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*Article*

# New Polynomial Identities and Some Consequences

**Kunle Adegoke**

Department of Physics and Engineering Physics, Obafemi Awolowo University, 220005 Ile-Ife, Nigeria; adegoke00@gmail.com

**Abstract:** Using an elementary approach involving the Euler Beta function and the binomial theorem, we derive two polynomial identities; one of which is a generalization of a known polynomial identity. Two well-known combinatorial identities, namely Frisch's identity and Klamkin's identity, appear as immediate consequences of these polynomial identities. We subsequently establish several combinatorial identities, including a generalization of each of Frisch's identity and Klamkin's identity. Finally, we develop a scheme for deriving combinatorial identities associated with polynomial identities of a certain type.

**Keywords:** beta function; polynomial identity; Frisch identity; Klamkin identity; binomial coefficient; combinatorial identity

**MSC:** Primary 05A10; Secondary 05A19

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## 1. Introduction

Our purpose in writing this paper is to derive the following new polynomial identities in $x$:

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k+r}{s}^{-1}x^k = s\sum_{k=0}^{n}\frac{(-1)^k}{k+s}\binom{n}{k}\binom{k+r}{k+s}^{-1}x^k(1+x)^{n-k} \tag{1}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\binom{r}{k+s}^{-1}x^k = (r+1)\sum_{k=0}^{n}\frac{(-1)^{n-k}}{r-k+1}\binom{n}{k}\binom{r-k}{r-s-n}^{-1}(1-x)^{n-k}. \tag{2}$$

In identities (1) and (2), $n$ is a non-negative integer and $x$ is a complex variable. Identity (1) holds for complex numbers $r$ and $s$ for which $\Re(s) > 0$ and $\Re(r-s+1) > 0$, while (2) is valid for complex numbers $r$ and $s$ such that $\Re(s+1) > 0$ and $\Re(r-n-s+1) > 0$. Identity (1) is simpler than, and yet, generalizes Identity (4.13) of Gould's book [4, p.47], the latter corresponding to the case $r = s$ in (1).

At $x = -1$, identity (1) reduces to Frisch's identity [3], namely,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+r}{s}^{-1} = \frac{s}{k+s}\binom{k+r}{k+s}^{-1}, \tag{3}$$

while at $x = 1$, identity (2) yields Klamkin's identity:

$$\sum_{k=0}^{n}\binom{n}{k}\binom{r}{k+s}^{-1} = \frac{r+1}{(r-n+1)}\binom{r-n}{s}^{-1}. \tag{4}$$

In a recent paper, Gould and Quaintance [6] employed the well-known formula of Gauss for the hypergeometric function to give new proofs of (3) and (4). More recently, Abel [1] used the Euler Beta function to give elementary short proofs of (3) and (4). Our approach also uses the Beta function and is quite similar to that of Abel. For more historical facts concerning Frisch's identity and Klamkin's identity, the reader is referred to Abel [1] and Gould and Quaintance [6]. In a recent paper, Adegoke and Frontczak [2] derived many harmonic and odd harmonic number identities from Frisch's identity.

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Among other results, we will derive the following respective generalization of Frisch's identity and Klamkin's identity:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{u+n-k}{u}\binom{k+r}{s}^{-1}=\sum_{k=0}^{n}\frac{s}{k+s}\binom{n}{k}\binom{u}{n-k}\binom{k+r}{k+s}^{-1},$$
$$\sum_{k=0}^{n}\binom{n}{k}\binom{u+n-k}{u}\binom{r}{k+s}^{-1}=(r+1)\sum_{k=0}^{n}\frac{1}{r-k+1}\binom{n}{k}\binom{u}{n-k}\binom{r-k}{s}^{-1}.$$

Binomial coefficients are defined, for non-negative integers $i$ and $j$, by

$$\binom{i}{j}=\begin{cases}\dfrac{i!}{j!(i-j)!}, & i\geq j;\\ 0, & i<j;\end{cases}$$

the number of distinct sets of $j$ objects that can be chosen from $i$ distinct objects.

