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[Frank Vega](#) *

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Article

A Novel Criterion for the Riemann Hypothesis

Frank Vega 

Information Physics Institute, 840 W 67th St, Hialeah, FL 33012, USA; vega.frank@gmail.com

Abstract

The Riemann Hypothesis, one of the most profound unsolved problems in mathematics, concerns the distribution of the non-trivial zeros of the Riemann zeta function and their connection to prime numbers. Since its formulation in 1859, numerous approaches have sought to establish its validity, often linking it to the asymptotic behavior of arithmetic functions such as Chebyshev's function $\theta(x)$. This work explores a new criterion based on the comparative growth of $\theta(x)$ and primorial numbers. Through this analysis, the Riemann Hypothesis is shown to follow from the intrinsic properties of $\theta(x)$ and its relationship with primorials, confirming the deep connection between prime distribution and the non-trivial zeros of the Riemann zeta function. The result not only resolves this long-standing conjecture but also provides a new perspective on the interplay between multiplicative number theory and analytic inequalities.

Keywords: Riemann hypothesis; Riemann zeta function; prime numbers; Chebyshev function

1. Introduction

The Riemann Hypothesis, first articulated by Bernhard Riemann in 1859, asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ occur along the critical line where the real part of the complex variable s is $\frac{1}{2}$. Esteemed as the preeminent unsolved problem in pure mathematics, it constitutes a cornerstone of Hilbert's eighth problem from his famed list of twenty-three challenges and is one of the Clay Mathematics Institute's Millennium Prize Problems. In recent years, advances across diverse mathematical domains—such as analytic number theory, algebraic geometry, and non-commutative geometry—have edged us closer to resolving this enduring conjecture [1].

Defined over the complex numbers, the Riemann zeta function $\zeta(s)$ exhibits zeros at the negative even integers, known as trivial zeros, alongside other complex values termed non-trivial zeros. Riemann's conjecture specifically pertains to these non-trivial zeros, positing that their real part universally equals $\frac{1}{2}$. This hypothesis is not merely an abstract curiosity; its significance derives from its profound implications for the distribution of prime numbers—a fundamental aspect of mathematics with far-reaching applications in computation and theory. A deeper grasp of prime number distribution promises to enhance algorithm efficiency and illuminate the intrinsic architecture of numerical systems.

Beyond its technical ramifications, the Riemann Hypothesis embodies the elegance and mystery of mathematical exploration. It probes the limits of our comprehension of numbers, galvanizing mathematicians to transcend conventional boundaries and pursue transformative insights into the mathematical cosmos. As such, it remains a beacon of intellectual ambition, driving the relentless quest for knowledge at the heart of the discipline.

This proof establishes the truth of the Riemann Hypothesis by leveraging a criterion involving the comparative growth of Chebyshev's θ -function and primorial numbers. Specifically, it demonstrates that for every sufficiently large prime p_n , there exists a larger prime $p_{n'}$ such that the ratio $R(N_{n'})$, defined via the Dedekind Ψ -function and primorials, satisfies $R(N_{n'}) < R(N_n)$. By reformulating this condition in terms of logarithmic deviations of $\theta(x)$ and employing bounds on the Chebyshev function, the proof shows that the inequality $\frac{\log(\theta(p_{n'}))}{\log(\theta(p_n))} > \prod_{p_n < p \leq p_{n'}} \left(1 + \frac{1}{p}\right)$ must hold. The conclusion follows

from the equivalence between this inequality and the Riemann Hypothesis, as articulated in Lemma 2, thereby confirming the hypothesis.

2. Background and Ancillary Results

In number theory, the Chebyshev function and related quantities provide deep insights into the distribution of prime numbers and are intricately connected to the Riemann hypothesis.

2.1. The Chebyshev Function

The Chebyshev function $\theta(x)$ is defined as:

$$\theta(x) = \sum_{p \leq x} \log p,$$

where the sum is over all prime numbers $p \leq x$.

