1}{(1-a)(1-aq)}$, holds.

$A=M, N=1, -a^1 \sum_{g=0}^{M} \frac{q^g}{(a;q)_{g+1}} + \frac{1}{(a;q)_{M+1}} = \frac{1-aq^M}{(a;q)_{M+1}} - a^1 \sum_{g=0}^{M-1} \frac{q^g}{(a;q)_{g+1}}$

$= \frac{(1-aq^M)(1-aq^{M-1})}{(a;q)_{M+1}} - a^1 \sum_{g=0}^{M-2} \frac{q^g}{(a;q)_{g+1}} = ... = 1$, holds. Suppose it's holds when N.

$\sum_{n=0}^{N} a^n G_M^{n+M} = \sum_{n=0}^{N} a^n (q^M G_M^{n-1+M} + G_{M-1}^{n-1+M}) = aq^M \sum_{n=0}^{N-1} a^n G_M^{n+M} + \sum_{n=0}^{N} a^n G_{M-1}^{n+M-1}$

$= -q^M a^{N+1} \sum_{g=0}^{M} \frac{q^{Ng} G_{M-g}^{N+M-1-g}}{(a;q)_{g+1}} + \frac{aq^M}{(a;q)_{M+1}} - a^{N+1} \sum_{g=0}^{M-1} \frac{q^{(N+1)g} G_{M-1-g}^{N+M-1-g}}{(a;q)_{g+1}} + \frac{1}{(a;q)_M}$

$= -a^{N+1} \sum_{g=0}^{M} \frac{q^{Ng+M} G_{M-g}^{N+M-1-g} + q^{(N+1)g} G_{M-1-g}^{N+M-1-g}}{(a;q)_{g+1}} + \frac{1}{(a;q)_{M+1}}$

$= -a^{N+1} \sum_{g=0}^{M} \frac{q^{(N+1)g} q^{M-g} G_{M-g}^{N+M-1-g} + q^{(N+1)g} G_{M-1-g}^{N+M-1-g}}{(a;q)_{g+1}} + \frac{1}{(a;q)_{M+1}}$

$= -a^{N+1} \sum_{g=0}^{M} \frac{q^{(N+1)g} G_{M-g}^{(N+1)+M-1-g}}{(a;q)_{g+1}} + \frac{1}{(a;q)_{M+1}}$. Proof of A=M completed.

$\sum_{n=0}^{N-1} a^n G_M^{n+A} = a^{M-A} \sum_{n=0}^{N-M+A-1} a^n G_M^{n+M}$, complete the remaining proofs.

□

Taking the limit $q \to 1, \sum_{n=0}^{N-1} a^n \binom{n+A}{M} = -a^N \sum_{g=0}^{M} \frac{\binom{N+A-1-g}{M-g}}{(1-a)^{g+1}} + \frac{a^{M-A}}{(1-a)^{M+1}}, 0 \le A \le M.$

$\sum_{n=0}^{N-1} q^n G_M^{n+A} = q^{M-A} G_{M+1}^{N+A} \to (q^{N+A-M};q)_{M+1} = -\sum_{g=0}^{M} q^{(N+A-M)(g+1)} G_{M-g}^{N+A-1-g} (q^{g+2};q)_{M-g} + 1.$

Theorem 4.1. $X = T_M - M - y \ge -1, 0 \le Y \le 1, f(g) = (aq^{2+X+g};q)_{M-g} = (aq^{2+T_M-M-y+g};q)_{M-g}$,

(1). $\sum_{g=0}^{M} H_1^q(g) a^g q^{-yg} f(g) = \sum_{g=0}^{M} H_2^q(g) f(g) = \sum_{g=0}^{M} H_3^q(g) a^g q^{-yg}$, *define as* $A_a^q(PS, PT, y)$.

(2). $\sum_{n=0}^{N-1} a^n \nabla_q^y SUM_q(n+Y) = -a^N \sum_{k=0}^{M} \frac{q^{(N+Y-1)k} \nabla_q^{y+k} SUM_q(N+Y-1)}{(a;q)_{k+1}} + \frac{a^{1-Y} A_a^q(PS,PT,y)}{(a;q)_{T_M+2-y}}$.

(3). $|a|,|q| < 1, \sum_{n=0}^{\infty} a^n \nabla_q^y SUM_q(n+Y) = \frac{a^{1-Y} A_a^q(PS,PT,y)}{(a;q)_{T_M+2-y}}$.

(4). $SUM_q(\infty) = \frac{A_q^q(PS,PT,1)}{(q;q)_{T_M+1}}, \nabla_q^y SUM_q(\infty) = \frac{A_q^q(PS,PT,1+y)}{(q;q)_{T_M+1-y}}$.