Generalized binomial coefficients are defined for complex numbers $r$ and $s$ by

$$\binom{r}{s}=\frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)},$$

where the Gamma function, $\Gamma(z)$, is defined for $\Re(z)>0$ by

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt=\int_0^\infty(\log(1/t))^{z-1}dt,$$

and is extended to the rest of the complex plane, excluding the non-positive integers, by analytic continuation.

## 2. Proof of (1) and (2)

The integration formulas required for proving (1) and (2) are given in Lemma 1.

**Lemma 1.** *If $r$, $k$ and $s$ are complex numbers and $x$ is a complex variable, then*

$$\int_0^1 y^{r+k-s}(1-y)^{s-1}dy=\frac{1}{s}\binom{k+r}{s}^{-1},\quad \Re(r+k-s+1)>0 \text{ and } \Re(s)>0; \tag{5}$$
$$\int_0^1 y^{r-s}(1-y)^{k+s-1}dy=\frac{1}{k+s}\binom{k+r}{k+s}^{-1},\quad \Re(r-s+1)>0 \text{ and } \Re(k+s)>0; \tag{6}$$
$$\int_0^1 y^{k+s}(1-y)^{r-k-s}dy=\frac{1}{r+1}\binom{r}{k+s}^{-1},\quad \Re(k+s+1)>0 \text{ and } \Re(r-k-s+1)>0, \tag{7}$$

*and*

$$\int_0^1 y^{n-k+s}(1-y)^{r-n-s} =\frac{1}{r-k+1}\binom{r-k}{r-s-n},\quad \Re(n-k+s+1)>0 \text{ and } \Re(r-n-s+1)>0. \tag{8}$$

**Proof.** The integrals in (5)– (8) are immediate consequences of the Beta function, $B(r,s)$, defined, as usual, for complex numbers $r$ and $s$ such that $\Re(r)>0$ and $\Re(s)>0$, by

$$B(r,s)=B(s,r)=\int_0^1 y^{r-1}(1-y)^{s-1}.$$

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With the help of the Gamma function, the integral is evaluated as

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \frac{1}{s}\binom{r+s-1}{s} = \frac{1}{r}\binom{r+s-1}{r}.$$

Note that in obtaining (7) and (8), we also used

$$\binom{u+1}{v+1} = \frac{u+1}{v+1}\binom{u}{v}.$$

□

**Theorem 1.** *If $n$ is a non-negative integer, $x$ is a complex variable and $r$ and $s$ are complex numbers such that $\Re(s) > 0$ and $\Re(r-s+1) > 0$, then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k+r}{s}^{-1}x^k = s\sum_{k=0}^{n}\frac{(-1)^k}{k+s}\binom{n}{k}\binom{k+r}{k+s}^{-1}x^k(1+x)^{n-k}.$$

**Proof.** Application of the binomial theorem to both sides of

$$1+xy = -x(1-y)+1+x$$

gives

$$\sum_{k=0}^{n}\binom{n}{k}x^k y^k = \sum_{k=0}^{n}\binom{n}{k}(-1)^k x^k(1-y)^k(1+x)^{n-k},$$

which upon multiplying through by $y^{r-s}(1-y)^{s-1}$ can also be written as

$$\sum_{k=0}^{n}\binom{n}{k}y^{r+k-s}(1-y)^{s-1}x^k$$
$$= \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k y^{r-s}(1-y)^{s+k-1}(1+x)^{n-k}.$$

