Article

Not peer-reviewed version

Harmonically *m*-Convex Set-Valued Function

Gabriel Alberto Santana * and José Benito Hernández

Posted Date: 7 April 2025

doi: 10.20944/preprints202504.0422.v1

Keywords: set-valued convex functions; harmonically convex functions; harmonically m-convex functions; hermite-hadamard type inequality; convex analysis

Article

Harmonically *m*-Convex Set-Valued Function

Gabriel Santana [1,*] and José Benito Hernández [2]

[1] Escuela de Matemáticas, Facultad de ciencias, Universidad Central de Venezuela, Caracas 1010, Venezuela
[2] Centro de Investigación en Matemáticas, A.C, Unidad Monterrey, Monterrey México
* Correspondence: gabriel.santana@ucv.ve

Abstract: This research introduces the concept of *harmonically m-convex set-valued functions*, combining harmonically m-convex functions and set-valued mappings. We establish fundamental properties and derive a Hermite-Hadamard-type inequality for these functions, generalizing classical results in convex analysis. The study provides a theoretical foundation with potential applications in optimization, variational analysis, and mathematical economics, where set-valued mappings are essential. This work advances the understanding of harmonic convexity in the context of set-valued analysis, offering new insights for both theoretical and applied mathematics.

Keywords: set-valued convex functions; harmonically convex functions; harmonically m-convex functions; hermite-hadamard type inequality; convex analysis

---

1. Introduction

Convexity is a fundamental concept in mathematics with profound applications across various fields of science and engineering. Over the past few decades, the study of convexity has expanded significantly, leading to the development of new generalizations and variants that have enriched the theory and its applications. Among these, harmonic convexity has emerged as a particularly interesting extension, offering unique properties and applications. Harmonic convex functions, introduced by Anderson *et al.* [1] and further explored by I. Iscan [5], exhibit properties analogous to those of classical convex functions, making them a valuable tool in mathematical analysis. Building on these foundations, Noor *et al.* [16] introduced the broader class of harmonic h-convex functions, which generalizes many known classes of harmonic convex functions and provides a unified framework for their study.

In parallel, the concept of m-convexity, introduced by G. Toader [19], has gained attention for its ability to model intermediate convexity properties between classical convexity and star-shapedness. This notion has been extended to set-valued functions, which have been a subject of intense research since their introduction by C. Berge [3]. Set-valued functions, which map points to sets rather than single values, have found applications in optimization, control theory, and economics, particularly in problems involving set constraints and inclusions. Recent works, such as those by T. Lara *et al.* [7], have explored m-convex set-valued functions, providing characterizations, algebraic properties, and examples that highlight their theoretical and practical significance.

Integral inequalities, such as the Hermite-Hadamard inequality, have been a central focus in the study of convex functions and their generalizations. For harmonically convex set-valued functions, Santana *et al.* [18] established important results, including Hermite-Hadamard and Fejér inequalities, as well as a Bernstein-Doetsch-type theorem. These results have opened new avenues for research and applications in areas such as optimization and variational analysis.

This research introduces the novel concept of *harmonically m-convex set-valued functions*, which combines the ideas of harmonic convexity, m-convexity, and set-valued mappings. We explore the fundamental properties and characteristics of these functions, providing a comprehensive theoretical framework. Additionally, we derive a new Hermite-Hadamard-type inequality for harmonically m-convex set-valued functions, generalizing classical results and offering new insights into their behavior.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2025 doi:10.20944/preprints202504.0422.v1

This work not only advances the theoretical understanding of convexity and its generalizations but also provides tools with potential applications in optimization, economics, and engineering. By bridging the gap between harmonic convexity and set-valued analysis, this research contributes to the growing body of knowledge in convex analysis and its interdisciplinary applications.

2. Preliminary

As part of our research it is necessary to provide the reader with some preliminary definitions used throughout this investigation in order to lay the foundations for the development of this work.

Definition 1 (see [6]). *Let X a linear space and $m \in (0,1]$. A nonempty subset D of X is said to be harmonic m-convex, if for all $x, y \in D$ and $t \in [0,1]$, we have:*

$$\frac{mxy}{tmx + (1-t)y} \in D.$$

In this case İ. İscan in [6] generalized the harmonically convex function definition introduced in [5] to harmonically (α, m)-convex function:

Definition 2 (see [6]). *Let $f : D \subset (0,\infty) \to \mathbb{R}$ a function. Then, f is said to be harmonically (α, m)-convex function if for all $\alpha, t \in [0,1]$, $m \in (0,1]$ and $x, y \in D$, we have:*

$$f\left(\frac{mxy}{tmx + (1-t)y}\right) \leq t^{\alpha} f(y) + m(1-t)^{\alpha} f(x). \tag{1}$$

Note that if we considered $\alpha = 1$ in (1), f is said to be a harmonically m-convex function and satisfies the following:

$$f\left(\frac{mxy}{tmx + (1-t)y}\right) \leq tf(y) + m(1-t)f(x).$$

For this kind of functions in [6] obtain the following result:

Theorem 1 (see [6]). *Let $f : (0,\infty) \to \mathbb{R}$ be a harmonically m-convex function with $m \in (0,1]$. If $0 < a < b < \infty$ and $f \in L[a,b]$, then one has the inequality:*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq min\left\{\frac{f(a) + f\left\{\frac{b}{m}\right\}}{2}, \frac{f(b) + f\left\{\frac{a}{m}\right\}}{2}\right\}.$$

In the other hand, G. Santana *et al.* [18] in 2018, introduced the definition of harmonically convex set-valued functions, extending the definition given by İ. İscan for real functions (see [5]).