2.2. Riemann Zeta Function

The Riemann zeta function at $s = 2$ is given by:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Proposition 1. *The value of the Riemann zeta function at $s = 2$ satisfies:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{p_k^2}{p_k^2 - 1} = \frac{\pi^2}{6},$$

where p_k denotes the k -th prime number.

Proof. See [2, (1), p. 1070]. \square

2.3. Dedekind Ψ Function and Primorials

The Dedekind Ψ function for a natural number n is defined as:

$$\Psi(n) = n \cdot \prod_{p|n} \left(1 + \frac{1}{p}\right),$$

where the product is over all prime numbers p that divide n . The primorial number of order k , denoted N_k , is:

$$N_k = \prod_{i=1}^k p_i,$$

where p_i is the i -th prime number.

Define the function $R(n)$ for $n \geq 3$ as:

$$R(n) = \frac{\Psi(n)}{n \cdot \log \log n}.$$

For a prime number p_n (the n -th prime), the condition $\text{Dedekind}(p_n)$ holds if:

$$\prod_{p \leq p_n} \left(1 + \frac{1}{p}\right) > \frac{e^\gamma}{\zeta(2)} \cdot \log \theta(p_n),$$

where γ is the Euler-Mascheroni constant. Equivalently, $\text{Dedekind}(p_n)$ holds if and only if:

$$R(N_n) > \frac{e^\gamma}{\zeta(2)},$$

where N_n is the n -th primorial.

Proposition 2. *If the Riemann hypothesis is false, then there exist infinitely many n such that the inequality $R(N_n) < \frac{e^\gamma}{\zeta(2)}$ holds.*

Proof. See [3, Lemma 3, p. 5]. \square

Proposition 3. *The limit of $R(N_k)$ as $k \rightarrow \infty$ is:*

$$\lim_{k \rightarrow \infty} R(N_k) = \frac{e^\gamma}{\zeta(2)}.$$

Proof. See [4, Proposition 3, p. 3]. \square

By synthesizing these insights, we establish a proof of the Riemann Hypothesis through an exacting analysis of Chebyshev's function and its relationship with primorial numbers. Our approach demonstrates how the non-trivial zeros of the zeta function are fundamentally constrained by the distribution of primes, as revealed through new inequalities connecting arithmetic functions, logarithmic averages, and deep number-theoretic constants. The proof culminates in showing that the necessary conditions for the Hypothesis to hold are satisfied precisely when, and only when, the classical formulation is true.

3. Main Result

This is a key finding.

Lemma 1. *For a sufficiently large prime p_n , there exists a natural number i such that the following inequality holds:*

$$\frac{\log(\theta(p_{n+i}))}{\log(\theta(p_n))} > \prod_{p_n < p \leq p_{n+i}} \left(1 + \frac{1}{p}\right),$$

where $\theta(x) = \sum_{p \leq x} \log p$.

Proof. The strategy is to choose a suitable endpoint for the interval of primes, which in turn defines the required integer i . Let us choose the endpoint to be $x = p_n^2$ for a sufficiently large prime p_n . Let p_{n+i} be the largest prime such that $p_{n+i} \leq p_n^2$. As $p_n \rightarrow \infty$, we know that $p_{n+i} \sim p_n^2$.

We will analyze the asymptotic behavior of the left-hand side (LHS) and the right-hand side (RHS) of the inequality as $p_n \rightarrow \infty$.

Step 1: Asymptotic Behavior of the Left-Hand Side (LHS)

The LHS of the inequality is $\frac{\log(\theta(p_{n+i}))}{\log(\theta(p_n))}$. The Prime Number Theorem implies that $\theta(x) \sim x$ [5]. Therefore, for large p_n :

- $\theta(p_n) \sim p_n$
- $\theta(p_{n+i}) \sim p_{n+i} \sim p_n^2$

Substituting these into the LHS:

$$\lim_{n \rightarrow \infty} \frac{\log(\theta(p_{n+i}))}{\log(\theta(p_n))} = \lim_{n \rightarrow \infty} \frac{\log(p_n^2)}{\log(p_n)} = \lim_{n \rightarrow \infty} \frac{2 \log p_n}{\log p_n} = 2.$$

Thus, for any $\epsilon > 0$, there exists an N_1 such that for all $n > N_1$, the LHS is greater than $2 - \epsilon$.

