Article

Not peer-reviewed version

Explanation of the Mass Pattern of the Low-Lying Scalar Nonet

Mihail Chizhov *, Emanuil Chizhov, Momchil Naydenov, Daniela Kirilova

Posted Date: 31 March 2025

doi: 10.20944/preprints202503.2281.v1

Keywords: symmetries in hadron physics; Bogoliubov – Nambu – Goldstone bosons; mass hierarchy in low-lying scalar nonet

Article

Explanation of the Mass Pattern of the Low-Lying Scalar Nonet

Mihail Chizhov [1,*,†], Emanuil Chizhov [2], Momchil Naydenov [1] and Daniela Kirilova [2]

1 Sofia University "St. Kliment Ohridski", Faculty of Physics;
2 Institute of Astronomy with NAO, Bulgarian Academy of Sciences
* Correspondence: mih@phys.sofia-uni.bg
† Current address: 5 James Bourchier Blvd., 1164 Sofia, Bulgaria

Abstract: The aim of this work is to propose an explanation of the inverse mass hierarchy of the low-lying nonet of the scalar mesons in the framework of the massless Nambu – Jona-Lasinio $U_R(3) \times U_L(3)$ quark model. The proposed explanation is based on symmetry principles. The collective meson states are described via quark-antiquark pairs, which condensates lead simultaneously to spontaneous breaking of the chiral and the flavour symmetry. It is shown that due to the flavour symmetry breaking two iso-doublets of $K_0^*(700)$ mesons play the role of Goldstone bosons. It is also proven that there exists a solution with degenerate masses of the $a_0(980)$ and $f_0(980)$ mesons and a zero mass of the $f_0(500)$ meson.

Keywords: symmetries in hadron physics; Bogoliubov – Nambu – Goldstone bosons; mass hierarchy in low-lying scalar nonet

1. Introduction

The symmetries and the symmetry breaking have a fundamental role in Physics. It is known that spontaneous symmetry breaking of the chiral $SU_R(2) \times SU_L(2)$ symmetry leads to massless Goldstone bosons, which role is played by the pions. In case of the spontaneous symmetry breaking of the $U_f(3)$ flavour symmetry $K_0^*(700)$ mesons play the role of massless Goldstone bosons. Futher on in this article we will call these bosons Bogoliubov – Nambu – Goldstone (BNG) bosons to highlight the legacy of Bogoliubov [1,2].

In this work we will consider simultaneous chiral and flavour spontaneous symmetry breaking and their important role for the explanation of the mass spectrum of the low-lying scalar mesons.

The pions, consisting of quark-antiquark pairs ($q\bar{q}$), have a very small mass in comparison with the other hadronic states. The smallness of pion mass would have been a problem of their constituent quark models, however, it is well known that the small mass of the pseudoscalar pions is explained by the mechanism of spontaneous breaking of the chiral $SU_R(2) \times SU_L(2)$ symmetry [3]. As a result of that symmetry breaking the pions play the role of massless BNG bosons [4,5].

There exists another problem connected with the mass of low-lying scalar mesons, namely the explanation of the inverse hierarchy mass in the low-lying scalar nonet. The constituent quark models, describing the mesons as quark-antiquark pairs could not explain this problem. Therefore, one of the first hypothesis for explanation of the inverse hierarchy was the work by Jaffe [6] proposing diquark-antidiquark structure ($qq\bar{q}\bar{q}$) for the scalar mesons. Another explanation, which allows to preserve the quark-antiquark structure of scalar mesons, see [7], uses the $U_A(1)$ anomaly term in dynamical chiral symmetry breaking of chiral effective theories. However, while this mechanism explains the difference between lowest-lying scalar meson $f_0(500)$ (or σ) and $a_0(980)$, it cannot explain the mass of the $K_0^*(700)$ (or κ) meson [8].

In this paper we propose an alternative explanation of the smallness of the mass of the $K_0^*(700)$ meson. It is based on the well-known mechanism of spontaneous symmetry breaking of the $U_f(3)$ flavour symmetry to $SU_f(2)$ isotopic symmetry. As a result of this symmetry breaking $K_0^*(700)$ mesons

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 March 2025 doi:10.20944/preprints202503.2281.v1

with isospin $I = 1/2$ play the role of massless BNG bosons. According to us this is a natural explanation of the small masses of the $K_0^*(700)$ mesons, similar to the explanation of the pion small masses. Note that the $SU_f(2)$ isotopic symmetry cannot be spontaneously broken, as shown in [9], i.e., there exists isotopic symmetry in nature.

In Section 2 we introduce a model of self-interaction of scalar quark currents, analogous to the Nambu – Jona-Lasinio (NJL) model [10,11]. We present a quantisation of this model and derive the effective potential for the scalar mesons nonet. In Section 3 we present the minimisation of this effective potential and we obtain the mass states of the scalar mesons. In the Conclusion we list the basic results and conclusions of this work.

2. The Model

Let us consider $U_R(3) \times U_L(3)$ chiral self-interaction of the scalar quark currents:

$$\mathcal{L} = i\overline{\Psi}\,\partial\!\!\!/\,\Psi + \frac{G_0}{2}\left(\overline{\Psi}\,\Psi\right)^2 + \frac{\widetilde{G}_0}{2}\sum_{a=1}^{8}\left(\overline{\Psi}\,\lambda_a\Psi\right)^2, \tag{1}$$

where, $\Psi = (u\ d\ s)^T$ is the triplet of massless color quarks[1], while λ_a $(1,...,8)$ are the Gell-Mann matrices. Here, G_0 and $\widetilde{G}_0$ are *positive* constants of the self-interaction of the singlet state quark scalar current and the octet states of the quark scalar currents, correspondingly, and both have a dimension $[mass]^{-2}$. We consider massless quarks, which will obtain masses due to spontaneous symmetry breaking.