Definition 3 (see [18]). *Let X and Y linear spaces, D a harmonically convex subset of X and $F : D \subset X \to n(Y)$ a set-valued function. Then F is said harmonically convex function if for all $x, y \in D$ and $t \in [0,1]$, we have:*

$$tF(y) + (1-t)F(x) \subseteq F\left(\frac{xy}{tx + (1-t)y}\right)$$

Remark 1. *Throughout this paper $n(Y)$ will denote the family of nonempty subsets of Y.*

For this kind of functions they obtain many results as algebraic properties, Hermite-Hadamard and Fejer type inequalities and Bernstein-Doetsch type result.

To prove certain algebraic properties of the results in this research, we use two definitions established by T. Lara *et al.* in 2014 [7].

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2025 doi:10.20944/preprints202504.0422.v1

Definition 4 (see [7]). *Let $F_1, F_2 : D \subseteq X \to n(Y)$ be two set-valued functions (or multifunctions) then:*

- *The union of F_1 and F_2 is a set-valued function $F_1 \cup F_2 : D \to n(Y)$ given by $(F_1 \cup F_2)(x) = F_1(x) \cup F_2(x)$ for each $x \in D$.*
- *The sum of F_1 and F_2 is the function $F_1 + F_2 : D \to n(Y)$ defined in its usual form $(F_1 + F_2)(x) = F_1(x) + F_2(x)$ for each $x \in D$.*

Definition 5 (see [7]). *Let X, Y, Z be linear spaces and D be a subset of X. Then:*

- *If $F_1 : D \to n(Y)$ and $F_2 : D \to n(Z)$, then the cartesian product function of F_1 and F_2 is the set-valued function $F_1 \times F_2 : D \to n(Y) \times n(Z)$ given by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in D$.*
- *If $F_1 : D \to n(Y)$ and $F_2 : n(Y) \to n(Z)$, then the composition function of F_1 and F_2 is the set-valued function $F_2 \circ F_1 : D \to n(Z)$ given by*

$$(F_2 \circ F_1)(x) = F_2(F_1(x)) = \bigcup_{y \in F_1(x)} F_2(y),$$

for each $x \in D$.

Following the idea establish in [18], in this paper we extend that definition and introduce a new concept of convexity, combining the definitions of harmonically m-convex functions and set-valued functions. Then we define the following:

Definition 6. *Let X and Y be linear spaces, D a harmonically m-convex subset of X and $F : D \subset X \to n(Y)$ a set-valued function. It said that F is said harmonically m-convex function if for all $x, y \in D$, $t \in [0,1]$, and $m \in (0,1]$, we have:*

$$tF(y) + m(1-t)F(x) \subseteq F\left(\frac{mxy}{tmx + (1-t)y}\right). \tag{2}$$

We have some examples of this kind of function.

Example 1. *Let $f_1, (-f_2) : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be harmonically m-convex functions with $f_1(x) \leq f_2(x)$ for all $x \in [a,b]$. Then, the set-valued function $F : D \subset X \to n(Y)$ defined by $F(x) = [f_1(x), f_2(x)]$ is harmonically m-convex.*

In fact, since f_1 and $-f_2$ are harmonically m-convex functions, then for all $x, y \in D$, $t \in [0,1]$ and $m \in (0,1]$, we have:

$$f_1\left(\frac{mxy}{tmx + (1-t)y}\right) \leq tf_1(y) + m(1-t)f_1(x),$$

and

$$-f_2\left(\frac{mxy}{tmx + (1-t)y}\right) \leq -(tf_2(y) + m(1-t)f_2(x)),$$

if we multiply (-1) to both sides of the last inequality, we have:

$$f_2\left(\frac{mxy}{tmx + (1-t)y}\right) \geq tf_2(y) + m(1-t)f_2(x),$$

Then,

$$[tf_1(y) + m(1-t)f_1(x), tf_2(y) + m(1-t)f_2(x)] \subset \left[f_1\left(\frac{mxy}{tmx + (1-t)y}\right), f_2\left(\frac{mxy}{tmx + (1-t)y}\right)\right] \tag{3}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2025 doi:10.20944/preprints202504.0422.v1

from (3) we obtain with a elementary calculus that

$$tF(y) + m(1-t)F(x) \subset F\left(\frac{mxy}{tmx + (1-t)y}\right).$$

Example 2. *Let $H \in \mathbb{R}^3$ a harmonically m-convex subset, $F : \mathbb{R} \to n(\mathbb{R})$ a set-valued function defined by $F(x) = f(x)H$, where $f(x) = x^2$ is a harmonically m-convex function.*

Then, since $f(x) = x^2$ is a harmonically m-convex function by definition, we have for all $x, y \in D$, $t \in [0,1]$ and $m \in (0,1]$ that

$$\left(\frac{mxy}{tmx + (1-t)y}\right)^2 \leq t(y)^2 + m(1-t)(x)^2,$$

using the harmonically m-convex properties of H we have to

$$(t(y)^2 + m(1-t)(x)^2)H \subseteq \left(\frac{mxy}{tmx + (1-t)y}\right)^2 H,$$

then,

$$\begin{aligned}
(t(y)^2 + m(1-t)(x)^2)H &= t(y)^2H + m(1-t)(x)^2H \\
&= tF(y) + m(1-t)F(x) \\
&\subseteq F\left(\frac{mxy}{tmx + (1-t)y}\right).
\end{aligned}$$

Thus F is a harmonically m-convex set-valued function.