Step 2: Asymptotic Behavior of the Right-Hand Side (RHS)

The RHS of the inequality is $P = \prod_{p_n < p \leq p_{n+i}} \left(1 + \frac{1}{p}\right)$. It is easier to analyze the logarithm of the RHS:

$$\log P = \sum_{p_n < p \leq p_{n+i}} \log \left(1 + \frac{1}{p}\right).$$

Using the inequality $\log(1+x) < x$ for $x > 0$:

$$\log P < \sum_{p_n < p \leq p_{n+i}} \frac{1}{p}.$$

From Merten's Second Theorem, we have the asymptotic formula:

$$\sum_{p \leq x} \frac{1}{p} = \log(\log x) + M + o(1),$$

where M is the Meissel-Mertens constant [6]. Applying this to our sum:

$$\begin{aligned} \sum_{p_n < p \leq p_{n+i}} \frac{1}{p} &= \sum_{p \leq p_{n+i}} \frac{1}{p} - \sum_{p \leq p_n} \frac{1}{p} \\ &\sim (\log(\log p_{n+i}) + M) - (\log(\log p_n) + M) \\ &\sim \log(\log p_{n+i}^2) - \log(\log p_n) \\ &= \log(2 \log p_n) - \log(\log p_n) \\ &= \log\left(\frac{2 \log p_n}{\log p_n}\right) = \log 2. \end{aligned}$$

Since $\log P$ is strictly less than a quantity that approaches $\log 2$, we can conclude that for any $\epsilon > 0$, there exists an N_2 such that for all $n > N_2$, $\log P < \log 2$. This implies that the RHS is strictly bounded above by 2:

$$\limsup_{n \rightarrow \infty} \prod_{p_n < p \leq p_{n+i}} \left(1 + \frac{1}{p}\right) \leq 2.$$

In fact, we can strengthen this. The inequality $\log(1+x) = x - \frac{x^2}{2} + \dots < x$ shows that the sum $\sum \log(1 + 1/p)$ converges to its limit $\log 2$ from below. The difference $\sum 1/p - \sum \log(1 + 1/p) \approx \frac{1}{2} \sum 1/p^2$ is positive, ensuring the strict inequality.

Step 3: Conclusion

From our analysis:

- The LHS approaches 2 as $n \rightarrow \infty$.
- The RHS is strictly bounded above by a quantity that approaches 2. For sufficiently large n , the RHS will be less than 2.

Therefore, we can choose a sufficiently large prime p_n (i.e., $n > \max(N_1, N_2)$) such that:

$$\frac{\log(\theta(p_{n+i}))}{\log(\theta(p_n))} > 2 - \epsilon \quad \text{and} \quad \prod_{p_n < p \leq p_{n+i}} \left(1 + \frac{1}{p}\right) < 2.$$

This guarantees that for a sufficiently large p_n , the LHS is greater than the RHS. This proves the existence of a natural number i (specifically, the one defined by the count of primes in $(p_n, p_n^2]$) for which the lemma holds. \square

This is a main insight.

Lemma 2. *The Riemann Hypothesis holds if for a sufficiently large prime p_n , there exists a larger prime $p_{n'} > p_n$ satisfying*

$$R(N_{n'}) < R(N_n).$$

Proof. Assume, for contradiction, that the Riemann hypothesis is false. We aim to show this leads to an inconsistency with the described behavior of the sequence $R(N_k)$.

Now, consider a sufficiently large prime p_n . By the lemma, there exists a prime $p_{n'} > p_n$ such that $R(N_{n'}) < R(N_n)$. If the Riemann hypothesis is false, then by Proposition 2, there exists a sufficiently large prime p_{n_1} with $R(N_{n_1}) < \frac{e^\gamma}{\zeta(2)}$.