It is obvious that the quantisation of this Lagrangian in perturbation theory on the dimensional constants G_0 and $\widetilde{G}_0$ will result into nonrenormalisable theory. Therefore, we will use the linearisation method of this Lagrangian developed in [12,13]. The linearisation of the Lagrangian (eq. 1) leads to the following equivalent on the classical level Lagrangian:

$$\mathcal{L} = i\overline{\Psi}\,\partial\!\!\!/\,\Psi + g_0\overline{\Psi}\,\Psi\,S_0 - \frac{g_0^2}{2G_0}\,S_0^2 + \sum_{a=1}^{8}\left(g_a\overline{\Psi}\,\lambda_a\Psi\,S_a - \frac{g_a^2}{2\widetilde{G}_0}\,S_a^2\right) \tag{2}$$

where we have introduced the auxiliary fields

$$S_0 = \frac{G_0}{g_0}\,\overline{\Psi}\Psi \quad \text{and} \quad S_a = \frac{\widetilde{G}_0}{g_a}\,\overline{\Psi}\,\lambda_a\Psi, \tag{3}$$

which will play the role of collective excitation states of the corresponding currents. Here, g_0 and g_a are the dimensionless Yukawa coupling constants.

All the collective modes become dynamical as a result of the self-energy quantum corrections from fermion loops (Figure 1a).

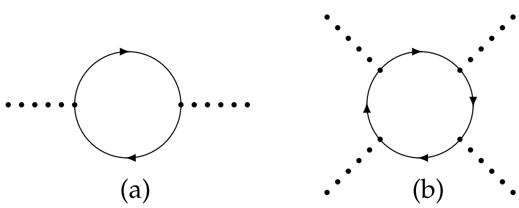

Figure 1. (a) Radiative correction to self-energy parts. (b) Radiative correction to self-interactions.

[1] Here the color indices are suppressed.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 March 2025 doi:10.20944/preprints202503.2281.v1

Let us consider the self-energy quantum corrections to the scalar field S_0:

$$\begin{aligned}\Pi_{00}(q) &= ig_0^2 N_C \mathrm{Tr}[\mathbb{1}] \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{Tr}\Big[(\not p - m_0)^{-1}(\not p - \not q - m_0)^{-1}\Big] \\ &= 12 i g_0^2 N_C \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 - m_0^2} - 6 i g_0^2 N_C \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{(p^2 - m_0^2)^2}\, q^2 + \text{finite terms } \mathcal{O}(q^4) \\ &= 12 g_0^2 N_C I_2 + 6 g_0^2 N_C I_0\, q^2 + \text{finite terms } \mathcal{O}(q^4), \end{aligned} \tag{4}$$

and to the scalar fields S_a

$$\begin{aligned}\Pi_{ab}(q) &= i g_a g_b N_C \mathrm{Tr}[\lambda_a \lambda_b] \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{Tr}\Big[(\not p - m_0)^{-1}(\not p - \not q - m_0)^{-1}\Big] \\ &= 8 i g_a^2 N_C \delta_{ab} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 - m_0^2} - 4 i g_a^2 N_C \delta_{ab} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{(p^2 - m_0^2)^2}\, q^2 + \text{finite terms } \mathcal{O}(q^4) \\ &= 8 g_a^2 N_C I_2 \delta_{ab} + 4 g_a^2 N_C I_0 \delta_{ab}\, q^2 + \text{finite terms } \mathcal{O}(q^4), \end{aligned} \tag{5}$$

where p is the internal and q is the external momentum, N_C denotes the number of colors, and

$$I_2 \equiv i \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 - m_0^2} = \int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} \frac{1}{p_E^2 + m_0^2} > 0 \tag{6}$$

is the quadratically divergent integral, while

$$I_0 \equiv -i \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{(p^2 - m_0^2)^2} = \int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} \frac{1}{(p_E^2 + m_0^2)^2} > 0 \tag{7}$$

is a logarithmically divergent integral, which are both positive in Euclidian momentum space.

Here we have introduced the small current mass m_0 for the quark, which will help to avoid the infrared divergences in the denominator and which will be neglected in the numerator. This kind of quark mass introduction explicitly breaks the chiral symmetry and is called *soft* symmetry breaking. Such symmetry breaking does not cause extra ultraviolet divergences in the scalar particles masses. Due to the dimensionless of the Yukawa coupling constants g_0 and g_a in the four dimensional space there exist only two types of divergent integrals (6) and (7). The ultraviolet divergences in (6) and (7) are removed using one of the known regularisation methods.

The first terms in the last lines of Equations (4) and (5) represent the corrections to the mass terms of the scalar fields S_0: $\mu^2 = g_0^2/G_0 - 12 g_0^2 N_C I_2$ and S_a: $\tilde{\mu}^2 = g_a^2/\widetilde{G}_0 - 8 g_a^2 N_C I_2$. The second terms represent the kinetic parts of the scalar fields. For a correct normalisation of the scalar fields wave functions the following requirements should be fulfilled:

$$6 g_0^2 N_C I_0 = 4 g_a^2 N_C I_0 = 1. \tag{8}$$

Due to the *dynamical origin* of the kinetic terms all the interactions in the NJL model are descried by a single dimensionless coupling constant

$$g \equiv g_0 = \sqrt{2/3}\, g_a. \tag{9}$$

Another essential point of the NJL model is the generation of the self-interactions of the scalar fields, which self-interactions lead to a spontaneous dynamical breaking of the chiral symmetry. Thus, at a quantum level, due to the radiative corrections (Figure 1b) in the effective Lagrangian local terms

