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Article

p-Adic Heisenberg-Robertson-Schrodinger and p-Adic Maccone-Pati Uncertainty Principles

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Abstract: Let $\mathcal{X}$ be a p-adic Hilbert space. Let $A : \mathcal{D}(A) \subseteq \mathcal{X} \rightarrow \mathcal{X}$ and $B : \mathcal{D}(B) \subseteq \mathcal{X} \rightarrow \mathcal{X}$ be possibly unbounded linear operators. For $x \in \mathcal{D}(A)$ with $\langle x, x \rangle = 1$, define $\Delta_x(A) := \|Ax - \langle Ax, x \rangle x\|$. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$, we show that

$$(1) \quad \max\{\Delta_x(A), \Delta_x(B)\} \geq \frac{\sqrt{\left| \langle [A, B]x, x \rangle^2 + (\langle \{A, B\}x, x \rangle - 2\langle Ax, x \rangle \langle Bx, x \rangle)^2 \right|}}{\sqrt{|2|}} \quad \text{and}$$

(2) $\max\{\Delta_x(A), \Delta_x(B)\} \geq |\langle (A + B)x, y \rangle|$, $\forall y \in \mathcal{X}$ satisfying $\|y\| \leq 1, \langle x, y \rangle = 0$. We call Inequality (1) as p-adic Heisenberg-Robertson-Schrodinger uncertainty principle and Inequality (2) as p-adic Maccone-Pati uncertainty principle.

Keywords: uncertainty principle; p-adic Hilbert space

MSC: 46S10; 12J25; 32P05

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and A be a possibly unbounded self-adjoint linear operator defined on the domain $\mathcal{D}(A) \subseteq \mathcal{H}$. For $h \in \mathcal{D}(A)$ with $\|h\| = 1$, define the uncertainty of A at the point h as

$$\Delta_h(A) := \|Ah - \langle Ah, h \rangle h\| = \sqrt{\|Ah\|^2 - \langle Ah, h \rangle^2}.$$

In 1929, Robertson [1] derived the following mathematical form of the uncertainty principle of Heisenberg derived in 1927 [2]. Recall that for two operators $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$, we define the commutator $[A, B] := AB - BA$ and anti-commutator $\{A, B\} := AB + BA$.

Theorem 1. [1–4] (*Heisenberg-Robertson Uncertainty Principle*) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

$$\frac{1}{2}(\Delta_h(A)^2 + \Delta_h(B)^2) \geq \left(\frac{\Delta_h(A) + \Delta_h(B)}{2} \right)^2 \geq \Delta_h(A)\Delta_h(B) \geq \frac{1}{2}|\langle [A, B]h, h \rangle|. \quad (1)$$

In 1930, Schrodinger made the following improvement of Inequality (1).

Theorem 2. [5,6] (*Heisenberg-Robertson-Schrodinger Uncertainty Principle*) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

$$\begin{aligned}\Delta_h(A)\Delta_h(B) &\geq |\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle| = \frac{\sqrt{|\langle [A, B]h, h \rangle|^2 + |\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|^2}}{2} \\ &= \frac{\sqrt{(\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle)^2 - \langle [A, B]h, h \rangle^2}}{2}.\end{aligned}$$

Surprisingly, in 2014, Maccone and Pati derived the following uncertainty principle which works for any unit vector which is orthogonal to given unit vector [7].

Theorem 3. [7] (*Maccone-Pati Uncertainty Principle*) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint linear operators. Then for all $h \in \mathcal{D}(A) \cap \mathcal{D}(B)$ with $\|h\| = 1$, we have

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq \frac{1}{2} \left(|\langle (A+B)h, k \rangle|^2 + |\langle (A-B)h, k \rangle|^2 \right), \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1, \langle h, k \rangle = 0.$$

As the study of p-adic Hilbert spaces is equally important as the study of Hilbert spaces, we naturally ask the following question.

Question 1.4. What are p-adic versions of Theorems 2 and 3?

In this paper, we answer Question 1.4.

2. p-Adic Heisenberg-Robertson-Schrodinger Uncertainty Principle and p-Adic Maccone-Pati Uncertainty Principle

We are going to consider the following notion of p-adic Hilbert space which is introduced by Kalisch [8] in 1947.