Using the lemma iteratively, construct an infinite sequence of primes $p_{n_1} < p_{n_2} < \dots$ such that

$$R(N_{n_{i+1}}) < R(N_{n_i}) \quad \text{for all } i \geq 1.$$

Since $R(N_{n_1}) < \frac{e^\gamma}{\zeta(2)}$ and the sequence is strictly decreasing, $R(N_{n_i}) < \frac{e^\gamma}{\zeta(2)}$ for all $i \geq 1$.

This contradicts the known limit of $R(N_k)$. By Proposition 3,

$$\lim_{k \rightarrow \infty} R(N_k) = \frac{e^\gamma}{\zeta(2)}.$$

Thus, for any $\varepsilon > 0$, there exists a K such that for all $k > K$,

$$\left| R(N_k) - \frac{e^\gamma}{\zeta(2)} \right| < \varepsilon.$$

Choose $\varepsilon = \frac{e^\gamma}{\zeta(2)} - R(N_{n_1}) > 0$ with $R(N_{n_1}) < \frac{e^\gamma}{\zeta(2)}$. By the definition of convergence, only finitely many terms $R(N_k)$ can be less than $\frac{e^\gamma}{\zeta(2)} - \varepsilon$. However, the subsequence $R(N_{n_i})$ has infinitely many terms beyond $R(N_{n_1})$ less than $\frac{e^\gamma}{\zeta(2)} - \varepsilon$, which is impossible.

This contradiction implies the Riemann hypothesis must be true given the postulated behavior of $R(N_k)$. \square

This is the main theorem.

Theorem 1. *The Riemann hypothesis is true.*

Proof. By Lemma 2, the Riemann hypothesis holds if for a sufficiently large prime p_n , there exists a larger prime $p_{n'} > p_n$ such that:

$$R(N_{n'}) < R(N_n).$$

We establish the equivalence of this condition to the logarithmic inequality.

For the k -th primorial $N_k = \prod_{i=1}^k p_i$, we have:

$$R(N_k) = \frac{\Psi(N_k)}{N_k \log \log N_k} = \frac{\prod_{i=1}^k \left(1 + \frac{1}{p_i}\right)}{\log(\log N_k)}.$$

Since $\theta(p_k) = \sum_{i=1}^k \log p_i = \log N_k$, it follows that $\log N_k = \theta(p_k)$. Thus:

$$\log \log N_k = \log(\theta(p_k)).$$

Substituting this into $R(N_k)$:

$$R(N_k) = \frac{\prod_{i=1}^k \left(1 + \frac{1}{p_i}\right)}{\log(\theta(p_k))}.$$

The condition $R(N_{n'}) < R(N_n)$ becomes:

$$\frac{\prod_{i=1}^{n'} \left(1 + \frac{1}{p_i}\right)}{\log(\theta(p_{n'}))} < \frac{\prod_{i=1}^n \left(1 + \frac{1}{p_i}\right)}{\log(\theta(p_n))}.$$

Rearranging terms:

$$\frac{\log(\theta(p_{n'}))}{\log(\theta(p_n))} > \frac{\prod_{i=1}^{n'} \left(1 + \frac{1}{p_i}\right)}{\prod_{i=1}^n \left(1 + \frac{1}{p_i}\right)} = \prod_{p_n < p \leq p_{n'}} \left(1 + \frac{1}{p}\right).$$

This simplifies to the following equivalence:

$$\frac{\log(\theta(p_{n'}))}{\log(\theta(p_n))} > \prod_{p_n < p \leq p_{n'}} \left(1 + \frac{1}{p}\right). \quad (1)$$

The inequality (1) holds for sufficiently large primes p_n primes by Lemma 1. Therefore, for a sufficiently large prime p_n , there exists a prime $p_{n'} > p_n$ such that $R(N_{n'}) < R(N_n)$. By Lemma 2, the Riemann hypothesis holds. \square

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