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 March 2025 doi:10.20944/preprints202503.2281.v1

appear for the self-interaction of the scalar mesons with zero external momentum and corresponding symmetry factors:

$$\Box_{0000}(0) = \frac{i}{4} g_0^4 N_C \mathrm{Tr}[\mathbb{1}]\Box = -3g_0^4 N_C I_0 + \mathcal{O}(m_0^2) \approx -\frac{1}{2}g^2; \quad (10)$$

$$\Box_{00aa}(0) = \frac{3i}{2} g_0^2 g_a^2 N_C \mathrm{Tr}\left[\lambda_a^2\right]\Box = -12 g_0^2 g_a^2 N_C I_0 + \mathcal{O}(m_0^2) \approx -3g^2; \quad (11)$$

$$\begin{aligned}\Box_{0aab}(0) &= 3i g_0 g_a^2 g_b N_C \mathrm{Tr}\left[\lambda_a^2 \lambda_b\right]\Box = -12 g_0 g_a^2 g_b N_C I_0 d^{aab} + \mathcal{O}(m_0^2) \\ &\approx \begin{cases} -3\sqrt{\frac{3}{2}}g^2, \text{ for} & a = 4,5 \text{ and } b = 3; \\ 3\sqrt{\frac{3}{2}}g^2, \text{ for} & a = 6,7 \text{ and } b = 3; \\ -3\sqrt{2}g^2, \text{ for} & a = 1,2,3 \text{ and } b = 8; \\ \frac{3}{\sqrt{2}}g^2, \text{ for} & a = 4,5,6,7 \text{ and } b = 8; \end{cases}\end{aligned} \quad (12)$$

$$\Box_{0888}(0) = i g_0 g_8^3 N_C \mathrm{Tr}\left[\lambda_8^3\right]\Box = \frac{8}{\sqrt{3}} g_0 g_8^3 N_C I_0 + \mathcal{O}(m_0^2) \approx \sqrt{2}g^2; \quad (13)$$

$$\Box_{0abc}(0) = 3i g_0 g_a g_b g_c N_C \mathrm{Tr}[\lambda_a\{\lambda_b, \lambda_c\}]\Box = -48 g_0 g_a g_b g_c N_C I_0 d^{abc} + \mathcal{O}(m_0^2) \approx -6\sqrt{6}g^2 d^{abc}; \quad (14)$$

$$\Box_{aaaa}(0) = \frac{i}{4} g_a^4 N_C \mathrm{Tr}\left[\lambda_a^4\right]\Box = -2 g_a^4 N_C I_0 + \mathcal{O}(m_0^2) \approx -\frac{3}{4}g^2; \quad (15)$$

$$\Box_{aabb}(0) = \frac{i}{2} g_a^2 g_b^2 N_C \mathrm{Tr}\left[\lambda_a^2 \lambda_b^2 + \lambda_a \lambda_b \{\lambda_a, \lambda_b\}\right]\Box = -4 g_a^2 g_b^2 N_C I_0 + \mathcal{O}(m_0^2) \approx -\frac{3}{2}g^2; \quad (16)$$

Here, d^{abc} are totally symmetric structure constants of $su(3)$ algebra, where the indices a, b and c obtain different values non equal to each others, and the integral on the internal momentum p

$$\Box = \int \frac{d^4 p}{(2\pi)^4} \mathrm{Tr}\left[(\not{p} - m_0)^{-1}(\not{p} - m_0)^{-1}(\not{p} - m_0)^{-1}(\not{p} - m_0)^{-1}\right] \quad (17)$$

comes from the fermionic loop in Figure 1b.

The last expressions (10, 11, 12, 13, 14, 15, 16) were obtained using the normalisation condition (8), the relation between coupling constants (9) and neglecting the small current quark mass m_0, which was introduced only to remove infrared divergencies. The normalisation condition (8) requires that the leading term in the effective Lagrangian consists only of the divergent terms of the diagrams with zero external momentum [14], depicted in Figure 1b.

Thus, the effective potential reads:

$$\begin{aligned} V_{\text{eff}} &= \frac{\mu^2}{2}S_0^2 + \frac{\tilde{\mu}^2}{2}\sum_{a=1}^{8} S_a^2 + \frac{g^2}{2}S_0^4 + 3g^2 S_0^2 \sum_{a=1}^{8} S_a^2 + \frac{3g^2}{4}\left(\sum_{a=1}^{8} S_a^2\right)^2 \\ &+ \frac{3\sqrt{3}}{\sqrt{2}} g^2 S_0 S_3 \left(S_4^2 + S_5^2 - S_6^2 - S_7^2\right) - \sqrt{2}g^2\, S_0 S_8^3 \\ &+ 3\sqrt{2}g^2\, S_0 S_8 \left(S_1^2 + S_2^2 + S_3^2 - \frac{S_4^2 + S_5^2 + S_6^2 + S_7^2}{2}\right) \\ &+ 3\sqrt{6}g^2 S_0 (S_1 S_4 S_6 + S_1 S_5 S_7 - S_2 S_4 S_7 + S_2 S_5 S_6). \end{aligned} \quad (18)$$

The advantage of this potential is that it depends only on one dimensionless constant g^2 and two dimensional parameters of the mass: μ^2 and $\tilde{\mu}^2$. This allows to find exact solutions of spontaneous

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 March 2025 doi:10.20944/preprints202503.2281.v1

symmetry breaking and the mass spectrum of the scalar mesons, which will be presented in the next section.

3. Symmetry Breaking and Mass Spectrum of Scalar Mesons

Due to the spontaneous symmetry breaking the scalar fields obtain non-zero vacuum expectation values. The physical vacuum must conserve the electric charge and the quark flavour. Therefore, only the scalar fields S_0, S_3 and S_8, which interact with the diagonal combinations of quark-antiquark flavours, can have non-zero vacuum expectation values. In order to find the minimum of the effective potential (18), we will differentiate the potential only on these degrees of freedom. The minimisation leads to the following system of equations:

$$\begin{cases} \left.\frac{\partial V_{\text{eff}}}{\partial S_0}\right|_{\substack{S_0=\langle S_0\rangle \\ S_3=\langle S_3\rangle \\ S_8=\langle S_8\rangle}} = \mu^2\langle S_0\rangle + 2g^2\langle S_0\rangle^3 + 6g^2\langle S_0\rangle\left(\langle S_3\rangle^2+\langle S_8\rangle^2\right) \\ \qquad\qquad -\sqrt{2}g^2\langle S_8\rangle^3 + 3\sqrt{2}g^2\langle S_3\rangle^2\langle S_8\rangle = 0, \\ \left.\frac{\partial V_{\text{eff}}}{\partial S_3}\right|_{\substack{S_0=\langle S_0\rangle \\ S_3=\langle S_3\rangle \\ S_8=\langle S_8\rangle}} = \left[\tilde{\mu}^2 + 3g^2\left(2\langle S_0\rangle^2+\langle S_3\rangle^2+\langle S_8\rangle^2+2\sqrt{2}\langle S_0\rangle\langle S_8\rangle\right)\right]\langle S_3\rangle = 0, \\ \left.\frac{\partial V_{\text{eff}}}{\partial S_8}\right|_{\substack{S_0=\langle S_0\rangle \\ S_3=\langle S_3\rangle \\ S_8=\langle S_8\rangle}} = \tilde{\mu}^2\langle S_8\rangle + 3g^2\left(2\langle S_0\rangle^2+\langle S_3\rangle^2+\langle S_8\rangle^2-\sqrt{2}\langle S_0\rangle\langle S_8\rangle\right)\langle S_8\rangle \\ \qquad\qquad +3\sqrt{2}g^2\langle S_0\rangle\langle S_3\rangle^2 = 0. \end{cases} \tag{19}$$

Note, that for $a = 1,2,4,5,6,7 : \langle S_a\rangle = 0$ due to neutrality of vacuum expectation values. It has also been shown in [9], that $SU_f(2)$ group cannot be spontaneously broken and $\langle S_3\rangle = 0$ is the right solution of the system (19), while $\langle S_0\rangle \neq 0$ and $\langle S_8\rangle \neq 0$ acquire non-zero vacuum expectation values.

The spontaneous symmetry breaking is possible only at strong coupling constants G_0 and $\widetilde{G}_0$, when the massive parameters have negative values, i.e., $\mu^2 < 0$ and $\tilde{\mu}^2 < 0$. To avoid the irrational coefficients and the dimensional parameters in (19), we introduce the dimensionless variables $x = 3\sqrt{2}g\langle S_0\rangle/\sqrt{-\tilde{\mu}^2}$, $z = 3g\langle S_8\rangle/\sqrt{-\tilde{\mu}^2}$ and $r^2 = \mu^2/\tilde{\mu}^2$. Then the constituent quark masses can be obtained from the following relations:

$$m_u = m_d = -\sqrt{\frac{-\tilde{\mu}^2}{18}}(x+z), \tag{20}$$

$$m_s = -\sqrt{\frac{-\tilde{\mu}^2}{18}}(x-2z). \tag{21}$$

The first and the third equations of the system (19) in the new dimensionless variables read:

$$\begin{cases} x^2 + 6z^2 - 2\frac{z^3}{x} = 9r^2, \\ x^2 - xz + z^2 = 3. \end{cases} \tag{22}$$

Before solving this system, let us find first the square of the masses of the scalar meson isotriplet $a_0^{\pm}(980) = (S_1 \mp iS_2)/\sqrt{2}$, $a_0^0(980) = S_3$:

$$\mathcal{M}_i^2 = \left.\frac{\partial^2 V_{\text{eff}}}{\partial S_i^2}\right|_{\substack{S_0=\langle S_0\rangle \\ S_8=\langle S_8\rangle}} = 9\sqrt{2}g^2\langle S_0\rangle\langle S_8\rangle = (-\tilde{\mu}^2)\,xz, \text{ where } i = 1,2,3. \tag{23}$$

This result shows that for non negative square masses the vacuum expectation values of S_0 and S_8 should have the same signs. We should note that the solutions of the system (22) are invariant with respect to simultaneous sign changing of the vacuum expectation values: $x \to -x$ and $z \to -z$. Therefore, for definiteness, we will search solutions for positive vacuum expectation values.

 doi:10.20944/preprints202503.2281.v1

The solutions of the second equation of the system (22) are:

$$z_{\pm} = \frac{x \pm \sqrt{3(4-x^2)}}{2}. \tag{24}$$

Substituting these solutions into the first equation of the system (22) the following expression is obtained:

$$\pm\sqrt{4-x^2}\left(x^2-1\right) = \sqrt{3}\,ax, \text{ where } a = r^2 - 1. \tag{25}$$

In order to solve this equation analytically it is necessary to square its right and left hand sides. Thus, using the substitution $x^2 = t$, a cubic equation follows:

$$4 - 9t + 6t^2 - t^3 = 3a^2 t. \tag{26}$$

The solutions of this equation are illustrated in Figure 2, where $t_1 = 2 - 2\sqrt[3]{6\sqrt{3}-10}$ and $t_0 = 1 + \sqrt{3}$.

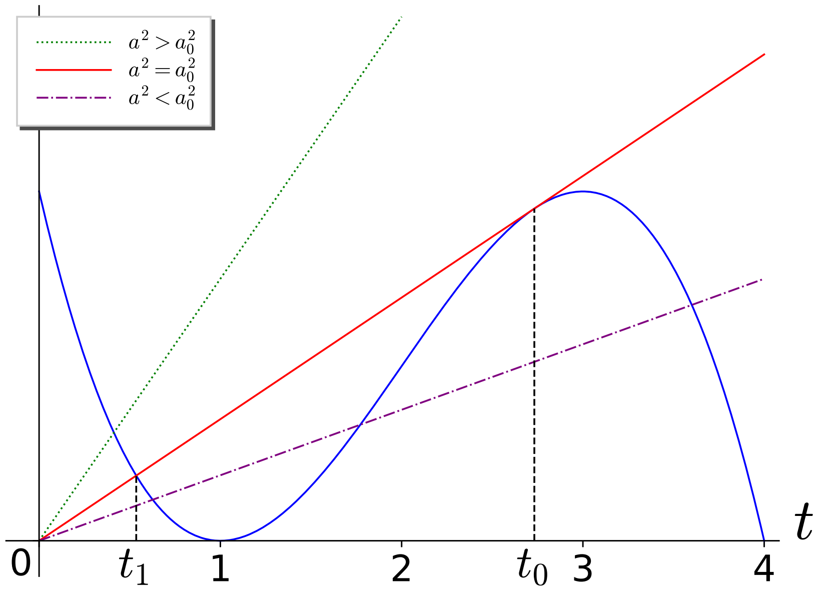

Figure 2. The graphical illustration of solutions of the cubic Equation (26)

It is obvious from Figure 2, that three real roots exist for $0 < a^2 < a_0^2$, where $a_0^2 = 2\sqrt{3} - 3$. Vice versa, for $a^2 > a_0^2$ only one real root exists. For $a^2 = 0$ and $a^2 = a_0^2$ we obtain only two real roots. From Equation (25) it follows that $x^2 \leq 4$.