Definition 1. [8] Let $\mathbb{K}$ be a non-Archimedean valued field (with valuation $|\cdot|$) and $\mathcal{X}$ be a non-Archimedean Banach space (with norm $\|\cdot\|$) over $\mathbb{K}$. We say that $\mathcal{X}$ is a **p-adic Hilbert space** if there is a map (called as p-adic inner product) $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{K}$ satisfying following.

- (i) If $x \in \mathcal{X}$ is such that $\langle x, y \rangle = 0$ for all $y \in \mathcal{X}$, then $x = 0$.
- (ii) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{X}$.
- (iii) $\langle x, \alpha y + z \rangle = \alpha \langle x, y \rangle + \langle x, z \rangle$ for all $\alpha \in \mathbb{K}$, for all $x, y, z \in \mathcal{X}$.
- (iv) $|\langle x, y \rangle| \leq \|x\| \|y\|$ for all $x, y \in \mathcal{X}$.

Following are standard examples.

Example 1. Let $d \in \mathbb{N}$ and $\mathbb{K}$ be a non-Archimedean valued field. Then $\mathbb{K}^d$ is a p-adic Hilbert space w.r.t. norm

$$\|(x_j)_{j=1}^d\| := \max_{1 \leq j \leq d} |x_j|, \quad \forall (x_j)_{j=1}^d \in \mathbb{K}^d$$

and p-adic inner product

$$\langle (x_j)_{j=1}^d, (y_j)_{j=1}^d \rangle := \sum_{j=1}^d x_j y_j, \quad \forall (x_j)_{j=1}^d, (y_j)_{j=1}^d \in \mathbb{K}^d.$$

Example 2. Let $\mathbb{K}$ be a non-Archimedean valued field. Define

$$c_0(\mathbb{N}, \mathbb{K}) := \{(x_n)_{n=1}^\infty : x_n \in \mathbb{K}, \forall n \in \mathbb{N}, \lim_{n \rightarrow \infty} x_n = 0\}.$$

Then $c_0(\mathbb{N}, \mathbb{K})$ is a p-adic Hilbert space w.r.t. norm

$$\|(x_n)_{n=1}^\infty\| := \sup_{n \in \mathbb{N}} |x_n|, \quad \forall (x_n)_{n=1}^\infty \in c_0(\mathbb{N}, \mathbb{K})$$

and p -adic inner product

$$\langle (x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \rangle := \sum_{n=1}^{\infty} x_n y_n, \quad \forall (x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \in c_0(\mathbb{N}, \mathbb{K}).$$

Let $\mathcal{X}, \mathcal{Y}$ be p -adic Hilbert spaces and $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a linear operator. We say that T is adjointable if there is a linear operator, denoted by $T^* : \mathcal{Y} \rightarrow \mathcal{X}$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$, $\forall x \in \mathcal{X}, \forall y \in \mathcal{Y}$. Note that (i) in Definition 1 says that adjoint, if exists, is unique. An adjointable linear operator $T : \mathcal{X} \rightarrow \mathcal{X}$ is said to be self-adjoint if $T^* = T$.

Let A be a possibly unbounded linear operator (need not be self-adjoint) defined on domain $\mathcal{D}(A) \subseteq \mathcal{X}$. For $x \in \mathcal{D}(A)$ with $\langle x, x \rangle = 1$, define the uncertainty of A at the point x as

$$\Delta_x(A) := \|Ax - \langle Ax, x \rangle x\|.$$

We now have the p -adic version of Theorem 2.

