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Posted Date: 30 April 2025

doi: 10.20944/preprints202504.2492.v1

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Article

Zero Extension for Dirichlet Problem of the Biharmonic Equation

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Abstract: In this paper we consider that extended function of solution by zero extension for the biharmonic equation of Dirichlet problem in a smaller domain is still the solution of the corresponding extension problem in a larger domain. We present a necessary and sufficient condition under the frameworks of classical solutions and strong solutions.

Keywords: Biharmonic; Dirichlet problem; Zero extension

MSC: 65N06; 65B99

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1. Introduction

The Dirichlet problem of Biharmonic equation is a classical problem from elasticity.([1–3]) Many person research this problem's Green function. They obtain many results.([4,5]) Here, We consider the zero extension Dirichlet Problem of the Biharmonic Equation.

Let Ω and $\tilde{\Omega}$ be two smooth domains in $\mathbb{R}^n$, and $\Omega \subset\subset \tilde{\Omega}$. Assume that $f(x)$ is a given function in the smaller domain Ω and u is a solution of the following Dirichlet problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Extend u and f from the smaller domain Ω to the larger domain $\tilde{\Omega}$ by zero extension and denote

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega, \\ 0 & x \in \bar{\tilde{\Omega}} \setminus \Omega, \end{cases} \qquad \tilde{f}(x) = \begin{cases} f(x) & x \in \Omega, \\ 0 & x \in \tilde{\Omega} \setminus \Omega. \end{cases} \tag{2}$$

Consider Dirichlet problem

$$\begin{cases} \Delta^2 v = \tilde{f} & \text{in } \tilde{\Omega}, \\ v = 0, \quad \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\tilde{\Omega} \end{cases} \tag{3}$$

in the larger domain $\tilde{\Omega}$.

General speaking, even if f is sufficiently smooth, the extended function $\tilde{u}$ of solution u may not be a solution of (3).

For example, let $\Omega = B(0, \frac{1}{2})$, $\tilde{\Omega} = B(0,1)$ $f \in C_0^\infty(B(0, \frac{1}{2}))$ be a nonnegative and nonzero function. It is obvious $\tilde{f} \in C_0^\infty(B(0,1))$. And then there exists a unique classical solution v for (3).

Let $\tilde{G}(x,y)$ be the Green's function of (3) in $\tilde{\Omega}$(see Definition 2.26. of [6]). We know

$$v(x) = \int_{B(0,1)} \tilde{G}(x,y)\tilde{f}(y)\,dy = \int_{B(0,\frac{1}{2})} \tilde{G}(x,y)f(y)\,dy.$$

Note

$$\tilde{G}(x,y) = \frac{1}{4ne_n}|x-y|^{4-n}\int_1^{\frac{\left||x|y-\frac{x}{|x|}\right|}{|x-y|}}(v^2-1)v^{1-n}dv > 0, \quad \forall x,y \in B(0,1),$$

where $e_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$(see Lemma 2.27 of [6]). Therefore, we have

$$v(x) > 0, \quad \forall x \in B(0,1),$$

which implies that $\tilde{u}(x)$ can not be a solution of (3) since $\tilde{u}(x) = 0$, $x \in \tilde{\Omega} \setminus \Omega$.

The interesting question is under what condition for function $f(x)$, $\tilde{u}(x)$ is still a solution for (3) and what conditions for function $f(x)$ must hold when the extended function $\tilde{u}(x)$ of $u(x)$ remains to be the solution for (3).

In this paper, we will give a complete answer to this question. We present a necessary and sufficient condition to guarantee that the extended function of the solution by zero extension for the biharmonic equation in the smaller domain is still the solution of the corresponding extension problem in the larger domain. We will prove the following conclusions under the frameworks of classical solutions and strong solutions.

We introduce a definition before stating our results.

Definition 1. *Let $f(x), g(x)$ be measurable in Ω. We say $f(x)$ is orthogonal to $g(x)$ if*

$$\int_\Omega f(x)g(x)\,dx = 0.$$

To make sure that (1) and (3) admit classical solutions and make the extension possible, we first assume f is Hölder continuous in $\bar{\Omega}$ and f equals 0 on the boundary $\partial\Omega$.