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Proof.

$$\sum_{n=0}^{N-1} a^n \nabla_q^y SUM_q(n+Y) = \sum_{g=0}^{M} H_1^q(g) \sum_{n=0}^{N-1} a^n G_{1+X+g}^{n+Y+X} q^{-yg}$$
$$= \sum_{g=0}^{M} H_1^q(g)(-a^N \sum_{k=0}^{1+X+g} \frac{q^{(N+Y-1-g)k} q^{-yg} G_{1+X+g-k}^{N+Y-1+X-k}}{(a;q)_{k+1}} + \frac{a^{1+g-Y} q^{-yg}}{(a;q)_{2+X+g}}).$$
$$= -a^N \sum_{k=0}^{M} \frac{q^{(N+Y-1)k} \sum_{g=0}^{M} H_1^q(g) q^{-(y+k)g} G_{1+X+g-k}^{N+Y-1+X-k}}{(a;q)_{k+1}} + \frac{\sum_{g=0}^{M} H_1^q(g) a^{1+g-Y} q^{-yg} f(g)}{(a;q)_{T_M+2-y}}$$
$$= -a^N \sum_{k=0}^{M} \frac{q^{(N+Y-1)k} \nabla_q^{y+k} SUM_q(N+Y-1)}{(a;q)_{k+1}} + \frac{\sum_{g=0}^{M} H_1^q(g) a^{1+g-Y} q^{-yg} f(g)}{(a;q)_{T_M+2-y}}.$$

$$\sum_{n=0}^{N-1} a^n \nabla_q^y SUM_q(n+Y) = \sum_{g=0}^{M} H_2^q(g) \sum_{n=0}^{N-1} a^n G_{1+X+g}^{n+Y+X+g}$$
$$= \sum_{g=0}^{M} H_2^q(g)(-a^N \sum_{k=0}^{1+X+g} \frac{q^{(N+Y-1)k} G_{1+X+g-k}^{N+Y-1+X+g-k}}{(a;q)_{k+1}} + \frac{a^{1-Y}}{(a;q)_{2+X+g}}).$$
$$= -a^N \sum_{k=0}^{M} \frac{q^{(N+Y-1)k} \nabla_q^{y+k} SUM_q(N+Y-1)}{(a;q)_{k+1}} + \frac{a^{1-Y} \sum_{g=0}^{M} H_2^q(g) f(g)}{(a;q)_{T_M+2-y}}.$$

$$\sum_{n=0}^{N-1} a^n \nabla_q^y SUM_q(n+Y) = \sum_{g=0}^{M} H_3^q(g) \sum_{n=0}^{N-1} a^n G_{T_M+1-p}^{n+Y+T_M-p-g} q^{-yg}$$
$$= \sum_{g=0}^{M} H_3^q(g)(-a^N \sum_{k=0}^{T_M+1-p} \frac{q^{(N+Y-1)k} q^{-(y+k)g} G_{T_M+1-p-k}^{N+Y-1+T_M-p-g-k}}{(a;q)_{T_M+2-p}} + \frac{a^{1+g-Y} q^{-yg}}{(a;q)_{T_M+2-y}}).$$
$$= -a^N \sum_{k=0}^{M} \frac{q^{(N+Y-1)k} \nabla_q^{y+k} SUM_q(N+Y-1)}{(a;q)_{T_M+2-y}} + \frac{\sum_{g=0}^{M} H_3^q(g) a^{1+g-Y} q^{-yg}}{(a;q)_{T_M+2-y}}.$$

The three summations are identical, which leads to (1) and (2). (3) is obvious.

$SUM_q(N) = \sum_{n=0}^{N-1} q^n \nabla_q^1 SUM_q(n+1) \to (4)$. $\lim_{N\to\infty} G_M^N = \dfrac{1}{(q;q)_M}$ can draw the same conclusion .

□

$\sum_{n=0}^{\infty} a^n [n]_q^M = \frac{E_M(a,q)}{(a;q)_{M+1}}$, $E_M(a,q)$ is q-Eularian polynomials.[2] pp 332. From [3.10], we can get three expressions for $E_M(a,q)$. In particular, we can get expressions for Eularian polynomials: $\sum_{n\geq 1} t^n n^M = \frac{tA_M(t)}{(1-t)^{M+1}}$. $A_M(a) = \sum_{g=0}^{M} H_1(g) a^g (1-a)^{M-g} = \sum_{g=0}^{M} H_2(g)(1-a)^{M-g} = \sum_{g=0}^{M} H_3(g) a^g$ $= \sum_{g=0}^{M} g! S_2(M,g) a^g (1-a)^{M-g} = \sum_{g=0}^{M} (-1)^{M-g} g! S_2(M,g)(1-a)^{M-g} = \sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle a^g$.
More clearly, we can reformulate (1) as: if $\sum_{g=0}^{M} a_g G_{1+p+g}^{X} = \sum_{g=0}^{M} b_g G_{1+p+g}^{X+g} = \sum_{g=0}^{M} c_g G_{1+p+M}^{X+M-g}$, then $\sum_{g=0}^{M} a_g q^{-yg} a^g (aq^{2+p-y+g};q)_{M-g} = \sum_{g=0}^{M} b_g (aq^{2+p-y+g};q)_{M-g} = \sum_{g=0}^{M} c_g q^{-yg} a^g, 2+p-y>0$.