3. Main Results

The results obtained in this paper are based on the developments and ideas of İ. Iscan en [6] and T. Lara *et al.* in [7]. The following proposition establishes a property over harmonic m-convex set.

Proposition 1. *Let harmonic m-convex ($m \neq 1$) subset D of X is said to be starshaped if, for all x in D and all t in the interval $(0,1]$, the point tx also belongs to D. That is:*

$$tD \subseteq D.$$

Proof. Let D be a harmonically m-convex subset of X. If D is an empty set, there is nothing to prove. If, on the contrary, we consider D a nonempty set, let $x \in D$ then the point $x = \frac{mab}{tma+(1-t)b} \in D$ for everything $a, b \in D$ and $t \in [0,1]$. Thus, $[m,1]x = \{rx : m \leq r \leq 1\} \subset D$, in particular $mx \in D$. If $m = 0$, then $[0,1]x \in D$, we got the desired result.

In the case $m > 0$, we similarly repeat the previous argument for mx (instead of x), in this case we have to $[m^2, m]x = [m,1]mx \subseteq D$.

Inductively, we have that $[m^n, m^{n-1}]x \subseteq D$ for all $n \in \mathbb{N}$. Therefore $(0,1]x = \bigcup_{n=1}^{\infty}[m^n, m^{n-1}]x \subseteq D$. Thus D satisfies $tD \subseteq D$ for $t \in (0,1]$. □

For harmonically m-convex set-valued functions we obtain the following results:

Proposition 2. *Let $F_1, F_2 : D \to n(Y)$ be harmonically m-convex set-valued functions with $F_1(x) \subseteq F_2(x)$ (or $F_2(x) \subseteq F_1(x)$) for each $x \in D$. Then the union of F_1 and F_2 ($F_1 \cup F_2$) is a harmonically m-convex set-valued function.*

Proof. Let F_1, F_2 be harmonically m-convex set-valued function with $x, y \in D$, $t \in [0,1]$ and $m \in (0,1]$. Let's assume $F_1(x) \subseteq F_2(x)$ (in the case $F_2(x) \subseteq F_1(x)$ is analogous) for each $x \in D$, then:

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2025 doi:10.20944/preprints202504.0422.v1

$$\begin{aligned} t(F_1 \cup F_2)(y) \quad + \quad & m(1-t)(F_1 \cup F_2)(x) \\ & = t(F_1(y) \cup F_2(y)) + m(1-t)(F_1(x) \cup F_2(x)) \\ & = tF_2(y) + m(1-t)F_2(x) \\ & \subseteq F_2\left(\frac{mxy}{tmx + (1-t)y}\right) \\ & = F_1\left(\frac{mxy}{tmx + (1-t)y}\right) \cup F_2\left(\frac{mxy}{tmx + (1-t)y}\right) \\ & = (F_1 \cup F_2)\left(\frac{mxy}{tmx + (1-t)y}\right). \end{aligned}$$

□

Proposition 3. *If $F : D \subset X \to n(Y)$ is a harmonically m-convex set-valued function, then the image of F of any harmonically m-convex subset of D is a m-convex set of Y.*

Proof. Let A be a harmonically m-convex subset of $D \subset X$ and $a, b \in F(A) = \cup_{z \in A} F(z)$. Then $a \in F(x)$ and $b \in F(y)$ for some $x, y \in A$. Thus, for all $t \in [0,1]$ and $m \in (0,1]$, we have to:

$$tb + m(1-t)a \in tF(y) + m(1-t)F(x) \subseteq F\left(\frac{mxy}{tmx + (1-t)y}\right).$$

Since A is harmonically m-convex set, we have $\frac{mxy}{tmx+(1-t)y} \in A$ and $tb + m(1-t)a \in F(A)$ for all $t \in [0,1]$. Which implies that $F(A)$ is a m-convex set of Y. □

Corollary 1. *If $F : D \subseteq X \to n(Y)$ is a harmonically m-convex set-valued function, then the range of F is a m-convex set of Y.*

Proof. If we consider $A = D$ in Proposition 10 we get that $Rang(F) = F(D)$. □

Proposition 4. *A set-valued function $F : D \to n(Y)$ is harmonically m-convex, if and only if,*

$$tF(B) + m(1-t)F(A) \subseteq F\left(\frac{mAB}{tmA + (1-t)B}\right), \tag{4}$$

for each $A, B \subseteq D$, $t \in [0,1]$ and $m \in (0,1]$.

Proof. $(\Rightarrow)$ Let A, B be arbitrary subsets of D, $t \in [0,1]$ and $m \in (0,1]$. Let $x \in t(F(B) = \cup_{b \in B} F(b)) + m(1-t)F((A) = \cup_{a \in A} F(a)$, that is to say $x \in tF(b) + m(1-t)F(a)$ for some $a \in A$ and $b \in B$. Since F is harmonically m-convex and $a, b \in D$, it follows that:

$$tF(b) + m(1-t)F(a) \subseteq F\left(\frac{mab}{tma + (1-t)b}\right),$$

moreover, $\frac{mab}{tma+(1-t)b} \in \frac{mAB}{tmA+(1-t)B}$ and, in consequence,

$$F\left(\frac{mab}{tma + (1-t)b}\right) \subset F\left(\frac{mAB}{tmA + (1-t)B}\right).$$

Therefore, $x \in F\left(\frac{mAB}{tmA+(1-t)B}\right)$.