Theorem 1. *Assume $f \in C^\alpha(\bar{\Omega})(0 < \alpha < 1)$ with $f|_{\partial\Omega} = 0$. Let $u \in C^{4,\alpha}(\bar{\Omega})$ be the unique solution of (1) and functions $\tilde{u}, \tilde{f}$ be defined in(2). Then $\tilde{u}$ is the classical solution of (3) if and only if f is orthogonal to any biharmonic function g in Ω satisfying that g can be continuously extended to $\partial\Omega$, i.e.,*

$$\int_\Omega f(x)g(x)\,dx = 0 \tag{4}$$

for any $g \in C^4(\Omega) \cap C^1(\bar{\Omega})$ satisfying $\Delta^2 g = 0$ in Ω.

Remark 1. *If $f \in \Delta^2 C_0^\infty(\Omega)$, which means that there exists a function $w \in C_0^\infty(\Omega)$ such that $f = \Delta^2 w$, then f is orthogonal to any function $g \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying $\Delta^2 g = 0$ in Ω.*

Next we assume f is in a Lebesgue space to guarantee that (1) and (3) admit strong solutions.

Theorem 2. *Assume $f \in L^p(\Omega)(1 < p < +\infty)$. Let $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ be the unique solution of (1) and functions $\tilde{u}, \tilde{f}$ be defined in(2). Then $\tilde{u}$ is the strong solution of (3) if and only if f is orthogonal to any biharmonic function g in Ω.*

Remark 2. *Let $f \in \Delta^2 W_0^{4,p}(\Omega)$, which means that there exists a function $w \in W_0^{4,p}(\Omega)$ such that $f = \Delta^2 w$, then f is orthogonal to any biharmonic function $g \in C^4(\Omega) \cap C^1(\bar{\Omega})$.*

2. Proof of Main Results

In this section we first prove a lemma, which will be used later.

Lemma 1. *Let $g \in C^4(\Omega) \cap C^1(\bar{\Omega})$ be a biharmonic function and Ω be a bounded $C^{4,\alpha}$ domain in $\mathbb{R}^n$. Then for any $\varepsilon > 0$, there exists a biharmonic function $g_\varepsilon \in C^4(\bar{\Omega})$ such that*

$$\max_{\bar{\Omega}} |g(x) - g_\varepsilon(x)| \leq \varepsilon.$$

Proof. Since $g \in C^1(\bar{\Omega})$, then for any $\delta > 0$, there exists a function $g_\delta \in C^\infty(\bar{\Omega})$ satisfies

$$\max_{\partial\Omega} |g(x) - g_\delta(x)| \leq \frac{\delta}{2} \quad \textit{and} \quad \max_{\partial\Omega} \left| \frac{\partial g(x)}{\partial \mathbf{n}} - \frac{\partial g_\delta(x)}{\partial \mathbf{n}} \right| \leq \frac{\delta}{2}. \tag{5}$$

We solve the following problem

$$\begin{cases} \Delta^2 g_\varepsilon = 0 & \textit{in} \quad \Omega, \\ g_\varepsilon = g_\delta, \dfrac{\partial g_\varepsilon}{\partial \mathbf{n}} = \dfrac{\partial g_\delta}{\partial \mathbf{n}} & \textit{on} \quad \partial\Omega. \end{cases} \tag{6}$$

The problem (6) is solvable and there exists a unique solution $g_\varepsilon \in C^4(\bar{\Omega})$([7]).
Then for any $x \in \Omega$, $g_\varepsilon(x)$ and $g(x)$ can be expressed as

$$g_\varepsilon(x) = \int_{\partial\Omega} K_0(x,y) g_\delta(y) d\sigma_y + \int_{\partial\Omega} K_1(x,y) \frac{\partial g_\delta(y)}{\partial \mathbf{n}} d\sigma_y,$$

$$g(x) = \int_{\partial\Omega} K_0(x,y) g(y) d\sigma_y + \int_{\partial\Omega} K_1(x,y) \frac{\partial g(y)}{\partial \mathbf{n}} d\sigma_y,$$

where K_0, K_1 are the Poisson kernels([5]).
Therefore, for any $x \in \Omega$ we have

$$\begin{aligned} |g(x) - g_\varepsilon(x)| \leq & \max_{\partial\Omega} |g - g_\delta| \int_{\partial\Omega} |K_0(x,y)| d\sigma_y \\ & + \max_{\partial\Omega} \left| \frac{\partial g}{\partial \mathbf{n}} - \frac{\partial g_\delta}{\partial \mathbf{n}} \right| \int_{\partial\Omega} |K_1(x,y)| d\sigma_y. \end{aligned} \tag{7}$$