Let us describe all the exact solutions of Equation (26) dividing them into three intervals, such that within each interval one single solution exists. The *first* case corresponds to the interval $0 < x^2 \leq 1$. This interval consists of two subintervals, namely $0 < x^2 \leq t_1$ and $t_1 \leq x^2 \leq 1$. In the first subinterval the solution with one real root for $a^2 \geq a_0^2$ is:

$$x^2 = 2 - \sqrt[3]{3a^2 - 1 + \sqrt{Q}} - \sqrt[3]{3a^2 - 1 - \sqrt{Q}}, \text{ where } Q = a^2(a^4 + 6a^2 - 3). \tag{27}$$

In the other subinterval, the solution for $0 \leq a^2 \leq a_0^2$ reads:

$$x^2 = 2 + 2\sqrt{1-a^2}\cos\left(\frac{1}{3}\arccos\frac{1-3a^2}{\sqrt{(1-a^2)^3}} + \frac{2\pi}{3}\right). \tag{28}$$

This solution corresponds to one of the three real roots of Equation (26).

 doi:10.20944/preprints202503.2281.v1

In the *second* case, corresponding to the interval $1 \leq x^2 \leq t_0$, the solution for $0 \leq a^2 \leq a_0^2$ reads:

$$x^2 = 2 + 2\sqrt{1-a^2}\cos\left(\frac{1}{3}\arccos\frac{1-3a^2}{\sqrt{(1-a^2)^3}} - \frac{2\pi}{3}\right). \tag{29}$$

In the *third* case, corresponding to the interval $t_0 \leq x^2 \leq 4$, the solution for $0 \leq a^2 \leq a_0^2$ reads:

$$x^2 = 2 + 2\sqrt{1-a^2}\cos\left(\frac{1}{3}\arccos\frac{1-3a^2}{\sqrt{(1-a^2)^3}}\right). \tag{30}$$

Let us now calculate all the masses of the scalar meson nonet. We have already done this for the isotriplet $a_0(980)$ (23). The squares of the masses of the scalar mesons κ with isospin $I = 1/2$: $\kappa^{\pm} = (S_4 \mp iS_5)/\sqrt{2}$, $\kappa^0 = (S_6 - iS_7)/\sqrt{2}$, $\bar{\kappa}^0 = (S_6 + iS_7)/\sqrt{2}$

$$\mathcal{M}_s^2 = \frac{\partial^2 V_{\text{eff}}}{\partial S_s^2}\bigg|_{\substack{S_0=\langle S_0\rangle \\ S_8=\langle S_8\rangle}} = \tilde{\mu}^2 + 3g^2\left(2\langle S_0\rangle^2 - \sqrt{2}\langle S_0\rangle\langle S_8\rangle + \langle S_8\rangle^2\right) = 0, \tag{31}$$

are equal to zero, where $s = 4, 5, 6, 7$ and the last equality follows from the third equation of the system (19) with $\langle S_3\rangle = 0$. The fact that the squares of the masses are equal to zero for these states is a direct consequence of the Goldstone's theorem [4,5].

Now let us calculate the squares of the masses of the isosinglet states with isospin $I = 0$, S_0 and S_8:

$$\mathcal{M}_0^2 = \frac{\partial^2 V_{\text{eff}}}{\partial S_0^2}\bigg|_{\substack{S_0=\langle S_0\rangle \\ S_8=\langle S_8\rangle}} = \mu^2 + 6g^2\left(\langle S_0\rangle^2 + \langle S_8\rangle^2\right) = (-\tilde{\mu}^2)\,\frac{2}{9}\left(x^2 + \frac{z^3}{x}\right), \tag{32}$$

$$\mathcal{M}_8^2 = \frac{\partial^2 V_{\text{eff}}}{\partial S_8^2}\bigg|_{\substack{S_0=\langle S_0\rangle \\ S_8=\langle S_8\rangle}} = \tilde{\mu}^2 + 3g^2\left(2\langle S_0\rangle^2 - 2\sqrt{2}\langle S_0\rangle\langle S_8\rangle + 3\langle S_8\rangle^2\right)$$
$$= (-\tilde{\mu}^2)\,\frac{z}{3}(2z - x), \tag{33}$$

$$\mathcal{M}_{08}^2 = \frac{\partial^2 V_{\text{eff}}}{\partial S_0 \partial S_8}\bigg|_{\substack{S_0=\langle S_0\rangle \\ S_8=\langle S_8\rangle}} = 3g^2\left(4\langle S_0\rangle - 2\sqrt{2}\langle S_8\rangle\right)\langle S_8\rangle = (-\tilde{\mu}^2)\,\frac{\sqrt{2}\,z}{3}(2x - z) \neq 0. \tag{34}$$

From the last equation it follows that there exists a non-trivial mixing between these states. This can be descried by the non-diagonal matrix:

$$\mathcal{M}_{I=0}^2 = (-\tilde{\mu}^2)\begin{pmatrix} A & B \\ B & C \end{pmatrix}, \tag{35}$$

where $A = \frac{2}{9}\left(x^2 + \frac{z^3}{x}\right)$, $B = \frac{\sqrt{2}}{3}z(2x - z)$, $C = \frac{1}{3}z(2z - x)$ The diagonalisation of this matrix leads to two eigenvalues:

$$\mathcal{M}_\sigma^2 = (-\tilde{\mu}^2)\,\frac{A + C - \sqrt{(A-C)^2 + 4B^2}}{2}, \tag{36}$$

$$\mathcal{M}_{f_0(980)}^2 = (-\tilde{\mu}^2)\,\frac{A + C + \sqrt{(A-C)^2 + 4B^2}}{2}. \tag{37}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 March 2025 doi:10.20944/preprints202503.2281.v1