($\Leftarrow$) For $x, y \in D$ $t \in [0,1]$ and $m \in (0,1]$, the result is obtained by replacing $A = \{x\}$ and $B = \{y\}$ in (7) and we obtain the desired result

$$tF(\{y\}) + m(1-t)F(\{x\}) \subseteq F\left(\frac{m\{x\}\{y\}}{tm\{x\} + (1-t)\{y\}}\right).$$

Then, F is a harmonically m-convex set-valued function. □

In the following, consider that for nonempty linear space subsets A, B, C, D and α a scalar, the following properties are true:

- $\alpha(A \times B) = \alpha A \times \alpha B$,
- $(A \times C) + (B \times D) = (A + B) \times (C + D)$,
- Si $A \subseteq B$ y $C \subseteq D$ then $A \times C \subseteq B \times D$.

Proposition 5. *Let $F_1 : D \to n(Y)$ and $F_2 : D \to n(Z)$ harmonically m-convex set-valued functions. Then the cartesian product $F_1 \times F_2$ is a harmonically m-convex set-valued function.*

Proof. Let $x, y \in D$ and $t \in [0,1]$, then:

$$\begin{aligned} t(F_1 \times F_2)(y) \quad + \quad & m(1-t)(F_1 \times F_2)(x) \\ &= [tF_1(y) \times tF_2(y)] + [m(1-t)F_1(x) \times m(1-t)F_2(x)] \\ &= (tF_1(y) + m(t-1)F_1(x)) \times (tF_2(y) + m(t-1)F_2(x)) \\ &\subseteq F_1\left(\frac{mxy}{tmx + (1-t)y}\right) \times F_2\left(\frac{mxy}{tmx + (1-t)y}\right) \\ &= (F_1 \times F_2)\left(\frac{mxy}{tmx + (1-t)y}\right). \end{aligned}$$

□

The following proposition establishes that the harmonically m-convex set-valued function are closed under the sum and the product by a scalar.

Proposition 6. *Let X, Y be two linear spaces. If D is a harmonically m-convex subset of X and $F, G : D \subset X \to n(Y)$ two harmonically m-convex set-valued functions. Then $\lambda F + G$ is a harmonically m-convex set-valued function, for all λ.*

Proof. Let $x, y \in D \subset X$, $t \in [0,1]$ and $m \in (0,1]$. Since F and G are harmonically m-convex set-valued functions, we have:

$$tF(y) + m(1-t)F(x) \subseteq F\left(\frac{mxy}{tmx + (1-t)y}\right),$$

and,

$$tG(y) + m(1-t)G(x) \subseteq G\left(\frac{mxy}{tmx + (1-t)y}\right).$$

Thus,

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2025 doi:10.20944/preprints202504.0422.v1

$$
\begin{aligned}
t(\lambda F+G)(y) \quad + \quad & m(1-t)(\lambda F+G)(x) \\
&= [t(\lambda F(y))+m(1-t)(\lambda F(x))]+[tG(y)+m(1-t)G(x)] \\
&\subseteq \lambda F\left(\frac{mxy}{tmx+(1-t)y}\right)+G\left(\frac{mxy}{tmx+(1-t)y}\right) \\
&= (\lambda F+G)\left(\frac{mxy}{tmx+(1-t)y}\right).
\end{aligned}
$$

□

Proposition 7. *Let x, Y be two linear spaces. If D is a harmonically m-convex subset of X and $F, G : D \subset X \to n(Y)$ two harmonically m-convex set-valued functions then $(F \cdot G)(x)$ is also harmonically m-convex set-valued function.*

Proof. First, from [18], we have that.

$$F(x_1)G(x_2)+F(x_2)G(x_1) \subset F(x_1)G(x_1)+F(x_2)G(x_2).$$

Then, for $t \in [0,1]$ and x_1, $x_2 \in D$:

$$
\begin{aligned}
(F\cdot G)\left(\frac{mx_1x_2}{mtx_1+(1-t)x_2}\right) &= F\left(\frac{mx_1x_2}{mtx_1+(1-t)x_2}\right)\cdot G\left(\frac{mx_1x_2}{mtx_1+(1-t)x_2}\right) \\
&\supseteq [tF(x_1)+m(1-t)F(x_2)][tG(x_1)+m(1-t)G(x_2)] \\
&= t^2F(x_1)G(x_1)+mt(1-t)F(x_1)G(x_2)+mt(1-t)F(x_2)G(x_1) \\
&\quad +m^2(1-t)^2F(x_2)G(x_2) \\
&= t^2F(x_1)G(x_1)+mt(1-t)[F(x_1)G(x_2)+F(x_2)G(x_1)] \\
&\quad m^2(1-t)^2F(x_2)G(x_2) \\
&\supseteq t^2F(x_1)G(x_1)+mt(1-t)[F(x_1)G(x_1)+F(x_2)G(x_2)] \\
&\quad +m^2(1-t)^2F(x_2)G(x_2) \\
&= t[t+m(1-t)]F(x_1)G(x_1)+[m(1-t)[t+m(1-t)]F(x_2)G(x_2)] \\
&\supseteq tF(x_1)G(x_1)+m(1-t)F(x_2)G(x_2).
\end{aligned}
$$

This shows that the product of two harmonically m-convex set-valued functions is again harmonically m-convex set-valued function. □

The following result follows the idea of İ. İscan in [6]. To integrate set-valued functions we use the definition given by R. J. Aumann, and if a function satisfies the requirements of being integrable under this integral definition given, we say that a set-valued function F is Aumann integrable under a certain domain (see [2]).

