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Anisotropic Four-Dimensional Spaces of Real Numbers

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Abstract: The article constructs all anisotropic spaces of four-dimensional numbers in which commutative and associative operations of addition and multiplication are defined. In this case, so-called "zero divisors" appear in these spaces. The structures of zero divisors in each space are described and their properties are investigated. It is shown that there are two types of zero divisors and they form a two-dimensional subspace of the four-dimensional space. A space of 4x4 matrices is constructed that is isomorphic to the space of four-dimensional numbers. The concept of the spectrum of a four-dimensional number is introduced and a bijective mapping between four-dimensional numbers and their spectra is constructed. Thanks to this, methods for solving linear and quadratic equations in four-dimensional spaces are developed. It is proved that a quadratic equation in a four-dimensional space generally has four roots. The concept of the spectral norm is introduced in the space of four-dimensional numbers and the equivalence of the spectral norm to the Euclidean norm is proved.

Keywords: four-dimensional mathematics; four-dimensional numbers; quaternions; fields; Frobenius theorem; zero divisors; associative and commutative operations; abstract algebra

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1. Introduction

Attempts of generalization of the concept of number were made in mathematics repeatedly. The very first, appeared successful, the concept of the imaginary which can be provided by ordered couple of real numbers is, the set of such numbers forms the field, as well as the set of real numbers. Imaginaries were extremely useful both in theoretical, and in applied application: many problems seeming unapproachable received transparent treatment in terms of the complex analysis and were solved in the general view. For example, the main theorem of algebra of availability at least of one root at the polynomial, other than the constant, with complex coefficients in the field of imaginaries can be the example. The complex analysis plays the important role in plane problems of mathematics and mechanics.

The following generalization of the concept of number is the hyper complex four-dimensional numbers offered by William Hamilton --- quaternions [1] which found important applications in physics. However it should be noted, quaternions owing to lack of commutativity of multiplication do not form the field, and, for this reason their application is limited. Because of not commutativity of multiplication it was not succeeded to construct the full-fledged four-dimensional calculus which would generalize one-dimensional and two-dimensional analogs.

The known theorem of Frobenius [2] claims that it is impossible to expand further the concept of number, without having offered some arithmetic property. At the beginning of the 21st century the Kazakhstan mathematician M. M. Abenov developed other than quaternions four-dimensional mathematics [3] in which multiplication of numbers is associative and commutative, but at the same time it is necessary to deal with zero divisors. Abenov managed to construct the harmonious theory which he called four-dimensional mathematics, and showed some of its applications to the solution

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of problems of hydrodynamics. Further in work [4] he together with M. B. Gabbasov received other four-dimensional spaces of numbers. In work [5] of M. B. Gabbasov with coauthors analytical solutions of the initial value problem for mathematical model of the theory of filtration in the three-dimensional non-stationary case are received. This article serves as the purpose of synthesis of this theory in the so-called anisotropic case when different measurements have different scales, at the same time the commutativity and associativity of multiplication remain. It is important to note that the offered vector spaces of four-dimensional numbers are natural generalization of spaces of one-dimensional and two-dimensional (complex) numbers.

1. *Various anisotropic spaces of four-dimensional numbers*

Let's consider space of four-dimensional numbers $x = (x_1, x_2, x_3, x_4)$where $x_i \in R, i = 1,2,3,4$.

Two numbers $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ are considered as equal if $x_i = y_i, i = 1,2,3,4$.

Let's enter addition and subtraction operations as coordinate-wise addition and subtraction which are associative and commutative.

Let's enter multiplication operation so that it was associative and commutative. Let's given four real numbers $\alpha, \beta, \gamma, \delta$, such are set $\alpha, \beta, \gamma, \delta$ that $\alpha \cdot \beta \cdot \gamma \cdot \delta \neq 0$. We will call the anisotropic product of numbers $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ a number $z = x \cdot y = (z_1, z_2, z_3, z_4)$ where

$$\begin{aligned} z_1 &= \alpha x_1 y_1 + \frac{\gamma\delta}{\alpha} x_2 y_2 + \frac{\beta\delta}{\alpha} x_3 y_3 + \frac{\beta\gamma}{\alpha} x_4 y_4 \\ z_2 &= \alpha x_2 y_1 + \alpha x_1 y_2 + \beta x_4 y_3 + \beta x_3 y_4 \\ z_3 &= \alpha x_3 y_1 + \gamma x_4 y_2 + \alpha x_1 y_3 + \gamma x_2 y_4 \\ z_4 &= \alpha x_4 y_1 + \delta x_3 y_2 + \delta x_2 y_3 + \alpha x_1 y_4 \end{aligned} \quad . \qquad (1)$$

Further instead of the anisotropic product we will often use just "product" or $(\alpha, \beta, \gamma, \delta)$-product.

Theorem 1. The entered operation of multiplication meets the following conditions:

1) $x \cdot y = y \cdot x$ (commutativity of multiplication);
2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (associativity of multiplication);
3) $(x + y) \cdot z = x \cdot z + y \cdot z$ (associativity of multiplication concerning addition),

for any $x, y, z \in R^4$.

Proof. It is carried out by direct check.

We will call $(x_1, 0,0,0)$ the real number.

Follows from ratios (1) that at multiplication of real number a by four-dimensional number there is the coordinate-wise multiplication on $\alpha \cdot a$, that is $(a, 0,0,0) \cdot (x_1, x_2, x_3, x_4) = (\alpha a x_1, \alpha a x_2, \alpha a x_3, \alpha a x_4)$.

The following four numbers are called basic numbers: $J_1 = (\frac{1}{\alpha}, 0,0,0)$, $J_2 = (0, \frac{1}{\alpha}, 0,0)$, $J_3 = (0,0, \frac{1}{\alpha}, 0)$, $J_4 = (0,0,0, \frac{1}{\alpha})$.

Let's construct the multiplication table of basic numbers (table 1):

Table 1. Products of basic numbers for general case.

	J_1	J_2	J_3	J_4
J_1	J_1	J_2	J_3	J_4
J_2	J_2	$\frac{\gamma\delta}{\alpha^3} J_1$	$\frac{\delta}{\alpha^2} J_4$	$\frac{\gamma}{\alpha^2} J_3$
J_3	J_3	$\frac{\delta}{\alpha^2} J_4$	$\frac{\beta\delta}{\alpha^3} J_1$	$\frac{\beta}{\alpha^2} J_2$
J_4	J_4	$\frac{\gamma}{\alpha^2} J_3$	$\frac{\beta}{\alpha^2} J_2$	$\frac{\beta\gamma}{\alpha^3} J_1$

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Then any four-dimensional number $x = (x_1, x_2, x_3, x_4)$ can be represented as an expantion by basic numbers $x = x_1 \cdot J_1 + x_2 \cdot J_2 + x_3 \cdot J_3 + x_4 \cdot J_4$.

The following four numbers are called unit numbers: $I_1 = (1,0,0,0)$, $I_2 = (0,1,0,0)$, $I_3 = (0,0,1,0)$, $I_4 = (0,0,0,1)$. For any number $x = (x_1, x_2, x_3, x_4)$ we have $x \cdot I_1 = (\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4)$.

Let $x = (x_1, x_2, x_3, x_4)$ be a four-dimensional number. Let's consider together with it the following numbers:

$$x^{(2)} = (x_1, x_2, -x_3, -x_4) = x_1 J_1 + x_2 J_2 - x_3 J_3 - x_4 J_4,$$

$$x^{(3)} = (x_1, -x_2, x_3, -x_4) = x_1 J_1 - x_2 J_2 + x_3 J_3 - x_4 J_4,$$

$$x^{(4)} = (x_1, -x_2, -x_3, x_4) = x_1 J_1 - x_2 J_2 - x_3 J_3 + x_4 J_4.$$

Let's calculate the product $x \cdot x^{(2)} \cdot x^{(3)} \cdot x^{(4)}$:

$$x \cdot x^{(2)} = \left(\alpha x_1^2 + \frac{\gamma\delta}{\alpha} x_2^2 - \frac{\beta\delta}{\alpha} x_3^2 - \frac{\beta\gamma}{\alpha} x_4^2,\ 2\alpha x_1 x_2 - 2\beta x_3 x_4,\ 0,\ 0\right),$$

$$x \cdot x^{(2)} \cdot x^{(3)} = \Big(\alpha^2 x_1^3 - \gamma\delta x_1 x_2^2 - \beta\delta x_1 x_3^2 - \beta\gamma x_1 x_4^2 + 2\frac{\beta\gamma\delta}{\alpha} x_2 x_3 x_4,\ \alpha^2 x_1^2 x_2 - \gamma\delta x_2^3 + \beta\delta x_2 x_3^2 +$$

$$\beta\gamma x_2 x_4^2 - 2\alpha\beta x_1 x_3 x_4,\ \alpha^2 x_1^2 x_3 + \gamma\delta x_2^2 x_3 - \beta\delta x_3^3 + \beta\gamma x_3 x_4^2 - 2\alpha\gamma x_1 x_2 x_4,\ 2\alpha\delta x_1 x_2 x_3 - \beta\delta x_3^2 x_4 -$$

$$\alpha^2 x_1^2 x_4 - \gamma\delta x_2^2 x_4 + \beta\gamma x_4^3\Big),$$

$$x \cdot x^{(2)} \cdot x^{(3)} \cdot x^{(4)} = \Big(\alpha^3 x_1^4 + \frac{\gamma^2\delta^2}{\alpha} x_2^4 + \frac{\beta^2\delta^2}{\alpha} x_3^4 + \frac{\beta^2\gamma^2}{\alpha} x_4^4 - 2\alpha\gamma\delta x_1^2 x_2^2 - 2\alpha\beta\delta x_1^2 x_3^2 - 2\alpha\beta\gamma x_1^2 x_4^2 -$$

$$2\frac{\beta\gamma\delta^2}{\alpha} x_2^2 x_3^2 - 2\frac{\beta\gamma^2\delta}{\alpha} x_2^2 x_4^2 - 2\frac{\beta^2\gamma\delta}{\alpha} x_3^2 x_4^2 + 8\beta\gamma\delta x_1 x_2 x_3 x_4,\ 0,\ 0,\ 0\Big).$$

Thus, $x \cdot x^{(2)} \cdot x^{(3)} \cdot x^{(4)}$ is the real number.

Definition. The symplectic module of four-dimensional number $x = (x_1, x_2, x_3, x_4)$ is called the real number

$$|x| = \big(\alpha^4 x_1^4 + \gamma^2\delta^2 x_2^4 + \beta^2\delta^2 x_3^4 + \beta^2\gamma^2 x_4^4 - 2\alpha^2\gamma\delta x_1^2 x_2^2 - 2\alpha^2\beta\delta x_1^2 x_3^2 - 2\alpha^2\beta\gamma x_1^2 x_4^2 - 2\beta\gamma\delta^2 x_2^2 x_3^2 -$$

$$2\beta\gamma^2\delta x_2^2 x_4^2 - 2\beta^2\gamma\delta x_3^2 x_4^2 + 8\alpha\beta\gamma\delta x_1 x_2 x_3 x_4\big)^{\frac{1}{4}}. \quad (2)$$

At the same time we assume that numbers $\alpha, \beta, \gamma, \delta$ are set so that expression under the radical is non-negative. As we will see below, such cases are possible, that is there are such values $\alpha, \beta, \gamma, \delta$ that the four-degree form in the right part of (2) is positively defined for any$x_i \in R, i = 1,2,3,4$.

Definition. The number $x^* = x^{(2)} \cdot x^{(3)} \cdot x^{(4)}$is called the conjugate number to number x.

Then $x \cdot x^* \cdot I_1 = |x|^4 \cdot J_1$. (3)

Direct calculation gives the conjugate number $x^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ to four-dimensional number $x = (x_1, x_2, x_3, x_4)$:

$$x_1^* = x_1(\alpha^2 x_1^2 - \gamma\delta x_2^2 - \beta\delta x_3^2 - \beta\gamma x_4^2) + \frac{2\beta\gamma\delta}{\alpha} x_2 x_3 x_4,$$

$$x_2^* = x_2(-\alpha^2 x_1^2 + \gamma\delta x_2^2 - \beta\delta x_3^2 - \beta\gamma x_4^2) + 2\alpha\beta x_1 x_3 x_4,$$

$$x_3^* = x_3(-\alpha^2 x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 - \beta\gamma x_4^2) + 2\alpha\gamma x_1 x_2 x_4,$$

$$x_4^* = x_4(-\alpha^2 x_1^2 - \gamma\delta x_2^2 - \beta\delta x_3^2 + \beta\gamma x_4^2) + 2\alpha\delta x_1 x_2 x_3.$$

Respectively, the conjugate numbers to basic numbers are:

$$J_1^* = J_1,\ J_2^* = \frac{\gamma\delta}{\alpha^3} \cdot J_2,\ J_3^* = \frac{\beta\delta}{\alpha^3} \cdot J_3,\ J_4^* = \frac{\beta\gamma}{\alpha^3} \cdot J_4.$$

The conjugate numbers to unit numbers are:

$$I_1^* = \alpha^2 I_1,\ I_2^* = \frac{\gamma\delta}{\alpha} \cdot I_2,\ I_3^* = \frac{\beta\delta}{\alpha} \cdot I_3,\ I_4^* = \frac{\beta\gamma}{\alpha} \cdot I_4.$$

Let $x = (x_1, x_2, x_3, x_4)$ be a four-dimensional number with non-zero symplectic module. Then there exist unique x^{-1} called the inverse to x, so that$x \cdot x^{-1} = x^{-1} \cdot x = J_1$. Multiplying both parts of the last equality by $x^* \cdot I_1$ we will get $x^{-1} \cdot x \cdot x^* \cdot I_1 = x^* \cdot I_1$ or $|x|^4 x^{-1} = x^* \cdot I_1$. Let's multiply both parts of this equality by number $\left(\frac{1}{\alpha \cdot |x|^4}, 0,0,0\right)$, then, taking into account that $\left(\frac{1}{\alpha \cdot |x|^4}, 0,0,0\right) \cdot |x|^4 = \left(\frac{1}{\alpha \cdot |x|^4}, 0,0,0\right) \cdot (|x|^4, 0,0,0) = I_1$, we will obtain $x^{-1} \cdot I_1 = x^* \cdot \left(\frac{1}{\alpha \cdot |x|^4}, 0,0,0\right) \cdot I_1$. Reducing both parts of equality by I_1 we have

$$x^{-1} = \frac{1}{\alpha |x|^4} \cdot x^* = \left(\frac{x_1^*}{|x|^4}, \frac{x_2^*}{|x|^4}, \frac{x_3^*}{|x|^4}, \frac{x_4^*}{|x|^4}\right) \quad (4)$$

Then we will define division operation of four-dimensional numbers as $\frac{y}{x} = y \cdot x^{-1} = \frac{1}{\alpha |x|^4} x^* y$ if $|x| \neq 0$.

We investigate for what values of constants $\alpha, \beta, \gamma, \delta$, the symplectic module (2) is non-negative for any $x \in R^4$. For this purpose we will consider various cases of signs of these constants.

1. Case $\beta > 0,\ \gamma > 0,\ \delta > 0$.

This case coincides with the general case considered above. In this case the symplectic module of four-dimensional number (2) can be represented in the following form:

$$|x|^4 = \alpha^4 x_1^4 + \gamma^2\delta^2 x_2^4 + \beta^2\delta^2 x_3^4 + \beta^2\gamma^2 x_4^4 - 2\alpha^2\gamma\delta x_1^2 x_2^2 - 2\alpha^2\beta\delta x_1^2 x_3^2 - 2\alpha^2\beta\gamma x_1^2 x_4^2 - 2\beta\gamma\delta^2 x_2^2 x_3^2 - 2\beta\gamma^2\delta x_2^2 x_4^2 - 2\beta^2\gamma\delta x_3^2 x_4^2 + 8\alpha\beta\gamma\delta x_1 x_2 x_3 x_4 = \left[\left(\alpha x_1 - \sqrt{\beta\delta} x_3\right)^2 - \left(\sqrt{\gamma\delta} x_2 - \sqrt{\beta\gamma} x_4\right)^2\right]\left[\left(\alpha x_1 + \sqrt{\beta\delta} x_3\right)^2 - \left(\sqrt{\gamma\delta} x_2 + \sqrt{\beta\gamma} x_4\right)^2\right],$$

or in the following forms

$$|x|^4 = \left[\left(\alpha x_1 - \sqrt{\gamma\delta} x_2\right)^2 - \left(\sqrt{\beta\delta} x_3 - \sqrt{\beta\gamma} x_4\right)^2\right]\left[\left(\alpha x_1 + \sqrt{\gamma\delta} x_2\right)^2 - \left(\sqrt{\beta\delta} x_3 + \sqrt{\beta\gamma} x_4\right)^2\right],$$

and

$$|x|^4 = \left[\left(\alpha x_1 - \sqrt{\beta\gamma} x_4\right)^2 - \left(\sqrt{\gamma\delta} x_2 - \sqrt{\beta\delta} x_3\right)^2\right]\left[\left(\alpha x_1 + \sqrt{\beta\gamma} x_4\right)^2 - \left(\sqrt{\gamma\delta} x_2 + \sqrt{\beta\delta} x_3\right)^2\right].$$

As seen from these formulas, the symplectic module of number in this case is not positively defined, so that is not the module. If the module of number is not well-defined, then further constructions do not make sense.

2. Case $\beta < 0,\ \gamma > 0,\ \delta > 0$.

In this case we will replace in definition of multiplication of numbers (1) β with $-\beta$ and we will consider $\beta > 0$ in further. Then multiplication of two four-dimensional numbers is defined by formulas

$$\begin{aligned} z_1 &= \alpha x_1 y_1 + \frac{\gamma\delta}{\alpha} x_2 y_2 - \frac{\beta\delta}{\alpha} x_3 y_3 - \frac{\beta\gamma}{\alpha} x_4 y_4 \\ z_2 &= \alpha x_2 y_1 + \alpha x_1 y_2 - \beta x_4 y_3 - \beta x_3 y_4 \\ z_3 &= \alpha x_3 y_1 + \gamma x_4 y_2 + \alpha x_1 y_3 + \gamma x_2 y_4 \\ z_4 &= \alpha x_4 y_1 + \delta x_3 y_2 + \delta x_2 y_3 + \alpha x_1 y_4 \end{aligned} \quad . \quad (5)$$

Then multiplication table of basic numbers takes the following form (table 2):

Table 2. Products of basic numbers in case of 2.