Fix any $x \in \Omega$ and denote $d_x = dist(x, \partial\Omega)$, it follows from ([5]) that

$$|K_j(x,y)| \leq C \frac{d_x^2}{|x-y|^{n-j+1}}, j = 0, 1, \; y \in \partial\Omega, \tag{8}$$

where C is a positive constant.
Using (8) and denote $x_0 \in \partial\Omega$ the point such that $d_x = |x - x_0|$, then

$$\int_{\partial\Omega} |K_0(x,y)| d\sigma_y \leq C \int_{\partial\Omega} \frac{d_x^2}{|x-y|^{n+1}} d\sigma_y.$$

Taking some $\delta_1 > 0$ to be determined later.

If $x \in \Omega_{\frac{\delta_1}{2}}$, where $\Omega_{\frac{\delta_1}{2}} = \{x \in \Omega | dist(x, \partial\Omega) \geq \frac{\delta_1}{2}\}$ then

$$\int_{\partial\Omega} \frac{d_x^2}{|x-y|^{n+1}} d\sigma_y \leq d_x^{1-n} |\partial\Omega| \leq \left(\frac{\delta_1}{2}\right)^{1-n} |\partial\Omega| \leq C_0, \tag{9}$$

where $|\partial\Omega|$ is the Lebesgue measure of $\partial\Omega$ and $C_0 = C_0(\delta_1, n, \partial\Omega) > 0$.

If $x \in \Omega \setminus \Omega_{\frac{\delta_1}{2}}$, then

$$\begin{aligned}\int_{\partial\Omega} \frac{d_x^2}{|x-y|^{n+1}} d\sigma_y \le & \int_{\partial\Omega\setminus B_{\delta_1}(x_0)} \frac{d_x^2}{|x-y|^{n+1}} d\sigma_y \\ & + \int_{\partial\Omega\cap B_{\delta_1}(x_0)} \frac{d_x^2}{|x-y|^{n+1}} d\sigma_y \\ = & I_1 + I_2.\end{aligned}$$

Since $|y-x| \ge |y-x_0| - d_x \ge \frac{\delta_1}{2}$ for any $y \in \partial\Omega \setminus B_{\delta_1}(x_0)$, we estimate I_1 by

$$I_1 \le \int_{\partial\Omega\setminus B_{\delta_1}(x_0)} \frac{d_x^2}{(\delta_1/2)^{n+1}} d\sigma_y \le \left(\frac{\delta_1}{2}\right)^{1-n} |\partial\Omega| \le C_0. \tag{10}$$

Now we need to estimate I_2, we use the method of "straighten out the boundary". Without loss of generality, we assume $x_0 = 0$ and x lies on the x_n-axis, that is, $x = (0, \dots, 0, d_x)$. Since Ω is a $C^{4,\alpha}$ domain, there exists a $C^{4,\alpha}$ mapping $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\partial\Omega \cap B_{\delta_1}(0) \subset \{y = (y_1, \dots, y_n) \in \mathbb{R}^n | y_n = \varphi(y_1, \dots, y_{n-1}), |y'| \le \tilde{C}\delta_1\},$$

where $y' = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$ and $\tilde{C} > 1$ is a constant.

For any $y \in \partial\Omega \cap B_{\delta_1}(0)$, we denote $\tilde{B}$ the ball in $\mathbb{R}^{n-1}$ and define

$$\begin{aligned}\psi : \partial\Omega \cap B_{\delta_1}(0) &\to \tilde{B}_{\tilde{C}\delta_1}(0) \\ y &\mapsto z\end{aligned}$$

in the following way

$$\begin{cases} z_i = y_i, \ i = 1, \dots, n-1, \\ z_n = y_n - \varphi(y_1, \dots, y_{n-1}). \end{cases}$$

It is obvious that ψ is a $C^{4,\alpha}$ mapping and $det(\psi) = det(\psi^{-1}) = 1$.