Theorem 2. *Let X, Y linear spaces, D be a harmonically m-convex subset of X and $F : D \subset X \to n(Y)$ a harmonically m-convex set-valued Aumann integrable function, then*

$$\min\left(\inf\left(\frac{F(a)+mF\left(\frac{b}{m}\right)}{2}\right), \inf\left(\frac{mF\left(\frac{a}{m}\right)+F(b)}{2}\right)\right) \subseteq \frac{ab}{b-a}\int_a^b \frac{F(x)}{x^2}dx.$$

Proof. Let $F : D \subset X \to n(Y)$ be a harmonically m-convex set-valued function, for every $x, y \in D$ we have to

$$tF(y)+m(1-t)F\left(\frac{x}{m}\right) \subset F\left(\frac{xy}{tx+(1-t)y}\right) = F\left(\frac{m\frac{x}{m}y}{tm\frac{x}{m}+(1-t)y}\right),$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2025 doi:10.20944/preprints202504.0422.v1

Then, we have the following:

$$tF(b) + m(1-t)F\left(\frac{a}{m}\right) \subset F\left(\frac{ab}{ta+(1-t)b}\right), \tag{5}$$

and

$$tF(a) + m(1-t)F\left(\frac{b}{m}\right) \subset F\left(\frac{ab}{tb+(1-t)a}\right),$$

for every $t \in [0,1]$, $m \in (0,1]$ and $a, b \in D$. Integrating both sides of (8) on $[0,1]$ with respect to t, we get that

$$\int_0^1 tF(b) + m(1-t)F\left(\frac{a}{m}\right)dt \subset \int_0^1 F\left(\frac{ab}{ta+(1-t)b}\right)dt. \tag{6}$$

Integrating the left side of (9) we have:

$$\int_0^1 tF(b) + m(1-t)F\left(\frac{a}{m}\right)dt = \frac{F(b) + mF\left(\frac{a}{m}\right)}{2}.$$

By the Aumann integral definition we get, that integral on the right hand of (9) is defined as:

$$\int_0^1 F\left(\frac{ab}{ta+(1-t)b}\right)dt = \left\{\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt : f(x) \in F(x) \wedge t \in [0,1]\right\}.$$

But,

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx,$$

then

$$\left\{\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx : f(x) \in F(x) \wedge x \in [a,b]\right\} = \frac{ab}{b-a}\int_a^b \frac{F(x)}{x^2}dx.$$

In consecuense:

$$\frac{F(b) + mF\left(\frac{a}{m}\right)}{2} \subseteq \frac{ab}{b-a}\int_a^b \frac{F(x)}{x^2}dx.$$

Similarly, we have that:

$$\frac{F(a) + mF\left(\frac{b}{m}\right)}{2} \subseteq \frac{ab}{b-a}\int_a^b \frac{F(x)}{x^2}dx,$$

so the required result is obtained.

$$\min\left(\inf\left(\frac{F(a) + mF\left(\frac{b}{m}\right)}{2}\right), \inf\left(\frac{mF\left(\frac{a}{m}\right) + F(b)}{2}\right)\right) \subseteq \frac{ab}{b-a}\int_a^b \frac{F(x)}{x^2}dx.$$

□

4. Results

The results obtained in this paper are based on the developments and ideas of İ. Iscan en [6] and T. Lara *et al.* in [7]. The following proposition establishes a property over harmonic m-convex set.

Proposition 8. *Let harmonic m-convex* ($m \neq 1$) *subset D of X is said to be starshaped if, for all x in D and all t in the interval* $(0,1]$, *the point tx also belongs to D. That is:*

$$tD \subseteq D.$$

Proof. Let D be a harmonically m-convex subset of X. If D is an empty set, there is nothing to prove. If, on the contrary, we consider D a nonempty set, let $x \in D$ then the point $x = \frac{mab}{tma+(1-t)b} \in D$ for

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2025 doi:10.20944/preprints202504.0422.v1

everything $a, b \in D$ and $t \in [0,1]$. Thus, $[m,1]x = \{rx : m \leq r \leq 1\} \subset D$, in particular $mx \in D$. If $m = 0$, then $[0,1]x \in D$, we got the desired result.

In the case $m > 0$, we similarly repeat the previous argument for mx (instead of x), in this case we have to $[m^2, m]x = [m,1]mx \subseteq D$.