At [2.4], some relationships have been obtained, and now the remaining ones can be deduced:

Theorem 4.2. $X \in \mathbb{N}$, *if* $\sum_{g=0}^{M} b_g G_{1+p+g}^{X+g} = \sum_{g=0}^{M} c_g G_{1+p+M}^{X+M-g}$, *then*
(1). $c_g q^{-pg} = (-1)^g q^{\frac{g(g+3)}{2}} \sum_{k=0}^{M-g} b_k G_g^{M-k} q^{gk} = (-1)^g q^{\frac{g(g+3)}{2}+Mg} \sum_{k=g}^{M} b_{M-k} G_g^k q^{-gk}$.
(2). $b_{M-g} = (-1)^g q^{\frac{g(g-1)}{2}} \sum_{k=g}^{M} c_k q^{-pk} G_g^k q^{-(M+1)k}$, $b_g = (-1)^{M-g} q^{\frac{(M-g)(M-g-1)}{2}} \sum_{k=M-g}^{M} c_k q^{-pk} G_{M-g}^k q^{-(M+1)k}$.

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Proof.

At [4.1], $PT=[1,2...M]$, $f(g)=(1-aq^{2+g})(1-aq^{3+g})...(1-aq^{M+1})$,
$\sum_{g=0}^{M} b_g f(g) = \sum_{g=0}^{M} b_g \sum_{k=0}^{M-g} G_k^{M-g}(-a)^k q^{\frac{g(g-1)}{2}+(2+g)k} = \sum_{g=0}^{M} c_g a^g q^{-pg}$.
Compare a^g on both sides and yields (1).
$\sum_{x=g}^{M} c_x = \sum_{x=g}^{M} (-1)^x q^{\frac{x(x+1)}{2}+(M+p+1)x} \sum_{k=x}^{M} b_{M-k} G_x^k q^{-xk}$.
$\sum_{x=g}^{M} c_x G_g^x q^{-\frac{x(x+1)}{2}-(M+p+1)x} q^{\frac{(x-g)(x-g-1)}{2}} = \sum_{x=g}^{M} c_x G_g^x q^{\frac{g(g+1)}{2}-(M+p+2)x-gx}$.
$\sum_{x=g}^{M} c_x G_g^x q^{\frac{g(g+1)}{2}-(M+p+2)x-gx} q^{gx+x} = \sum_{x=g}^{M} (-1)^x G_g^x q^{\frac{(x-g)(x-g-1)}{2}+gx+x} \sum_{k=x}^{M} b_{M-k} G_x^k q^{-xk}$.
$\sum_{x=g}^{M} c_x G_g^x q^{\frac{g(g+1)}{2}-(M+p+1)x} = \sum_{k=x}^{M} b_{M-k} \sum_{x=g}^{k} (-1)^x G_g^x G_x^k q^{\frac{(x-g)(x-g-1)}{2}} q^{-(k-g-1)x} = (-1)^k b_{M-k} q^k$. [3.2(3)]

□

Theorem 4.3. $0 \le g \le M$,
(1). $a_g = \sum_{k=g}^{M} b_k G_g^k \leftrightarrow b_g = \sum_{k=g}^{M} (-1)^{k+g} G_g^k q^{\binom{k-g}{2}} a_k$.
(2). $a_g = \sum_{k=0}^{g} c_k G_{M-g}^{M-k} q^{-(g-k)k} \leftrightarrow c_g = \sum_{k=0}^{g} (-1)^{k+g} G_{M-g}^{M-k} q^{-(g-k)k-\binom{g-k}{2}} a_k$.
(3). $c_g = \sum_{k=g}^{M} b_{M-k} G_g^k q^{-gk} \leftrightarrow b_{M-g} q^{-\frac{g(g-1)}{2}} = \sum_{k=g}^{M} (-1)^{k+g} c_k G_g^k q^{\frac{k(k+1)}{2}}$.

Proof.

By combining [2.4] and [4.2], the inversion formula can be obtained.
$a_g q^{-pg} = \sum_{k=0}^{g} c_k G_{M-g}^{M-k} q^{(g+1)(g-k)} q^{-pk}, c_g q^{-pg} = \sum_{k=0}^{g} (-1)^{k+g} G_{M-g}^{M-k} q^{(g+1)(g-k)-\binom{g-k}{2}} a_k q^{-pk}$.
Replace a_g with $a_g q^{-pg}$, c_g with $c_g q^{-pg}$, then
$a_g = \sum_{k=0}^{g} c_k G_{M-g}^{M-k} q^{(g+1)(g-k)} = q^{(g+1)g} \sum_{k=0}^{g} c_k G_{M-g}^{M-k} q^{-(g+1)k} = q^{(g+1)g} \sum_{k=0}^{g} q^{-(g-k)k} G_{M-g}^{M-k} c_k q^{-(k+1)k}$,
$c_g = \sum_{k=0}^{g} (-1)^{k+g} G_{M-g}^{M-k} q^{(g+1)(g-k)-\binom{g-k}{2}} a_k = q^{(g+1)g} \sum_{k=0}^{g} (-1)^{k+g} G_{M-g}^{M-k} q^{-(g-k)k-\binom{g-k}{2}} a_k q^{-(k+1)k}$.
Replace a_g with $a_g q^{-(g+1)g}$, c_g with $c_g q^{-(g+1)g} \to$ (2). Other proofs are similar.