Note that Cauchy inequality yields

$$\frac{1}{2}|x-y|^2 \le |x|^2 + |y|^2 \le 5|y-x|^2 \ for \ any \ y \in \partial\Omega \cap B_{\delta_1}(0).$$

Therefore, after changing of variables we have

$$\begin{aligned} I_2 \le & C \int_{\partial\Omega\cap B_{\delta_1}(0)} \frac{d_x^2}{\left(d_x^2 + |y|^2\right)^{\frac{n+1}{2}}} d\sigma_y \\ \le & C \int_{\tilde{B}_{\tilde{C}\delta_1}(0)} \frac{d_x^2}{\left(d_x^2 + |\psi^{-1}(z)|^2\right)^{\frac{n+1}{2}}} d\tilde{\sigma}_z \\ \le & C \int_0^{\tilde{C}\delta_1} \frac{d_x^2 r^{n-2}}{(d_x^2 + r^2)^{\frac{n+1}{2}}} dr \\ \le & C d_x^2 \int_0^{\infty} \frac{1}{(d_x + r)^3} dr \\ \le & C_1. \end{aligned} \tag{11}$$

where $C_1 = C_1(n) > 0$.

doi:10.20944/preprints202504.2492.v1

By the same argument, using (8) for $j = 1$ we can compute

$$\int_{\partial\Omega} |K_1(x,y)| d\sigma_y \le C_2, \tag{12}$$

where $C_2 = C_2(\delta_1, n) > 0$.

Taking $C = \max(1, C_0, C_1, C_2)$, it following from (9)-(12) that

$$\int_{\partial\Omega} |K_0(x,y)| d\sigma_y \le C \quad and \quad \int_{\partial\Omega} |K_1(x,y)| d\sigma_y \le C.$$

Hence, by virtue of (5) and (7), choosing $\delta = \frac{\varepsilon}{C}$ we obtain

$$\max_{\bar{\Omega}} |g(x) - g_\varepsilon(x)| \le \varepsilon.$$

□

2.1. *Classical Solutions*

Now we are ready to prove Theorem 1.

Proof. (1)Necessity.

As $f \in C^\alpha(\bar{\Omega})$ and $f|_{\partial\Omega} = 0$, we know $\tilde{f} \in C^\alpha(\bar{\tilde{\Omega}})$. Then there exist an unique solution $v \in C^{4,\alpha}(\bar{\tilde{\Omega}}) \cap C^1(\bar{\tilde{\Omega}})$ for (3) ([7]).

Let $u \in C^{4,\alpha}(\bar{\Omega})$ be the classical solution of (1). If $\tilde{u}$ is the classical solution of (3), then $\tilde{u} = v \in C^4(\bar{\tilde{\Omega}})$, which implies $D^3u|_{\partial\Omega} = 0$, $D^2u|_{\partial\Omega} = 0$, $Du|_{\partial\Omega} = 0$.

First we assume that $g \in C^4(\bar{\Omega})$ satisfies $\Delta^2 g = 0$. Integration by parts yields that

$$\begin{aligned}
\int_\Omega f(x)g(x)\,dx &= \int_\Omega \Delta^2 u(x) g(x)\,dx \\
&= -\int_\Omega \nabla(\Delta u(x)) \cdot \nabla g(x)\,dx \\
&= \int_\Omega \Delta u(x) \Delta g(x)\,dx \\
&= -\int_\Omega \nabla u(x) \cdot \nabla(\Delta g(x))\,dx \\
&= \int_\Omega u(x) \Delta^2 g(x)\,dx \\
&= 0.
\end{aligned}$$

Next for $g \in C^4(\Omega) \cap C^1(\bar{\Omega})$ satisfying that $\Delta^2 g = 0$ in Ω, by Lemma 1 we find $g_\varepsilon \in C^4(\bar{\Omega})$ satisfy $\Delta^2 g_\varepsilon = 0$ such that

$$\max_{\bar{\Omega}} |g(x) - g_\varepsilon(x)| < \varepsilon.$$

Recalling

$$\int_\Omega f(x) g_\varepsilon(x)\,dx = 0,$$

and sending $\varepsilon \to 0$, we have

$$\int_\Omega f(x) g(x)\,dx = 0.$$

(2)Sufficiency

Now f is orthogonal to any biharmonic function in Ω, which can be continuously extended to $\partial\Omega$. Then f is orthogonal to any harmonic function in Ω, which is continuous on $\bar{\Omega}$.