Theorem 4. (p -adic Heisenberg-Robertson-Schrodinger Uncertainty Principles) Let $\mathcal{X}$ be a p -adic Hilbert space. Let $A : \mathcal{D}(A) \subseteq \mathcal{X} \rightarrow \mathcal{X}$ and $B : \mathcal{D}(B) \subseteq \mathcal{X} \rightarrow \mathcal{X}$ be linear operators. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$, we have

(i)

$$\Delta_x(A)\Delta_x(B) \geq |\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle| = |\langle Bx, Ax \rangle - \langle Ax, x \rangle \langle Bx, x \rangle|.$$

In particular, if A and B are self-adjoint, then

$$\Delta_x(A)\Delta_x(B) \geq |\langle BAx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle| = |\langle ABx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle|.$$

(ii)

$$\max\{\Delta_x(A), \Delta_x(B)\} \geq \frac{\sqrt{|\langle [A, B]x, x \rangle|^2 + (\langle \{A, B\}x, x \rangle - 2\langle Ax, x \rangle \langle Bx, x \rangle)^2}}{\sqrt{|2|}}.$$

(iii) If A and B are adjointable, then

$$\max\{\Delta_x(A), \Delta_x(B)\} \geq \sqrt{\left| \langle (A^*A + B^*B)x, x \rangle - \frac{\langle (A+B)x, x \rangle^2 + \langle (A-B)x, x \rangle^2}{2} \right|}.$$

In particular, if A and B are self-adjoint, then

$$\begin{aligned} \max\{\Delta_x(A), \Delta_x(B)\} &\geq \sqrt{\left| \langle (A^2 + B^2)x, x \rangle - \frac{\langle (A+B)x, x \rangle^2 + \langle (A-B)x, x \rangle^2}{2} \right|} \\ &= \sqrt{\left| \frac{\langle (A+B)^2x, x \rangle + \langle (A-B)^2x, x \rangle - \langle (A+B)x, x \rangle^2 - \langle (A-B)x, x \rangle^2}{2} \right|}. \end{aligned}$$

(iv) If A and B are adjointable, then

$$\max\{\Delta_x(A), \Delta_x(B)\} \geq \sqrt{|\langle (A^*A - B^*B)x, x \rangle - \langle (A+B)x, x \rangle \langle (A-B)x, x \rangle|}.$$

In particular, if A and B are self-adjoint, then

$$\max\{\Delta_x(A), \Delta_x(B)\} \geq \sqrt{|\langle (A^2 - B^2)x, x \rangle - \langle (A + B)x, x \rangle \langle (A - B)x, x \rangle|}.$$

(v)

$$\max\{\Delta_x(A), \Delta_x(B)\} \geq \sqrt{|\langle (A + B)x, (A + B)x \rangle - \langle (A + B)x, x \rangle^2|}.$$

(vi)

$$\max\{\Delta_x(A), \Delta_x(B)\} \geq \sqrt{|\langle (A - B)x, (A - B)x \rangle - \langle (A - B)x, x \rangle^2|}.$$

Proof. Let $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ be such that $\langle x, x \rangle = 1$.

(i) By using the definition of p-adic inner product,

$$\begin{aligned} \Delta_x(A)\Delta_x(B) &= \|Ax - \langle Ax, x \rangle x\| \|Bx - \langle Bx, x \rangle x\| \\ &\geq |\langle Ax - \langle Ax, x \rangle x, Bx - \langle Bx, x \rangle x \rangle| \\ &= |\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle|. \end{aligned}$$

(ii) By making a direct expansion and simplification, we see that

$$\begin{aligned} &\langle [A, B]x, x \rangle^2 + (\langle \{A, B\}x, x \rangle - 2\langle Ax, x \rangle \langle Bx, x \rangle)^2 \\ &= (\langle ABx, x \rangle - \langle BAx, x \rangle)^2 + \langle \{A, B\}x, x \rangle^2 + 4\langle Ax, x \rangle^2 \langle Bx, x \rangle^2 - 4\langle \{A, B\}x, x \rangle \langle Ax, x \rangle \langle Bx, x \rangle \\ &= (\langle ABx, x \rangle - \langle BAx, x \rangle)^2 + (\langle ABx, x \rangle + \langle BAx, x \rangle)^2 + 4\langle Ax, x \rangle^2 \langle Bx, x \rangle^2 \\ &\quad - 4\langle ABx, x \rangle \langle Ax, x \rangle \langle Bx, x \rangle - 4\langle BAx, x \rangle \langle Ax, x \rangle \langle Bx, x \rangle \\ &= 2\langle ABx, x \rangle^2 + 2\langle BAx, x \rangle^2 + 4\langle Ax, x \rangle^2 \langle Bx, x \rangle^2 - 4\langle ABx, x \rangle \langle Ax, x \rangle \langle Bx, x \rangle - 4\langle BAx, x \rangle \langle Ax, x \rangle \langle Bx, x \rangle \\ &= 2(\langle ABx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle)^2 + 2(\langle BAx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle)^2. \end{aligned}$$