□

Theorem 4.4. $1+p \ge 0$. *If* $\sum_{g=0}^{M} a_g G_{1+p+g}^{X} = \sum_{g=0}^{M} b_g G_{1+p+g}^{X+g} = \sum_{g=0}^{M} c_g G_{1+p+M}^{X+M-g}, a_g^* = a_g q^{-pg}$, $c_g^* = c_g p^{-pg}$, *then*
(1). $\sum_{g=0}^{M} a_g^* q^{-\frac{g(g+1)}{2}+Ag} a^{M-g} z^g = a^M \sum_{g=0}^{M} b_g (-q^{A+1}\frac{z}{a};q)_g = \sum_{g=0}^{M} c_g^* q^{-\frac{g(g+1)}{2}+Ag} a^{M-g} z^g (-q^{A+1}\frac{z}{a};q)_{M-g}$.
(2). $\sum_{g=0}^{M} b_g q^{Ag} a^{M-g} z^g = a^M \sum_{g=0}^{M} (-1)^g a_g^* q^{-\frac{g(g+3)}{2}} (q^{A-g+1}\frac{z}{a};q)_g = q^{AM} z^M \sum_{g=0}^{M} c_g^* q^{-(M+1)g} (q^{-A}\frac{a}{z};q)_g$.
(3). $\sum_{g=0}^{M} c_g^* q^{Ag} a^{M-g} z^g = \sum_{g=0}^{M} a_g^* q^{Ag} z^g a^{M-g} (q^{A+2+g}\frac{z}{a};q)_{M-g} = a^M \sum_{g=0}^{M} b_g (q^{A+2+g}\frac{z}{a};q)_{M-g}$.

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Proof. From [2.4] and [4.2]:

$\sum_{g=0}^{M} a_g^* q^{-g(g+1)} q^{\frac{g(g-1)}{2}+(A+1)g} a^{M-g} z^g = \sum_{g=0}^{M}\sum_{k=0}^{M} b_k G_g^k q^{\frac{g(g-1)}{2}+(A+1)g} a^{M-g} z^g$

$= \sum_{k=0}^{M} b_k a^{M-k} \sum_{g=0}^{M} G_g^k q^{\frac{g(g-1)}{2}+(A+1)g} a^{k-g} z^g = \sum_{k=0}^{M} b_k a^{M-k} \prod_{i=1}^{k} (a+q^{A+i}z).$

$\sum_{g=0}^{M} a_g^* a^{M-g} z^g = \sum_{g=0}^{M}\sum_{k=0}^{M} c_k^* G_{M-g}^{M-k} q^{(g+1)(g-k)} a^{M-g} z^g, q^{(g+1)(g-k)} = q^{\binom{g-k}{2}+\frac{g^2+3g-k^2-3k}{2}}.$

$\sum_{g=0}^{M} a_g^* q^{-\frac{g(g+3)}{2}+(A+1)g} a^{M-g} z^g = \sum_{k=0}^{M} c_k^* q^{-\frac{k(k+3)}{2}+(A+1)k} z^k \sum_{g=0}^{M} G_{g-k}^{M-k} q^{\binom{g-k}{2}+(A+1)(g-k)} a^{(M-k)-(g-k)} z^{g-k}$

$= \sum_{k=0}^{M} c_k^* q^{-\frac{k(k+1)}{2}+Ak} z^k \prod_{i=1}^{M-k} (a+q^{A+i}z) \to (1).$

$\sum_{g=0}^{M} b_g q^{Ag} a^{M-g} z^g = \sum_{g=0}^{M} q^{Ag} a^{M-g} z^g \sum_{k=0}^{M} (-1)^{k+g} G_g^k q^{-k(k+1)+\frac{(k-g)(k-g-1)}{2}} a_k^*$

$= \sum_{k=0}^{M} (-1)^k a_k^* q^{-\frac{k(k+3)}{2}} a^{M-k} \sum_{g=0}^{M} G_g^k q^{\binom{g}{2}+(A-k+1)g} a^{k-g}(-z)^g = \sum_{k=0}^{M} (-1)^k a_k^* q^{-\frac{k(k+3)}{2}} a^{M-k} \prod_{i=1}^{k} (a-q^{A-k+i}z).$

$\sum_{g=0}^{M} b_g q^{Ag} a^{M-g} z^g = \sum_{g=0}^{M} q^{Ag} a^{M-g} z^g (-1)^{M-g} q^{\frac{(M-g)(M-g-1)}{2}} \sum_{k=0}^{M} G_{M-g}^k q^{-(M+1)k} c_k^*$

$= \sum_{k=0}^{M} q^{-(M+1)k} c_k^* \sum_{g=0}^{M} a^{M-g} z^g (-1)^{M-g} G_{M-g}^k q^{\frac{(M-g)(M-g-1)}{2}+Ag}$, replace g with M-g $\to$

$= \sum_{k=0}^{M} q^{-(M+1)k} c_k^* \sum_{g=0}^{M} z^{M-g}(-a)^g G_g^k q^{\frac{g(g-1)}{2}+A(M-g)} = q^{AM} \sum_{k=0}^{M} q^{-(M+1)k} c_k^* z^{M-k} \prod_{i=1}^{k} (z-q^{-A-1+i}a) \to (2).$