	J_1	J_2	J_3	J_4
J_1	J_1	J_2	J_3	J_4

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J_2	J_2	$\frac{\gamma\delta}{\alpha^3}J_1$	$\frac{\delta}{\alpha^2}J_4$	$\frac{\gamma}{\alpha^2}J_3$
J_3	J_3	$\frac{\delta}{\alpha^2}J_4$	$-\frac{\beta\delta}{\alpha^3}J_1$	$-\frac{\beta}{\alpha^2}J_2$
J_4	J_4	$\frac{\gamma}{\alpha^2}J_3$	$-\frac{\beta}{\alpha^2}J_2$	$-\frac{\beta\gamma}{\alpha^3}J_1$

In this case the module of four-dimensional number (2) can be transformed as follows:

$$|x|^4 = \alpha^4x_1^4 + \gamma^2\delta^2x_2^4 + \beta^2\delta^2x_3^4 + \beta^2\gamma^2x_4^4 - 2\alpha^2\gamma\delta x_1^2x_2^2 + 2\alpha^2\beta\delta x_1^2x_3^2 + 2\alpha^2\beta\gamma x_1^2x_4^2 + 2\beta\gamma\delta^2x_2^2x_3^2 + 2\beta\gamma^2\delta x_2^2x_4^2 - 2\beta^2\gamma\delta x_3^2x_4^2 - 8\alpha\beta\gamma\delta x_1x_2x_3x_4 = \left[\left(\alpha x_1 - \sqrt{\gamma\delta}x_2\right)^2 + \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)^2\right]\cdot \left[\left(\alpha x_1 + \sqrt{\gamma\delta}x_2\right)^2 + \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)^2\right]. \quad (6)$$

As seen from table 2 the basic numbers J_1 and J_2 are real, and basic numbers J_3 and J_4 are imaginary. At the same time the module of number $(x_1, 0, x_3, 0)$ is defined by the formula $|x| = \sqrt{\alpha^2x_1^2 + \beta\delta x_3^2}$, that is the number $(x_1, 0, x_3, 0)$ can be accepted as the imaginary.

The conjugate number $x^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ to the number $x = (x_1, x_2, x_3, x_4)$ is defined by the formula:

$$x_1^* = x_1(\alpha^2x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2) - \frac{2\beta\gamma\delta}{\alpha}x_2x_3x_4,$$
$$x_2^* = x_2(-\alpha^2x_1^2 + \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2) - 2\alpha\beta x_1x_3x_4,$$
$$x_3^* = x_3(-\alpha^2x_1^2 - \gamma\delta x_2^2 - \beta\delta x_3^2 + \beta\gamma x_4^2) + 2\alpha\gamma x_1x_2x_4,$$
$$x_4^* = x_4(-\alpha^2x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 - \beta\gamma x_4^2) + 2\alpha\delta x_1x_2x_3.$$

Respectively, the conjugate numbers to basic numbers have the form:

$$J_1^* = J_1,\ J_2^* = \frac{\gamma\delta}{\alpha^3}\cdot J_2,\ J_3^* = -\frac{\beta\delta}{\alpha^3}\cdot J_3,\ J_4^* = -\frac{\beta\gamma}{\alpha^3}\cdot J_4.$$

The conjugate numbers to unit numbers have the form:

$$I_1^* = \alpha^2 I_1,\ I_2^* = \frac{\gamma\delta}{\alpha}\cdot I_2,\ I_3^* = -\frac{\beta\delta}{\alpha}\cdot I_3,\ I_4^* = -\frac{\beta\gamma}{\alpha}\cdot I_4.$$

The conjugate number to the imaginary number $(x_1, 0, x_3, 0)$ has the form $\frac{|x|^2}{\alpha}(x_1, 0, -x_3, 0)$, that easily follows from the last formulas.

Let's denote the received space of four-dimensional numbers by $M_2(\alpha, \beta, \gamma, \delta)$ where the index 2 stands for the number of the considered case and we will call it anisotropic space of four-dimensional numbers.

Let's associate with each four-dimensional number $x = (x_1, x_2, x_3, x_4)$ some matrix $F(x)$ of the following form:

$$F(x) = \begin{pmatrix} \alpha x_1 & \frac{\gamma\delta}{\alpha}x_2 & -\frac{\beta\delta}{\alpha}x_3 & -\frac{\beta\gamma}{\alpha}x_4 \\ \alpha x_2 & \alpha x_1 & -\beta x_4 & -\beta x_3 \\ \alpha x_3 & \gamma x_4 & \alpha x_1 & \gamma x_2 \\ \alpha x_4 & \delta x_3 & \delta x_2 & \alpha x_1 \end{pmatrix}. \quad (7)$$

The mapping $F: x \to F(x)$ is one-to-one and surjection. Indeed, for two different numbers x and y there correspond different matrices and for any matrix of the specified form it is possible to find the corresponding four-dimensional number.

Then multiplication of two four-dimensional numbers $x = (x_1, x_2, x_3, x_4)$and $y = (y_1, y_2, y_3, y_4)$ in the space $M_2(\alpha, \beta, \gamma, \delta)$ can be represented in the form $x \cdot y = F(x) \cdot y$ where the multiplication sign in the left part is understood in the sense of (5), and multiplication sign in the right part is treated as multiplication of the matrix by the vector.

Thus, we defined alternative definition of multiplication of four-dimensional numbers by the matrix (7).

The inverse number to four-dimensional number $x = (x_1, x_2, x_3, x_4)$ in the sense of multiplication is defined by the formula (4), if $|x| \neq 0$. Then we will define division operation of four-dimensional numbers as $\frac{y}{x} = y \cdot x^{-1} = \frac{1}{\alpha|x|^4}x^*y$, if $|x| \neq 0$.

We will call space $M_2(1,1,1,1)$ isotropic space M_2. In isotropic space the multiplication of two numbers is defined by equalities

$$\begin{aligned} z_1 &= x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4 \\ z_2 &= x_2y_1 + x_1y_2 - x_4y_3 - x_3y_4 \\ z_3 &= x_3y_1 + x_4y_2 + x_1\quad_3 + x_2y_4' \\ z_4 &= x_4y_1 + x_3y_2 + x_2y_3 + \alpha x_1y_4 \end{aligned} \qquad (8)$$

and the matrix $F(x)$ has the form

$$F(x) = \begin{pmatrix} x_1 & x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}. \qquad (9)$$

The symplectic module of four-dimensional number in isotropic space M_2 has the form

$$|x| = \sqrt[4]{[(x_1 - x_2)^2 + (x_3 - x_4)^2]\cdot[(x_1 + x_2)^2 + (x_3 + x_4)^2]}. \qquad (10)$$

3. Case $\beta > 0,\ \gamma < 0,\ \delta > 0$.

In this case we will replace in definition of multiplication of numbers (1) γ with $-\gamma$ and we will consider$\gamma > 0$ in further. Then multiplication of two four-dimensional numbers is defined as follows

$$\begin{aligned} z_1 &= \alpha x_1y_1 - \frac{\gamma\delta}{\alpha}x_2y_2 + \frac{\beta\delta}{\alpha}x_3y_3 - \frac{\beta\gamma}{\alpha}x_4y_4 \\ z_2 &= \alpha x_2y_1 + \alpha x_1y_2 + \beta x_4y_3 + \beta x_3y_4 \\ z_3 &= \alpha x_3y_1 - \gamma x_4y_2 + \alpha x_1y_3 - \gamma x_2y_4 \\ z_4 &= \alpha x_4y_1 + \delta x_3y_2 + \delta x_2y_3 + \alpha x_1y_4 \end{aligned}. \qquad (11)$$

The multiplication table of basic numbers it is provided in table 3.

Table 3. Products of basic numbers in case of 3.

	J_1	J_2	J_3	J_4
J_1	J_1	J_2	J_3	J_4
J_2	J_2	$-\frac{\gamma\delta}{\alpha^3}J_1$	$\frac{\delta}{\alpha^2}J_4$	$-\frac{\gamma}{\alpha^2}J_3$
J_3	J_3	$\frac{\delta}{\alpha^2}J_4$	$\frac{\beta\delta}{\alpha^3}J_1$	$\frac{\beta}{\alpha^2}J_2$
J_4	J_4	$-\frac{\gamma}{\alpha^2}J_3$	$\frac{\beta}{\alpha^2}J_2$	$-\frac{\beta\gamma}{\alpha^3}J_1$

In this case the module of four-dimensional number (2) can be transformed as follows:

$$|x|^4 = \alpha^4x_1^4 + \gamma^2\delta^2x_2^4 + \beta^2\delta^2x_3^4 + \beta^2\gamma^2x_4^4 + 2\alpha^2\gamma\delta x_1^2x_2^2 - 2\alpha^2\beta\delta x_1^2x_3^2 + 2\alpha^2\beta\gamma x_1^2x_4^2 + 2\beta\gamma\delta^2x_2^2x_3^2 - 2\beta\gamma^2\delta x_2^2x_4^2 + 2\beta^2\gamma\delta x_3^2x_4^2 - 8\alpha\beta\gamma\delta x_1x_2x_3x_4 = \left[\left(\alpha x_1 - \sqrt{\beta\delta}x_3\right)^2 + \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\gamma}x_4\right)^2\right]\cdot\left[\left(\alpha x_1 + \sqrt{\beta\delta}x_3\right)^2 + \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\gamma}x_4\right)^2\right]. \qquad (12)$$

As seen from table 3 the basic numbers J_1 and J_3 are real, and basic numbers J_2 and J_4 are imaginary. Then in this case as the imaginary we will accept number $(x_1, x_2, 0,0)$. Respectively, the module of the imaginary is defined by the formula $|x| = \sqrt{\alpha^2x_1^2 + \gamma\delta x_2^2}$.

The conjugate number $x^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ to number $x = (x_1, x_2, x_3, x_4)$ is defined by the formula:

$$\begin{aligned} x_1^* &= x_1(\alpha^2x_1^2 + \gamma\delta x_2^2 - \beta\delta x_3^2 + \beta\gamma x_4^2) - \frac{2\beta\gamma\delta}{\alpha}x_2x_3x_4, \\ x_2^* &= x_2(-\alpha^2x_1^2 - \gamma\delta x_2^2 - \beta\delta x_3^2 + \beta\gamma x_4^2) + 2\alpha\beta x_1x_3x_4, \\ x_3^* &= x_3(-\alpha^2x_1^2 + \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2) - 2\alpha\gamma x_1x_2x_4, \\ x_4^* &= x_4(-\alpha^2x_1^2 + \gamma\delta x_2^2 - \beta\delta x_3^2 - \beta\gamma x_4^2) + 2\alpha\delta x_1x_2x_3. \end{aligned}$$

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Respectively, the conjugate numbers to basic numbers have the form:
$J_1^* = J_1,\ J_2^* = -\frac{\gamma\delta}{\alpha^3}\cdot J_2,\ J_3^* = \frac{\beta\delta}{\alpha^3}\cdot J_3,\ J_4^* = -\frac{\beta\gamma}{\alpha^3}\cdot J_4.$
The conjugate numbers to unit numbers have the form:
$I_1^* = \alpha^2 I_1,\ I_2^* = -\frac{\gamma\delta}{\alpha}\cdot I_2,\ I_3^* = \frac{\beta\delta}{\alpha}\cdot I_3,\ I_4^* = -\frac{\beta\gamma}{\alpha}\cdot I_4.$
The conjugate number to the imaginary number $x = (x_1, x_2, 0{,}0)$ has the form $\frac{|x|^2}{\alpha}(x_1, -x_2, 0{,}0)$.

Let's denote the received anisotropic space of four-dimensional numbers, similarly to the previous case, by $M_3(\alpha, \beta, \gamma, \delta)$.

The matrix $F(x)$ from (7) in this case looks as follows:

$$F(x) = \begin{pmatrix} \alpha x_1 & -\frac{\gamma\delta}{\alpha}x_2 & \frac{\beta\delta}{\alpha}x_3 & -\frac{\beta\gamma}{\alpha}x_4 \\ \alpha x_2 & \alpha x_1 & \beta x_4 & \beta x_3 \\ \alpha x_3 & -\gamma x_4 & \alpha x_1 & -\gamma x_2 \\ \alpha x_4 & \delta x_3 & \delta x_2 & \alpha x_1 \end{pmatrix}. \quad (13)$$

The mapping $F: x \to F(x)$ is one-to-one and onto, and multiplication of two four-dimensional numbers $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in space $M_3(\alpha, \beta, \gamma, \delta)$ can be represented in the form $x \cdot y = F(x) \cdot y$. Division operation is defined similarly to the Case 2.

We will call space $M_3(1{,}1{,}1{,}1)$ isotropic spaceM_3. In isotropic space the multiplication of two numbers is defined by equalities

$$\begin{aligned} z_1 &= x_1y_1 - x_2y_2 + x_3y_3 - x_4y_4 \\ z_2 &= x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 &= x_3y_1 - x_4y_2 + x_1y_3 - x_2y_4, \\ z_4 &= x_4y_1 + x_3y_2 + x_2y_3 + \alpha x_1y_4 \end{aligned} \quad (14)$$

and the matrix $F(x)$ has the form

$$F(x) = \begin{pmatrix} x_1 & -x_2 & x_3 & -x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}. \quad (15)$$

The symplectic module of four-dimensional number in isotropic space M_3 has the form

$$|x| = \sqrt[4]{[(x_1 - x_3)^2 + (x_2 - x_4)^2] \cdot [(x_1 + x_3)^2 + (x_2 + x_4)^2]}. \quad (16)$$

It is isotropic space M_3 is considered in [1].

Case $\beta > 0,\ \gamma > 0,\ \delta < 0$.

In this case we will replace in definition of multiplication of numbers (1) δ with $-\delta$ and we will consider $\delta > 0$ in further. Then multiplication of two four-dimensional numbers is defined by formulas

$$\begin{aligned} z_1 &= \alpha x_1y_1 - \frac{\gamma\delta}{\alpha}x_2y_2 - \frac{\beta\delta}{\alpha}x_3y_3 + \frac{\beta\gamma}{\alpha}x_4y_4 \\ z_2 &= \alpha x_2y_1 + \alpha x_1y_2 + \beta x_4y_3 + \beta x_3y_4 \\ z_3 &= \alpha x_3y_1 + \gamma x_4y_2 + \alpha x_1y_3 + \gamma x_2y_4 \\ z_4 &= \alpha x_4y_1 - \delta x_3y_2 - \delta x_2y_3 + \alpha x_1y_4 \end{aligned}. \quad (17)$$

The multiplication table of basic numbers takes the form given in table 4.

Table 4. Products of basic numbers in case of 4.

	J_1	J_2	J_3	J_4
J_1	J_1	J_2	J_3	J_4
J_2	J_2	$-\frac{\gamma\delta}{\alpha^2}J_1$	$-\frac{\delta}{\alpha}J_4$	$\frac{\gamma}{\alpha}J_3$
J_3	J_3	$-\frac{\delta}{\alpha}J_4$	$-\frac{\beta\delta}{\alpha^2}J_1$	$\frac{\beta}{\alpha}J_2$
J_4	J_4	$\frac{\gamma}{\alpha}J_3$	$\frac{\beta}{\alpha}J_2$	$\frac{\beta\gamma}{\alpha^2}J_1$

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In this case the module of four-dimensional number (2) can be transformed as follows:

$$|x|^4 = \alpha^4 x_1^4 + \gamma^2\delta^2 x_2^4 + \beta^2\delta^2 x_3^4 + \beta^2\gamma^2 x_4^4 + 2\alpha^2\gamma\delta x_1^2 x_2^2 + 2\alpha^2\beta\delta x_1^2 x_3^2 - 2\alpha^2\beta\gamma x_1^2 x_4^2 - 2\beta\gamma\delta^2 x_2^2 x_3^2 + 2\beta\gamma^2\delta x_2^2 x_4^2 + 2\beta^2\gamma\delta x_3^2 x_4^2 - 8\alpha\beta\gamma\delta x_1 x_2 x_3 x_4 = \left[\left(\alpha x_1 - \sqrt{\beta\gamma} x_4\right)^2 + \left(\sqrt{\gamma\delta} x_2 - \sqrt{\beta\delta} x_3\right)^2\right] \cdot \left[\left(\alpha x_1 + \sqrt{\beta\gamma} x_4\right)^2 + \left(\sqrt{\gamma\delta} x_2 + \sqrt{\beta\delta} x_3\right)^2\right]. \quad (18)$$

As seen from table 4 the basic numbers J_1 and J_4 are real, and basic numbers J_2 and J_3 are imaginary. Then in this case as the imaginary number we will accept number $(x_1, x_2, 0, 0)$. Respectively, the module of the imaginary is defined by the formula $|x| = \sqrt{\alpha^2 x_1^2 + \gamma\delta x_2^2}$.

The conjugate number $x^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ to number $x = (x_1, x_2, x_3, x_4)$ is defined by the formula:

$$x_1^* = x_1(\alpha^2 x_1^2 + \gamma\delta x_2^2 + \beta\delta x_3^2 - \beta\gamma x_4^2) - \frac{2\beta\gamma\delta}{\alpha} x_2 x_3 x_4,$$
$$x_2^* = x_2(-\alpha^2 x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 - \beta\gamma x_4^2) + 2\alpha\beta x_1 x_3 x_4,$$
$$x_3^* = x_3(-\alpha^2 x_1^2 + \gamma\delta x_2^2 - \beta\delta x_3^2 - \text{짌}\gamma x_4^2) + 2\alpha\gamma x_1 x_2 x_4,$$
$$x_4^* = x_4(-\alpha^2 x_1^2 + \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2) - 2\alpha\delta x_1 x_2 x_3.$$

Respectively, the conjugate numbers to basic numbers have the form:

$$J_1^* = J_1,\ J_2^* = -\frac{\gamma\delta}{\alpha^3} \cdot J_2,\ J_3^* = -\frac{\beta\delta}{\alpha^3} \cdot J_3,\ J_4^* = \frac{\beta\gamma}{\alpha^3} \cdot J_4.$$

The conjugate numbers to unit numbers have the form:

$$I_1^* = \alpha^2 I_1,\ I_2^* = -\frac{\gamma\delta}{\alpha} \cdot I_2,\ I_3^* = -\frac{\beta\delta}{\alpha} \cdot I_3,\ I_4^* = \frac{\beta\gamma}{\alpha} \cdot I_4.$$

The conjugate number to the imaginary number $x = (x_1, x_2, 0, 0)$ has the form $\frac{|x|^2}{\alpha}(x_1, -x_2, 0, 0)$.

Let's denote the obtained anisotropic space of four-dimensional numbers by $M_4(\alpha, \beta, \gamma, \delta)$.

The matrix $F(x)$ from (7) in this case looks as follows:

$$F(x) = \begin{pmatrix} \alpha x_1 & -\frac{\gamma\delta}{\alpha} x_2 & -\frac{\beta\delta}{\alpha} x_3 & \frac{\beta\gamma}{\alpha} x_4 \\ \alpha x_2 & \alpha x_1 & \beta x_4 & \beta x_3 \\ \alpha x_3 & \gamma x_4 & \alpha x_1 & \gamma x_2 \\ \alpha x_4 & -\delta x_3 & -\delta x_2 & \alpha x_1 \end{pmatrix}. \quad (19)$$

It is also easily proved that the mapping $F: x \to F(x)$ is one-to-one and on. Alternative definitions of multiplication and division operations are defined as well as in the previous cases.