Therefore

$$\begin{aligned} &\left| \langle [A, B]x, x \rangle^2 + (\langle \{A, B\}x, x \rangle - 2\langle Ax, x \rangle \langle Bx, x \rangle)^2 \right| \\ &= |2| \left| (\langle ABx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle)^2 + (\langle BAx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle)^2 \right| \\ &\leq |2| \max \left\{ \left| (\langle ABx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle)^2 \right|, \left| (\langle BAx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle)^2 \right| \right\} \\ &= |2| \max \left\{ |\langle ABx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle|^2, |\langle BAx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle|^2 \right\} \\ &\leq |2| \max \left\{ \max\{\Delta_x(A)^2, \Delta_x(B)^2\}, \max\{\Delta_x(B)^2, \Delta_x(A)^2\} \right\} = |2| \max\{\Delta_x(A)^2, \Delta_x(B)^2\}. \end{aligned}$$

(iii) By using the non-Archimedean triangle inequality and the definition of p-adic inner product,

$$\begin{aligned}
 \max\{\Delta_x(A), \Delta_x(B)\} &= \max\{\|Ax - \langle Ax, x \rangle x\|, \|Bx - \langle Bx, x \rangle x\|\} \\
 &\geq \max\{\sqrt{|\langle Ax - \langle Ax, x \rangle x, Ax - \langle Ax, x \rangle x \rangle|}, \sqrt{|\langle Bx - \langle Bx, x \rangle x, Bx - \langle Bx, x \rangle x \rangle|}\} \\
 &= \max\{\sqrt{|\langle Ax, Ax \rangle - \langle Ax, x \rangle^2|}, \sqrt{|\langle Bx, Bx \rangle - \langle Bx, x \rangle^2|}\} \\
 &= \sqrt{\max\{|\langle Ax, Ax \rangle - \langle Ax, x \rangle^2|, |\langle Bx, Bx \rangle - \langle Bx, x \rangle^2|\}} \\
 &\geq \sqrt{|\langle Ax, Ax \rangle - \langle Ax, x \rangle^2 + \langle Bx, Bx \rangle - \langle Bx, x \rangle^2|} \\
 &= \sqrt{|\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle - (\langle Ax, x \rangle^2 + \langle Bx, x \rangle^2)|} \\
 &= \sqrt{\left| \langle (A^*A + B^*B)x, x \rangle - \frac{\langle (A+B)x, x \rangle^2 + \langle (A-B)x, x \rangle^2}{2} \right|}.
 \end{aligned}$$

(iv) Using initial calculations in (iii),

$$\begin{aligned}
 \max\{\Delta_x(A), \Delta_x(B)\} &\geq \sqrt{\max\{|\langle Ax, Ax \rangle - \langle Ax, x \rangle^2|, |\langle Bx, Bx \rangle - \langle Bx, x \rangle^2|\}} \\
 &\geq \sqrt{|\langle Ax, Ax \rangle - \langle Ax, x \rangle^2 - \langle Bx, Bx \rangle + \langle Bx, x \rangle^2|} \\
 &= \sqrt{|\langle A^*Ax, x \rangle - \langle B^*Bx, x \rangle - (\langle Ax, x \rangle^2 - \langle Bx, x \rangle^2)|} \\
 &= \sqrt{|\langle (A^*A - B^*B)x, x \rangle - \langle (A+B)x, x \rangle \langle (A-B)x, x \rangle|}.
 \end{aligned}$$