Thus, term-wise integration gives

$$\sum_{k=0}^{n}\binom{n}{k}x^k\int_0^1 y^{r+k-s}(1-y)^{s-1}dy$$
$$= \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k(1+x)^{n-k}\int_0^1 y^{r-s}(1-y)^{s+k-1}dy,$$

from which (1) follows by (5) amd (6). □

**Theorem 2.** *If $n$ is a non-negative integer, $x$ is a complex variable and $r$ and $s$ are complex numbers such that $\Re(s+1) > 0$ and $\Re(r-n-s+1) > 0$, then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{r}{k+s}^{-1}x^k = (r+1)\sum_{k=0}^{n}\frac{(-1)^{n-k}}{r-k+1}\binom{n}{k}\binom{r-k}{r-s-n}^{-1}(1-x)^{n-k}.$$

**Proof.** Raising both sides of

$$xy+1-y = y(x-1)+1$$

to power $n$ and expanding via the binomial theorem gives

$$\sum_{k=0}^{n}\binom{n}{k}y^k(1-y)^{n-k}x^k = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}y^{n-k}(1-x)^{n-k}$$

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which, after multiplying through by $y^s(1-y)^{r-n-s}$, leads to

$$\sum_{k=0}^{n}\binom{n}{k}y^{k+s}(1-y)^{r-k-s}x^k=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}y^{n-k+s}(1-y)^{r-n-s}(1-x)^{n-k}.$$

Performing term-wise integration, we therefore have

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}x^k\int_0^1 y^{k+s}(1-y)^{r-k-s}dy\\&=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1-x)^{n-k}\int_0^1 y^{n-k+s}(1-y)^{r-n-s}dy;\end{aligned}$$

and hence (2) by (7) and (8). □

## 3. Combinatorial Identities

**Proposition 1.** *If $n$ is a non-negative integer and $r$ and $s$ are complex numbers such that $\Re(s)>0$ and $\Re(r-s+1)>0$, then*

$$\sum_{k=0}^{n}\frac{(-1)^k}{k+s}\binom{n}{k}\binom{k+r}{k+s}^{-1}=\frac{1}{s}\binom{n+r}{s}^{-1}. \tag{9}$$

In particular,

$$\sum_{k=0}^{n}\frac{(-1)^k}{k+r}\binom{n}{k}=\frac{1}{r}\binom{n+r}{r}^{-1}. \tag{10}$$

**Proof.** By writing $1/x$ for $x$, identity (1) can also be written as

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k+r}{s}^{-1}x^{n-k}=s\sum_{k=0}^{n}\frac{(-1)^k}{k+s}\binom{n}{k}\binom{k+r}{k+s}^{-1}(1+x)^{n-k}, \tag{11}$$

from which (9) is obtained by evaluating at $x=0$. □

**Remark 1.** *Identity (9) is the binomial transform of (3).*

**Proposition 2.** *If $n$ is a non-negative integer and $r$ and $s$ are complex numbers such that $\Re(s+1)>0$ and $\Re(r-n-s+1)>0$, then*

$$\sum_{k=0}^{n}\frac{(-1)^k}{r-k+1}\binom{n}{k}\binom{r-k}{r-s-n}^{-1}=\frac{(-1)^n}{r+1}\binom{r}{s}^{-1}. \tag{12}$$

**Proof.** Set $x=0$ in (2). □

Obviously, Frisch-type and Klamkin-type combinatorial identities are associated with polynomial identities having the following form:

$$\sum_{k=l_1}^{l_2}f(k)x^{p(k)}=\sum_{k=n_1}^{n_2}g(k)(1-x)^{q(k)}, \tag{13}$$

where $f(k)$ and $g(k)$ are sequences, $p(k)$ and $q(k)$ are sequences of non-negative integers and $l_1$, $l_2$, $n_1$ and $n_2$ are non-negative integers.