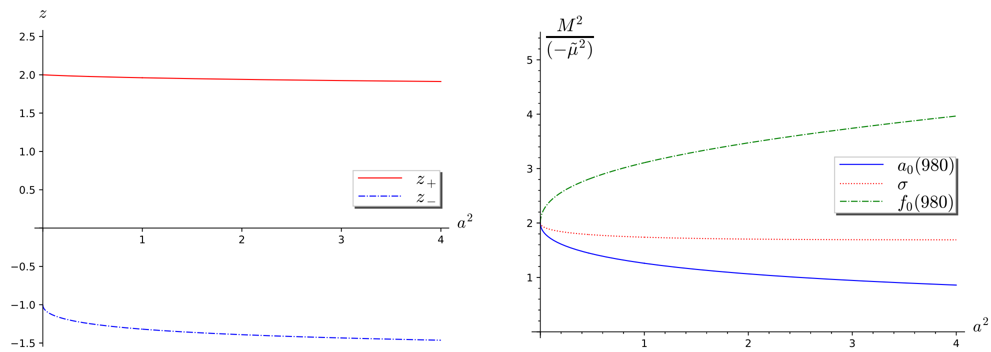

Figure 3. The left panel presents the solutions of Equation (24). The right panel presents the reduced squares of masses of σ, $f_0(980)$ and $a_0(980)$ mesons. The figure corresponds to the *first* case.

They correspond to the squares of the masses of the physical states σ and $f_0(980)$.

Let us analyse the obtained solutions (27, 28, 29, 30), keeping in mind that the squares of masses of σ, $f_0(980)$ and $a_0(980)$ mesons should be non-negative. In the *first* case, the solutions (27, 28), are illustrated in Figure 3. The left panel of the figure presents the solutions of Equation (24). From the panel it is obvious that only the solution z_+ is positive. For positive x this leads to positive masses of the scalar isotriplet $a_0(980)$. The right panel presents the reduced squares of the masses of the meson isotriplet $a_0(980)$ and the two isosinglets, σ and $f_0(980)$. Thus, it is seen from the right panel that this case does not correspond to physical masses, due to the unphysical pattern $\mathcal{M}_{a_0(980)} < \mathcal{M}_\sigma < \mathcal{M}_{f_0(980)}$.

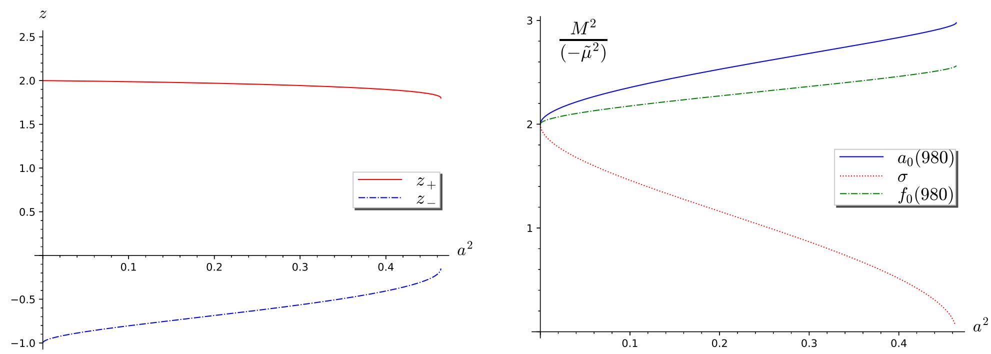

Figure 4. The left panel presents the solutions of Equation (24). The right panel presents the reduced squares of masses of σ, $f_0(980)$ and $a_0(980)$ mesons. The figure corresponds to the *second* case.

The *second* case, solution (29), is illustrated in Figure 4. The left panel of the figure presents the solutions of Equation (24). Again, only solution z_+ is positive. The right panel presents the reduced squares of the masses of the meson isotriplet and the two isosinglets. This result is more attractive from physical point of view. Namely, for $a^2 \to a_0^2$ the masses of the isotriplet $a_0(980)$ and isosinglet $f_0(980)$ are almost degenerate and heavy, while the σ meson mass tends to zero, and it equals zero for $a^2 = a_0^2$. From Equation (25) it follows that for the case $z_+ > 0$ the parameter a is positive: $a > 0$. So, for the *second* case the mass ratio $\mu^2/\tilde{\mu}^2 = 1 + a > 1$. This mass ratio depends on the initial coupling constants in the Lagrangian (1) and means, that G_0 is greater than $\widetilde{G}_0$.

Let us analyse the mixing at $a = a_0 = \sqrt{2\sqrt{3} - 3}$. The physical states σ and $f_0(980)$ are defined by the states S_0 and S_8 from the following relation:

$$\begin{pmatrix} \sigma \\ f_0 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} S_0 \\ S_8 \end{pmatrix}, \tag{38}$$

where the mixing angle φ is

$$\varphi = \frac{1}{2} \arccos \frac{C - A}{\sqrt{(C-A)^2 + 4B^2}}. \tag{39}$$

 doi:10.20944/preprints202503.2281.v1

The corresponding illustration of this mixing angle is presented in Figure 5.