Let $G(x,y)$ be the Green's function of (1) in Ω. We know

$$G(x,y) = \Gamma(y - x) - \phi(x,y),$$

where $\Gamma(y-x)$ is the fundamental solution(see chapter1 [8]) and

$$\begin{cases} \Delta_y^2\phi(x,y)=0, & y\in\Omega, \\ \phi(x,y)=\Gamma(y-x), \dfrac{\partial\phi(x,y)}{\partial n}=\dfrac{\partial\Gamma(y-x)}{\partial n} & y\in\partial\Omega. \end{cases}$$

Therefore, it follows from (4) that

$$u(x)=\int_\Omega G(x,y)f(y)\,dy=\int_\Omega \Gamma(x-y)f(y)\,dy, \quad \forall x\in\Omega \tag{13}$$

.

Let $\tilde{G}(x,y)$ be the Green's function of (3) in $\tilde{\Omega}$, which is

$$\tilde{G}(x,y)=\Gamma(x-y)-\tilde{\phi}(x,y),$$

where $\tilde{\phi}(x,y)$ is the solution of the boundary value problem $\Delta_y^2\tilde{\phi}(x,y)=0 \texttt{ in } \tilde{\Omega};\ \tilde{\phi}(x,y)=\Gamma(x-y), \frac{\partial\tilde{\phi}(x,y)}{\partial n}=\frac{\partial\Gamma(x-y)}{\partial n}, \texttt{ on } \partial\tilde{\Omega}$. Then,

$$\begin{aligned} v(x) &= \int_{\tilde{\Omega}} \tilde{G}(x,y)\tilde{f}(y)\,dy \\ &= \int_\Omega (\Gamma(x-y)-\tilde{\phi}(x,y)f(y)\,dy. \end{aligned}$$

Therefore, it follows from (4) that,

$$v(x)=\int_\Omega \Gamma(x-y)f(y)\,dy, \quad \forall x\in\tilde{\Omega}, \tag{14}$$

Case 1. $x\in\Omega$.

It follows from (13) and (14) that $v(x)=u(x)$.

Case 2. $x\in\tilde{\Omega}\setminus\Omega$.

When $y\in\Omega$, $\Gamma(x-y)$ is a biharmonic function in Ω.

It follows from (14), (4) that

$$v(x)=0, \quad \forall x\in\tilde{\Omega}\setminus\bar{\Omega}.$$

In view of the continuity of $v(x)$, we have

$$v(x)=0, \quad \forall x\in\tilde{\Omega}\setminus\Omega.$$

Combining the two cases above, we find that

$$\tilde{u}(x)=v(x), \quad x\in\tilde{\Omega},$$

which implies that $\tilde{u}(x)$ is the unique classical solution of (3). □

3. Strong Solutions

Next we use an approximation argument to prove Theorem 2.

Proof. Now $f\in L^p(\Omega)$ and then $\tilde{f}\in L^p(\tilde{\Omega})$. There exist the unique solution $u\in W^{4,p}(\Omega)\cap W_0^{2,p}(\Omega)$ for (1) (see chapter.3 [4] or see chapter.9 [7]).

(1)Necessity

Now $v(x)\in W^{4,p}(\tilde{\Omega})\cap W_0^{2,p}(\tilde{\Omega})$ is the strong solution of (3).

Let $\eta(x):\mathbb{R}^n\to[0,\infty)$ be a mollifier satisfying

(i) $\eta(x)\in C_0^\infty(\mathbb{R}^n)$,

(ii) $\int_{B_1} \eta(x)\, dx = 1$, where B_1 is the unit ball centered at the origin.

For $\varepsilon > 0$, Denote

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon}).$$

Then Supp $\eta_\varepsilon(x) \subset B_\varepsilon$, and $\int_{B_\varepsilon} \eta_\varepsilon(x)\, dx = 1$, where B_ε is the ball with radius ε, centered at the origin.

We extend $\tilde{u}$, $\tilde{f}$ from $\tilde{\Omega}$ to $\mathbb{R}^n$ by setting $\tilde{u}(x) = 0$, $\tilde{f}(x) = 0$, $x \in \mathbb{R}^n \setminus \tilde{\Omega}$. We define

$$\begin{aligned} u_\varepsilon(x) &= \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\tilde{u}(y)\, dy = \int_\Omega \eta_\varepsilon(x-y)u(y)\, dy; \\ f_\varepsilon(x) &= \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\tilde{f}(y)\, dy = \int_\Omega \eta_\varepsilon(x-y)f(y)\, dy. \end{aligned}$$

Choose $\varepsilon < \frac{1}{2}\text{dist}(\partial\tilde{\Omega}, \partial\Omega)$, and denote $\Omega_\varepsilon = \{x \in \mathbb{R}^n | \mathtt{dist}(x, \Omega) < \varepsilon\}$.