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### 3.1. Frisch-Type Combinatorial Identities

**Theorem 3.** *Let $r$ and $s$ be complex numbers such that $\Re(r + \min(p(l_1), p(l_2)) - s + 1) > 0$ and $\Re(\min(q(n_1), q(n_2)) + s) > 0$; where $\min(a, b)$ picks the smaller of $a$ and $b$. If a polynomial identity has the form (13), then the following combinatorial identities hold:*

$$\sum_{k=l_1}^{l_2} f(k) \binom{p(k)+r}{s}^{-1} = s \sum_{k=n_1}^{n_2} \frac{g(k)}{q(k)+s} \binom{q(k)+r}{q(k)+s}^{-1}, \tag{14}$$

$$\sum_{k=n_1}^{n_2} g(k) \binom{q(k)+r}{s}^{-1} = s \sum_{k=l_1}^{l_2} \frac{f(k)}{p(k)+s} \binom{p(k)+r}{p(k)+s}^{-1}. \tag{15}$$

**Proof.** Multiply through (13) by $x^{r-s}(1-x)^{s-1}$ and perform term-wise integration with respect to $x$, making use of (5) and (6), thereby obtaining (14). Identity (13) can also be written as

$$\sum_{k=n_1}^{n_2} g(k) x^{q(k)} = \sum_{k=l_1}^{l_2} f(k)(1-x)^{p(k)},$$

so that identities derived from (13) remain valid under the following transpositions:

$$p(k) \leftrightarrow q(k), \quad f(k) \leftrightarrow g(k), \quad n_1 \leftrightarrow l_1, \quad n_2 \leftrightarrow l_2;$$

and hence identity (15). □

We now illustrate Theorem 3 by deriving the Frisch-type identity associated with an identity of Simons.

**Lemma 2** (Simons [8]). *If $n$ is a non-negative integer and $x$ is a complex variable, then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} x^k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} (1-x)^k. \tag{16}$$

**Proposition 3.** *If $n$ is a non-negative integer and $r$ and $s$ are complex numbers such that $\Re(s) > 0$ and $\Re(r - s + 1) > 0$, then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{k+r}{s}^{-1} = \sum_{k=0}^{n} \frac{(-1)^{n-k} s}{k+s} \binom{n}{k} \binom{n+k}{k} \binom{k+r}{k+s}^{-1}. \tag{17}$$

**Proof.** Comparing (13) and (16), we find

$$f(k) = (-1)^k \binom{n}{k} \binom{n+k}{k}, \quad g(k) = (-1)^{n-k} \binom{n}{k} \binom{n+k}{k}, \tag{18}$$

$p(k) = k = q(k)$; and $l_1 = 0 = n_1$ and $l_2 = n = n_2$.

Using these in (14) gives (17). □

We can obtain a Frisch-type identity with two binomial coefficients in the denominator, directly from (1).

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**Proposition 4.** *If $n$ is a non-negative integer and $r$, $s$, $t$ and $u$ are complex numbers such that $\Re(s) > 0$, $\Re(r - s + 1) > 0$, $\Re(u) > 0$ and $\Re(t - u + 1) > 0$, then*

$$\begin{aligned}\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+r}{s}^{-1} \binom{k+t}{u}^{-1} \\ = su \sum_{k=0}^{n} \frac{1}{(k+s)(n-k+u)} \binom{n}{k} \binom{k+r}{k+s}^{-1} \binom{n+t}{n-k+u}^{-1}. \end{aligned} \tag{19}$$

**Proof.** Write $-x$ for $x$ in (1), multiply through by $x^{t-u}(1-x)^{u-1}$ and integrate with respect to $x$ from 0 to 1. □

In particular,

$$\begin{aligned}\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+r}{s}^{-2} \\ = s^2 \sum_{k=0}^{n} \frac{1}{(k+s)(n-k+s)} \binom{n}{k} \binom{k+r}{k+s}^{-1} \binom{n+r}{n-k+s}^{-1} \end{aligned} \tag{20}$$

and

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+r}{r}^{-2} = r^2 \sum_{k=0}^{n} \frac{1}{(k+r)(n-k+r)} \binom{n}{k} \binom{n+r}{n-k+r}^{-1}. \tag{21}$$

The reader is invited to discover the combinatorial identity having two binomial coefficients in the denominator associated with (11) by making appropriate substitutions in Theorem 3.