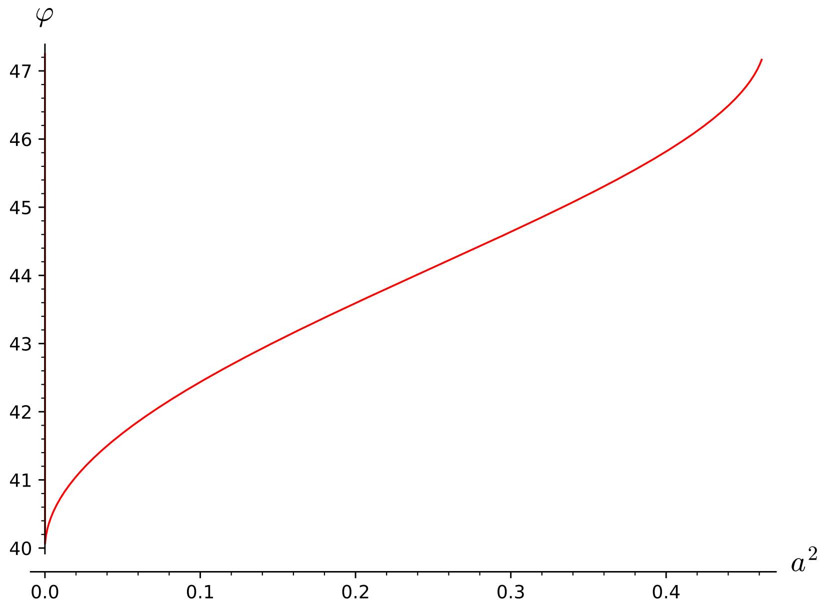

Figure 5. The figure presents the mixing angle φ corresponding to the *second* case.

The formula connecting the pure states $(u\bar{u} + d\bar{d})/\sqrt{2}$ and $s\bar{s}$ to the physical states σ and $f_0(980)$ reads:

$$\begin{pmatrix} \sigma \\ f_0 \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} s\bar{s} \\ \frac{u\bar{u}+d\bar{d}}{\sqrt{2}} \end{pmatrix}, \tag{40}$$

where the mixing angle ω is related to the mixing angle φ by the relation:

$$\omega = \arctan\sqrt{2} - \varphi. \tag{41}$$

It is interesting to note, that at point $a = a_0$ $\omega = \omega_0$, where $\omega_0 \approx 54.7^\circ - 47.5^\circ = 7.2^\circ$. This very small angle corresponds to a nearly ideal mixing. However, contrary to the common assumption, that $f_0(980)$ meson is almost pure $s\bar{s}$ state, we obtain that actually it is the σ meson which has almost pure $s\bar{s}$ state, while $f_0(980)$ meson is close to $(u\bar{u} + d\bar{d})/\sqrt{2}$ state.

In the *third* case, solution (30), both the solutions z_+ and z_- can be positive. Therefore, we will consider separately both cases. Figure 6 presents the solution z_+, while in Figure 7 z_- solution is presented. The solution z_- is nonnegative for $0 \le a^2 \le a_1^2 = (-2/3)^2$.

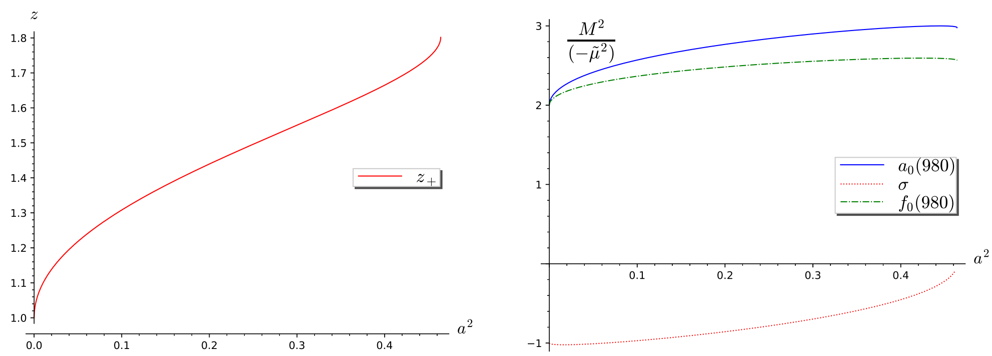

Figure 6. The left panel presents the solution z_+ of Equation (24). The right panel presents the reduced squares of masses of σ, $f_0(980)$ and $a_0(980)$ mesons. The figure corresponds to the *third* case.

 doi:10.20944/preprints202503.2281.v1

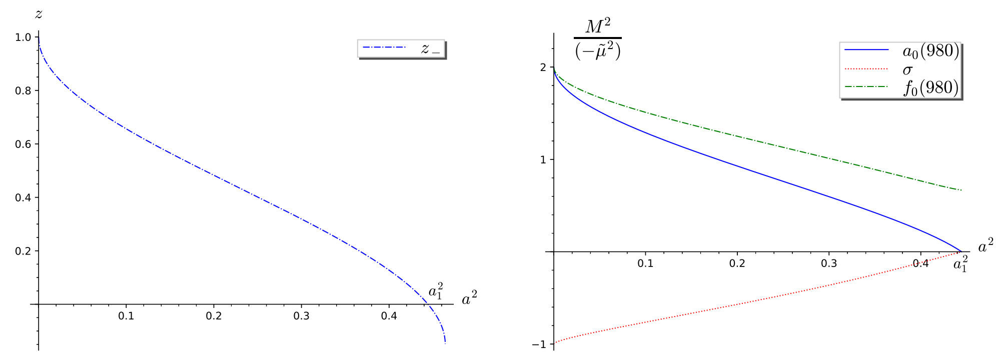

Figure 7. The left panel presents the solution z_- of the Equation (24). The right panel presents the reduced squares of masses of σ, $f_0(980)$ and $a_0(980)$ mesons. The figure corresponds to the *third* case.

However, as it is seen from the right panels of these figures, the square of the mass of the σ meson is negative in the whole range of variation of parameter a^2, except of point $a^2 = a_0^2$ in Figure 6 and point $a^2 = a_1^2$ in Figure 7, at which the mass of the σ meson is zero. The first point is included in the second case, as intermediate point between the second and the third cases, and already have been considered as physical solution, while at the second point in the right panel of Figure 7 the mass of $a_0(980)$ meson is also zero, which does not correspond to a physical case. Negative values of the mass of the σ meson do not correspond to the minimum of the potential.

In conclusion, only the *second* case, illustrated in Figure 4, corresponds to a physical solution.