We will call space $M_4(1,1,1,1)$ isotropic space M_4. In isotropic space the multiplication of two numbers is defined by equalities

$$\begin{aligned} z_1 &= x_1 y_1 - x_2 y_2 - x_3 y_3 + x_4 y_4 \\ z_2 &= x_2 y_1 + x_1 y_2 + x_4 y_3 + x_3 y_4 \\ z_3 &= x_3 y_1 + x_4 y_2 + x_1 y_3 + x_2 y_4, \\ z_4 &= x_4 y_1 - x_3 y_2 - x_2 y_3 + \alpha x_1 y_4 \end{aligned} \quad (20)$$

and the matrix $F(x)$ has the form

$$F(x) = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}. \quad (21)$$

The symplectic module of four-dimensional number in isotropic space M_4 has the form

$$|x| = \sqrt[4]{[(x_1 - x_4)^2 + (x_2 - x_3)^2] \cdot [(x_1 + x_4)^2 + (x_2 + x_3)^2]}. \quad (22)$$

4. Case $\beta < 0,\ \gamma < 0,\ \delta > 0$.

In this case multiplication of two four-dimensional numbers is defined by formulas

$$\begin{aligned} z_1 &= \alpha x_1 y_1 - \frac{\gamma\delta}{\alpha} x_2 y_2 - \frac{\beta\delta}{\alpha} x_3 y_3 + \frac{\beta\gamma}{\alpha} x_4 y_4 \\ z_2 &= \alpha x_2 y_1 + \alpha x_1 y_2 - \beta x_4 y_3 - \beta x_3 y_4 \\ z_3 &= \alpha x_3 y_1 - \gamma x_4 y_2 + \alpha x_1 y_3 - \gamma x_2 y_4 \\ z_4 &= \alpha x_4 y_1 + \delta x_3 y_2 + \delta x_2 y_3 + \alpha x_1 y_4 \end{aligned}. \quad (23)$$

At the same time further we consider that $\beta > 0,\ \gamma > 0,\ \delta > 0$.

The multiplication table of basic numbers takes the form given in table 5.

Table 5. Products of basic numbers in case of 5.

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	J_1	J_2	J_3	J_4
J_1	J_1	J_2	J_3	J_4
J_2	J_2	$-\frac{\gamma\delta}{\alpha^3}J_1$	$\frac{\delta}{\alpha^2}J_4$	$-\frac{\gamma}{\alpha^2}J_3$
J_3	J_3	$\frac{\delta}{\alpha^2}J_4$	$-\frac{\beta\delta}{\alpha^3}J_1$	$-\frac{\beta}{\alpha^2}J_2$
J_4	J_4	$-\frac{\gamma}{\alpha^2}J_3$	$-\frac{\beta}{\alpha^2}J_2$	$\frac{\beta\gamma}{\alpha^3}J_1$

In this case the module of four-dimensional number (2) can be transformed as follows:

$$|x|^4 = \alpha^4x_1^4 + \gamma^2\delta^2x_2^4 + \beta^2\delta^2x_3^4 + \beta^2\gamma^2x_4^4 + 2\alpha^2\gamma\delta x_1^2x_2^2 + 2\alpha^2\beta\delta x_1^2x_3^2 - 2\alpha^2\beta\gamma x_1^2x_4^2 - 2\beta\gamma\delta^2x_2^2x_3^2 + 2\beta\gamma^2\delta x_2^2x_4^2 + 2\beta^2\gamma\delta x_3^2x_4^2 + 8\alpha\beta\gamma\delta x_1x_2x_3x_4 = \left[\left(\alpha x_1 - \sqrt{\beta\gamma}x_4\right)^2 + \left(\sqrt{\text{档}\delta}x_2 + \sqrt{\beta\delta}x_3\right)^2\right]\cdot\left[\left(\alpha x_1 + \sqrt{\beta\gamma}x_4\right)^2 + \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\delta}x_3\right)^2\right]. \qquad (24)$$

In this case the module of the imaginary number is defined by the formula $|x| = \sqrt{\alpha^2x_1^2 + \gamma\delta x_2^2}$ because as imaginarie numbers we take $(x_1, x_2, 0,0)$, as well as in case 4.

Notice that if modules of four-dimensional numbers in cases 2, 3 and 4 were expressed similarly, then the number module in this fifth case significantly differs from the previous cases.

The conjugate number $x^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ to number $x = (x_1, x_2, x_3, x_4)$ is defined by the formula:

$$x_1^* = x_1\left(\alpha^2x_1^2 + \gamma\delta x_2^2 + \beta\,\text{℧}x_3^2 - \beta\gamma x_4^2\right) + \frac{2\beta\gamma\delta}{\alpha}x_2x_3x_4,$$

$$x_2^* = x_2\left(-\alpha^2x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 - \beta\gamma x_4^2\right) - 2\alpha\beta x_1x_3x_4,$$
$$x_3^* = x_3\left(-\alpha^2x_1^2 + \gamma\delta x_2^2 - \beta\delta x_3^2 - \beta\gamma x_4^2\right) - 2\alpha\gamma x_1x_2x_4,$$
$$x_4^* = x_4\left(-\alpha^2x_1^2 + \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2\right) + 2\alpha\delta x_1x_2x_3.$$

Respectively, the conjugate numbers to basic numbers have the form:

$$J_1^* = J_1,\ J_2^* = -\frac{\gamma\delta}{\alpha^3}\cdot J_2,\ J_3^* = -\frac{\beta\delta}{\alpha^3}\cdot J_3,\ J_4^* = \frac{\beta\gamma}{\alpha^3}\cdot J_4.$$

The conjugate numbers to unit numbers have the form:

$$I_1^* = \alpha^2I_1,\ I_2^* = -\frac{\gamma\delta}{\alpha}\cdot I_2,\ I_3^* = -\frac{\beta\delta}{\alpha}\cdot I_3,\ I_4^* = \frac{\beta\gamma}{\alpha}\cdot I_4.$$

The conjugate number to the imaginary number $x = (x_1, x_2, 0,0)$ has the form $\frac{|x|^2}{\alpha}(x_1, -x_2, 0,0)$.

Let's denote the obtained space by $M_5(\alpha,\beta,\gamma,\delta)$.

The matrix $F(x)$ from (7) in this case looks as follows:

$$F(x) = \begin{pmatrix} \alpha x_1 & -\frac{\gamma\delta}{\alpha}x_2 & -\frac{\beta\delta}{\alpha}x_3 & \frac{\beta\gamma}{\alpha}x_4 \\ \alpha x_2 & \alpha x_1 & -\beta x_4 & -\beta x_3 \\ \alpha x_3 & -\gamma x_4 & \alpha x_1 & -\gamma x_2 \\ \alpha x_4 & \delta x_3 & \delta x_2 & \alpha x_1 \end{pmatrix}. \qquad (25)$$

It is also easily proved that mapping $F: x \to F(x)$ is one-to-one and on. Alternative definitions of multiplication and division operations are defined as in the previous cases.

We will call space $M_5(1,1,1,1)$ isotropic spaceM_5. In isotropic space the multiplication of two numbers is defined by equalities

$$\begin{matrix} z_1 = x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4 \\ z_2 = x_2y_1 + x_1y_2 - x_4y_3 - x_3y_4 \\ z_3 = x_3y_1 - x_4y_2 + x_1y_3 - x_2y_4\,' \\ z_4 = x_4y_1 + x_3y_2 + x_2y_3 + \alpha x_1y_4 \end{matrix} \qquad (26)$$

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and the matrix $F(x)$ has the form

$$F(x)=\begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}. \quad (27)$$

The symplectic module of four-dimensional number in isotropic space M_5 has the form

$$|x|=\sqrt[4]{[(x_1-x_4)^2+(x_2+x_3)^2]\cdot[(x_1+x_4)^2+(x_2-x_3)^2]}. \quad (28)$$

In [5, 6] properties of isotropic space M_5 are investigated.

5. Case $\beta<0, \gamma>0, \delta<0$.

In this case multiplication of two four-dimensional numbers is defined by formulas

$$\begin{aligned} z_1 &= \alpha x_1 y_1 - \frac{\gamma\delta}{\alpha} x_2 y_2 + \frac{\beta\delta}{\alpha} x_3 y_3 - \frac{\beta\gamma}{\alpha} x_4 y_4 \\ z_2 &= \alpha x_2 y_1 + \alpha x_1 y_2 - \beta x_4 y_3 - \beta x_3 y_4 \\ z_3 &= \alpha x_3 y_1 + \gamma x_4 y_2 + \alpha x_1 y_3 + \gamma x_2 y_4 \\ z_4 &= \alpha x_4 y_1 - \delta x_3 y_2 - \delta x_2 y_3 + \alpha x_1 y_4 \end{aligned}. \quad (29)$$

Here we consider that $\beta>0, \gamma>0, \delta>0$ in further.

The multiplication table of basic numbers is provided in the following table 6.

Table 6. Products of basic numbers in case 6.

	J_1	J_2	J_3	J_4
J_1	J_1	J_2	J_3	J_4
J_2	J_2	$-\frac{\gamma\delta}{\alpha^3}J_1$	$-\frac{\delta}{\alpha^2}J_4$	$\frac{\gamma}{\alpha^2}J_3$
J_3	J_3	$-\frac{\delta}{\alpha^2}J_4$	$\frac{\beta\delta}{\alpha^3}J_1$	$-\frac{\beta}{\alpha^2}J_2$
J_4	J_4	$\frac{\gamma}{\alpha^2}J_3$	$-\frac{\beta}{\alpha^2}J_2$	$-\frac{\beta\gamma}{\alpha^3}J_1$

In this case the module of four-dimensional number (2) can be transformed as follows:

$$|x|^4=\alpha^4x_1^4+\gamma^2\delta^2x_2^4+\beta^2\delta^2x_3^4+\beta^2\gamma^2x_4^4+2\alpha^2\gamma\delta x_1^2x_2^2-2\alpha^2\beta\delta x_1^2x_3^2+2\alpha^2\beta\gamma x_1^2x_4^2+2\beta\gamma\delta^2x_2^2x_3^2-2\beta\gamma^2\delta x_2^2x_4^2+2\beta^2\gamma\delta x_3^2x_4^2+8\alpha\beta\gamma\delta x_1x_2x_3x_4=\left[\left(\alpha x_1-\sqrt{\beta\delta}x_3\right)^2+\left(\sqrt{\gamma\delta}x_2+\sqrt{\beta\gamma}x_4\right)^2\right]\cdot\left[\left(\alpha x_1+\sqrt{\beta\delta}x_3\right)^2+\left(\sqrt{\gamma\delta}x_2-\sqrt{\beta\gamma}x_4\right)^2\right]. \quad (30)$$

In this case the module of the imaginary number is defined by the formula $|x|=\sqrt{\alpha^2x_1^2+\gamma\delta x_2^2}$. Definition of the module is somewhat similar to the previous case 5.

The conjugate number $x^*=(x_1^*,x_2^*,x_3^*,x_4^*)$ to number $x=(x_1,x_2,x_3,x_4)$ is defined by the formula:

$$x_1^*=x_1\left(\alpha^2x_1^2+\gamma\delta x_2^2-\beta\delta x_3^2+\beta\gamma x_4^2\right)+\frac{2\beta\gamma\delta}{\alpha}x_2x_3x_4,$$

$$x_2^*=x_2\left(-\alpha^2x_1^2-\gamma\delta x_2^2-\beta\delta x_3^2+\beta\gamma x_4^2\right)-2\alpha\beta x_1x_3x_4,$$
$$x_3^*=x_3\left(-\alpha^2x_1^2+\gamma\delta x_2^2+\beta\delta x_3^2+\beta\gamma x_4^2\right)+2\alpha\gamma x_1x_2x_4,$$
$$x_4^*=x_4\left(-\alpha^2x_1^2+\gamma\delta x_2^2-\beta\delta x_3^2-\beta\gamma x_4^2\right)-2\alpha\delta x_1x_2x_3.$$

Respectively, the conjugate numbers to basic numbers have the form:

$J_1^*=J_1,\ J_2^*=-\frac{\gamma\delta}{\alpha^3}\cdot J_2,\ J_3^*=\frac{\beta\delta}{\alpha^3}\cdot J_3,\ J_4^*=-\frac{\beta\gamma}{\alpha^3}\cdot J_4.$

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The conjugate numbers to unit numbers have the form:

$$I_1^* = \alpha^2 I_1,\ I_2^* = -\frac{\gamma\delta}{\alpha}\cdot I_2,\ I_3^* = \frac{\beta\delta}{\alpha}\cdot I_3,\ I_4^* = -\frac{\beta\gamma}{\alpha}\cdot I_4.$$

The conjugate number to the imaginary number $x = (x_1, x_2, 0{,}0)$ has the form $\frac{|x|^2}{\alpha}(x_1, -x_2, 0{,}0)$.
Let's denote the obtained space by $M_6(\alpha,\beta,\gamma,\delta)$.
The matrix $F(x)$ from (7) in this case looks as follows:

$$F(x) = \begin{pmatrix} \alpha x_1 & -\frac{\gamma\delta}{\alpha}x_2 & \frac{\beta\delta}{\alpha}x_3 & -\frac{\beta\gamma}{\alpha}x_4 \\ \alpha x_2 & \alpha x_1 & -\beta x_4 & -\beta x_3 \\ \alpha x_3 & \gamma x_4 & \alpha x_1 & \gamma x_2 \\ \alpha x_4 & -\delta x_3 & -\delta x_2 & \alpha x_1 \end{pmatrix}. \quad (31)$$

Obviously, mapping $F: x \to F(x)$is one-to-one and onto, and defines alternative definition of multiplication of four-dimensional numbers in space $M_6(\alpha,\beta,\gamma,\delta)$.

We will call space $M_6(1{,}1{,}1{,}1)$ isotropic space M_6. In isotropic space the multiplication of two numbers is defined by equalities

$$\begin{aligned} z_1 &= x_1y_1 - x_2y_2 + x_3y_3 - x_4y_4 \\ z_2 &= x_2y_1 + x_1y_2 - x_4y_3 - x_3y_4 \\ z_3 &= x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4, \\ z_4 &= x_4y_1 - x_3y_2 - x_2y_3 + \alpha x_1y_4 \end{aligned} \quad (32)$$

and the matrix $F(x)$ has the form

$$F(x) = \begin{pmatrix} x_1 & -x_2 & x_3 & -x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}. \quad (33)$$

The symplectic module of four-dimensional number in isotropic space M_6 has the form

$$|x| = \sqrt[4]{[(x_1-x_3)^2 + (x_2+x_4)^2]\cdot[(x_1+x_3)^2 + (x_2-x_4)^2]}. \quad (34)$$

6. Case $\beta > 0, \gamma < 0, \delta < 0$.

In this case multiplication of two four-dimensional numbers is defined by formulas

$$\begin{aligned} z_1 &= \alpha x_1y_1 + \frac{\gamma\delta}{\alpha}x_2y_2 - \frac{\beta\delta}{\alpha}x_3y_3 - \frac{\beta\gamma}{\alpha}x_4y_4 \\ z_2 &= \alpha x_2y_1 + \alpha x_1y_2 + \beta x_4y_3 + \beta x_3y_4 \\ z_3 &= \alpha x_3y_1 - \gamma x_4y_2 + \alpha x_1y_3 - \gamma x_2y_4 \\ z_4 &= \alpha x_4y_1 - \delta x_3y_2 - \delta x_2y_3 + \alpha x_1y_4 \end{aligned}. \quad (35)$$

Here we consider that $\beta > 0, \gamma > 0, \delta > 0$ in further.
The multiplication table of basic numbers takes the form given in table 7.

Table 7. Products of basic numbers in case of 7.

	J_1	J_2	J_3	J_4
J_1	J_1	J_2	J_3	J_4
J_2	J_2	$\frac{\gamma\delta}{\alpha^3}J_1$	$-\frac{\delta}{\alpha^2}J_4$	$-\frac{\gamma}{\alpha^2}J_3$
J_3	J_3	$-\frac{\delta}{\alpha^2}J_4$	$-\frac{\beta\delta}{\alpha^3}J_1$	$\frac{\beta}{\alpha^2}J_2$
J_4	J_4	$-\frac{\gamma}{\alpha^2}J_3$	$\frac{\beta}{\alpha^2}J_2$	$-\frac{\beta\gamma}{\alpha^3}J_1$

In this case the module of four-dimensional number (2) can be transformed as follows:

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$|x|^4 = \alpha^4 x_1^4 + \gamma^2\delta^2 x_2^4 + \beta^2\delta^2 x_3^4 + \beta^2\gamma^2 x_4^4 - 2\alpha^2\gamma\delta x_1^2 x_2^2 + 2\alpha^2\beta\delta x_1^2 x_3^2 + 2\alpha^2\beta\gamma x_1^2 x_4^2 + 2\beta\gamma\delta^2 x_2^2 x_3^2 + 2\beta\gamma^2\delta x_2^2 x_4^2 - 2\beta^2\gamma\delta x_3^2 x_4^2 + 8\alpha\beta\gamma\delta x_1 x_2 x_3 x_4 = \left[\left(\alpha x_1 - \sqrt{\gamma\delta} x_2\right)^2 + \left(\sqrt{\beta\delta} x_3 + \sqrt{\beta\gamma} x_4\right)^2\right] \cdot \left[\left(\alpha x_1 + \sqrt{\gamma\delta} x_2\right)^2 + \left(\sqrt{\beta\delta} x_3 - \sqrt{\beta\gamma} x_4\right)^2\right].$ (36)

Here the module of the imaginary number $x = (x_1, 0, x_3, 0)$ is defined by the formula $|x| = \sqrt{|\alpha^2 x_1^2 + \beta\delta x_3^2|}$.

The conjugate number $x^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ to number $x = (x_1, x_2, x_3, x_4)$ is defined by the formula:

$x_1^* = x_1(\alpha^2 x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2) + \frac{2\beta\gamma\delta}{\alpha} x_2 x_3 x_4,$

$x_2^* = x_2(-\alpha^2 x_1^2 + \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2) + 2\alpha\beta x_1 x_3 x_4,$
$x_3^* = x_3(-\alpha^2 x_1^2 - \gamma\delta x_2^2 - \beta\delta x_3^2 + \beta\gamma x_4^2) - 2\alpha\gamma x_1 x_2 x_4,$
$x_4^* = x_4(-\alpha^2 x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 - \beta\Delta x_4^2) - 2\alpha\delta x_1 x_2 x_3.$

Respectively, the conjugate numbers to basic numbers have the form:

$J_1^* = J_1,\ J_2^* = \frac{\gamma\delta}{\alpha^3} \cdot J_2,\ J_3^* = -\frac{\beta\delta}{\alpha^3} \cdot J_3,\ J_4^* = -\frac{\beta\gamma}{\alpha^3} \cdot J_4.$

The conjugate numbers to unit numbers have the form:

$I_1^* = \alpha^2 I_1,\ I_2^* = \frac{\gamma\delta}{\alpha} \cdot I_2,\ I_3^* = -\frac{\beta\delta}{\alpha} \cdot I_3,\ I_4^* = -\frac{\beta\gamma}{\alpha} \cdot I_4.$

The conjugate number to the imaginary number $(x_1, 0, x_3, 0)$ has the form $\frac{|x|^2}{\alpha}(x_1, 0, -x_3, 0)$ that easily follows from the last formulas.

Let's denote the obtained anisotropic space of four-dimensional numbers by $M_7(\alpha, \beta, \gamma, \delta)$.