It is a simple fact that $u_\varepsilon(x), f_\varepsilon(x) \in C_0^\infty(\Omega_{2\varepsilon})$. Recalling

$$\Delta^2 \tilde{u} = \tilde{f} \ \mathtt{in}\ \tilde{\Omega},$$

we have

$$\Delta^2 u_\varepsilon = f_\varepsilon \ \mathtt{in}\ \mathbb{R}^n.$$

First Let $g \in C^4(\bar{\Omega})$ be a biharmonic function in Ω. By Whitney's extension theorem, we extend g to be $\tilde{g}$ from $\bar{\Omega}$ to $\bar{\tilde{\Omega}}$ such that $\tilde{g} \in C^4(\bar{\tilde{\Omega}})$(see [9]) and [10]). By using the fact that $u_\varepsilon(x) \in C_0^\infty(\Omega_{2\varepsilon})$, we find that

$$\int_{\Omega_{2\varepsilon}} \tilde{g}\Delta^2 u_\varepsilon\, dx = \int_{\Omega_{2\varepsilon}} u_\varepsilon \Delta^2 \tilde{g}\, dx,$$

which implies

$$\int_{\Omega_{2\varepsilon}} f_\varepsilon \tilde{g}\, dx = \int_{\Omega_{2\varepsilon}\setminus\Omega} u_\varepsilon \Delta^2 \tilde{g}\, dx.$$

On the other hand, we have

$$\begin{aligned} \int_\Omega fg\, dx &= \int_\Omega (f - f_\varepsilon)g\, dx + \int_{\Omega_{2\varepsilon}} f_\varepsilon \tilde{g}\, dx - \int_{\Omega_{2\varepsilon}\setminus\Omega} f_\varepsilon \tilde{g}\, dx \\ &= \int_\Omega (f - f_\varepsilon)g\, dx + \int_{\Omega_{2\varepsilon}\setminus\Omega} u_\varepsilon \Delta^2 \tilde{g}\, dx - \int_{\Omega_{2\varepsilon}\setminus\Omega} f_\varepsilon \tilde{g}\, dx \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By Hölder inequality, we estimate the three terms I_1, I_2 and I_3 as below:

$$\begin{aligned} I_1 &\le \max_{\bar{\Omega}} |g| \Big(\int_\Omega |f - f_\varepsilon|^p\, dx \Big)^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}}; \\ I_2 &\le \max_{\bar{\tilde{\Omega}}} |\Delta^2 \tilde{g}| \Big(\int_{\Omega_{2\varepsilon}\setminus\Omega} |u_\varepsilon|^p\, dx \Big)^{\frac{1}{p}} |\Omega_{2\varepsilon} \setminus \Omega|^{1-\frac{1}{p}} \\ &\le \max_{\bar{\tilde{\Omega}}} |\Delta^2 \tilde{g}| \Big(\int_\Omega |u|^p\, dx \Big)^{\frac{1}{p}} |\Omega_{2\varepsilon} \setminus \Omega|^{1-\frac{1}{p}}; \\ I_3 &\le \max_{\bar{\tilde{\Omega}}} |\tilde{g}| \Big(\int_{\Omega_{2\varepsilon}\setminus\Omega} |f_\varepsilon|^p\, dx \Big)^{\frac{1}{p}} |\Omega_{2\varepsilon} \setminus \Omega|^{1-\frac{1}{p}} \\ &\le \max_{\bar{\tilde{\Omega}}} |\tilde{g}| \Big(\int_\Omega |f|^p\, dx \Big)^{\frac{1}{p}} |\Omega_{2\varepsilon} \setminus \Omega|^{1-\frac{1}{p}}. \end{aligned}$$

Sending $\varepsilon \to 0$, we conclude that

$$\int_\Omega fg\, dx = 0,$$

doi:10.20944/preprints202504.2492.v1

which is (4).