4. Conclusions and Discussion

In this article we propose an explanation of the inverted mass hierarchy of the low-lying nonet of the scalar mesons. Such type of explanation is proposed for a first time in literature. The explanation is provided in the framework of quark-antiquark NJL model. In particular, the chiral symmetry $U_R(3) \times U_L(3)$ is broken to $U_V(3)$, which on its turn is broken simultaneously to $SU(2)$. The latter type of spontaneous symmetry breaking has not been considered before. Remarkably, this type of symmetry breaking leads to the appearance of BNG bosons, which results into zero masses of the κ mesons. Moreover, our model suggests also a zero mass for the σ meson. We have also determined that the σ meson state consists almost completely of $s\bar{s}$ quark combination. In previous studies it was assumed that the σ meson consists predominantly of light quarks. We have also shown $f_0(980)$ consists predominantly of light quarks, contrary to the previous assumptions that it consists mainly of strange quarks. Thus, the experimentally observed decay of $f_0(980)$ to two pions is nicely described by our model.

Future analysis should consider the realistic case of non-zero current masses. In order to explain the real mass spectrum it is necessary to introduce explicit breaking of $SU(3)$ flavour symmetry due to non-zero current quark masses. The method for introduction of heavy current mass of the strange quark has been developed in the framework of the chiral model $U(1)$ with a spontaneous symmetry breaking [15]. The results obtained in this work confirmed the predicted in ref. [16] mass value of the $h_1(1415)$ meson. This meson was experimentally discovered 15 years later [17] exactly with the predicted mass value. Such analysis will be provided elsewhere. Here we want only to note that during the $U_R(3) \times U_L(3)$ symmetry breaking besides pseudoscalar pions, K mesons, which contain a strange quark, obtain zero mass, as well.

It is known that the account for the current quark masses, in the case of dominant strange quark mass, leads to a real mass of the pseudoscalar K mesons of the order of 500 MeV. We expect that similarly to the K mesons case, σ and κ mesons, which contain strange quarks, will obtain masses of the order of K mesons.

 doi:10.20944/preprints202503.2281.v1

Author Contributions: Conceptualization, M.C.; methodology, M.C.; software, E.C. and M.N.; formal analysis, D.K. and E.C.; writing – original draft preparation, D.K. and M.C.; writing – review and editing, D.K. and M.C.; visualization, E.C. and M.N.. All authors have read and agreed to the published version of the manuscript.

Funding: This work was partially funded by the project D01-326 04.12.2023 of the Bulgarian Ministry of Education and Science.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: E.C. was partially financially supported by the project D01-326 04.12.2023 of the Bulgarian Ministry of Education and Science. M.N. acknowledges partial support within the program FNI-SU/80-10-190 and by the Bulgarian Ministry of Education and Science under the National Program Young Scientists and Posdoctoral Students – 2. The research of D.K. was carried out with the help of infrastructure purchased under the National Roadmap for Scientific Infrastructure, financially coordinated by the Ministry of Education and Science of the Republic of Bulgaria (agreement D01-326/04.12.2023).

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

BNG:	Bogoliubov – Nambu – Goldstone
NJL:	Nambu – Jona-Lasinio

References

1. Bogoliubov, N.N. "A New method in the theory of superconductivity. I". *Sov. Phys. JETP* **1958** *34(7)* 41–46.
2. Bogoliubov, N.N. "A New method in the theory of superconductivity. III". *Sov. Phys. JETP* **1958** *34 (7)* 51–55.
3. Nambu Y. "Quasiparticles and Gauge Invariance in the Theory of Superconductivity". *Physical Review* **1960**, *117(3)*, 648–663.
4. Goldstone J. "Field Theories with "Superconductor" Solutions". *Nuovo Cimento* **1961** , *19 (1)*, 154–164.
5. Goldstone J., Salam Abdus, Weinberg Steven. "Broken Symmetries". *Physical Review* **1962**, *127 (3)*, 965–970.
6. Jaffe R.L. "Multi-Quark Hadrons. I. The Phenomenology of $Q^2\bar{Q}^2$ Mesons". *Phys. Rev. D* **1977**, *15*, 267.
7. Kuroda Y., Harada M., Matsuzaki Sh., Jido D. "Inverse Mass Hierarchy of Light Scalar Mesons Driven by Anomaly-Induced Flavor Breaking". *Prog. Theor. Exp. Phys.* **2020**, *5*, 053D02.
8. Naito K., Oka M., Takizawa M., Umekawa T. "$U_A(1)$ breaking effects on the light scalar meson spectrum". *Prog. Theor. Phys.* **2003**, *109*, 969–980 .
9. Naydenov M.N., Chizhov M.V. "On the Problem of Spontaneous Isotopic Symmetry Breaking in Nambu – Jona-Lasinio Models". *Physics of Particles and Nuclei Letters* **2020**, *17(2)*, 120–125.
10. Nambu Y., Jona-Lasinio G. "Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I". *Phys. Rev.* **1961**, *122* , 345–358.
11. Nambu Y., Jona-Lasinio G. "Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. II". *Phys. Rev.* **1961**, *124* , 246–254.
12. Bender C.M., Cooper F., Guralnik G.S. "Path Integral Formulation of Mean Field Perturbation Theory." em Annals Phys. **1977**, *109*, 165.
13. Tamvakis K., Guralnik G.S. "Renormalization of Four Fermion Theories in a Mean Field Expansion." *Phys. Rev. D* **1978**, *18*, 4551.
14. Eguchi T. "A New Approach to Collective Phenomena in Superconductivity Models." *Phys. Rev. D* **1976**, *14*, 2755.
15. Chizhov M.V., Naydenov M.N. "Mass prediction for the last discovered member of the axial-vector nonet with quantum numbers $J^{PC} = 1^{+-}$." *JETP Lett.* **2020**, *112(3)*, 133–137.

 doi:10.20944/preprints202503.2281.v1

16. Chizhov M.V. "Vector meson couplings to vector and tensor currents in extended NJL quark model." *JETP Lett.* **2004**, 80, 73–77.
17. Ablikim M., Achasov M. N., Ahmed S., Albrecht M., Amoroso A., An F. F., An Q., Bai J. Z., Bai Y., Bakina O. et al. "Observation of h1(1380) in the $J/\psi \to \eta' K\bar{K}\pi$ Decay." *Phys. Rev. D* **2018**, *98 (7)*, 072005.