$\sum_{g=0}^{M} c_g^* q^{Ag} a^{M-g} z^g = \sum_{g=0}^{M}\sum_{k=0}^{M} (-1)^{k+g} a_k^* G_{g-k}^{M-k} q^{\frac{g(g+3)}{2}-\frac{k(k+3)}{2}+Ag} a^{M-g} z^g$

$= \sum_{k=0}^{M} (-1)^k a_k^* q^{-\frac{k(k+3)}{2}} \sum_{g=0}^{M} (-1)^g G_{g-k}^{M-k} q^{\frac{g(g+3)}{2}+Ag} a^{M-g} z^g$, replace g with g+k $\to$

$= \sum_{k=0}^{M} (-1)^k a_k^* q^{-\frac{k(k+3)}{2}} \sum_{g=0}^{M} (-1)^{g+k} G_g^{M-k} q^{\frac{(g+k)(g+k+3)}{2}+A(g+k)} z^{g+k} a^{M-k-g}$

$= \sum_{k=0}^{M} a_k^* q^{Ak} z^k \sum_{g=0}^{M} G_g^{M-k} q^{\frac{g(g-1)}{2}+(A+k+2)g} (-z)^g a^{M-k-g} = \sum_{k=0}^{M} a_k^* q^{Ak} z^k \prod_{i=1}^{M-k} (a-q^{A+k+1+i}z).$

$\sum_{g=0}^{M} c_g^* q^{Ag} a^{M-g} z^g = \sum_{k=0}^{M} b_k \sum_{g=0}^{M} (-1)^g q^{\frac{g(g+3)}{2}} G_g^{M-k} q^{gk+Ag} a^{M-g} z^g$

$= \sum_{k=0}^{M} b_k a^k \sum_{g=0}^{M} (-1)^g q^{\frac{g(g-1)}{2}+(A+k+2)g} G_g^{M-k} a^{M-k-g} z^g = \sum_{k=0}^{M} b_k a^k \prod_{i=1}^{M-k} (a-q^{A+k+1+i}z) \to (3).$

□

$a \neq z \neq 0$, if $\sum_{g=0}^{M} a_g \binom{X}{1+p+g} = \sum_{g=0}^{M} b_g \binom{X+g}{1+p+g} = \sum_{g=0}^{M} c_g \binom{X+M-g}{1+p+M}$, then
$\sum_{g=0}^{M} a_g a^{M-g} z^g = \sum_{g=0}^{M} b_g a^{M-g} (a+z)^g = \sum_{g=0}^{M} c_g z^g (a+z)^{M-g}.$

From [4.4][4.1][4.4] and the mutual transformation of a_g, b_g, c_g [2.4][4.2], many equations can be obtained.

We can arbitrarily specify a set of values, calculate the other two sets, and obtain the corresponding relationships.

$\sum_{g=0}^{M} a_g G_{1+p+g}^{X} = \sum_{g=0}^{M} b_g G_{1+p+g}^{X+g} = \sum_{g=0}^{M} c_g G_{1+p+M}^{X+M-g}.\ 1+p \geq 0.$

$a_M = c_M = 1, a_{g<M} = c_{g<M} = 0 \to b_g = (-1)^{M+g} G_g^M q^{-M(1+M)+\frac{(M-g)(M-g-1)}{2}}$

$\to \sum_{g=0}^{M} b_g (-q^{A+1}\frac{z}{a};q)_g = q^{AM-\frac{M(M+1)}{2}} z^M a^{-M}; \sum_{g=0}^{M} b_g (q^{A+g+2}\frac{z}{a};q)_{M-g} = q^{AM} z^M a^{-M}.$

$b_M = 1, b_{g<M} = 0 \to a_g = q^{g(g+p+1)} G_g^M$

$\to \sum_{g=0}^{M} a_g z^g (zq^{2+p+g};q)_{M-g} = 1.\ \sum_{g=0}^{M} (-1)^g a_g q^{-pg} q^{-\frac{g(g+3)}{2}} (q^{A-g+1}\frac{z}{a};q)_g = q^{AM} z^M a^{-M}.$

$a_0^* = b_0 = 1, a_{g>0}^* = b_{g>0} = 0 \to c_g^* = (-1)^g q^{\frac{g(g+1)}{2}+g} G_g^M$[3.2] $\to \sum_{g=0}^{M} c_g^* q^{-(M+1)g} (q^{-A}\frac{a}{z};q)_g = q^{-AM} z^{-M} a^M.$

$\to \sum_{k=0}^{g} (-1)^k G_k^M G_{M-g}^{M-k} q^{\frac{k(k+1)}{2}-kg} = 0, g > 0;\ \sum_{k=g}^{M} (-1)^k G_k^M G_g^k q^{\frac{k(k+1)}{2}-Mk} = 0, g < M \to$ special case of [3.2(3)(4)].

$b_g = (-1)^g G_g^M, a_g^* = q^{g(g+1)} \sum_{k=0}^{M} b_k G_g^k$, [3.5] will lead to some special identities.