Inductively, we have that $[m^n, m^{n-1}]x \subseteq D$ for all $n \in \mathbb{N}$. Therefore $(0,1]x = \bigcup_{n=1}^{\infty}[m^n, m^{n-1}]x \subseteq D$. Thus D satisfies $tD \subseteq D$ for $t \in (0,1]$. □

For harmonically m-convex set-valued functions we obtain the following results:

Proposition 9. *Let $F_1, F_2 : D \to n(Y)$ be harmonically m-convex set-valued functions with $F_1(x) \subseteq F_2(x)$ (or $F_2(x) \subseteq F_1(x)$) for each $x \in D$. Then the union of F_1 and F_2 ($F_1 \cup F_2$) is a harmonically m-convex set-valued function.*

Proof. Let F_1, F_2 be harmonically m-convex set-valued function with $x, y \in D$, $t \in [0,1]$ and $m \in (0,1]$. Let's assume $F_1(x) \subseteq F_2(x)$ (in the case $F_2(x) \subseteq F_1(x)$ is analogous) for each $x \in D$, then:

$$
\begin{aligned}
t(F_1 \cup F_2)(y) \quad + \quad & m(1-t)(F_1 \cup F_2)(x) \\
& = t(F_1(y) \cup F_2(y)) + m(1-t)(F_1(x) \cup F_2(x) \\
& = tF_2(y) + m(1-t)F_2(x) \\
& \subseteq F_2\left(\frac{mxy}{tmx + (1-t)y}\right) \\
& = F_1\left(\frac{mxy}{tmx + (1-t)y}\right) \cup F_2\left(\frac{mxy}{tmx + (1-t)y}\right) \\
& = (F_1 \cup F_2)\left(\frac{mxy}{tmx + (1-t)y}\right).
\end{aligned}
$$

□

Proposition 10. *If $F : D \subset X \to n(Y)$ is a harmonically m-convex set-valued function, then the image of F of any harmonically m-convex subset of D is a m-convex set of Y.*

Proof. Let A be a harmonically m-convex subset of $D \subset X$ and $a, b \in F(A) = \cup_{z \in A} F(z)$. Then $a \in F(x)$ and $b \in F(y)$ for some $x, y \in A$. Thus, for all $t \in [0,1]$ and $m \in (0,1]$, we have to:

$$tb + m(1-t)a \in tF(y) + m(1-t)F(x) \subseteq F\left(\frac{mxy}{tmx + (1-t)y}\right).$$

Since A is harmonically m-convex set, we have $\frac{mxy}{tmx+(1-t)y} \in A$ and $tb + m(1-t)a \in F(A)$ for all $t \in [0,1]$. Which implies that $F(A)$ is a m-convex set of Y. □

Corollary 2. *If $F : D \subseteq X \to n(Y)$ is a harmonically m-convex set-valued function, then the range of F is a m-convex set of Y.*

Proof. If we consider $A = D$ in Proposition 10 we get that $Rang(F) = F(D)$. □

Proposition 11. *A set-valued function $F : D \to n(Y)$ is harmonically m-convex, if and only if,*

$$tF(B) + m(1-t)F(A) \subseteq F\left(\frac{mAB}{tmA + (1-t)B}\right), \tag{7}$$

for each $A, B \subseteq D$, $t \in [0,1]$ and $m \in (0,1]$.

 doi:10.20944/preprints202504.0422.v1

Proof. ($\Rightarrow$) Let A, B be arbitrary subsets of D, $t \in [0,1]$ and $m \in (0,1]$. Let $x \in t(F(B) = \cup_{b\in B} F(b)) + m(1-t)F((A) = \cup_{a\in A} F(a)$, that is to say $x \in tF(b) + m(1-t)F(a)$ for some $a \in A$ and $b \in B$. Since F is harmonically m-convex and $a, b \in D$, it follows that:

$$tF(b) + m(1-t)F(a) \subseteq F\left(\frac{mab}{tma + (1-t)b}\right),$$

moreover, $\frac{mab}{tma+(1-t)b} \in \frac{mAB}{tmA+(1-t)B}$ and, in consequence,

$$F\left(\frac{mab}{tma + (1-t)b}\right) \subset F\left(\frac{mAB}{tmA + (1-t)B}\right).$$

Therefore, $x \in F\left(\frac{mAB}{tmA+(1-t)B}\right)$.

($\Leftarrow$) For $x, y \in D$ $t \in [0,1]$ and $m \in (0,1]$, the result is obtained by replacing $A = \{x\}$ and $B = \{y\}$ in (7) and we obtain the desired result

$$tF(\{y\}) + m(1-t)F(\{x\}) \subseteq F\left(\frac{m\{x\}\{y\}}{tm\{x\} + (1-t)\{y\}}\right).$$

Then, F is a harmonically m-convex set-valued function. □

In the following, consider that for nonempty linear space subsets A, B, C, D and α a scalar, the following properties are true:

- $\alpha(A \times B) = \alpha A \times \alpha B$,
- $(A \times C) + (B \times D) = (A + B) \times (C + D)$,
- Si $A \subseteq B$ y $C \subseteq D$ then $A \times C \subseteq B \times D$.

Proposition 12. *Let $F_1 : D \to n(Y)$ and $F_2 : D \to n(Z)$ harmonically m-convex set-valued functions. Then the cartesian product $F_1 \times F_2$ is a harmonically m-convex set-valued function.*

Proof. Let $x, y \in D$ and $t \in [0,1]$, then:

$$\begin{aligned} t(F_1 \times F_2)(y) \quad + \quad & m(1-t)(F_1 \times F_2)(x) \\ & = [tF_1(y) \times tF_2(y)] + [m(1-t)F_1(x) \times m(1-t)F_2(x)] \\ & = (tF_1(y) + m(t-1)F_1(x)) \times (tF_2(y) + m(t-1)F_2(x)) \\ & \subseteq F_1\left(\frac{mxy}{tmx + (1-t)y}\right) \times F_2\left(\frac{mxy}{tmx + (1-t)y}\right) \\ & = (F_1 \times F_2)\left(\frac{mxy}{tmx + (1-t)y}\right). \end{aligned}$$

□

The following proposition establishes that the harmonically m-convex set-valued function are closed under the sum and the product by a scalar.