The matrix $F(x)$ from (7) in this case looks as follows:

$$F(x) = \begin{pmatrix} \alpha x_1 & \frac{\gamma\delta}{\alpha} x_2 & -\frac{\beta\delta}{\alpha} x_3 & -\frac{\beta\gamma}{\alpha} x_4 \\ \alpha x_2 & \alpha x_1 & \beta x_4 & \beta x_3 \\ \alpha x_3 & -\gamma x_4 & \alpha x_1 & -\gamma x_2 \\ \alpha x_4 & -\delta x_3 & -\delta x_2 & \alpha x_1 \end{pmatrix}. \quad (37)$$

We will call space $M_7(1,1,1,1)$ isotropic space M_7. In isotropic space the multiplication of two numbers is defined by equalities

$$\begin{aligned} z_1 &= x_1 y_1 + x_2 y_2 - x_3 y_3 - x_4 y_4 \\ z_2 &= x_2 y_1 + x_1 y_2 + x_4 y_3 + x_3 y_4 \\ z_3 &= x_3 y_1 - x_4 y_2 + x_1 y_3 - x_2 y_4 \\ z_4 &= x_4 y_1 - x_3 y_2 - x_2 y_3 + \alpha x_1 y_4 \end{aligned}, \quad (38)$$

and the matrix $F(x)$ has the form

$$F(x) = \begin{pmatrix} x_1 & -x_2 & x_3 & -x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}. \quad (39)$$

The symplectic module of four-dimensional number in isotropic space M_7 has the form

$$|x| = \sqrt[4]{[(x_1 - x_3)^2 + (x_2 + x_4)^2] \cdot [(x_1 + x_3)^2 + (x_2 - x_4)^2]}. \quad (40)$$

7. Case $\beta < 0, \gamma < 0, \delta < 0$.

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In this case multiplication of two four-dimensional numbers is defined by formulas

$$\begin{aligned} z_1 &= \alpha x_1 y_1 + \frac{\gamma\delta}{\alpha} x_2 y_2 + \frac{\beta\delta}{\alpha} x_3 y_3 + \frac{\beta\gamma}{\alpha} x_4 y_4 \\ z_2 &= \alpha x_2 y_1 + \alpha x_1 y_2 - \beta x_4 y_3 - \beta x_3 y_4 \\ z_3 &= \alpha x_3 y_1 - \gamma x_4 y_2 + \alpha x_1 y_3 - \gamma x_2 y_4 \\ z_4 &= \alpha x_4 y_1 - \delta x_3 y_2 - \delta x_2 y_3 + \alpha x_1 y_4 \end{aligned} .$$

Here we consider that $\beta > 0, \gamma > 0, \delta > 0$ in further.

In this case the module of four-dimensional number (2) can be transformed as follows:

$$|x|^4 = \alpha^4 x_1^4 + \gamma^2\delta^2 x_2^4 + \beta^2\delta^2 x_3^4 + \beta^2\gamma^2 x_4^4 - 2\alpha^2\gamma\delta x_1^2 x_2^2 - 2\alpha^2\beta\delta x_1^2 x_3^2 - 2\alpha^2\beta\gamma x_1^2 x_4^2 - 2\beta\gamma\delta^2 x_2^2 x_3^2 - 2\beta\gamma^2\delta x_2^2 x_4^2 - 2\beta^2\gamma\delta x_3^2 x_4^2 - 8\alpha\beta\gamma\delta x_1 x_2 x_3 x_4 = \left[\left(\alpha x_1 + \sqrt{\beta\delta}x_3\right)^2 - \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\gamma}x_4\right)^2\right]\left[\left(\alpha x_1 - \sqrt{\beta\delta}x_3\right)^2 - \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\gamma}x_4\right)^2\right],$$

or, as follows

$$|x|^4 = \left[\left(\alpha x_1 + \sqrt{\gamma\delta}x_2\right)^2 - \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)^2\right]\left[\left(\alpha x_1 - \sqrt{\gamma\delta}x_2\right)^2 - \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)^2\right],$$

or, as follows

$$|x|^4 = \left[\left(\alpha x_1 + \sqrt{\beta\gamma}x_4\right)^2 - \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\delta}x_3\right)^2\right]\left[\left(\alpha x_1 - \sqrt{\beta\gamma}x_4\right)^2 - \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\delta}x_3\right)^2\right].$$

For this case the symplectic module is not non-negatively defined form.

Thus, we have six various spaces of four-dimensional numbers in which operation of multiplication and the corresponding modules of numbers are defined by various formulas. Further we investigate properties of these spaces.

2. Degenerate Numbers in Spaces of Four-Dimensional Numbers

Definition. The four-dimensional number is called nondegenerate if $|x| > 0$, and degenerate if $|x| = 0$.

We investigate solutions of the equation $|x| = 0$ or degenerate numbers. This equation has explicit solutions thanks to which we can describe the general structure of degenerate numbers in all spaces $M_i(\alpha,\beta,\gamma,\delta), i = 2,3,4,5,6,7$.

As it follows from definition of the module of number (6) in space $M_2(\alpha,\beta,\gamma,\delta)$ there are two types of degenerate numbers, namely, numbers of the kind $\left(c_1, \frac{\alpha}{\sqrt{\gamma\delta}}c_1, c_2, \sqrt{\frac{\delta}{\gamma}}c_2\right)$ and $\left(c_1, -\frac{\alpha}{\sqrt{\gamma\delta}}c_1, c_2, -\sqrt{\frac{\delta}{\gamma}}c_2\right)$ for any real c_1 and c_2.

We will call numbers of the kind $\left(c_1, \frac{\alpha}{\sqrt{\gamma\delta}}c_1, c_2, \sqrt{\frac{\delta}{\gamma}}c_2\right)$ degenerate numbers of the first type, and numbers $\left(c_1, -\frac{\alpha}{\sqrt{\gamma\delta}}c_1, c_2, -\sqrt{\frac{\delta}{\gamma}}c_2\right)$ degenerate numbers of the second type.

Obviously, the only degenerate number belonging both to the first and the second type is the number 0 = (0,0,0,0). Let's denote the set of all degenerate numbers of the first type by O_I, and the set of all degenerate numbers of the second type by O_{II}.

The module of a number in space $M_3(\alpha,\beta,\gamma,\delta)$ is defined by the formula (12), therefore, in this space there are also two types of degenerate numbers $\left(c_1, c_2, \frac{\alpha}{\sqrt{\beta\delta}}c_1, \sqrt{\frac{\delta}{\beta}}c_2\right)$ and $\left(c_1, c_2, -\frac{\alpha}{\sqrt{\beta\delta}}c_1, -\sqrt{\frac{\delta}{\beta}}c_2\right)$, for any real c_1 and c_2, which respectively we will call degenerate numbers of the first type and degenerate numbers of the second type and them will also denote by O_I and O_{II}.

Number 0 = (0,0,0,0) is the only degenerate number belonging to both types.

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In space $M_4(\alpha,\beta,\gamma,\delta)$ the set of degenerate numbers of the first type O_I consists from the numbers$\left(c_1, c_2, \sqrt{\frac{\gamma}{\beta}}c_2, \frac{\alpha}{\sqrt{\beta\gamma}}c_1\right)$, and the set of degenerate numbers of the second type O_{II} consists from the numbers $\left(c_1, c_2, -\sqrt{\frac{\gamma}{\beta}}c_2, -\frac{\alpha}{\sqrt{\beta\gamma}}c_1\right)$ for any real c_1 and c_2 that follows from definition of symplectic module (18).

Similarly, in space $M_5(\alpha,\beta,\gamma,\delta)$ the set of degenerate numbers of the first type O_I consists from the numbers $\left(c_1, c_2, -\sqrt{\frac{\gamma}{\beta}}c_2, \frac{\alpha}{\sqrt{\beta\gamma}}c_1\right)$, and the set of degenerate numbers of the second type O_{II} consists from the numbers $\left(c_1, c_2, \sqrt{\frac{\gamma}{\beta}}c_2, -\frac{\alpha}{\sqrt{\beta\gamma}}c_1\right)$ for any real c_1 and c_2.

In space $M_6(\alpha,\beta,\gamma,\delta)$ the symplectic module of number is determined by the formula (30), therefore, the set of degenerate numbers of the first type O_I consists of the numbers $\left(c_1, c_2, \frac{\alpha}{\sqrt{\beta\gamma}}c_1, -\sqrt{\frac{\delta}{\text{瑣}}}c_2\right)$, and the set of degenerate numbers of the second type O_{II} consists of the numbers $\left(c_1, c_2, -\frac{\alpha}{\sqrt{\beta\gamma}}c_1, \sqrt{\frac{\delta}{\beta}}c_2\right)$ for any real c_1 and c_2.

In space $M_7(\alpha,\beta,\gamma,\delta)$ we will get from the equation $|x| = 0$ that there are two types of degenerate numbers, namely, numbers of the kind $\left(c_1, \frac{\alpha}{\sqrt{\gamma\delta}}c_1, c_2, -\sqrt{\frac{\delta}{\gamma}}c_2\right)$ and $\left(c_1, -\frac{\alpha}{\sqrt{\gamma\delta}}c_1, c_2, \sqrt{\frac{\delta}{\gamma}}c_2\right)$ for any real c_1 and c_2 which are respectively degenerate numbers of the first and second types.

Theorem 2. Degenerate numbers in spaces $M_i(\alpha,\beta,\gamma,\delta), i = 2,3,4,5,6,7$, have the following properties:

1) If $x, y \in O_I$, then $x + y \in O_I$, $x - y \in O_I$, $x \cdot y \in O_I$.
2) If $x, y \in O_{II}$, then $x + y \in O_{II}$, $x - y \in O_{II}$, $x \cdot y \in O_{II}$.
3) If $x \in O_I, y \in O_{II}$, then $x \cdot y = 0 = (0,0,0,0)$.
4) If $x \in O_I$, $y \notin O_I \cup O_{II}$, then $x \cdot y \in O_I$.
5) If $x \in O_{II}$, $y \notin O_I \cup O_{II}$, then $x \cdot y \in O_{II}$.

The proof is easily follows from definitions of addition and multiplication in the corresponding spaces.

This theorem describes properties of degenerate numbers in spaces$M_i(\alpha,\beta,\gamma,\delta), i = 2,3,4,5,6,7$. In particular, it follows from the first two properties that sets O_I and O_{II} are subspaces of spaces $M_i(\alpha,\beta,\gamma,\delta), i = 2,3,4,5,6,7$.

The third property claims that points of subspaces O_I and O_{II} are zero divisors. Moreover, as it will become obvious from further discussions, there are no other zero divisors. Such obvious description of structure of zero divisors allows to control influence of zero divisors on various mathematical structures. For example, to build the full-fledged calculus, the differential and integral calculus and other constructions.

The last approvals of the theorem say that at multiplication of any four-dimensional number by degenerate number, we will always receive degenerate number.

3. **Range of Four-Dimensional Numbers**

To each four-dimensional number $x \in M_i, i = 2,3,\ldots,7$, we compared some matrix *of* $F(x)$, determined respectively by formulas (7), (13), (19), (25), (31), (37) by means of which multiplication of two numbers can be reduced to multiplication of this matrix by the vector determined by the second multiplier.

Theorem 3. The set of matrices $F(x)$ is closed with respect to matrix operations of addition, subtraction and multiplication and also multiplication of the matrix by the scalar. The inverse matrix to the nonsingular matrix has the same form.

It can be proved by direct check.

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Theorem 4. For each space $M_i(\alpha,\beta,\gamma,\delta), i = 2,3,4,5,6,7$, mapping $F: x \to F(x)$ for any four-dimensional numbers x, y has the following properties:

1) $F(x \pm y) = F(x) \pm F(y)$;
2) $F(cx) = cF(x)$ for any $c \in R$;
3) $F(xy) = F(x)F(y)$;
4) $F(x^{-1}) = F^{-1}(x)$;
5) $\det(F(x)) = |x|^4$;
6) $\det(F(x) \pm F(y)) = |x \pm y|^4$;
7) $\det(F(\alpha x)) = |\alpha x|^4$;
8) $\det(F(x)F(y)) = |xy|^4$;
9) $\det\left(F^{-1}(x)\right) = \left|x^{-1}\right|^4$, where x is nondegenerate number.

Proof. Let's prove the theorem for space$M_2(\alpha,\beta,\gamma,\delta)$, for other spaces the proof is carried out in a similar way.

Properties 1) and 2) are obvious. Let's prove property 3).

$$F(x)F(y) = \begin{pmatrix} \alpha x_1 & \frac{\gamma\delta}{\alpha}x_2 & -\frac{\beta\delta}{\alpha}x_3 & -\frac{\beta\gamma}{\alpha}x_4 \\ \alpha x_2 & \alpha x_1 & -\beta x_4 & -\beta x_3 \\ \alpha x_3 & \gamma x_4 & \alpha x_1 & \gamma x_2 \\ \alpha x_4 & \delta x_3 & \delta x_2 & \alpha x_1 \end{pmatrix} \begin{pmatrix} \alpha y_1 & \frac{\gamma\delta}{\alpha}y_2 & -\frac{\beta\delta}{\alpha}y_3 & -\frac{\beta\gamma}{\alpha}y_4 \\ \alpha y_2 & \alpha y_1 & -\beta y_4 & -\beta y_3 \\ \alpha y_3 & \gamma y_4 & \alpha y_1 & \gamma y_2 \\ \alpha y_4 & \delta y_3 & \delta y_2 & \alpha y_1 \end{pmatrix} = B,$$

where B is the resulting matrix. Let's calculate elements of the matrix B.

$$b_{11} = \alpha^2 x_1 y_1 + \gamma\delta x_2 y_2 - \beta\delta x_3 y_3 - \beta\gamma x_4 y_4 = \alpha z_1;$$

$$b_{12} = \gamma\delta x_1 y_2 + \gamma\delta x_2 y_1 - \frac{\beta\gamma\delta}{\alpha}x_3 y_4 - \frac{\beta\gamma\delta}{\alpha}x_4 y_3 = \frac{\gamma\delta}{\alpha}z_2;$$

$$b_{13} = -\beta\delta x_1 y_3 - \frac{\beta\gamma\delta}{\alpha}x_2 y_4 - \beta\delta x_3 y_1 - \frac{\beta\gamma\delta}{\alpha}x_4 y_2 = -\frac{\beta\delta}{\alpha}z_3;$$

$$b_{14} = -\beta\gamma x_1 y_4 - \frac{\beta\gamma\delta}{\alpha}x_2 y_3 - \frac{\beta\gamma\delta}{\alpha}x_3 y_2 - \beta\gamma x_4 y_1 = -\frac{\beta\gamma}{\alpha}z_4;$$

$$b_{21} = \alpha^2 x_2 y_1 + \alpha^2 x_1 y_2 - \alpha\beta x_4 y_3 - \alpha\beta x_3 y_4 = \alpha z_2;$$
$$b_{22} = \gamma\delta x_2 y_2 + \alpha^2 x_1 y_1 - \beta\gamma x_4 y_4 - \beta\delta x_3 y_3 = \alpha z_1;$$
$$b_{23} = -\beta\delta x_2 y_3 - \alpha\beta x_1 y_4 - \alpha\beta x_4 y_1 - \beta\delta x_3 y_2 = -\beta z_4;$$
$$b_{24} = -\beta\gamma x_2 y_4 - \alpha\beta x_1 y_3 - \beta\gamma x_4 y_2 - \alpha\beta x_3 y_1 = -\beta z_3;$$
$$b_{31} = \alpha^2 x_3 y_1 + \alpha\gamma x_4 y_2 + \alpha^2 x_1 y_3 + \alpha\gamma x_2 y_4 = \alpha z_3;$$
$$b_{32} = \gamma\delta x_3 y_2 + \alpha\gamma x_4 y_1 + \alpha\gamma x_1 y_4 + \gamma\delta x_2 y_3 = \gamma z_4;$$
$$b_{33} = -\beta\delta x_3 y_3 - \beta\gamma x_4 y_4 + \alpha^2 x_1 y_1 + \gamma\delta x_2 y_2 = \alpha z_1;$$
$$b_{34} = -\beta\gamma x_3 y_4 - \beta\gamma x_4 y_3 + \alpha\gamma x_1 y_2 + \alpha\gamma x_2 y_1 = \gamma z_2;$$
$$b_{41} = \alpha^2 x_4 y_1 + \alpha\delta x_3 y_2 + \alpha\delta x_2 y_3 + \alpha^2 x_1 y_4 = \alpha z_4;$$
$$b_{42} = \gamma\delta x_4 y_2 + \alpha\delta x_3 y_1 + \gamma\delta x_2 y_4 + \alpha\delta x_1 y_3 = \delta z_3;$$
$$b_{43} = -\beta\delta x_4 y_3 - \beta\delta x_3 y_4 + \alpha\delta x_2 y_1 + \alpha\delta x_1 y_2 = \delta z_2;$$
$$b_{44} = -\beta\gamma x_4 y_4 - \beta\delta x_3 y_3 - \gamma\delta x_2 y_2 + \alpha^2 x_1 y_1 = \alpha z_1,$$

where $z = (z_1, z_2, z_3, z_4) = x \cdot y$.

Let's prove property 4). According to the formula (4), $x^{-1} = \frac{1}{\alpha \cdot |x|^4}x^*$, therefore

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$$(x^{-1})_1 = \frac{1}{|x|^4} x_1(\alpha^2 x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2) - \frac{2\beta\gamma\delta}{\alpha|x|^4} x_2 x_3 x_4,$$

$$(x^{-1})_2 = \frac{1}{|x|^4} x_2(-\alpha^2 x_1^2 + \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2) - \frac{2\alpha\beta}{|x|^4} x_1 x_3 x_4,$$

$$(x^{-1})_3 = \frac{1}{|x|^4} x_3(-\alpha^2 x_1^2 - \gamma\delta x_2^2 - \beta\delta x_3^2 + \beta\gamma x_4^2) + \frac{2\alpha\gamma}{|x|^4} x_1 x_2 x_4,$$

$$(x^{-1})_4 = \frac{1}{|x|^4} x_4(-\alpha^2 x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 - \beta\gamma x_4^2) + \frac{2\alpha\delta}{|x|^4} x_1 x_2 x_3,$$

respectively, we have

$$F(x^{-1}) = \begin{pmatrix} \alpha(x^{-1})_1 & \frac{\gamma\delta}{\alpha}(x^{-1})_2 & -\frac{\beta\delta}{\alpha}(x^{-1})_3 & -\frac{\beta\gamma}{\alpha}(x^{-1})_4 \\ \alpha(x^{-1})_2 & \alpha(x^{-1})_1 & -\beta(x^{-1})_4 & -\beta(x^{-1})_3 \\ \alpha(x^{-1})_3 & \gamma(x^{-1})_4 & \alpha(x^{-1})_1 & \gamma(x^{-1})_2 \\ \alpha(x^{-1})_4 & \delta(x^{-1})_3 & \delta(x^{-1})_2 & \alpha(x^{-1})_1 \end{pmatrix},$$

and

$$F(x) = \begin{pmatrix} \alpha x_1 & \frac{\gamma\delta}{\alpha}x_2 & -\frac{\beta\delta}{\alpha}x_3 & -\frac{\beta\gamma}{\alpha}x_4 \\ \alpha x_2 & \alpha x_1 & -\beta x_4 & -\beta x_3 \\ \alpha x_3 & \gamma x_4 & \alpha x_1 & \gamma x_2 \\ \alpha x_4 & \delta x_3 & \delta x_2 & \alpha x_1 \end{pmatrix}.$$

Multiplying these matrices at each other, we will receive that $F(x^{-1}) \cdot F(x) = E$ where E is the identity matrix that completes proof.