Next for $g \in C^4(\Omega) \cap C(\bar{\Omega})$ satisfying that $\Delta^2 g = 0$ in Ω. We use the same approximation argument as in the proof of Theorem 1 to obtain

$$\int_\Omega f(x)g(x)\,dx = 0.$$

(2)Sufficiency

Let $\{f_n\} \subset C_0^\infty(\Omega)$ be a sequence satisfying

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(\Omega)} = 0.$$

Define

$$\tilde{f}_n(x) = \begin{cases} f_n(x), & x \in \Omega, \\ 0, & x \in \tilde{\Omega} \setminus \Omega. \end{cases}$$

We know that $\tilde{f}_n(x) \in C_0^\infty(\tilde{\Omega})$ and

$$\lim_{n\to\infty} \|\tilde{f}_n - \tilde{f}\|_{L^p(\tilde{\Omega})} = 0.$$

Let $\{u_n\}$ be the classical solutions of Dirichlet problems

$$\begin{cases} \Delta^2 u_n = f_n(x) & \text{in } \Omega, \\ u_n = 0, \ \dfrac{\partial u_n}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

It is obvious that $u_n \in W^{2,p}(\Omega) \cap W_0^{2,p}(\Omega)$ Let $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ be the unique solution of (1). It implies that(see [4])

$$\|u_n - u\|_{W^{4,p}(\Omega)} \le C\|f_n - f\|_{L^p(\Omega)}, \quad n = 1, 2, \cdots.$$

Let v_n be the classical solutions of Dirichlet problems

$$\begin{cases} -\Delta^2 v_n = \tilde{f}_n & \texttt{in } \tilde{\Omega}, \\ v_n = 0, \ \frac{\partial v_n}{\partial n} = 0 & \texttt{on } \partial\tilde{\Omega}. \end{cases} \tag{16}$$

It is obvious that $v_n \in W^{4,p}(\tilde{\Omega}) \cap W_0^{2,p}(\tilde{\Omega})$ Let $v \in W^{4,p}(\tilde{\Omega}) \cap W_0^{2,p}(\tilde{\Omega})$, be the unique solution of system (3).

It implies that

$$\begin{aligned} \|v_n - v\|_{W^{4,p}(\tilde{\Omega})} &\le C\|\tilde{f}_n - \tilde{f}\|_{L^p(\tilde{\Omega})} \\ &\le C\|f_n - f\|_{L^p(\Omega)}. \quad n = 1, 2, \cdots. \end{aligned}$$

Let $G(x,y)$ and $\tilde{G}(x,y)$ be the Green's functions of Δ^2 in Ω and $\tilde{\Omega}$. We know

$$G(x,y) = \Gamma(y-x) - \phi(x,y), \quad \tilde{G}(x,y) = \Gamma(y-x) - \tilde{\phi}(x,y), \tag{17}$$

where $\Gamma(y-x)$ is the fundamental solution, and $\phi(x,y)$ is the solution of the boundary value problem $\Delta_y^2\phi(x,y) = 0 \texttt{ in } \Omega;\ \phi(x,y) = \Gamma(x-y), \frac{\partial\phi(x,y)}{\partial n} = \frac{\partial\Gamma(x-y)}{\partial n}, \texttt{ on } \partial\Omega$. and $\tilde{\phi}(x,y)$ is the solution of the boundary value problem $\Delta_y^2\tilde{\phi}(x,y) = 0 \texttt{ in } \tilde{\Omega};\ \tilde{\phi}(x,y) = \Gamma(x-y), \frac{\partial\tilde{\phi}(x,y)}{\partial n} = \frac{\partial\Gamma(x-y)}{\partial n}, \texttt{ on } \partial\tilde{\Omega}$. Thus we have $\phi(x,y) \in C^4(\bar{\Omega})$ and $\tilde{\phi}(x,y) \in C^4(\bar{\tilde{\Omega}})$.