$b_g = q^g, \sum_{k=0}^{M} G_g^k q^{k-g} = G_{g+1}^{M+1} \to a_g^* = q^{g(g+1)} \sum_{k=g}^{M} q^k G_g^k = q^{g(g+2)} G_{g+1}^{M+1}.$

calculate b_g using $a_g^* \to \sum_{k=0}^{M} (-1)^{k+g} G_g^k q^{\binom{k-g}{2}} q^k G_{k+1}^{M+1} = q^g.$

$\sum_{g=0}^{M} q^g (q^{1+g};q)_{M-g} = \sum_{g=0}^{M} q^{g(g+1)} G_{g+1}^{M+1} (q^{1+g};q)_{M-g} = (q;q)_{M+1} \sum_{g=0}^{M} \frac{q^{g(g+1)} G_g^M}{(q;q)_{g+1}}.$

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From [3.2(1)], $G_{M+1}^{N+M} = \sum_{g=0}^{M} q^{(g+1)g} G_g^M G_{1+g}^N \rightarrow \sum_{g=0}^{M} \frac{q^{g(g+1)} G_g^M}{(q;q)_{g+1}} = \frac{1}{(q;q)_{M+1}}$

$\rightarrow \sum_{g=0}^{M} q^g (q^{1+g};q)_{M-g} = 1 \rightarrow \sum_{g=0}^{M} \frac{q^g}{(q;q)_g} = \frac{1}{(q;q)_M}(*)$. It is a known formula [2].

It can be proven through induction that: $\sum_{k=0}^{M} G_g^{M-k} q^{(g+1)k} = G_{g+1}^{M+1} \rightarrow c_g^* = (-1)^g q^{\frac{g(g+3)}{2}} G_{g+1}^{M+1}$

$\rightarrow \sum_{g=0}^{M} c_g^* q^{-g} = \sum_{g=0}^{M} (-1)^g q^{\frac{g(g+1)}{2}} G_{g+1}^{M+1} = -\sum_{g=1}^{M+1} (-1)^g q^{\frac{g(g-1)}{2}} G_g^{M+1} = 1 - (q^0;q)_{M+1} = 1$, get (*) again.

$\sum_{g=0}^{M} c_g^* q^{-g} a^g = a^{-1} - a^{-1}(a;q)_{M+1}$, [4.1(2)] $\rightarrow 1 + \sum_{g=0}^{M} \frac{aq^g}{(a;q)_{g+1}} = \frac{1}{(a;q)_{M+1}}$, generalization of (*) [2] pp 113.

a_g^* and $c_g^* \rightarrow G_{g+1}^{M+1} = \sum_{k=0}^{M} (-1)^k G_{M-g}^{M-k} G_{k+1}^{M+1} q^{\frac{k(k+1)}{2}} = \sum_{k=0}^{M} (-1)^k G_{M-g}^{M-k} G_{k+1}^{M+1} q^{\frac{k(k+1)}{2} - g(k+1)}$.

[4.4(2)] $\rightarrow \sum_{g=0}^{M} b_g q^{-g} = M + 1 = q^{-M} \sum_{g=0}^{M} c_g^* q^{-(M+1)g} (q;q)_g = \sum_{g=0}^{M} (-1)^g G_{g+1}^{M+1} q^{\frac{(g+1)g}{2} - M(g+1)} (q;q)_g$.

If we replace q by q^{-1}, we will still get $M+1$, which is Euler's identity: $\sum_{g=1}^{M} G_g^M (q;q)_{g-1} = M$ [2] pp 83.

$a_g^* = G_g^M q^{g^2+g} \rightarrow c_g^* = (-1)^g q^{\frac{g(g+3)}{2}} \sum_{k=0}^{M} (-1)^k G_{M-g}^{M-k} G_k^M q^{\frac{k(k-1)}{2}}$, [3.2] $\rightarrow c_0^* = 1, c_{g>0}^* = 0$.

$\rightarrow \frac{\sum_{g=0}^{M} G_g^M q^{g^2+g-yg} (aq)^g (aq \times q^{2-y+g};q)_{M-g}}{(aq;q)_{M+2-y}} = \frac{1}{(aq;q)_{M+2-y}} \rightarrow \frac{\sum_{g=0}^{M} G_g^M q^{g^2} a^g (aq^{1+g};q)_{M-g}}{(aq;q)_M} = \frac{1}{(aq;q)_M}$.

It's a finite form of Jacobi's Durfee square identity [2] pp 158-159: $\sum_{g=0}^{\infty} \frac{q^{g^2+gr} a^g}{(q;q)_g (aq;q)_{g+r}} = \frac{1}{(aq;q)_\infty}, r = 2 - y \geq 0$.

[3.2(4)], $g > x, \sum_{k=0}^{g} (-1)^k G_{M-g}^{M-k} G_k^M q^{\binom{k}{2} - xk} = 0$, further promotion can be done. Calculating b_g is also acceptable.

[4.4] is flexible. We can consider A, a, z as an independent variable or as a part of a_g, b_g, c_g, for example:

$a_g^{*,1} = q^{g(g+1)+gx} G_g^M, a_g^{*,2} = a_g^{*,1} z^g$ [3.4] $\rightarrow$

$b_g^{*,1} = q^{xg} G_g^M (q^x;q)_{M-g}, b_g^{*,2} = q^{xg} G_g^M (zq^x;q)_{M-g} z^g, c_g^{*,1} = (-1)^g q^{\frac{g(g+3)}{2}} G_g^M (q^x;q)_g, c_g^{*,2} = (-1)^g q^{\frac{g(g+3)}{2}} G_g^M (zq^x;q)_g$.