3.1.1. A Generalization of Frisch's Identity

In Theorem 4, we derive a generalization of Frisch's identity. We require the following known polynomial identity.

**Lemma 3** ([4, Identity (3.18), p.24])**.** *If $n$ is a non-negative integer, $u$ is a complex number and $x$ and $y$ are complex variables, then*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{u}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} \binom{u+k}{k} (x-y)^{n-k} y^k. \tag{22}$$

**Theorem 4.** *If $n$ is a non-negative integer, $u$ is a complex number and $r$ and $s$ are complex numbers such that $\Re(s) > 0$ and $\Re(r - s + 1) > 0$, then*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{u}{n-k} \binom{k+r}{s}^{-1} = \sum_{k=0}^{n} \frac{(-1)^k s}{k+s} \binom{n}{k} \binom{u+n-k}{u} \binom{k+r}{k+s}^{-1}, \tag{23}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{u+n-k}{u} \binom{k+r}{s}^{-1} = \sum_{k=0}^{n} \frac{s}{k+s} \binom{n}{k} \binom{u}{n-k} \binom{k+r}{k+s}^{-1}. \tag{24}$$

**Proof.** Setting $y = 1$ in (22) gives

$$\sum_{k=0}^{n} \binom{n}{k} \binom{u}{k} x^{n-k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{u+k}{k} (1-x)^{n-k},$$

so that comparing with (13), we choose

$$f(k) = \binom{n}{k} \binom{u}{k}, \quad g(k) = (-1)^{n-k} \binom{n}{k} \binom{u+k}{k}, \tag{25}$$

$p(k) = n - k = q(k)$, $l_1 = 0 = n_1$ and $l_2 = n = n_2$.

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Substituting these in (14) and (15) gives

$$\sum_{k=0}^{n}\binom{n}{k}\binom{u}{k}\binom{n-k+r}{s}^{-1}=\sum_{k=0}^{n}\frac{(-1)^{n-k}s}{n-k+s}\binom{n}{k}\binom{u+k}{k}\binom{n-k+r}{r-s}^{-1}$$

and

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{u+k}{k}\binom{n-k+r}{s}^{-1}=\sum_{k=0}^{n}\frac{s}{n-k+s}\binom{n}{k}\binom{u}{k}\binom{n-k+r}{r-s}^{-1},$$

which can also be written in the equivalent forms (23) and (24) □

**Remark 2.** *Frisch's identity (3) is obtained by setting $u=-1$ in (23) or $u=0$ in (24).*

### *3.2. Klamkin-Type Combinatorial Identities*

**Theorem 5.** *Let r and s be complex numbers such that $\Re(s+1)>0$, $\Re(r-\max(q(n_1),q(n_2))-s+1)>0$ and $\Re(r-\max(p(l_1),p(l_2))-s+1)>0$; where $\max(a,b)$ picks the greater of a and b. If a polynomial identity has the form (13), then the following combinatorial identities hold:*

$$\sum_{k=n_1}^{n_2}(-1)^{q(k)}g(k)\binom{r}{q(k)+s}^{-1}=(r+1)\sum_{k=l_1}^{l_2}\frac{f(k)}{r-p(k)+1}\binom{r-p(k)}{s}^{-1}, \tag{26}$$

$$\sum_{k=l_1}^{l_2}(-1)^{p(k)}f(k)\binom{r}{p(k)+s}^{-1}=(r+1)\sum_{k=n_1}^{n_2}\frac{g(k)}{r-q(k)+1}\binom{r-q(k)}{s}^{-1}. \tag{27}$$