Proposition 13. *Let X, Y be two linear spaces. If D is a harmonically m-convex subset of X and $F, G : D \subset X \to n(Y)$ two harmonically m-convex set-valued functions. Then $\lambda F + G$ is a harmonically m-convex set-valued function, for all λ.*

Proof. Let $x, y \in D \subset X$, $t \in [0,1]$ and $m \in (0,1]$. Since F and G are harmonically m-convex set-valued functions, we have:

$$tF(y) + m(1-t)F(x) \subseteq F\left(\frac{mxy}{tmx + (1-t)y}\right),$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2025 doi:10.20944/preprints202504.0422.v1

and,

$$tG(y)+m(1-t)G(x)\subseteq G\left(\frac{mxy}{tmx+(1-t)y}\right).$$

Thus,

$$\begin{aligned}t(\lambda F+G)(y) \quad + \quad & m(1-t)(\lambda F+G)(x)\\ &= [t(\lambda F(y))+m(1-t)(\lambda F(x))]+[tG(y)+m(1-t)G(x)]\\ &\subseteq \lambda F\left(\frac{mxy}{tmx+(1-t)y}\right)+G\left(\frac{mxy}{tmx+(1-t)y}\right)\\ &= (\lambda F+G)\left(\frac{mxy}{tmx+(1-t)y}\right).\end{aligned}$$

□

Proposition 14. *Let x, Y be two linear spaces. If D is a harmonically m-convex subset of X and $F, G : D \subset X \to n(Y)$ two harmonically m-convex set-valued functions then $(F\cdot G)(x)$ is also harmonically m-convex set-valued function.*

Proof. First, from [18], we have that.

$$F(x_1)G(x_2)+F(x_2)G(x_1)\subset F(x_1)G(x_1)+F(x_2)G(x_2).$$

Then, for $t\in[0,1]$ and x_1, $x_2\in D$:

$$\begin{aligned}(F\cdot G)\left(\frac{mx_1x_2}{mtx_1+(1-t)x_2}\right) &= F\left(\frac{mx_1x_2}{mtx_1+(1-t)x_2}\right)\cdot G\left(\frac{mx_1x_2}{mtx_1+(1-t)x_2}\right)\\ &\supseteq [tF(x_1)+m(1-t)F(x_2)][tG(x_1)+m(1-t)G(x_2)]\\ &= t^2F(x_1)G(x_1)+mt(1-t)F(x_1)G(x_2)+mt(1-t)F(x_2)G(x_1)\\ &\quad +m^2(1-t)^2F(x_2)G(x_2)\\ &= t^2F(x_1)G(x_1)+mt(1-t)[F(x_1)G(x_2)+F(x_2)G(x_1)]\\ &\quad m^2(1-t)^2F(x_2)G(x_2)\\ &\supseteq t^2F(x_1)G(x_1)+mt(1-t)[F(x_1)G(x_1)+F(x_2)G(x_2)]\\ &\quad +m^2(1-t)^2F(x_2)G(x_2)\\ &= t[t+m(1-t)]F(x_1)G(x_1)+[m(1-t)[t+m(1-t)]F(x_2)G(x_2)]\\ &\supseteq tF(x_1)G(x_1)+m(1-t)F(x_2)G(x_2).\end{aligned}$$

This shows that the product of two harmonically m-convex set-valued functions is again harmonically m-convex set-valued function. □

The following result follows the idea of İ. İscan in [6]. To integrate set-valued functions we use the definition given by R. J. Aumann, and if a function satisfies the requirements of being integrable under this integral definition given, we say that a set-valued function F is Aumann integrable under a certain domain (see [2]).

Theorem 3. *Let X, Y linear spaces, D be a harmonically m-convex subset of X and $F : D \subset X \to n(Y)$ a harmonically m-convex set-valued Aumann integrable function, then*

$$\min\left(\inf\left(\frac{F(a)+mF\left(\frac{b}{m}\right)}{2}\right),\inf\left(\frac{mF\left(\frac{a}{m}\right)+F(b)}{2}\right)\right)\subseteq\frac{ab}{b-a}\int_a^b\frac{F(x)}{x^2}dx.$$

Proof. Let $F : D \subset X \to n(Y)$ be a harmonically m-convex set-valued function, for every $x, y \in D$ we have to

$$tF(y) + m(1-t)F\left(\frac{x}{m}\right) \subset F\left(\frac{xy}{tx + (1-t)y}\right) = F\left(\frac{m\frac{x}{m}y}{tm\frac{x}{m} + (1-t)y}\right),$$

Then, we have the following:

$$tF(b) + m(1-t)F\left(\frac{a}{m}\right) \subset F\left(\frac{ab}{ta + (1-t)b}\right), \tag{8}$$

and

$$tF(a) + m(1-t)F\left(\frac{b}{m}\right) \subset F\left(\frac{ab}{tb + (1-t)a}\right),$$

for every $t \in [0,1]$, $m \in (0,1]$ and $a, b \in D$. Integrating both sides of (8) on $[0,1]$ with respect to t, we get that

$$\int_0^1 tF(b) + m(1-t)F\left(\frac{a}{m}\right)dt \subset \int_0^1 F\left(\frac{ab}{ta + (1-t)b}\right)dt. \tag{9}$$