$\sum_{g=0}^{M} a_g^{*,1} q^{-\frac{g(g+1)}{2}} z^g = \sum_{g=0}^{M} a_g^{*,2} q^{-\frac{g(g+1)}{2}} = (-zq^{1+x};q)_M$

$= \sum_{g=0}^{M} q^{xg} G_g^M (q^x;q)_{M-g} (-zq;q)_g \ (*) = \sum_{g=0}^{M} q^{xg} z^g G_g^M (zq^x;q)_{M-g} (-q;q)_g$

$= \sum_{g=0}^{M} (-1)^g q^g z^g G_g^M (q^x;q)_g (-zq;q)_{M-g} = \sum_{g=0}^{M} (-1)^g q^g G_g^M (zq^x;q)_g (-q;q)_{M-g}$.

$x = 1$, replace zq by z, $(*) \rightarrow \frac{(-zq;q)_\infty}{(q;q)_\infty} = \sum_{g=0}^{\infty} \frac{q^g (-z;q)_g}{(q;q)_g}$.

[4.4(1)] $a^M \sum_{g=0}^{M} b_g (-q^{A+1} \frac{z}{a};q)_g = \sum_{g=0}^{M} c_g^* q^{-\frac{g(g+1)}{2} + Ag} a^{M-g} z^g (-q^{A+1} \frac{z}{a};q)_{M-g}$

$= \sum_{g=0}^{M} q^{-\frac{g(g+1)}{2} + Ag} a^{M-g} z^g (-q^{A+1} \frac{z}{a};q)_{M-g} (-1)^g q^{\frac{g(g+3)}{2}} \sum_{k=0}^{M} b_k G_g^{M-k} q^{gk}$, let $b_{g \neq k} = 0 \rightarrow$

$a^M (-q^{A+1} \frac{z}{a};q)_k = \sum_{g=0}^{M} q^{g+Ag} a^{M-g} (-q^{A+1} \frac{z}{a};q)_{M-g} (-z)^g G_g^{M-k} q^{gk}; (z;q)_k = \sum_{g=0}^{M} (z;q)_{M-g} z^g G_g^{M-k} q^{gk}$.

(3) $\rightarrow \sum_{g=0}^{M} c_g^* = \sum_{g=0}^{M} a_g^* (q^{2+g};q)_{M-g} = \sum_{g=0}^{M} b_g (q^{2+g};q)_{M-g}$. Let $a_g^* = b_g = x_g$, calculate c_g by a_g^* and b_g $\rightarrow$

For any x_g, $\sum_{g=0}^{M} (-1)^g q^{\frac{g(g+3)}{2}} \sum_{k=0}^{M} (-1)^k G_{M-g}^{M-k} q^{\frac{-k(k+3)}{2}} x_k = \sum_{g=0}^{M} (-1)^g q^{\frac{g(g+3)}{2}} \sum_{k=0}^{M} (-1)^k G_g^{M-k} q^{gk} x_k$.

More basically, $x_{g \neq k} = 0 \rightarrow q^{\frac{-k(k+3)}{2}} \sum_{g=0}^{M} (-1)^g q^{\frac{g(g+3)}{2}} G_{M-g}^{M-k} = \sum_{g=0}^{M} (-1)^g q^{\frac{g(g+3)}{2}} G_g^{M-k} q^{gk}$.

In the previous text, $T_i \in \mathbb{N}$, but excluding the actual meaning of $SUM_q(N)$, T_i can be any number.

$PS = [bq^{-T}, bq^{-T} ... bq^{-T}] : bq^{-T}(q-1), PT = [T, T+1 .... T+M-1], a = bq^{-T}$, due to T, a and b are independent.

At $H_1^q(g), B_i = \begin{cases} q^{X_T} G_1^{T+i-1-X_{K-1}} (q-1) bq^{-T} = q^{X_T} (bq^{X_T - 1} - a), \ X_i = T_i \\ q^{(T_i - T_{i-1} - 1) X_{T-1}} (K_i + G_1^{X_{T-1}} D_i) = aq^{X_T}, X_i = K_i \end{cases}, H_1^q(g) = G_g^M q^{\frac{g(g+1)}{2}} (b-a)(bq-a)...(bq^{g-1}-a) a^{M-g}$.

At $H_3^q(g), B_i = \begin{cases} q^1 \{(q^{X_{T-1}} G_1^{T_i} - q^{T_i} G_1^{X_{T-1}}) D_i - K_i q^{T_i}\} = -aq^{X_T}, \ X_i = T_i \\ (K_i + G_1^{X_{T-1}} D_i) = aq^{X_T}, \ X_i = K_i \end{cases}, H_3^q(g) = G_g^M (-1)^g q^{\frac{g(g+1)}{2}} a^M$.