Let's prove property 5). By definition of the determinant we have

$$detF(x) = \alpha x_1 \begin{vmatrix} \alpha x_1 & -\beta x_4 & -\beta x_3 \\ \gamma x_4 & \alpha x_1 & \gamma x_2 \\ \delta x_3 & \delta x_2 & \alpha x_1 \end{vmatrix} - \frac{\gamma\delta}{\alpha} x_2 \begin{vmatrix} \alpha x_2 & -\beta x_4 & -\beta x_3 \\ \alpha x_3 & \alpha x_1 & \gamma x_2 \\ \alpha x_4 & \delta x_2 & \alpha x_1 \end{vmatrix} -$$

$$-\frac{\beta\delta}{\alpha} x_3 \begin{vmatrix} \alpha x_2 & \alpha x_1 & -\beta x_3 \\ \alpha x_3 & \gamma x_4 & \gamma x_2 \\ \alpha x_4 & \delta x_3 & \alpha x_1 \end{vmatrix} + \frac{\beta\gamma}{\alpha} x_4 \begin{vmatrix} \alpha x_2 & \alpha x_1 & -\beta x_4 \\ \alpha x_3 & \gamma x_4 & \alpha x_1 \\ \alpha x_4 & \delta x_3 & \delta x_2 \end{vmatrix}.$$

Calculating determinants in the last equality, we make sure that

$$\begin{vmatrix} \alpha x_1 & -\beta x_4 & -\beta x_3 \\ \gamma x_4 & \alpha x_1 & \gamma x_2 \\ \delta x_3 & \delta x_2 & \alpha x_1 \end{vmatrix} = \alpha x_1^*, \quad \begin{vmatrix} \alpha x_2 & -\beta x_4 & -\beta x_3 \\ \alpha x_3 & \alpha x_1 & \gamma x_2 \\ \alpha x_4 & \delta x_2 & \alpha x_1 \end{vmatrix} = -\alpha x_2^*, \quad \begin{vmatrix} \alpha x_2 & \alpha x_1 & -\beta x_3 \\ \alpha x_3 & \gamma x_4 & \gamma x_2 \\ \alpha x_4 & \delta x_3 & \alpha x_1 \end{vmatrix} = \alpha x_3^*,$$

$$\begin{vmatrix} \alpha x_2 & \alpha x_1 & -\beta x_4 \\ \alpha x_3 & \gamma x_4 & \alpha x_1 \\ \alpha x_4 & \delta x_3 & \delta x_2 \end{vmatrix} = -\alpha x_4^*.$$

Then $detF(x) = \alpha^2 x_1 x_1^* + \gamma\delta x_2 x_2^* - \beta\delta x_3 x_3^* - \beta\gamma x_4 x_4^* = \alpha \cdot (x \cdot x^*)_1 = |x|^4$.

Let's prove property 6). $\det(F(x) \pm F(y)) = \det(F(x \pm y)) = |x \pm y|^4$

Proofs of properties 7), 8), 9) are obvious.

Thus, there is the bijection between space of four-dimensional numbers and space of 4 x 4 matrices which maintains arithmetic operations, that is the existing bijection is homomorphism.

Definition. The set of characteristic numbers of the corresponding matrix $F(x)$ is called the spectrum of four-dimensional number x.

Let's work out characteristic equation for definition of the spectrum of four-dimensional number $x = (x_1, x_2, x_3, x_4)$ in space $M_2(\alpha, \beta, \gamma, \delta)$:

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$$\begin{vmatrix} \alpha x_1-\mu & \frac{\gamma\delta}{\alpha}x_2 & -\frac{\beta\delta}{\alpha}x_3 & -\frac{\beta\gamma}{\alpha}x_4 \\ \alpha x_2 & \alpha x_1-\mu & -\beta x_4 & -\beta x_3 \\ \alpha x_3 & \gamma x_4 & \alpha x_1-\mu & \gamma x_2 \\ \alpha x_4 & \delta x_3 & \delta x_2 & \alpha x_1-\mu \end{vmatrix} = 0.$$

Let's calculate minors $M(a_{1i})$ of elements of the first row of the matrix:

$$M(\alpha x_1-\mu) = (\alpha x_1-\mu)\begin{vmatrix} \alpha x_1-\mu & \gamma x_2 \\ \delta x_2 & \alpha x_1-\mu \end{vmatrix} + \beta x_4 \begin{vmatrix} \gamma x_4 & \gamma x_2 \\ \delta x_3 & \alpha x_1-\mu \end{vmatrix} - \beta x_3 \begin{vmatrix} \gamma x_4 & \alpha x_1-\mu \\ \delta x_3 & \delta x_2 \end{vmatrix}.$$

$$M(\alpha x_1-\mu) = -\mu^3 + 3\alpha x_1\mu^2 + \left(-3\alpha^2x_1^2 + \gamma\delta x_2^2 - \beta\delta x_3^2 - \beta\gamma x_4^2\right)\cdot\mu + \alpha^3x_1^3 - \alpha\gamma\delta x_1x_2^2 + \alpha\beta\delta x_1x_3^2 + \alpha\beta\gamma x_1x_4^2 - 2\beta\gamma\delta x_2x_3x_4.$$

$$M\left(\frac{\gamma\delta}{\alpha}x_2\right) = \alpha x_2\begin{vmatrix} \alpha x_1-\mu & \gamma x_2 \\ \delta x_2 & \alpha x_1-\mu \end{vmatrix} + \beta x_4 \begin{vmatrix} \alpha x_3 & \gamma x_2 \\ \alpha x_4 & \alpha x_1-\mu \end{vmatrix} - \beta x_3 \begin{vmatrix} \alpha x_3 & \alpha x_1-\mu \\ \alpha x_4 & \delta x_2 \end{vmatrix}.$$

$$M\left(\frac{\gamma\delta}{\alpha}x_2\right) = \alpha x_2\mu^2 - 2\alpha(赤x_1x_2 + \beta x_3x_4)\cdot\mu + \alpha\left(\alpha^2x_1^2x_2 - \gamma\delta x_2^3 - \beta\delta x_2x_3^2 - \beta\gamma x_2x_4^2 + 2\alpha\beta x_1x_3x_4\right).$$

$$M\left(-\frac{\beta\delta}{\alpha}x_3\right) = \alpha x_2\begin{vmatrix} \gamma x_4 & \gamma x_2 \\ \delta x_3 & \alpha x_1-\mu \end{vmatrix} - (\alpha x_1-\mu)\begin{vmatrix} \alpha x_3 & \gamma x_2 \\ \alpha x_4 & \alpha x_1-\mu \end{vmatrix} - \beta x_3 \begin{vmatrix} \alpha x_3 & \gamma x_4 \\ \alpha x_4 & \delta x_3 \end{vmatrix}.$$

$$M\left(-\frac{\beta\delta}{\alpha}x_3\right) = -\alpha x_3\mu^2 + 2\alpha(\alpha x_1x_3 - \gamma x_2x_4)\cdot\mu + \alpha\left(2\alpha\gamma x_1x_2x_4 - \alpha^2x_1^2x_3 - \gamma\delta x_2^2x_3 - \beta\delta x_3^3 + \beta\gamma x_3x_4^2\right).$$

$$M\left(-\frac{\beta\gamma}{\alpha}x_4\right) = \alpha x_2\begin{vmatrix} \gamma x_4 & \alpha x_1-\mu \\ \delta x_3 & \delta x_2 \end{vmatrix} - (\alpha x_1-\mu)\begin{vmatrix} \alpha x_3 & \alpha x_1-\mu \\ \alpha x_4 & \delta x_2 \end{vmatrix} - \beta x_4 \begin{vmatrix} \alpha x_3 & \gamma x_4 \\ \alpha x_4 & \delta x_3 \end{vmatrix}.$$

$$M\left(-\frac{\beta\gamma}{\alpha}x_4\right) = \alpha x_4\mu^2 + 2\alpha(\delta x_2x_3 - \alpha x_1x_4)\cdot\mu + \alpha\left(\alpha^2x_1^2x_4 + \gamma\delta x_2^2x_4 - \beta\delta x_3^2x_4 + \beta\gamma x_4^3 - 2\alpha\delta x_1x_2x_3\right).$$

Using these values of minors, we will calculate determinant of characteristic equation:

$$\mu^4 - 4\alpha x_1\mu^3 + 2\left(3\alpha^2x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2\right)\mu^2 + \left(-4\alpha^3x_1^3 + 4\alpha\gamma\delta x_1x_2^2 - 4\alpha\beta\delta x_1x_3^2 - 4\alpha\beta\gamma x_1x_4^2 + 8\beta\gamma\delta x_2x_3x_4\right)\cdot\mu + \alpha^4x_1^4 + \gamma^2\delta^2x_2^4 + \beta^2\delta^2x_3^4 + \beta^2\gamma^2x_4^4 - 2\alpha^2\gamma\delta x_1^2x_2^2 + 2\alpha^2\beta\delta x_1^2x_3^2 + 2\alpha^2\beta\gamma x_1^2x_4^2 + 2\beta\gamma\delta^2x_2^2x_3^2 + 2\beta\gamma^2\delta x_2^2x_4^2 - 2\beta^2\gamma\delta x_3^2x_4^2 - 8\alpha\beta\gamma\delta x_1x_2x_3x_4 = 0. \quad (41)$$

Taking into account the equation (6) we will rewrite the equation (41) as follows:

$$\mu^4 - 4\alpha x_1\mu^3 + 2\left(3\alpha^2x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2\right)\mu^2 + \left(-4\alpha^3x_1^3 + 4\alpha\gamma\delta x_1x_2^2 - 4\alpha\beta\delta x_1x_3^2 - 4\alpha\beta\gamma x_1x_4^2 + 8\beta\gamma\delta x_2x_3x_4\right)\cdot\mu + \left[\left(\alpha x_1 - \sqrt{\gamma\delta}x_2\right)^2 + \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)^2\right]\cdot\left[\left(\alpha x_1 + \sqrt{\gamma\delta}x_2\right)^2 + \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)^2\right] = 0.$$

Proceeding from the type of free term, we will consider the possibility of decomposition of the equation on the product of two square trinomials, namely, we will present the last equation in the form $(\mu^2 + t_1\mu + u_1)\cdot(\mu^2 + t_2\mu + u_2) = 0$ where $u_1 = \left(\alpha x_1 - \sqrt{\gamma\delta}x_2\right)^2 + \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)^2 u_2 = \left(\alpha x_1 + \sqrt{\gamma\delta}x_2\right)^2 + \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)^2$. Expanding the brackets and equating coefficients of degrees μ, we find for t_1 and t_2 the following system:

$$\begin{cases} t_1 + t_2 = -4\alpha x_1 \\ t_1t_2 + u_1 + u_2 = 2\left(3\alpha^2x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2\right) \\ u_2t_1 + u_1t_2 = -4\alpha^3x_1^3 + 4\alpha\gamma\delta x_1x_2^2 - 4\alpha\beta\delta x_1x_3^2 - 4\alpha\beta\gamma x_1x_4^2 + 8\beta\gamma\delta x_2x_3x_4 \end{cases}.$$

This redefined system has unique solution $t_1 = -2\left(\alpha x_1 - \sqrt{\gamma\delta}x_2\right)$, $t_2 = -2\left(\alpha x_1 + \sqrt{\gamma\delta}x_2\right)$. Therefore, the equation (8) is equivalent to the equation

$$\left(\mu^2 - 2\left(\alpha x_1 - \sqrt{\gamma\delta}x_2\right)\mu + u_1\right)\cdot\left(\mu^2 - 2\left(\alpha x_1 + \sqrt{\gamma\delta}x_2\right)\mu + u_2\right) = 0. \quad (42)$$

The equation (42) breaks up to two quadratic equations, solving which it is found four characteristic numbers of four-dimensional number x:

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$$\begin{cases} \mu_1 = \alpha x_1 - \sqrt{\gamma\delta}x_2 + \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)i \\ \mu_2 = \alpha x_1 - \sqrt{\gamma\delta}x_2 - \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)i \\ \mu_3 = \alpha x_1 + \sqrt{\gamma\delta}x_2 + \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)i \\ \mu_4 = \alpha x_1 + \sqrt{\gamma\delta}x_2 - \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)i \end{cases} \quad (43)$$

Thus, the spectrum of four-dimensional number $x = (x_1, x_2, x_3, x_4)$ in space $M_2(\alpha,\beta,\gamma,\delta)$ consists of four pairwise complex conjugate numbers of the form (43).

We will find in the similar way spectra of numbers in spaces$M_i(\alpha,\beta,\gamma,\delta), i = 3,4,5,6,7$. The spectrum of a number $x = (x_1, x_2, x_3, x_4)$ in space $M_3(\alpha,\beta,\gamma,\delta)$ is

$$\begin{cases} \mu_1 = \alpha x_1 - \sqrt{\beta\delta}x_3 + \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\gamma}x_4\right)i \\ \mu_2 = \alpha x_1 - \sqrt{\beta\delta}x_3 - \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\gamma}x_4\right)i \\ \mu_3 = \alpha x_1 + \sqrt{\beta\delta}x_3 + \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\gamma}x_4\right)i \\ \mu_4 = \alpha x_1 + \sqrt{\beta\delta}x_3 - \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\gamma}x_4\right)i \end{cases} \quad (44)$$

The spectrum of a number $x = (x_1, x_2, x_3, x_4)$ in space $M_4(\alpha,\beta,\gamma,\delta)$ is

$$\begin{cases} \mu_1 = \alpha x_1 - \sqrt{\beta\gamma}x_4 + \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\delta}x_3\right)i \\ \mu_2 = \alpha x_1 - \sqrt{\beta\gamma}x_4 - \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\delta}x_3\right)i \\ \mu_3 = \langle x_1 + \sqrt{\beta\gamma}x_4 + \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\delta}x_3\right)i \\ \mu_4 = \alpha x_1 + \sqrt{\beta\gamma}x_4 - \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\delta}x_3\right)i \end{cases} \quad (45)$$

The spectrum of a number $x = (x_1, x_2, x_3, x_4)$ in space $M_5(\alpha,\beta,\gamma,\delta)$ is

$$\begin{cases} \mu_1 = \alpha x_1 - \sqrt{\beta\gamma}x_4 + \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\delta}x_3\right)i \\ \mu_2 = \alpha x_1 - \sqrt{\beta\gamma}x_4 - \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\delta}x_3\right)i \\ \mu_3 = \alpha x_1 + \sqrt{\beta\gamma}x_4 + \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\delta}x_3\right)i \\ \mu_4 = \alpha x_1 + \sqrt{\beta\gamma}x_4 - \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\delta}x_3\right)i \end{cases} \quad (46)$$

The spectrum of a number $x = (x_1, x_2, x_3, x_4)$ in space $M_6(\alpha,\beta,\gamma,\delta)$ is

$$\begin{cases} \mu_1 = \alpha x_1 - \sqrt{\beta\delta}x_3 + \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\gamma}x_4\right)i \\ \mu_2 = \alpha x_1 - \sqrt{\beta\delta}x_3 - \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\gamma}x_4\right)i \\ \mu_3 = \alpha x_1 + \sqrt{\beta\delta}x_3 + \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\gamma}x_4\right)i \\ \mu_4 = \alpha x_1 + \sqrt{\beta\delta}x_3 - \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\gamma}x_4\right)i \end{cases} \quad (47)$$

The spectrum of a number $x = (x_1, x_2, x_3, x_4)$ in space $M_7(\alpha,\beta,\gamma,\delta)$ is

$$\begin{cases} \mu_1 = \alpha x_1 - \sqrt{\gamma\delta}x_2 + \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)i \\ \mu_2 = \alpha x_1 - \sqrt{\gamma\delta}x_2 - \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)i \\ \mu_3 = \alpha x_1 + \sqrt{\gamma\delta}x_2 + \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)i \\ \mu_4 = \alpha x_1 + \sqrt{\gamma\delta}x_2 - \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)i \end{cases} \quad (48)$$

Let's denote the spectrum of a number x by $\Lambda(x)$ and consider mapping $S: x \to \Lambda(x)$ in spaces $M_i(\alpha,\beta,\gamma,\delta), i = 2,3,4,5,6,7$.

Theorem 5. The mapping S is one-to-one and onto, so it is a bijection.

Proof. Let's prove the theorem only for space $M_2(\alpha,\beta,\gamma,\delta)$, for other spaces the proof is similar.

Let$x \neq y$, we will show that then $\Lambda(x) \neq \Lambda(y)$. Let's allow opposite, then it means that $\mu_i(x) = \mu_i(y), i = 1,2,3,4$, therefore

$$\alpha x_1 - \sqrt{\gamma\delta}x_2 + \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)i = \alpha y_1 - \sqrt{\gamma\delta}y_2 + \left(\sqrt{\beta\delta}y_3 - \sqrt{\beta\gamma}y_4\right)i,$$
$$\alpha x_1 - \sqrt{\gamma\delta}x_2 - \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)i = \text{赤}y_1 - \sqrt{\gamma\delta}y_2 - \left(\sqrt{\beta\delta}y_3 - \sqrt{\beta\gamma}y_4\right)i,$$
$$\alpha x_1 + \sqrt{\gamma\delta}x_2 + \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)i = \alpha y_1 + \sqrt{\gamma\delta}y_2 + \left(\sqrt{\beta\delta}y_3 + \sqrt{\beta\gamma}y_4\right)i,$$
$$\alpha x_1 + \sqrt{\gamma\delta}x_2 - \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)i = \alpha y_1 + \sqrt{\gamma\delta}y_2 - \left(\sqrt{\beta\delta}y_3 + \sqrt{\beta\gamma}y_4\right)i.$$

Moving the right parts to the left part and having collecting similar terms, we get

$$(\alpha x_1 - \alpha y_1) - \left(\sqrt{\gamma\delta}x_2 - \sqrt{\gamma\delta}y_2\right) + \left(\left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\delta}y_3\right) - \left(\sqrt{\beta\gamma}x_4 - \sqrt{\beta\delta}y_4\right)\right)i = 0,$$
$$(\alpha x_1 - \alpha y_1) - \left(\sqrt{\gamma\delta}x_2 - \sqrt{\gamma\delta}y_2\right) - \left(\left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\delta}y_3\right) - \left(\sqrt{\beta\gamma}x_4 - \sqrt{\beta\delta}y_4\right)\right)i = 0,$$

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$$(\alpha x_1-\alpha y_1)+(\sqrt{\gamma\delta}x_2-\sqrt{\gamma\delta}y_2)+\left((\sqrt{\beta\delta}x_3-\sqrt{\beta\delta}y_3)+(\sqrt{\beta\gamma}x_4-\sqrt{\beta\delta}y_4)\right)i=0,$$
$$(\alpha x_1-\alpha y_1)+(\sqrt{\gamma\delta}x_2-\sqrt{\gamma\delta}y_2)-\left((\sqrt{\beta\delta}x_3-\sqrt{\beta\delta}y_3)+(\sqrt{\beta\gamma}x_4-\sqrt{\beta\delta}y_4)\right)i=0.$$

From here we obtain

$$\begin{cases}\alpha x_1-\alpha y_1=0,\\ \sqrt{\gamma\delta}x_2-\sqrt{\gamma\delta}y_2=0,\\ \sqrt{\beta\delta}x_3-\sqrt{\beta\delta}y_3=0,\\ \sqrt{\beta\gamma}x_4-\sqrt{\beta\delta}y_4=0.\end{cases}$$

or, $x-y=0$. We received the contradiction with the condition $x\neq y$.