Let $\varphi(x) \in C_0^\infty(\Omega)$ is a arbitrary. Since u_k and v_k are the classical solutions of (15) and (16). By virtue of Fubini's theorem, we have

$$\begin{aligned}\int_\Omega u_k(x)\varphi(x)dx &= \int_\Omega \Big[\int_\Omega G(x-y)f_k(y)dy\Big]\varphi(x)dx \\ &= \int_\Omega \Big[\int_\Omega G(x-y)\varphi(x)dx\Big]f_k(y)dy.\end{aligned}$$

Sending $k \to \infty$,we obtain that

$$\begin{aligned}\int_\Omega u(x)\varphi(x)dx &= \int_\Omega \Big[\int_\Omega G(x-y)\varphi(x)dx\Big]f(y)dy \\ &= \int_\Omega \Big[\int_\Omega (\Gamma(x-y)-\phi(x,y))\varphi(x)dx\Big]f(y)dy \\ &= \int_\Omega \Big[\int_\Omega \Gamma(x-y)\varphi(x)dx\Big]f(y)dy \\ &\quad -\int_\Omega \Big[\int_\Omega \phi(x,y)f(y)dy\Big]\varphi(x)dx.\end{aligned}$$

It follows from (4) that

$$\int_\Omega u(x)\varphi(x)dx = \int_\Omega \Big[\int_\Omega \Gamma(x-y)\varphi(x)dx\Big]f(y)dy \tag{18}$$

Let $\psi(x) \in C_0^\infty(\tilde{\Omega})$is a arbitrary.By the same argument as above, we conclude that

$$\int_{\tilde\Omega} v(x)\psi(x)dx = \int_\Omega \Big[\int_{\tilde\Omega} \Gamma(x-y)\psi(x)dx\Big]f(y)dy \tag{19}$$

Case 1. $x \in \Omega$.

Let$\psi(x) = \varphi(x), x \in \Omega$ and $\psi(x) = 0, x \in \tilde\Omega \setminus \bar\Omega$. By virtue of (18),(19) we obtain that

$$u(x) = v(x), \ \ \texttt{a.e.} \quad x \in \Omega,$$

Case 2. $x \in \tilde\Omega \setminus \bar\Omega$.

Choose $\varepsilon < \frac{1}{2}\text{dist}(\partial\tilde\Omega, \partial\Omega)$, and denote $\Omega_\varepsilon = \{x \in \mathbb{R}^n | \text{dist}(x,\Omega) < \varepsilon\}$. Let $\varphi(x) \in C_0^\infty(\tilde\Omega \setminus \Omega_{2\varepsilon})$ is a arbitrary. By the same argument as above, we conclude that

$$\begin{aligned}\int_{\tilde\Omega\setminus\Omega_{2\varepsilon}} v(x)\varphi(x)dx &= \int_\Omega \Big[\int_{\tilde\Omega\setminus\Omega_{2\varepsilon}} \tilde\Gamma(x,y)\varphi(x)dx\Big]f(y)dy \\ &= \int_{\tilde\Omega\setminus\Omega_{2\varepsilon}} \Big[\int_\Omega \tilde\Gamma(x,y)f(y)dy\Big]\varphi(x)dx\end{aligned}$$

By virtue of(4) we have

$$\int_{\tilde\Omega\setminus\Omega_{2\varepsilon}} v(x)\varphi(x)dx = 0$$

sending $\varepsilon \to 0$,we obtain that

$$\int_{\tilde\Omega\setminus\Omega} v(x)\varphi(x)dx = 0$$

which implies that

$$v(x) = 0, \ \ \texttt{a.e.} \quad x \in \tilde\Omega \setminus \bar\Omega,$$

Combining the two cases above, we find that

$$\tilde{u}(x) = v(x), \quad \texttt{a.e.} \quad x \in \tilde{\Omega},$$

which implies that $\tilde{u}(x)$ is the unique strong solution of (3). ☐

4. Generalization

For the following boundary problem of the Δ^m equation

$$\begin{cases} \Delta^m u = f & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = \cdots = \dfrac{\partial^{m-1} u}{\partial n^{m-1}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $m \geq 3$ is an integer, we may consider the zero extension problem.

The similar conclusion may be drawn and proved by the same argument as above.

5. Conclusions

This paper focuses on zero extension for the biharmonic equation of Dirichlet problem. We established the Theorem 1 and the Theorem 2, which is the necessary and sufficient condition under the frameworks of classical solutions and strong solutions.

Author Contributions: Xu, S. carried out the mathematical studies and Yu, C.drafted the manuscript. All authors contributed equally to the preparation of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Hainan Provincial Natural Science Foundation of China(No. 122MS002). and Hainan Provincial Natural Science Foundation of China(No. 125RC625).

Data Availability Statement: The original contributions presented in the study are included in the article; further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

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