$\frac{\sum_{g=0}^{M} q^{-yg} G_g^M q^{\frac{g(g+1)}{2}} (b-a)(bq-a)...(bq^{g-1}-a)(bq^{-T})^{M-g} (bx)^g (bx \times q^{2+T_M-M-y+g};q)_{M-g}}{(bx;q)_{T_M+2-y}}, T_M = T + M - 1,$

$T_M + 2 - y = M, y = T + 1 \rightarrow \frac{b^M q^{-TM} \sum_{g=0}^{M} q^{-g} G_g^M q^{\frac{g(g+1)}{2}} (b-a)(bq-a)...(bq^{g-1}-a) x^g (bx \times q^g;q)_{M-g}}{(bx;q)_M}$

$= \frac{\sum_{g=0}^{M} H_3^q(g) (bx)^g q^{-yg}}{(bx;q)_M} = \frac{a^M \sum_{g=0}^{M} G_g^M (-1)^g q^{\frac{g(g+1)}{2}} (bx)^g q^{-yg}}{(bx;q)_M} = \frac{a^M \sum_{g=0}^{M} G_g^M (-1)^g q^{\frac{g(g-1)}{2}} (bq^{-T} x)^g}{(bx;q)_M}$.

That is to say: $\frac{\sum_{g=0}^{M} G_g^M q^{\frac{g(g-1)}{2}}(b-a)(bq-a)...(bq^{g-1}-a)x^g(bx\times q^g;q)_{M-g}}{(bx;q)_M} = \frac{(ax;q)_M}{(bx;q)_M}$.

This proves Cauchy's identity [2] pp-260 : $\frac{(ax;q)_\infty}{(bx;q)_\infty} = \sum_{g=0}^{\infty} \frac{q^{\frac{g(g-1)}{2}} x^g(b-a)(bq-a)...(bq^{g-1}-a)}{(q;q)_g(bx;q)_g}$.

5. An Example

We can verify various propositions and theorems. PS=[A,B,C] ,PT=[1,3,5], M=3, p=5-M=2.

$\sum_{n_3=0}^{N-1}\sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2} q^{n_1+n_2+n_3}(A+[n_1]_q)(B+[n_2]_q)(C+[n_3]_q)$

$=\sum_{g=0}^{3} H_1^q(g)G_{3+g}^{N+2} = \sum_{g=0}^{3} H_2^q(g)G_{3+g}^{N+2+g} = \sum_{g=0}^{3} H_3^q(g)G_6^{N+5-g}$.

$H_1^q(0) = ABC$.

$H_1^q(1) = ABq^1G_1^3 + Aq^1G_1^2(C+1) + q^1G_1^1q^1(B+1)q^1(C+1)$.

$H_1^q(2) = q^1G_1^1q^3G_1^3q^2(C+G_1^2) + q^1G_1^1q^1(B+1)q^3G_1^4 + Aq^1G_1^2q^3G_1^4$.

$H_1^q(3) = q^1G_1^1q^3G_1^3q^5G_1^5$.

$H_2^q(0) = (A-q^{-1}G_1^1)(B-q^{-2}G_1^2)(C-q^{-3}G_1^3)$.

$H_2^q(1) = (A-q^{-1}G_1^1)(B-q^{-2}G_1^2)q^{-3}G_1^3 + (A-q^{-1}G_1^1)q^{-2}G_1^2(C-q^{-4}G_1^4) + q^{-1}G_1^1(B-q^{-3}G_1^3)(C-q^{-4}G_1^4)$.

$H_2^q(2) = q^{-1}G_1^1q^{-3}G_1^3(C-q^{-5}G_1^5) + q^{-1}G_1^1(B-q^{-3}G_1^3)q^{-4}G_1^4 + (A-q^{-1}G_1^1)q^{-2}G_1^2q^{-4}G_1^4$.

$H_2^q(3) = q^{-1}G_1^1q^{-3}G_1^3q^{-5}G_1^5$.

$H_3^q(0) = ABC$.

$H_3^q(1) = ABq^1(G_1^5 - q^5C) + Aq^2(G_1^3 - q^3B)(C+1) + q^3(1-q^1A)(B+1)(C+1)$.

$H_3^q(2) = q^5(1-q^1A)(q^1G_1^3 - q^3 - q^3B)(C+G_1^2) + q^4(1-q^1A)(B+1)(q^1G_1^5 - q^5 - q^5C) + Aq^3(G_1^3 - q^3B)(q^1G_1^5 - q^5 - q^5C)$.

$H_3^q(3) = q^6(1-q^1A)(q^1G_1^3 - q^3 - q^3B)(q^2G_1^5 - q^5G_1^2 - q^5C)$.

$PS = [A,B], PT = [1,3]$.

$\sum_{n_2=0}^{N-1}\sum_{n_1=0}^{n_2}(A+n_1)(B+n_2) = \sum_{g=0}^{2} H_1(g)\binom{N+1}{2+g} = \sum_{g=0}^{2} H_2(g)\binom{N+1+g}{2+g} = \sum_{g=0}^{2} H_3(g)\binom{N+3-g}{4}$.

$H_1(0) = AB, H_1(1) = A\times 2 + 1\times(B+1), H_1(2) = 1\times 3$.

$H_2(0) = (A-1)(B-2), H_2(1) = (A-1)\times 2 + 1\times(B-3), H_2(2) = 1\times 3$.

$H_3(0) = AB, H_3(1) = A(3-B) + (1-A)(B+1), H_3(2) = (1-A)(2-B)$.

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