Back, we will show that one and only one four-dimensional number can correspond to any spectrum consisting of numbers of the form (43). Indeed, let $\mu_1=a+bi,\mu_2=a-bi,\mu_3=c+di,\mu_4=c-di$ is the spectrum of some four-dimensional number. Then it follows from formulas (43) that

$$\begin{cases}\alpha x_1=\frac{a+c}{2}\\ \sqrt{\gamma\delta}x_2=\frac{c-a}{2}\\ \sqrt{\beta\delta}x_3=\frac{b+d}{2}\\ \sqrt{\beta\gamma}x_4=\frac{d-c}{2}\end{cases}$$

Corollary. The only number having zero spectrum is the number $0=(0,0,0,0)$.

Corollary. Spectra of basic numbers in space $M_2(\alpha,\beta,\gamma,\delta)$ are $\Lambda(J_1)=(1,1,1,1)$, $\Lambda(J_2)=\frac{\sqrt{\gamma\delta}}{\alpha}(-1,-1,1,1)$, $\Lambda(J_3)=\frac{\sqrt{\beta\delta}}{\alpha}(i,-i,i,-i)$, $\Lambda(J_4)=\frac{\sqrt{\beta\gamma}}{\alpha}(-i,i,i,-i)$. Spectra of basic numbers in other spaces are written out similarly.

Theorem 6. For any four-dimensional number $x=(x_1,x_2,x_3,x_4)$ the equality is hold

$$|x|^4=\mu_1\cdot\mu_2\cdot\mu_3\cdot\mu_4. \quad (49)$$

Proof. Let's carry out the proof of the theorem also only for space $M_2(\alpha,\beta,\gamma,\delta)$. For other spaces the proof is absolutely similar.

It follows from equations (43) that $\mu_1\cdot\mu_2=\left(\alpha x_1-\sqrt{\gamma\delta}x_2\right)^2+\left(\sqrt{\beta\delta}x_3-\sqrt{\beta\gamma}x_4\right)^2\mu_3\cdot\mu_4=\left(\alpha x_1+\sqrt{\gamma\delta}x_2\right)^2+\left(\sqrt{\beta\delta}x_3+\sqrt{\beta\gamma}x_4\right)^2$. Therefore, it follows from equality (6) that $\mu_1\cdot\mu_2\cdot\mu_3\cdot\mu_4=|x|^4$.

Corollary. $|J_1|=1$, $|J_2|=\frac{\sqrt{\gamma\delta}}{\alpha}$, $|J_3|=\frac{\sqrt{\beta\delta}}{\alpha}$, $|J_4|=\frac{\sqrt{\beta\gamma}}{\alpha}$ in any space $M_i(\alpha,\beta,\gamma,\delta), i=2,3,4,5,6,7$.

Theorem 7. In each of six anisotropic spaces of four-dimensional numbers the following relations hold:

1) $\mu_i(x\pm y)=\mu_i(x)\pm\mu_i(y)$ for any $x\in R^4$, $y\in R^4$, $i=1,2,3,4$;
2) $\mu_i(x\cdot y)=\mu_i(x)\cdot\mu_i(y)$ for any $x\in R^4$, $y\in R^4$, $i=1,2,3,4$;
3) $\mu_i(b\cdot x)=b\cdot\mu_i(x)$, for any $x\in R^4$, $b\in R^1$, $i=1,2,3,4$;
4) $\mu_i(x^{-1})=(\mu_i(x))^{-1}$ for any nondegenerate $x\in R^4$, $i=1,2,3,4$,
5) $\mu_i(x^n)=\mu_i^n(x)$, for any $x\in R^4$, $i=1,2,3,4$, $n\in N$;

where $\mu_i(x)$ is the *i-th* component of the spectrum of four-dimensional number x.

Proof. Let's carry out only for space $M_2(\alpha,\beta,\gamma,\delta)$.

Relations 1) are obvious. Let's prove the relation 2). According with (5) and (43) ,

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$$\mu_1(x\cdot y)=\left(\alpha^2x_1y_1+\gamma\delta x_2y_2-\beta\delta x_3y_3-\beta\gamma x_4y_4-\alpha\sqrt{\gamma\delta}x_2y_1-\alpha\sqrt{\gamma\delta}x_1y_2+\beta\sqrt{\gamma\delta}x_4y_3+\beta\sqrt{\gamma\delta}x_3y_4\right)+\left(\alpha\sqrt{\beta\delta}x_3y_1+\gamma\sqrt{\beta\delta}x_4y_2+\alpha\sqrt{\beta\delta}x_1y_3+\gamma\sqrt{\beta\delta}x_2y_4-\alpha\sqrt{\beta\gamma}x_4y_1-\delta\sqrt{\beta\gamma}x_3y_2-\delta\sqrt{\beta\gamma}x_2y_3-\alpha\sqrt{\beta\gamma}x_1y_4\right)\cdot i,$$

$$\mu_2(x\cdot y)=\left(\alpha^2x_1y_1+\gamma\delta x_2y_2-\beta\delta x_3y_3-\beta\gamma x_4y_4-\alpha\sqrt{\gamma\delta}x_2y_1-\alpha\sqrt{\gamma\delta}x_1y_2+\beta\sqrt{\gamma\delta}x_4y_3+\beta\sqrt{\gamma\delta}x_3y_4\right)-\left(\alpha\sqrt{\beta\delta}x_3y_1+\gamma\sqrt{\beta\delta}x_4y_2+\alpha\sqrt{\beta\delta}x_1y_3+\gamma\sqrt{\beta\delta}x_2y_4-\alpha\sqrt{\beta\gamma}x_4y_1-\delta\sqrt{\beta\gamma}x_3y_2-\delta\sqrt{\beta\gamma}x_2y_3-\alpha\sqrt{\beta\gamma}x_1y_4\right)\cdot i,$$

$$\mu_3(x\cdot y)=\left(\alpha^2x_1y_1+\gamma\delta x_2y_2-\beta\delta x_3y_3-\beta\gamma x_4y_4+\alpha\sqrt{\gamma\delta}x_2y_1+\alpha\sqrt{\gamma\delta}x_1y_2-\beta\sqrt{\gamma\delta}x_4y_3-\beta\sqrt{\gamma\delta}x_3y_4\right)+\left(\alpha\sqrt{\beta\delta}x_3y_1+\gamma\sqrt{\beta\delta}x_4y_2+\alpha\sqrt{\beta\delta}x_1y_3+\gamma\sqrt{\beta\delta}x_2y_4+\alpha\sqrt{\beta\gamma}x_4y_1+\delta\sqrt{\beta\gamma}x_3y_2+\delta\sqrt{\beta\gamma}x_2y_3+\alpha\sqrt{\beta\gamma}x_1y_4\right)\cdot i,$$

$$\mu_4(x\cdot y)=\left(\alpha^2x_1y_1+\gamma\delta x_2y_2-\beta\delta x_3y_3-\beta\gamma x_4y_4+\alpha\sqrt{\gamma\delta}x_2y_1+\alpha\sqrt{\gamma\delta}x_1y_2-\beta\sqrt{\gamma\,\langle}x_4y_3-\beta\sqrt{\gamma\delta}x_3y_4\right)-\left(\alpha\sqrt{\beta\delta}x_3y_1+\gamma\sqrt{\beta\delta}x_4y_2+\alpha\sqrt{\beta\delta}x_1y_3+\gamma\sqrt{\beta\delta}x_2y_4+\alpha\sqrt{\beta\gamma}x_4y_1+\delta\sqrt{\beta\gamma}x_3y_2+\delta\sqrt{\beta\gamma}x_2y_3+\alpha\sqrt{\beta\gamma}x_1y_4\right)\cdot i.$$

Similarly,

$$\mu_1(x)=\left(\alpha x_1-\sqrt{\gamma\delta}x_2\right)+\left(\sqrt{\beta\delta}x_3-\sqrt{\beta\gamma}x_4\right)\cdot i,$$
$$\mu_2(x)=\left(\alpha x_1-\sqrt{\gamma\delta}x_2\right)-\left(\sqrt{\beta\delta}x_3-\sqrt{\beta\gamma}x_4\right)\cdot i,$$
$$\mu_3(x)=\left(\alpha x_1+\sqrt{\gamma\delta}x_2\right)+\left(\sqrt{\beta\delta}x_3+\sqrt{\beta\gamma}x_4\right)\cdot i,$$
$$\mu_4(x)=\left(\alpha x_1+\sqrt{\gamma\delta}x_2\right)-\left(\sqrt{\beta\delta}x_3+\sqrt{\beta\gamma}x_4\right)\cdot i.$$
$$\mu_1(y)=\left(\alpha y_1-\sqrt{\gamma\delta}y_2\right)+\left(\sqrt{\beta\delta}y_3-\sqrt{\beta\gamma}y_4\right)\cdot i,$$
$$\mu_2(y)=\left(\alpha y_1-\sqrt{\gamma\delta}y_2\right)-\left(\sqrt{\beta\delta}y_3-\sqrt{\beta\gamma}y_4\right)\cdot i,$$
$$\mu_3(y)=\left(\alpha y_1+\sqrt{\gamma\delta}y_2\right)+\left(\sqrt{\beta\delta}y_3+\sqrt{\beta\gamma}y_4\right)\cdot i,$$
$$\mu_4(y)=\left(\alpha y_1+\sqrt{\gamma\delta}y_2\right)-\left(\sqrt{\beta\delta}y_3+\sqrt{\beta\gamma}y_4\right)\cdot i.$$

Then

$$\mu_1(x)\cdot\mu_1(y)=\left(\alpha^2x_1y_1+\gamma\delta x_2y_2-\beta\delta x_3y_3-\beta\gamma x_4y_4-\alpha\sqrt{\gamma\delta}x_2y_1-\alpha\sqrt{\gamma\delta}x_1y_2+\beta\sqrt{\gamma\delta}x_4y_3+\beta\sqrt{\gamma\delta}x_3y_4\right)+\left(\alpha\sqrt{\beta\delta}x_3y_1+\gamma\sqrt{\beta\delta}x_4y_2+\alpha\sqrt{\beta\delta}x_1y_3+\gamma\sqrt{\beta\delta}x_2y_4-\alpha\sqrt{\beta\gamma}x_4y_1-\delta\sqrt{\beta\gamma}x_3y_2-\delta\sqrt{\beta\gamma}x_2y_3-\alpha\sqrt{\beta\gamma}x_1y_4\right)\cdot i=\mu_1(x\cdot y),$$

$$\mu_2(x)\cdot\mu_2(y)=\left(\alpha^2x_1y_1+\gamma\delta x_2y_2-\beta\delta x_3y_3-\beta\gamma x_4y_4-\alpha\sqrt{\gamma\delta}x_2y_1-\alpha\sqrt{\gamma\,\langle}x_1y_2+\beta\sqrt{\gamma\delta}x_4y_3+\beta\sqrt{\gamma\delta}x_3y_4\right)-\left(\alpha\sqrt{\beta\delta}x_3y_1+\gamma\sqrt{\beta\delta}x_4y_2+\alpha\sqrt{\beta\delta}x_1y_3+\gamma\sqrt{\beta\delta}x_2y_4-\alpha\sqrt{\beta\gamma}x_4y_1-\delta\sqrt{\beta\gamma}x_3y_2-\delta\sqrt{\beta\gamma}x_2y_3-\alpha\sqrt{\beta\gamma}x_1y_4\right)\cdot i=\mu_2(x\cdot y),$$

$$\mu_3(x)\cdot\mu_3(y)=\left(\alpha^2x_1y_1+\gamma\delta x_2y_2-\beta\delta x_3y_3-\beta\gamma x_4y_4+\alpha\sqrt{\gamma\delta}x_2y_1+\alpha\sqrt{\gamma\delta}x_1y_2-\beta\sqrt{\gamma\delta}x_4y_3-\beta\sqrt{\gamma\delta}x_3y_4\right)+\left(\alpha\sqrt{\beta\delta}x_3y_1+\gamma\sqrt{\beta\delta}x_4y_2+\alpha\sqrt{\beta\delta}x_1y_3+\gamma\sqrt{\beta\delta}x_2y_4+\alpha\sqrt{\beta\gamma}x_4y_1+\delta\sqrt{\beta\gamma}x_3y_2+\delta\sqrt{\beta\gamma}x_2y_3+\alpha\sqrt{\beta\gamma}x_1y_4\right)\cdot i=\mu_3(x\cdot y),$$

$$\mu_4(x)\cdot\mu_4(y)=\left(\alpha^2x_1y_1+\gamma\delta x_2y_2-\beta\delta x_3y_3-\beta\gamma x_4y_4+\alpha\sqrt{\gamma\delta}x_2y_1+\alpha\sqrt{\gamma\delta}x_1y_2-\beta\sqrt{\gamma\delta}x_4y_3-\beta\sqrt{\gamma\delta}x_3y_4\right)-\left(\alpha\sqrt{\beta\delta}x_3y_1+\gamma\sqrt{\beta\delta}x_4y_2+\alpha\sqrt{\beta\delta}x_1y_3+\gamma\sqrt{\beta\delta}x_2y_4+\alpha\sqrt{\beta\gamma}x_4y_1+\delta\sqrt{\beta\gamma}x_3y_2+\delta\sqrt{\beta\gamma}x_2y_3+\alpha\sqrt{\beta\gamma}x_1y_4\right)\cdot i=\mu_4(x\cdot y).$$

The ratio 3) follows from the ratio 2). Let's prove the ratio 4). 2) follows from the ratio that$\mu_i(x\cdot x^{-1})=\mu_i(x)\cdot\mu_i(x^{-1})$. On the other hand$x\cdot x^{-1}=J_1$. Follows from ratios (43) that $\mu_i(J_1)=1$for all$i=1,2,3,4$. Therefore$\mu_i(x^{-1})=\frac{1}{\mu_i(x)}$.

The relation 5) follows from the relation 2) as well.

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The proved theorems show that there are three various approaches for work with four-dimensional numbers: four-dimensional numbers, 4x4 matrices and four-dimensional imaginary numbers in the form of the spectrum. For carrying out arithmetic operations these approaches are equivalent. For the solution of the algebraic equations the most convenient is the spectra approach.

4. Application to the Solution of Systems of the Algebraic Equations

Let's consider the linear algebraic equation $ax = b$ where $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$ are given four-dimensional numbers. Let's assume at first that a *is* nondegenerate number, then this equation has the only solution$x = ba^{-1}$. Of course, we consider this equation in one of the spaces$M_i(\alpha, \beta, \gamma, \delta), i = 2,3,4,5,6,7$. Let's find the solution of the equation in isotropic space M_2.

For finding of the inverse number a^{-1} we will use the theorem 7, according to which we get $\mu_i(a^{-1}) = (\mu_i(a))^{-1}$ for all *i=1,2,3,4*. Therefore, in isotropic space M_2 we have

$$\mu_1^{-1}(a) = \frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} - \frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} i = (\mu_1(a))^{-1},$$

$$\mu_2^{-1}(a) = \frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} + \frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} i = (\mu_2(a))^{-1},$$

$$\mu_3^{-1}(a) = \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2} - \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2} i = (\mu_3(a))^{-1},$$

$$\mu_4^{-1}(a) = \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2} + \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2} i = (\mu_4(a))^{-1}.$$

Knowing the spectrum, we will restore number a^{-1}:

$$a^{-1} = \left(\frac{1}{2}\left(\frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} + \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2}\right), \frac{1}{2}\left(-\frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} + \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2}\right), \frac{1}{2}\left(-\frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} - \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2}\right), \frac{1}{2}\left(\frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} - \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2}\right)\right).$$

Then the solution $x = (x_1, x_2, x_3, x_4)$ of linear equation $ax = b$ in the space M_2 is:

$$x_1 = \frac{b_1}{2}\left(\frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} + \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2}\right) - \frac{b_2}{2}\left(\frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} - \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2}\right) + \frac{b_3}{2}\left(\frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} + \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2}\right) - \frac{b_4}{2}\left(\frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} - \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2}\right),$$

$$x_2 = \frac{b_2}{2}\left(\frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} + \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2}\right) - \frac{b_1}{2}\left(\frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} - \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2}\right) + \frac{b_4}{2}\left(\frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} + \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2}\right) - \frac{b_3}{2}\left(\frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} - \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2}\right),$$

$$x_3 = \frac{b_3}{2}\left(\frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} + \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2}\right) - \frac{b_4}{2}\left(\frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} - \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2}\right) - \frac{b_1}{2}\left(\frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} + \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2}\right) + \frac{b_2}{2}\left(\frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} - \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2}\right),$$

$$x_4 = \frac{b_4}{2}\left(\frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} + \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2}\right) - \frac{b_3}{2}\left(\frac{a_1-a_2}{(a_1-a_2)^2+(a_3-a_4)^2} - \frac{a_1+a_2}{(a_1+a_2)^2+(a_3+a_4)^2}\right) - \frac{b_2}{2}\left(\frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} + \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2}\right) + \frac{b_1}{2}\left(\frac{a_3-a_4}{(a_1-a_2)^2+(a_3-a_4)^2} - \frac{a_3+a_4}{(a_1+a_2)^2+(a_3+a_4)^2}\right).$$

We received the analog of formulas of Kramer for system

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$$\begin{cases} a_1x_1 + a_2x_2 - a_3x_3 - a_4x_4 = b_1 \\ a_2x_1 + a_1x_2 - a_4x_3 - a_3x_4 = b_2 \\ a_3x_1 + a_4x_2 + a_1x_3 + a_2x_4 = b_3{}' \\ a_4x_1 + a_3x_2 + a_2x_3 + a_1x_4 = b_4 \end{cases} \tag{50}$$

which is equivalent to the four-dimensional equation $ax = b$ in isotropic space M_2. Similar formulas can be obtained also for other four-dimensional spaces.

Let's assume now that a is a degenerate number. Then if $b = (0,0,0,0)$, then, according to the theorem 2, considered linear equation have infinitely many solutions, namely if a is a degenerate number of type II, then the solutions of the equation are all degenerate numbers of type I, if a is a degenerate number of type I, then the solutions of the equation are all degenerate numbers of type II.

Let's assume now that a and b be degenerate numbers, and $b \neq (0,0,0,0)$. Then if a and b are degenerate numbers of different types, then according to the same theorem 2, considered equation has no solution. If a and b are degenerate numbers of the same type, then the equation has infinitely many solutions. Indeed, according to the theorem 7 we have

$$\mu_i(a)\mu_i(x) = \mu_i(b), i = 1,2,3,4.$$

Suppose that a and b are degenerate numbers of the first type, that is $a = (c_1, c_1, c_2, c_2)$, $b = (d_1, d_1, d_2, d_2)$. Then $\mu(a) = (0,0,2c_1 + 2c_2i, 2c_1 - 2c_2i)$, $\mu(b) = (0,0,2d_1 + 2d_2i, 2d_1 - 2d_2i)$, but$\mu(x) = (x_1 - x_2 + (x_3 - x_4)i, x_1 - x_2 - (x_3 - x_4)i, x_1 + x_2 + (x_3 + x_4)i, x_1 + x_2 - (x_3 + x_4)i)$. Substituting these expressions in the last equations, we have

$$\begin{cases} (c_1 + c_2i)(x_1 + x_2 + (x_3 + x_4)i) = d_1 + d_2i \\ (c_1 - c_2i)(x_1 + x_2 - (x_3 + x_4)i) = d_1 - d_2i \end{cases}.$$

This system has infinitely many solutions

$$x_1 + x_2 = \frac{c_1d_1 + c_2d_2}{c_1^2 + c_2^2},\ x_3 + x_4 = \frac{c_1d_2 - c_2d_1}{c_1^2 + c_2^2}.$$

Respectively, solutions of linear equation are four-dimensional numbers$\left(x_1, -x_1 + \frac{c_1d_1 + c_2d_2}{c_1^2 + c_2^2}, x_3, -x_3 + \frac{c_1d_2 - c_2d_1}{c_1^2 + c_2^2}\right)$, for any real x_1 and x_2.