Integrating the left side of (9) we have:

$$\int_0^1 tF(b) + m(1-t)F\left(\frac{a}{m}\right)dt = \frac{F(b) + mF(\frac{a}{m})}{2}.$$

By the Aumann integral definition we get, that integral on the right hand of (9) is defined as:

$$\int_0^1 F\left(\frac{ab}{ta + (1-t)b}\right)dt = \left\{\int_0^1 f\left(\frac{ab}{ta + (1-t)b}\right)dt : f(x) \in F(x) \wedge t \in [0,1]\right\}.$$

But,

$$\int_0^1 f\left(\frac{ab}{ta + (1-t)b}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx,$$

then

$$\left\{\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx : f(x) \in F(x) \wedge x \in [a,b]\right\} = \frac{ab}{b-a}\int_a^b \frac{F(x)}{x^2}dx.$$

In consecuense:

$$\frac{F(b) + mF(\frac{a}{m})}{2} \subseteq \frac{ab}{b-a}\int_a^b \frac{F(x)}{x^2}dx.$$

Similarly, we have that:

$$\frac{F(a) + mF\left(\frac{b}{m}\right)}{2} \subseteq \frac{ab}{b-a}\int_a^b \frac{F(x)}{x^2}dx,$$

so the required result is obtained.

$$\min\left(\inf\left(\frac{F(a) + mF\left(\frac{b}{m}\right)}{2}\right), \inf\left(\frac{mF(\frac{a}{m}) + F(b)}{2}\right)\right) \subseteq \frac{ab}{b-a}\int_a^b \frac{F(x)}{x^2}dx.$$

□

Acknowledgments: Special thanks to Dr. Nelson Merentes for his invaluable guidance on the problem and his contributions of articles for review.

References

1. Anderson, G. D., Vamanamurthy, M. K. and Vuorinen, M. (2007). Generalized convexity and inequalities. Journal of Mathematical Analysis and Applications, **335**(2), 1294-1308.

2. Aumann, R. J. (1965). *Integrals of set-valued functions*. Journal of mathematical analysis and applications, **12**(1), 1-12.
3. Berge, C. (1963). *Topological Space: Including a treatment of Multi-Valued functions, vector spaces and convexity*. Dover Publications, INC. Mineola, New York.
4. Geletu, A. (2006). Introduction to topological spaces and set-valued maps (Lecture notes). Department of Operations Research & Stochastics Ilmenau University of Technology. August 25.
5. İscan, İ. (2014) *Hermite-Hadamard type inequalities for harmonically convex functions*. Hacettepe Journal of Mathematics and statistics , **43** (6), 935-942.
6. İscan, İ. (2016) *Hermite-Hadamard type inequalities for harmonically* (α, m)*-convex functions*. Hacettepe Journal of Mathematics and Statistics, **45** (2), 381-390.
7. Lara . T. , Merentes N., Quintero R. and Rosales E., (2019) *On m-convexity of set-valued functions*, Advance in Operator Theory, **4** (14), 767-783.
8. Leiva, H., Merentes, N., Nikodem, K., and Sánchez, J. L. (2013). Strongly convex set-valued maps. Journal of Global Optimization, 57(3), 695-705.
9. Matkowski J. and Nikodem K. (1998), Convex set-valued functions on $(0, +1)$ and their conjugate, Rocznik Nauk.-Dydakt. Prace Mat. No. 15 , 103-107.
10. Mejıa, O., Merentes, N., and Nikodem, K. (2014). Strongly concave set-valued maps. Mathematica Aeterna, **4**, 477-487.
11. Merentes N., Ribas S. (2013).*El desarrollo del concepto de función convexa*. Ediciones IVIC, Caracas-Venezuela.
12. Narváez D. X., Restrepo G. (2011).*Funciones Multivaluadas*. Facultad de Ciencias Naturales y Exactas Universidad del Valle. Revista de Ciencias, septiembre 20.
13. Nikodem K. (1987), On concave and midpoint concave set-valued functions, Glasnik Mat. Ser. III **22**(42), no. 1, 69-76.
14. Nikodem, K. (1989). A characterization of midconvex set-valued functions. Acta Universitatis Carolinae. Mathematica et Physica, 30(2), 125-129.
15. Nikodem, K., Sánchez, J. L., and Sánchez, L. (2014). Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps. Mathematica Aeterna, **4**(8), 979-987.
16. Noor, M. A., Noor, K. I., Awan, M. U., and Costache, S. (2015). Some integral inequalities for harmonically h-convex functions. Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys, 77(1), 5-16.
17. Sadowska, E. (1996). A sandwich with convexity for set-valued functions. Mathematica Pannonica, **7**(1) 163-169.
18. Santana G., González L. and Merentes N. (2019) *Funciones conjunto valuadas armónicamente convexas*, Divulgaciones Matemáticas, **19**(1), pp. 20-33.
19. Toader, G. H. (1984, October). Some generalizations of the convexity. In Proc. Colloq. Approx. Optim (Vol. 329, p. 338). Cluj-Napoca.