(v) Using ultrametric inequality first and then using p-adic inner product we get

$$\begin{aligned}
 \max\{\Delta_x(A), \Delta_x(B)\} &= \max\{\|Ax - \langle Ax, x \rangle x\|, \|Bx - \langle Bx, x \rangle x\|\} \\
 &\geq \|Ax - \langle Ax, x \rangle x + Bx - \langle Bx, x \rangle x\| \\
 &\geq \sqrt{|\langle Ax - \langle Ax, x \rangle x + Bx - \langle Bx, x \rangle x, Ax - \langle Ax, x \rangle x + Bx - \langle Bx, x \rangle x \rangle|} \\
 &= \sqrt{|\langle Ax, Ax \rangle + \langle Bx, Bx \rangle + 2\langle Ax, Bx \rangle - 2\langle Ax, x \rangle \langle Bx, x \rangle - \langle Ax, x \rangle^2 - \langle Bx, x \rangle^2|} \\
 &= \sqrt{|\langle (A+B)x, (A+B)x \rangle - \langle (A+B)x, x \rangle^2|}.
 \end{aligned}$$

(vi) Using initial calculations in (v),

$$\begin{aligned}
 \max\{\Delta_x(A), \Delta_x(B)\} &= \max\{\|Ax - \langle Ax, x \rangle x\|, \|Bx - \langle Bx, x \rangle x\|\} \\
 &\geq \|Ax - \langle Ax, x \rangle x - Bx + \langle Bx, x \rangle x\| \\
 &\geq \sqrt{|\langle Ax - \langle Ax, x \rangle x - Bx + \langle Bx, x \rangle x, Ax - \langle Ax, x \rangle x - Bx + \langle Bx, x \rangle x \rangle|} \\
 &= \sqrt{|\langle (A-B)x, (A-B)x \rangle - \langle (A-B)x, x \rangle^2|}.
 \end{aligned}$$

□

Note that for self-adjoint operators A and B , we have

$$\langle [A, B]x, x \rangle = \langle ABx, x \rangle - \langle BAx, x \rangle = \langle Bx, Ax \rangle - \langle Ax, Bx \rangle = \langle Bx, Ax \rangle - \langle Bx, Ax \rangle = 0$$

and

$$\begin{aligned}
 \langle \{A, B\}x, x \rangle &= \langle ABx, x \rangle + \langle BAx, x \rangle = \langle Bx, Ax \rangle + \langle Ax, Bx \rangle = \langle Bx, Ax \rangle + \langle Bx, Ax \rangle \\
 &= 2\langle Bx, Ax \rangle = 2\langle ABx, x \rangle.
 \end{aligned}$$

We next derive p-adic version of Theorem 3.

Theorem 5. (*p*-adic Maccone-Pati Uncertainty Principle) Let $\mathcal{X}$ be a *p*-adic Hilbert space. Let $A : \mathcal{D}(A) \subseteq \mathcal{X} \rightarrow \mathcal{X}$ and $B : \mathcal{D}(B) \subseteq \mathcal{X} \rightarrow \mathcal{X}$ be linear operators. Then for all $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$ with $\langle x, x \rangle = 1$, we have

$$\max\{\Delta_x(A), \Delta_x(B)\} \geq |\langle (A+B)x, y \rangle|, \quad \forall y \in \mathcal{X} \text{ satisfying } \|y\| \leq 1, \langle x, y \rangle = 0$$

and

$$\max\{\Delta_x(A), \Delta_x(B)\} \geq |\langle (A-B)x, y \rangle|, \quad \forall y \in \mathcal{X} \text{ satisfying } \|y\| \leq 1, \langle x, y \rangle = 0.$$

Proof. Let $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$ be such that $\langle x, x \rangle = 1$. Let $y \in \mathcal{X}$ satisfies $\|y\| \leq 1$ and $\langle x, y \rangle = 0$. Then

$$\begin{aligned} |\langle (A+B)x, y \rangle| &= |\langle Ax - \langle Ax, x \rangle x + Bx - \langle Bx, x \rangle x, y \rangle| \\ &\leq \|Ax - \langle Ax, x \rangle x + Bx - \langle Bx, x \rangle x\| \|y\| \\ &\leq \|Ax - \langle Ax, x \rangle x + Bx - \langle Bx, x \rangle x\| \\ &\leq \max\{\|Ax - \langle Ax, x \rangle x\|, \|Bx - \langle Bx, x \rangle x\|\} \\ &= \max\{\Delta_x(A), \Delta_x(B)\}. \end{aligned}$$

□

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