Similar calculations can be carried out in case when a and b are degenerate numbers of the second type too.

Thus, in four-dimensional spaces, linear equation $ax = b$ may have no solution, unique solution, or, infinite set of solutions. We showed this fact on the example of isotropic spaceM_2. Check in other isotropic and anisotropic spaces does not cause difficulties.

Let's consider the quadratic equation $x^2 = a$where $a = (a_1, a_2, a_3, a_4)$ is a given four-dimensional number. As the example we will consider this equation in isotropic spaceM_2. In this case this equation can be rewritten by the definition of multiplication (6) in space M_2 in the following form:

$$\begin{cases} x_1^2 + x_2^2 - x_3^2 - x_4^2 = a_1 \\ 2x_1x_2 - 2x_3x_4 = a_2 \\ 2x_1x_3 + 2x_2x_4 = a_3 \\ 2x_1x_4 + 2x_2x_3 = a_4 \end{cases}. \tag{51}$$

For the solution of this equation we will use the spectra approach. Let's assume at first that a is a nondegenerate number. According to the theorem 7, $\mu_i(x^2) = \mu_i^2(x) = \mu_i(a), i = 1,2,3,4$. By definition of the spectrum in space M_2

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$$\begin{gathered}(x_1 - x_2 + (x_3 - x_4)i)^2 = a_1 - a_2 + (a_3 - a_4)i \\ (x_1 - x_2 - (x_3 - x_4)i)^2 = a_1 - a_2 - (a_3 - a_4)i \\ (x_1 + x_2 + (x_3 + x_4)i)^2 = a_1 + a_2 + (a_3 + a_4)i' \\ (x_1 + x_2 - (x_3 + x_4)i)^2 = a_1 + a_2 - (a_3 + a_4)i\end{gathered}$$

or

$$\begin{gathered}(x_1 - x_2)^2 - (x_3 - x_4)^2 + 2(x_1 - x_2)(x_3 - x_4)i = a_1 - a_2 + (a_3 - a_4)i \\ (x_1 - x_2)^2 - (x_3 - x_4)^2 - 2(x_1 - x_2)(x_3 - x_4)i = a_1 - a_2 - (a_3 - a_4)i \\ (x_1 + x_2)^2 - (x_3 + x_4)^2 + 2(x_1 + x_2)(x_3 + x_4)i = a_1 + a_2 + (a_3 + a_4)i^{\cdot} \\ (x_1 + x_2)^2 - (x_3 + x_4)^2 - 2(x_1 + x_2)(x_3 + x_4)i = a_1 + a_2 - (a_3 + a_4)i\end{gathered}$$

From here, equating the real and imaginary parts, we have

$$\begin{gathered}(x_1 - x_2)^2 - (x_3 - x_4)^2 = a_1 - a_2 \\ 2(x_1 - x_2)(x_3 - x_4) = a_3 - a_4 \\ (x_1 + x_2)^2 - (x_3 + x_4)^2 = a_1 + a_2 . \\ 2(x_1 + x_2)(x_3 + x_4) = a_3 + a_4\end{gathered} \qquad (52)$$

This system has the following four solutions (if all composed the formulas given below make sense): $\left(\frac{b_+}{2\sqrt{2}} + \frac{b_-}{2\sqrt{2}}, \frac{b_+}{2\sqrt{2}} - \frac{b_-}{2\sqrt{2}}, \frac{a_3+a_4}{2\sqrt{2}b_+} + \frac{a_3-a_4}{2\sqrt{2}b_-}, \frac{a_3+a_4}{2\sqrt{2}b_+} - \frac{a_3-a_4}{2\sqrt{2}b_-}\right)$, $\left(-\frac{b_+}{2\sqrt{2}} + \frac{b_-}{2\sqrt{2}}, -\frac{b_+}{2\sqrt{2}} - \frac{b_-}{2\sqrt{2}}, -\frac{a_3+a_4}{2\sqrt{2}b_+} + \frac{a_3-a_4}{2\sqrt{2}b_-}, -\frac{a_3+a_4}{2\sqrt{2}b_+} - \frac{a_3-a_4}{2\sqrt{2}b_-}\right)$, $\left(\frac{b_+}{2\sqrt{2}} - \frac{b_-}{2\sqrt{2}}, \frac{b_+}{2\sqrt{2}} + \frac{b_-}{2\sqrt{2}}, \frac{a_3+a_4}{2\sqrt{2}b_+} - \frac{a_3-a_4}{2\sqrt{2}b_-}, \frac{a_3+a_4}{2\sqrt{2}b_+} + \frac{a_3-a_4}{2\sqrt{2}b_-}\right)$, $\left(-\frac{b_+}{2\sqrt{2}} - \frac{b_-}{2\sqrt{2}}, -\frac{b_+}{2\sqrt{2}} + \frac{b_-}{2\sqrt{2}}, -\frac{a_3+a_4}{2\sqrt{2}b_+} - \frac{a_3-a_4}{2\sqrt{2}b_-}, -\frac{a_3+a_4}{2\sqrt{2}b_+} + \frac{a_3-a_4}{2\sqrt{2}b_-}\right)$, where $b_+ = \sqrt{a_1 + a_2 + \sqrt{(a_1 + a_2)^2 + (a_3 + a_4)^2}}$, $b_- = \sqrt{a_1 - a_2 + \sqrt{(a_1 - a_2)^2 + (a_3 - a_4)^2}}$.

The obtained solutions are the square root from four-dimensional number a. In particular, in space M_2 solutions of the equation $x^2 = J_1 = (1,0,0,0)$ are four four-dimensional numbers (1,0,0,0), (0,1,0,0), (-1,0,0,0) and (0,-1,0,0). And solutions of the equation $x^2 = J_3 = (0,0,1,0)$ are the following numbers: $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0\right), \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$. If $a = J_2$, then the second terms of formulas for x_3 and x_4 are undefined. In this case solving the system (52) for this case we will receive the following solutions: $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$, $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

Thus, system of type (51) have four roots.

Let's assume now that $a = (a_1, a_1, a_3, a_3)$ is nonzero degenerate number of the first type. Then system (52) will take the form

$$\begin{gathered}(x_1 - x_2)^2 - (x_3 - x_4)^2 = 0 \\ 2(x_1 - x_2)(x_3 - x_4) = 0 \\ (x_1 + x_2)^2 - (x_3 + x_4)^2 = 2a_1 . \\ 2(x_1 + x_2)(x_3 + x_4) = 2a_3\end{gathered} \qquad (53)$$

System (53) have the following two solutions: $\frac{1}{2}\left(b_+, b_+, \frac{a_3}{b_+}, \frac{a_3}{b_+}\right)$ and $-\frac{1}{2}\left(b_+, b_+, \frac{a_3}{b_+}, \frac{a_3}{b_+}\right)$, where $b_+ = \sqrt{a_1 + \sqrt{a_1^2 + a_3^2}}$. Similarly, if $a = (a_1, -a_1, a_3, -a_3)$ is nonzero degenerate number of the second type,

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then the quadratic equation $x^2 = a$ has the following two solutions: $\frac{1}{2}\left(b_+, -b_+, \frac{a_3}{b_+}, -\frac{a_3}{b_+}\right)$ and $\frac{1}{2}\left(-b_+, b_+, -\frac{a_3}{b_+}, \frac{a_3}{b_+}\right)$.

That is, if a is degenerate number, then the considered quadratic equation has two solutions which are also degenerate numbers of the same type.
And at last, if $a=(0,0,0,0)$, then the considered equation has the only zero solution. Thus, we found all solutions of the equation $x^2 = a$ in isotropic spaceM_2. In other spaces everything is similar.
Moreover, in the four-dimensional space any quadratic equation $ax^2 + bx + c = 0$ has four roots which are defined by the formula

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad (54)$$

if a is nondegenerate number what will easily make sure of, having substituted this expression in the equation. Thus, we can draw the following conclusion: if a and $b^2 - 4ac$ are nondegenerate numbers, then the quadratic equation $ax^2 + bx + c = 0$ has 4 roots if a is nondegenerate number, and $b^2 - 4ac$ is the degenerate number, the quadratic equation has two roots and if$b^2 - 4ac = 0$, the quadratic equation has one root. Solutions of an equation are expressed by formula (54).
Let's assume now that a is degenerate number. Then we will consider various options of numbers b and c. If b is degenerate number of this kind, as a, and c is degenerate number of other type, then the equation has no solution. Indeed, then according to the theorem 2 $ax^2 + bx$is degenerate number of the same type, as a, and c is degenerate number of other type. Let's give various options of coefficients of the equation at which the quadratic equation has no solution, in table 7.

Table 7. Options of coefficients *of a, b, c*.

№	*a* coefficient	*b* coefficient	*c* coefficient	Solution
1	Degenerate number of the first type	Degenerate number of the first type	Nondegenerate number	There is no solution
2	Degenerate number of the first type	Degenerate number of the first type	Degenerate number of the second type	There is no solution
3	Degenerate number of the first type	Degenerate number of the second type	Degenerate number of the first type	There is no solution
4	Degenerate number of the first type	Degenerate number of the second type	Degenerate number of the second type	There is no solution
5	Degenerate number of the second type	Degenerate number of the first type	Degenerate number of the first type	There is no solution
6	Degenerate number of the second type	Degenerate number of the first type	Degenerate number of the second type	There is no solution
7	Degenerate number of the second type	degenerate number of the second type	Nondegenerate number	There is no solution
8	Degenerate number of the second type	Degenerate number of the second type	Degenerate number of the first type	There is no solution

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In other cases when the coefficient *of* a is degenerate number, the quadratic equation has two solutions. Let's consider, for example the case when $a = (a_1, a_1, a_3, a_3)$ is degenerate number of the first type, $b = (b_1, b_2, b_3, b_4)$ is nondegenerate number, $c = (c_1, -c_1, c_3, -c_3)$ is degenerate number of the second type, in isotropic spaceM_2. Having passed to the spectrum, we have

$$\mu_i(a)\mu_i^2(x) + \mu_i(b)\mu_i(x) + \mu_i(c) = 0, i = 1,2,3,4.$$

This system will be transformed to the following form:

$$\begin{array}{c} (b_1 - b_2)(x_1 - x_2) - (b_3 - b_4)(x_3 - x_4) + 2c_1 = 0 \\ (b_3 - b_4)(x_1 - x_2) + (b_1 - b_2)(x_3 - x_4) + 2c_3 = 0 \\ 2a_1[(x_1 + x_2)^2 - (x_3 + x_4)^2] - 4a_3(x_1 + x_2)(x_3 + x_4) + \\ +(b_1 + b_2)(x_1 + x_2) - (b_3 + b_4)(x_3 + x_4) = 0 \\ 2a_3[(x_1 + x_2)^2 - (x_3 + x_4)^2] + 4a_1(x_1 + x_2)(x_3 + x_4) + \\ +(b_3 + b_4)(x_1 + x_2) + (b_1 + b_2)(x_3 + x_4) = 0 \end{array}. \quad (55)$$

Let's introduce the following denotes:

$$\begin{array}{c} y_1 = x_1 - x_2 \\ y_2 = x_3 - x_4 \\ y_3 = x_1 + x_2 \\ y_4 = x_3 + x_4 \end{array}.$$

Then the system (55) will be rewritten as

$$\begin{array}{c} (b_1 - b_2)y_1 - (b_3 - b_4)y_2 + 2c_1 = 0 \\ (b_3 - b_4)y_1 + (b_1 - b_2)y_2 + 2c_3 = 0 \\ 2a_1[y_3^2 - y_4^2] - 4a_3y_3y_4 + (b_1 + b_2)y_3 - (b_3 + b_4)y_4 = 0. \\ 2a_3[y_3^2 - y_4^2] + 4a_1y_3y_4 + (b_3 + b_4)y_3 + (b_1 + b_2)y_4 = 0 \end{array} \quad (56)$$

From the first two equations we find

$$y_1 = -\frac{2c_1(b_1 - b_2) + 2c_3(b_3 - b_4)}{(b_1 - b_2)^2 + (b_3 - b_4)^2},$$

$$y_2 = \frac{2c_1(b_3 - b_4) - 2c_3(b_1 - b_2)}{(b_1 - b_2)^2 + (b_3 - b_4)^2}.$$

Multiply the third equation by a_3, the fourth equation by a_1 and subtract the third equation from the fourth:

$$4(a_1^2 + a_3^2)y_3y_4 + [a_1(b_3 + b_4) - a_3(b_1 + b_2)]y_3 + [a_1(b_1 + b_2) + a_3(b_3 + b_4)]y_4 = 0.$$

Further we multiply the third equation by a_1, the fourth equation by a_3 and add these equations:

$$2(a_1^2 + a_3^2)(y_3^2 - y_4^2) + [a_1(b_1 + b_2) + a_3(b_3 + b_4)]y_3 - [a_1(b_3 + b_4) - a_3(b_1 + b_2)]y_4 = 0.$$

Now we divide both parts of the last equations by $2(a_1^2 + a_3^2)$:

$$2y_3y_4 + \frac{[a_1(b_3 + b_4) - a_3(b_1 + b_2)]}{2(a_1^2 + a_3^2)}y_3 + \frac{[a_1(b_1 + b_2) + a_3(b_3 + b_4)]}{2(a_1^2 + a_3^2)}y_4 = 0.$$

$$y_3^2 - y_4^2 + \frac{[a_1(b_1 + b_2) + a_3(b_3 + b_4)]}{2(a_1^2 + a_3^2)}y_3 - \frac{[a_1(b_3 + b_4) - a_3(b_1 + b_2)]}{2(a_1^2 + a_3^2)}y_4 = 0.$$

Let's introduce new variables

$$z_3 = y_3 + \frac{[a_1(b_1 + b_2) + a_3(b_3 + b_4)]}{4(a_1^2 + a_3^2)},$$

$$z_4 = y_4 + \frac{[a_1(b_3 + b_4) - a_3(b_1 + b_2)]}{4(a_1^2 + a_3^2)}.$$

Then the last equations in new variables will take the form

$$z_3 z_4 = \frac{\left(a_1^2 - a_3^2\right)(b_1+b_2)(b_3+b_4) - a_1 a_3\left[(b_1+b_2)^2 - (b_3+b_4)^2\right]}{16\left(a_1^2+a_3^2\right)^2},$$

$$z_3^2 - z_4^2 = \frac{\left(a_1^2 - a_3^2\right)\left[(b_1+b_2)^2 - (b_3+b_4)^2\right] + 4a_1 a_3 (b_1+b_2)(b_3+b_4)}{16\left(a_1^2+a_3^2\right)^2}.$$

This system has two solutions

$$z_3 = \pm\frac{[a_1(b_1+b_2)+a_3(b_3+b_4)]}{4(a_1^2+a_3^2)},$$

$$z_4 = \pm\frac{[a_1(b_3+b_4)-a_3(b_1+b_2)]}{4(a_1^2+a_3^2)}.$$

From where

$$(y_3, y_4) = (0,0),\ (y_3, y_4) = \left(-\frac{[a_1(b_1+b_2)+a_3(b_3+b_4)]}{2(a_1^2+a_3^2)}, -\frac{[a_1(b_3+b_4)-a_3(b_1+b_2)]}{2(a_1^2+a_3^2)}\right).$$

Respectively,

$(x_1, x_2, x_3, x_4) = (-d_1, d_1, d_3, -d_4)$, $(x_1, x_2, x_3, x_4) = (-d_1 - d_2, d_1 - d_2, d_3 - d_4, -d_3 - d_4)$,

where $d_1 = \frac{c_1(b_1-b_2)+c_3(b_3-b_4)}{(b_1-b_2)^2+(b_3-b_4)^2}$, $d_2 = \frac{[a_1(b_1+b_2)+a_3(b_3+b_4)]}{4(a_1^2+a_3^2)}$, $d_3 = \frac{c_1(b_3-b_4)-c_3(b_1-b_2)}{(b_1-b_2)^2+(b_3-b_4)^2}$, $d_4 = \frac{[a_1(b_3+b_4)-a_3(b_1+b_2)]}{4(a_1^2+a_3^2)}$.

Thus we found two solutions of the quadratic equation in case when a is degenerate number of the first type, b is nondegenerate number, c is degenerate number of the second type, in isotropic spaceM_2. Other cases are investigated the same way.

Let's notice that in isotropic space M_2 the four-dimensional quadratic equation in components of four-dimensional number $x = (x_1, x_2, x_3, x_4)$ is written as

$$\begin{aligned}
&a_1\left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right) + 2a_2(x_1x_2 - x_3x_4) - 2a_3(x_1x_3 + x_2x_4) - 2a_4(x_1x_4 + x_2x_3) + \\
&\qquad + b_1x_1 + b_2x_2 - b_3x_3 - b_4x_4 + c_1 = 0 \\
&a_2\left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right) + 2a_1(x_1x_2 - x_3x_4) - 2a_4(x_1x_3 + x_2x_4) - 2a_3(x_1x_4 + x_2x_3) + \\
&\qquad + b_2x_1 + b_1x_2 - b_4x_3 - b_3x_4 + c_2 = 0 \\
&a_3\left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right) + 2a_4(x_1x_2 - x_3x_4) + 2a_1(x_1x_3 + x_2x_4) + 2a_2(x_1x_4 + x_2x_3) + \\
&\qquad + b_3x_1 + b_4x_2 + b_1x_3 + b_2x_4 + c_3 = 0 \\
&a_4\left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right) + 2a_3(x_1x_2 - x_3x_4) + 2a_2(x_1x_3 + x_2x_4) + 2a_1(x_1x_4 + x_2x_3) + \\
&\qquad + b_4x_1 + b_3x_2 + b_2x_3 + b_1x_4 + c_4 = 0
\end{aligned}, \quad (57)$$

and in one-dimensional mathematics there are no methods of solution of systems (57) type. We stated the simple method of finding of all roots of such systems here.

Similar to (57) systems can be written also in other four-dimensional spaces (anisotropic and isotropic) and all roots of such systems can be found.

5. Norms of Four-Dimensional Numbers

The symplectic module of four-dimensional number is not norm as the whole space of degenerate numbers have the zero module. The possibility of determination of norm of four-dimensional numbers turns spaces $M_i(\alpha, \beta, \gamma, \delta), i = 2,3,4,5,6,7$ of four-dimensional numbers into normed spaces that opens huge opportunities for expansion of results of one-dimensional mathematics on the four-dimensional case. Therefore the concept of norm plays the important role in definition of topology of four-dimensional spaces and building of the four-dimensional calculus.