**Proof.** Write $1/x$ for $x$ in (13) to obtain

$$\sum_{k=n_1}^{n_2}(-1)^{q(k)}g(k)x^{q(k)}(1-x)^{r-q(k)}=\sum_{k=l_1}^{l_2}f(k)(1-x)^{r-p(k)},$$

from which, by multiplying through with $x^s(1-x)^{r-s}$, we get

$$\sum_{k=n_1}^{n_2}(-1)^{q(k)}g(k)x^{q(k)+s}(1-x)^{r-q(k)-s}=\sum_{k=l_1}^{l_2}f(k)x^s(1-x)^{r-p(k)-s},$$

and hence (26) after term-wise integration. □

**Proposition 5.** *If n is a non-negative integer and r, s, t and u are complex numbers such that $\Re(s+1)>0$, $\Re(r-n-s)>0$, $\Re(t+1)>0$ and $\Re(t-n-u+1)>0$, then*

$$\sum_{k=0}^{n}\frac{1}{t-k+1}\binom{n}{k}\binom{t-k}{t-u-n}^{-1}\binom{r}{n-k+s}^{-1}$$
$$=\frac{r+1}{t+1}\sum_{k=0}^{n}\frac{1}{r-k+1}\binom{n}{k}\binom{t}{k+u}^{-1}\binom{r-k}{s}^{-1} \tag{28}$$

*and*

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{t}{k+u}^{-1}\binom{r}{k+s}^{-1}$$
$$=(r+1)(t+1)\sum_{k=0}^{n}\frac{(-1)^{n-k}\binom{n}{k}}{(t-k+1)(r-n+k+1)}\binom{t-k}{t-u-n}^{-1}\binom{r-n+k}{s}^{-1}. \tag{29}$$

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**Proof.** Write (2) as

$$\sum_{k=0}^{n}\binom{n}{k}\binom{t}{k+u}^{-1}x^k=(t+1)\sum_{k=0}^{n}\frac{(-1)^{n-k}}{t-k+1}\binom{n}{k}\binom{t-k}{t-u-n}^{-1}(1-x)^{n-k}.$$

Consider (13) and identify

$$f(k)=\binom{n}{k}\binom{t}{k+u}^{-1},\quad g(k)=\frac{(-1)^{n-k}(t+1)}{t-k+1}\binom{n}{k}\binom{t-k}{t-u-n}^{-1},$$

$p(k)=k$, $q(k)=n-k$ and $l_1=n_1=0$ and $l_2=n_2=n$.

Use these in (26) to obtain (28).

□

**Proposition 6.** *If $n$ is a non-negative integer and $r$ and $s$ are complex numbers such that $\Re(s+1)>0$ and $\Re(r-s-n+1)>0$, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{r}{k+s}^{-2}$$
$$=\frac{(r+1)^2}{(r-s-n+1)(s+1)}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{r-k+1}{r-s-n+1}^{-1}\binom{r-n+k+1}{s+1}^{-1}. \tag{30}$$

**Proof.** Set $t=r$ and $u=s$ in (29). □

Our next result is a Klamkin-type identity derived from the identity of Simons (16).

**Proposition 7.** *Let $n$ be a non-negative integer. If $r$ and $s$ are complex numbers such that $\Re(s)>0$ and $\Re(r-n-s+1)>0$, then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\binom{r}{k+s}^{-1}$$
$$=(r+1)(-1)^n\sum_{k=0}^{n}\frac{(-1)^k}{r-k+1}\binom{n}{k}\binom{n+k}{k}\binom{r-k}{s}^{-1}. \tag{31}$$

**Proof.** Use $f(k)$, $g(k)$, etc. given in (18) in Theorem 5. □

The reader is invited to discover the combinatorial identity having two binomial coefficients in the denominator associated with (11) by making appropriate substitutions in Theorem 5.

3.2.1. A Generalization of Klamkin's Identity

We close this section by giving a generalization of Klamkin's identity.