In this section we will define the concept of norm and we investigate its properties.

Definition. Spectral norm of four-dimensional number $x = (x_1, x_2, x_3, x_4)$ is called the number

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$$\|x\| = \frac{1}{4}(|\mu_1(x)| + |\mu_2(x)| + |\mu_3(x)| + |\mu_4(x)|), \tag{58}$$

where $(\mu_1(x), \mu_2(x), \mu_3(x), \mu_4(x))$ is the spectrum of number x.

This definition is universal in the sense that it does not depend on index of the four-dimensional space.

Let's paint this definition for various spaces of four-dimensional numbers.

In space $M_2(\alpha,\beta,\gamma,\delta)$ the expression (58) is

$$\|x\| = \frac{1}{2}\left(\sqrt{\left(\alpha x_1 - \sqrt{\gamma\delta}x_2\right)^2 + \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)^2} + \sqrt{\left(\alpha x_1 + \sqrt{\gamma\delta}x_2\right)^2 + \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)^2}\right). \tag{59}$$

Similarly, the norm in space $M_3(\alpha,\beta,\gamma,\delta)$ has the form

$$\|x\| = \frac{1}{2}\left(\sqrt{\left(\alpha x_1 - \sqrt{\beta\delta}x_3\right)^2 + \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\gamma}x_4\right)^2} + \sqrt{\left(\alpha x_1 + \sqrt{\beta\delta}x_3\right)^2 + \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\gamma}x_4\right)^2}\right). \tag{60}$$

In space $M_4(\alpha,\beta,\gamma,\delta)$ the norm (58) is

$$\|x\| = \frac{1}{2}\left(\sqrt{\left(\alpha x_1 - \sqrt{\beta\gamma}x_4\right)^2 + \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\delta}x_3\right)^2} + \sqrt{\left(\alpha x_1 + \sqrt{\beta\gamma}x_4\right)^2 + \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\delta}x_3\right)^2}\right). \tag{61}$$

Further in spaces $M_5(\alpha,\beta,\gamma,\delta)$, $M_6(\alpha,\beta,\gamma,\delta)$, $M_7(\alpha,\beta,\gamma,\delta)$ norms have the form, respectively (62), (63) and (64):

$$\|x\| = \frac{1}{2}\left(\sqrt{\left(\alpha x_1 - \sqrt{\beta\gamma}x_4\right)^2 + \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\delta}x_3\right)^2} + \sqrt{\left(\alpha x_1 + \sqrt{\beta\gamma}x_4\right)^2 + \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\delta}x_3\right)^2}\right). \tag{62}$$

$$\|x\| = \frac{1}{2}\left(\sqrt{\left(\alpha x_1 - \sqrt{\beta\delta}x_3\right)^2 + \left(\sqrt{\gamma\delta}x_2 + \sqrt{\beta\gamma}x_4\right)^2} + \sqrt{\left(\alpha x_1 + \sqrt{\beta\delta}x_3\right)^2 + \left(\sqrt{\gamma\delta}x_2 - \sqrt{\beta\gamma}x_4\right)^2}\right). \tag{63}$$

$$\|x\| = \frac{1}{2}\left(\sqrt{\left(\alpha x_1 - \sqrt{\gamma\delta}x_2\right)^2 + \left(\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4\right)^2} + \sqrt{\left(\alpha x_1 + \sqrt{\gamma\delta}x_2\right)^2 + \left(\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4\right)^2}\right). \tag{64}$$

Theorem 8. The spectral norm (58) is norm in spaces $M_i(\alpha,\beta,\gamma,\delta), i = 2,3,\dots,7$.

Proof. For the proof it is necessary to prove that expression (58) satisfies the following conditions:

1) $\|x\| \geq 0$, $\|x\| = 0$ if and only if when $x = (0,0,0,0)$.
2) $\|a \cdot x\| = |a|\|x\|$ for any real number $a \in R$.
3) $\|x + y\| \leq \|x\| + \|y\|$ for any four-dimensional numbers x and y (triangle inequality).

The first two conditions are obvious. Let's prove triangle inequality, following [6]. For this purpose we will consider two separate inequalities:

$$(L_1M_2 - L_2M_1)^2 \geq 0,$$
$$(N_1K_2 - N_2K_1)^2 \geq 0,$$

where L_1, L_2, M_1, M_2, N_1, N_2, K_1, K_2 are arbitrary real numbers.

Expand the brackets and after algebraic transformations we receive

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$$2L_1L_2M_1M_2 \le {L_1}^2{M_2}^2 + {L_2}^2{M_1}^2$$
$$2N_1N_2K_1K_2 \le {N_1}^2{K_2}^2 + {N_2}^2{K_1}^2$$

Let's add to both parts of the first inequality $(L_1L_2)^2 + (M_1M_2)^2$, and the second inequality $(N_1N_2)^2 + (K_1K_2)^2$, and, taking the root, we will write

$$L_1L_2 + M_1M_2 \le \sqrt{\left({L_1}^2 + {M_1}^2\right)\left({L_2}^2 + {M_2}^2\right)}$$

$$N_1N_2 + K_1K_2 \le \sqrt{\left({N_1}^2 + {K_1}^2\right)\left({N_2}^2 + {K_2}^2\right)}$$

Let's double and will add to both parts of the first and second inequalities ${L_1}^2 + {L_2}^2 + {M_1}^2 + {M_2}^2$ and ${N_1}^2 + {N_2}^2 + {K_1}^2 + {K_2}^2$ respectively and take the root:

$$\sqrt{(L_1 + L_2)^2 + (M_1 + M_2)^2} \le \sqrt{{L_1}^2 + {M_1}^2} + \sqrt{{L_2}^2 + {M_2}^2},$$

$$\sqrt{(N_1 + N_2)^2 + (K_1 + K_2)^2} \le \sqrt{{N_1}^2 + {K_1}^2} + \sqrt{{N_2}^2 + {K_2}^2},$$

therefore

$$\frac{1}{2}\sqrt{(L_1 + L_2)^2 + (M_1 + M_2)^2} + \frac{1}{2}\sqrt{(N_1 + N_2)^2 + (K_1 + K_2)^2} \le$$
$$\frac{1}{2}\sqrt{{L_1}^2 + {M_1}^2} + \frac{1}{2}\sqrt{{N_1}^2 + {K_1}^2} + \frac{1}{2}\sqrt{{L_2}^2 + {M_2}^2} + \frac{1}{2}\sqrt{{N_2}^2 + {K_2}^2}. \quad (65)$$

Now depending on that in what space we work, we will enter the corresponding changes of variables, for example, for space $M_2(\alpha, \beta, \gamma, \delta)$ we will enter the following changes of variables:

$$L_1 = \alpha x_1 - \sqrt{\gamma\delta}x_2, L_2 = \alpha y_1 - \sqrt{\gamma\delta}y_2,$$
$$M_1 = \sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4, M_2 = \sqrt{\beta\delta}y_3 - \sqrt{\beta\gamma}y_4,$$
$$N_1 = \alpha x_1 + \sqrt{\gamma\delta}x_2, N_2 = \alpha y_1 + \sqrt{\gamma\delta}y_2,$$
$$K_1 = \sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4, K_2 = \sqrt{\beta\delta}y_3 + \sqrt{\beta\gamma}y_4.$$

Then inequality (65) turns into inequality $\|x + y\| \le \|x\| + \|y\|$.

Definition. Euclidean norm of four-dimensional number $x = (x_1, x_2, x_3, x_4)$ in space $M_i(\alpha, \beta, \gamma, \delta), i = 2,3, \dots, 7$ is called the number

$$\|x\|_E = \sqrt{\alpha^2 x_1^2 + \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2}. \quad (66)$$

Let's notice that there are following relations between spectral and Euclidean norms.

$M_2(\alpha, \beta, \gamma, \delta)$:

$$\|x\| = \frac{1}{2}\left(\sqrt{\|x\|_E^2 - 2\sqrt{\gamma\delta}(\alpha x_1x_2 + \beta x_3x_4)} + \sqrt{\|x\|_E^2 + 2\sqrt{\gamma\delta}(\alpha x_1x_2 + \beta x_3x_4)}\right). \quad (67)$$

$M_3(\alpha, \beta, \gamma, \delta)$:

$$\|x\| = \frac{1}{2}\left(\sqrt{\|x\|_E^2 - 2\sqrt{\beta\delta}(\alpha x_1x_3 + \gamma x_2x_4)} + \sqrt{\|x\|_E^2 + 2\sqrt{\beta\delta}(\alpha x_1x_3 + \gamma x_2x_4)}\right). \quad (68)$$

$M_4(\alpha, \beta, \gamma, \delta)$:

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$$\|x\| = \frac{1}{2}\left(\sqrt{\|x\|_E^2 - 2\sqrt{\beta\gamma}(\alpha x_1 x_4 + \delta x_2 x_3)} + \sqrt{\|x\|_E^2 + 2\sqrt{\beta\gamma}(\alpha x_1 x_4 + \delta x_2 x_3)}\right). \quad (69)$$

$M_5(\alpha,\beta,\gamma,\delta)$:

$$\|x\| = \frac{1}{2}\left(\sqrt{\|x\|_E^2 - 2\sqrt{\beta\gamma}(\alpha x_1 x_4 - \delta x_2 x_3)} + \sqrt{\|x\|_E^2 + 2\sqrt{\beta\gamma}(\alpha x_1 x_4 - \delta x_2 x_3)}\right). \quad (70)$$

$M_6(\alpha,\beta,\gamma,\delta)$:

$$\|x\| = \frac{1}{2}\left(\sqrt{\|x\|_E^2 - 2\sqrt{\beta\delta}(\alpha x_1 x_3 - \gamma x_2 x_4)} + \sqrt{\|x\|_E^2 + 2\sqrt{\beta\delta}(\alpha x_1 x_3 - \gamma x_2 x_4)}\right). \quad (71)$$

$M_7(\alpha,\beta,\gamma,\delta)$:

$$\|x\| = \frac{1}{2}\left(\sqrt{\|x\|_E^2 - 2\sqrt{\gamma\delta}(\_x_1 x_2 - \beta x_3 x_4)} + \sqrt{\|x\|_E^2 + 2\sqrt{\gamma\delta}(\alpha x_1 x_2 - \beta x_3 x_4)}\right). \quad (72)$$

Theorem 9. Spectral and Euclidean norms are equivalent in spaces $M_i(\alpha,\beta,\gamma,\delta), i = 2,3,\dots,7$, namely

$$\frac{1}{\sqrt{2}}\|x\|_E \le \|x\| \le \|x\|_E. \quad (73)$$

Proof. Let's carry out following [6]. Consider inequality

$$0 \le \sqrt{L^2 - M^2} \le L,$$

where $0 \le |M| \le L$. Let's add to both parts of inequality L and multiply all parts of inequality by 2:

$$2L \le L + M + 2\sqrt{L^2 - M^2} + L - M \le 4L.$$

From here after simple transformations we get

$$\frac{1}{\sqrt{2}}\sqrt{L} \le \frac{1}{2}\left(\sqrt{L+M} + \sqrt{L-M}\right) \le \sqrt{L}.$$

Out of these inequalities and out of equalities (67) – (72) it is easily brought (73) for all spaces $M_i(\alpha,\beta,\gamma,\delta), i = 2,3,\dots,7$.

Let's denote by N_0 the set of all points of the four-dimensional space for which $\|x\| = \|x\|_E$.

It follows from relations (67) – (72) that in spaces $M_i(\alpha,\beta,\gamma,\delta), i = 2,3,\dots,7$ the coordinates of points from the set N_0 satisfy to the following relations:

In $M_2(\alpha,\beta,\gamma,\delta)$: $\alpha x_1 x_2 + \beta x_3 x_4 = 0$.

In $M_3(\alpha,\beta,\gamma,\delta)$: $\alpha x_1 x_3 + \gamma x_2 x_4 = 0$.

In $M_4(\alpha,\beta,\gamma,\delta)$: $\alpha x_1 x_4 + \delta x_2 x_3 = 0$.

In $M_5(\alpha,\beta,\gamma,\delta)$: $\alpha x_1 x_4 - \delta x_2 x_3 = 0$.

In $M_6(\alpha,\beta,\gamma,\delta)$: $\alpha x_1 x_3 - \gamma x_2 x_4 = 0$.

In $M_7(\alpha,\beta,\gamma,\delta)$: $\alpha x_1 x_2 - \beta x_3 x_4 = 0$.

Theorem 10. In spaces $M_i(\alpha,\beta,\gamma,\delta), i = 2,3,\dots,7$ the set N_0 is closed with respect to multiplication and the following equalities hold:

1) For any $x, y \in N_0$, $\|xy\| = \|x\|\|y\|$, $\|xy\|_E = \|x\|_E\|y\|_E$.
2) For any $x, y \in O_I$, $\|x\| = \frac{1}{\sqrt{2}}\|x\|_E$, $\|xy\| = 2\|x\|\|y\|$, $\|xy\|_E = \sqrt{2}\|x\|_E\|y\|_E$.
3) For any $x, y \in O_{II}$, $\|x\| = \frac{1}{\sqrt{2}}\|x\|_E$, $\|xy\| = 2\|x\|\|y\|$, $\|xy\|_E = \sqrt{2}\|x\|_E\|y\|_E$.

Proof. We will carry out the proof for space $M_2(\alpha,\beta,\gamma,\delta)$, for other spaces it is proved the same way. Let $x, y \in N_0$ and $z = xy$. Then $\alpha z_1 z_2 + \beta z_3 z_4 = (\alpha x_1 x_2 + \beta x_3 x_4)\left(\alpha^2 y_1^2 + \gamma\delta y_2^2 + \beta\delta y_3^2 + \beta\gamma y_4^2\right) +$

$(\alpha y_1 y_2 + \beta y_3 y_4)(\alpha^2 x_1^2 + \gamma\delta x_2^2 + \beta\delta x_3^2 + \beta\gamma x_4^2) = 0$, that is $z \in N_0$. It means that in space $M_2(\alpha,\beta,\gamma,\delta)$ the set N_0 is closed with respect to multiplication.

Let's prove equality $\|xy\| = \|x\|\|y\|$. Represent $\|xy\|$ in the form

$$\|xy\| = \frac{1}{2}\left(\sqrt{A^2+B^2} + \sqrt{C^2+D^2}\right),$$

where $A = (\alpha x_1 - \sqrt{\gamma\delta}x_2)(\alpha y_1 - \sqrt{\gamma\delta}y_2) - (\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4)(\sqrt{\beta\delta}y_3 - \sqrt{\beta\gamma}y_4)$,
$B = (\sqrt{\beta\delta}x_3 - \sqrt{\beta\gamma}x_4)(\alpha y_1 - \sqrt{\gamma\delta}y_2) + (\alpha x_1 - \sqrt{\gamma\delta}x_2)(\sqrt{\beta\delta}y_3 - \sqrt{\beta\gamma}y_4)$,
$C = (\alpha x_1 + \sqrt{\gamma\delta}x_2)(\alpha y_1 + \sqrt{\gamma\delta}y_2) - (\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4)(\sqrt{\beta\delta}y_3 + \sqrt{\beta\gamma}y_4)$,
$D = (\sqrt{\beta\delta}x_3 + \sqrt{\beta\gamma}x_4)(\alpha y_1 + \sqrt{\gamma\delta}y_2) + (\alpha x_1 + \sqrt{\gamma\delta}x_2)(\sqrt{\beta\delta}y_3 + \sqrt{\beta\gamma}y_4)$.

Then removing the brackets and considering that $x, y \in N_0$ we have $\sqrt{A^2+B^2} = \|x\|_E\|y\|_E$ and $\sqrt{C^2+D^2} = \|x\|_E\|y\|_E$. Then, it follows from definition of the set N_0 that $\|xy\| = \|x\|\|y\|$.

Equality $\|xy\|_E = \|x\|_E\|y\|_E$ follows from the proved equality.

Now prove equality $\|x\| = \frac{1}{\sqrt{2}}\|x\|_E$ for degenerate numbers of the first type. Let $x = \left(c_1, \frac{\alpha}{\sqrt{\gamma\delta}}c_1, c_2, \sqrt{\frac{\delta}{\gamma}}c_2\right)$ and $y = \left(d_1, \frac{\alpha}{\sqrt{\gamma\delta}}d_1, d_2, \sqrt{\frac{\delta}{\gamma}}d_2\right)$ are degenerate numbers of the first type. Then $\|x\| = \sqrt{\alpha^2 c_1^2 + \beta\delta c_2^2}$, $\|x\|_E = \sqrt{2}\sqrt{\alpha^2 c_1^2 + \beta\delta c_2^2}$. Similarly $\|y\| = \sqrt{\alpha^2 d_1^2 + \beta\delta d_2^2}$, $\|y\|_E = \sqrt{2}\sqrt{\alpha^2 d_1^2 + \beta\delta d_2^2}$. $xy = \left(2\alpha c_1 d_1 - 2\frac{\beta\delta}{\alpha}c_2 d_2, \frac{\alpha}{\sqrt{\gamma\delta}}\left(2\alpha c_1 d_1 - 2\frac{\beta\delta}{\alpha}c_2 d_2\right), 2\alpha c_1 d_2 + 2\alpha c_2 d_1, \sqrt{\frac{\delta}{\gamma}}(2\alpha c_1 d_2 + 2\alpha c_2 d_1)\right)$. Respectively,

$$\|xy\| = 2\sqrt{\alpha^4 c_1^2 d_1^2 + \alpha^2\beta\delta c_1^2 d_2^2 + \alpha^2\beta\delta c_2^2 d_1^2 + \beta^2\delta^2 c_2^2 d_2^2} = 2\|x\|\|y\|,$$

$$\|xy\|_E = 2\sqrt{2}\sqrt{\alpha^4 c_1^2 d_1^2 + \alpha^2\beta\delta c_1^2 d_2^2 + \alpha^2\beta\delta c_2^2 d_1^2 + \beta^2\delta^2 c_2^2 d_2^2} = \sqrt{2}\|x\|_E\|y\|_E.$$

For degenerate numbers of the second type the approval of the theorem is proved similarly.

Thus, in the subspace N_0 the spectral norm accepts the greatest values coinciding with Euclidean norm, and in the subspace of degenerate numbers its value is less than values of Euclidean norm.

6. Discussions

In this work various anisotropic, including isotropic, spaces of four-dimensional numbers with associative and commutative multiplication are constructed. At the same time, in these spaces according to Frobenius [2] theorem, there are so-called "zero divisors". But good news is that the general form of zero divisors is described explicitly that allows "to fight" against them effectively.

In the article the algebra in four-dimensional numerical spaces is described and important feature of the constructed algebras is that they are natural generalization one-dimensional and two-dimensional (complex) algebras that favourably distinguishes them from the theory of quaternions. The given common solutions of linear and quadratic equations allowed to solve explicitly the system of four square equations with four unknown. It is shown that such system has generally four roots. Generally, the modern mathematics has no effective methods of solution of such systems.

And the most important, spectral and Euclidean norms are introduced that turns four-dimensional spaces into normed spaces. Further it is possible to define and investigate convergence of the sequences of four-dimensional numbers and to consider series of four-dimensional numbers. It is possible to enter various topology and to define functions of four-dimensional numbers.

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