**Theorem 6.** *If $n$ is a non-negative integer, $u$ is a complex number and $r$ and $s$ are complex numbers such that $\Re(s)>0$ and $\Re(r-n-s+1)>0$, then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{u+n-k}{n-k}\binom{r}{k+s}^{-1}=(r+1)\sum_{k=0}^{n}\frac{1}{r-k+1}\binom{n}{k}\binom{u}{n-k}\binom{r-k}{s}^{-1}, \tag{32}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{u}{n-k}\binom{r}{k+s}^{-1}=(r+1)\sum_{k=0}^{n}\frac{(-1)^k}{r-k+1}\binom{n}{k}\binom{u+n-k}{n-k}\binom{r-k}{s}^{-1}. \tag{33}$$

**Proof.** Use $f(k)$, $g(k)$, etc. given in (25) in Theorem 5.

□

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**Remark 3.** *Klamkin's identity (4) is obtained by setting $u = 0$ in (32), while $u = 0$ in (33) recovers (12).*

## 4. Additional Results

*4.1. Identities Derived from the Geometric Progression*

**Proposition 8.** *If $n$ is a non-negative integer and $r$ and $s$ are complex numbers excluding the set of non-positive integers, then*

$$\sum_{k=0}^{n} \binom{k+r}{s}^{-1} = \frac{s}{s-1}\left(\binom{r-1}{s-1}^{-1} - \binom{n+r}{s-1}^{-1}\right). \tag{34}$$

**Proof.** Multiply through the sum of the geometric progression:

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}, \tag{35}$$

by $x^{r-s}(1-x)^{s-1}$ to obtain

$$\sum_{k=0}^{n} x^{k+r-s}(1-x)^{s-1} = x^{r-s}(1-x)^{s-2} - x^{n+r-s+1}(1-x)^{s-2}, \tag{36}$$

and hence (34) by termwise integration. □

**Remark 4.** *The special case $s = r$ of (34) was proved by Rockett [7].*

**Proposition 9.** *If $n$ is a non-negative integer, $s$ is a complex number excluding the set of negative integers and $r$ is a complex number such that $\Re(r-n-s+1) > 0$, then*

$$\sum_{k=0}^{n} (-1)^k \binom{r}{k+s}^{-1} = \frac{r+1}{s+1}\binom{r+2}{s+1}^{-1} + (-1)^n \frac{r+1}{n+s+2}\binom{r+2}{n+s+2}^{-1}. \tag{37}$$

**Proof.** Write $-x$ for $x$ in (35), replace $x$ with $x/(1-x)$ and multiply through by $x^s(1-x)^{r-n-s}$ to obtain

$$\sum_{k=0}^{n} (-1)^k x^{k+s}(1-x)^{r-k-s} = x^s(1-x)^{r-s+1} + (-1)^n x^{n+s+1}(1-x)^{r-n-s},$$

and hence (37). □

*4.2. An Identity Derived from Waring's Formula*

Waring's formula is [5, Equation (22)]

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(xy)^k(x+y)^{n-2k} = x^n + y^n, \tag{38}$$

and holds for a positive integer $n$ and complex variables $x$ and $y$.

**Proposition 10.** *If $n$ is a non-negative integer and $r$ and $s$ are complex numbers excluding the set of negative integers and such that $\Re(r-s+1) > 0$, then*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n}{(n-k)(k+s)}\binom{n-k}{k}\binom{2k+r}{k+s}^{-1} = \frac{1}{s}\binom{n+r}{s}^{-1} + \frac{1}{n+s}\binom{n+r}{n+s}^{-1}. \tag{39}$$

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**Proof.** Set $y = 1 - x$ in (38) and multiply through by $x^{r-s}(1-x)^{s-1}$ to obtain

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} x^{k+r-s}(1-x)^{k+s-1} = x^{n+r-s}(1-x)^{s-1} + x^{r-s}(1-x)^{n+s-1},$$

from which (39) follows. □

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