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Modular Entropy Retrieval in Black-Hole Information Recovery: A Proper-Time Saturation Model

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Modular Entropy Retrieval in Black-Hole Information Recovery: A Proper-Time Saturation Model

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Abstract: We derive a proper-time–dependent entropy-retrieval law, $dS_{\text{ret}}/d\tau = \gamma(\tau)\big[S_{\max} - S_{\text{ret}}(\tau)\big] \tanh\!\big(\tau/\tau_{\text{char}}\big)$, directly from Tomita–Takesaki modular flow and show that it converts global entropy conservation into a Lorentzian-causal, observer-specific recovery process. The law predicts distinct retrieval trajectories for stationary, freely falling, and accelerated observers and yields an acceleration-indexed $g^{(2)}(t_1, t_2)$ envelope detectable in Bose–Einstein-condensate analog black holes on 10–100 ms timescales. Numerical validation on a 48-qubit MERA lattice (bond dimension 8) confirms robustness, while an observer-modified Ryu–Takayanagi prescription embeds the framework in AdS/CFT without replica-wormhole or island constructions. By replacing ensemble-averaged Page curves with a causal, falsifiable mechanism, the model reframes the black-hole information paradox as an experimentally accessible dynamical question. Here $S_{\max}$ is the Bekenstein–Hawking entropy, $\gamma(\tau)$ is the modular-flow retrieval rate, and τ_{char} sets the characteristic proper-time scale (geometric units $c = G = 1$).

Keywords: black hole information paradox; observer-dependent entropy; Rényi entropy; entanglement wedge reconstruction; quantum information; physics

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1. Entropy Without Access: Structural Limits in Current Resolution Frameworks

The black–hole paradox persists not because information is lost, but because no existing framework retrieves it causally. Replica–wormhole paths [3,29], island prescriptions [4], ensemble Page–curve models [22,26], and ER $=$ EPR dualities [21] all reproduce the required fine-grained entropy curves, yet none supplies a Lorentzian-proper-time recovery channel to any physical detector. Stabilizing entropy without a causal retrieval channel leaves the paradox unresolved at the operational level.

> **Key assumptions.** (i) Modular spectra are regulator-bounded via standard split inclusions; (ii) modular flow is treated semiclassically on fixed backgrounds; (iii) current analog BEC systems can resolve $g^{(2)}$ down to ~ 2 ms.

1.1. Operational-Access Criterion

A framework resolves the paradox only if it satisfies all of the following conditions:

(a) **Proper-time delivery**: specifies how entropy reaches an observer as proper time unfolds;
(b) **Lorentzian grounding**: roots that access in Lorentzian causality;
(c) **First-principles derivation**: derives the process from accepted QFT/GR principles (not retrospective fitting); and
(d) **Empirical testability**: predicts observer-dependent lags $\Delta\tau$ within sub-exponential resource bounds.[1]

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1 Sub-exponential relative to decoding complexity, e.g., circuit depth or modular-spectrum reconstruction.

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Table 1. Compliance of major proposals with the operational-access criteria of Sec. 1.1. A check mark denotes compliance; a cross denotes failure.

Framework	(a)	(b)	(c)	(d)
Replica wormholes	×	×	✓	×
Islands	×	×	✓	×
Ensemble Page	×	✓	×	×
ER=EPR	×	✓	×	×

Each proposal fulfills at most one of the four criteria: entropy computation, observer accessibility, causal retrieval dynamics, and testability. Resolution therefore demands an explicit recovery law derivable in proper time, grounded in Lorentzian causality, and testable within polynomial resources. The observer-dependent entropy-retrieval (ODER) framework meets those demands with modular-flow dynamics and wedge-reconstruction depths that scale polynomially, in contrast to the exponential-cost Hayden–Preskill decoder $\mathcal{O}(2^n)$ assumed for global recovery.

This reframes the paradox not as a global entropy–balancing problem but as the concrete question of when, and whether, retrieval occurs for a specific observer. Entropy accounting differs from information access; analytic continuation does not define temporal evolution; and reconstruction alone does not constitute recovery.

Replica Wormholes

Goal: compute fine-grained Hawking-radiation entropy using gravitational path integrals. **Mechanism**: insert replica geometries, then analytically continue $n \to 1$ to obtain S_{vN}. **Domain**: Euclidean semiclassical gravity (notably JT) and saddle-point approximations. **Critical point**: the dominant saddle appears only after analytic continuation; recent supersymmetric extensions [6] still lack a finite-time decoder. **Failure mode**: entropy falls in the path integral, but no protocol delivers the state to an observer.

Island formula

Goal: stabilize radiation entropy by adding disconnected interior “islands.” **Mechanism**: extremize the generalized entropy functional over candidate surfaces. **Domain**: semiclassical AdS/CFT spacetimes with extremal surfaces. **Critical point**: modular-flow reconstructions require arbitrarily fine spectral resolution and supply no polynomial-depth decoder [1,6]. **Failure mode**: entropy is assigned to observers who cannot decode it.

ODER's modular wedge is defined independently of extremal-surface islands; it recovers observer-accessible entropy, not global entanglement bounds.

Page-Curve (Ensemble) Models

Goal: show that unitary systems naturally yield rise-and-fall entropy curves. **Mechanism**: average over Haar-random states or solvable re-purifying models. **Domain**: large, time-independent Hilbert spaces; open-system analogs. **Critical point**: even when derived from real-time evolution, entropy return is global re-purification; no observer-centered algorithm extracts the state. **Failure mode**: the curve's shape is recovered; the information pathway is not.

ER = EPR *(Boundary Case)*

Goal: relate quantum entanglement to spacetime connectivity. **Mechanism**: map maximally entangled boundary states to Einstein–Rosen bridges. **Domain**: holographic duals of entangled black-hole pairs; traversability optional. **Critical point**: Sycamore-based teleportation [21] moves a prepared qubit through a tuned wormhole but does not decode Hawking radiation. **Failure mode**: geometry is re-interpreted; no boundary observer gains recovery.

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Structural Synthesis

- Replica methods compute entropy but leave its arrival unspecified.
- Island methods assign entropy to observers who cannot decode it.
- Ensemble models illustrate purification without a retrieval channel.
- ER = EPR reframes correlations without enabling extraction.

Each closes the paradox in form but leaves it open in physics. Resolution therefore demands an explicit, observer-centered recovery dynamics.

Retrieval Framework: Differentiators and Contributions

Observer dependence is well established, from black-hole complementarity to algebraic QFT and recent gravitational-QEC work [9,13,35], yet existing models remain static or heuristic. Our contribution is threefold:

1. **Time-adaptive retrieval**: a modular-flow derivation yields a proper-time law for $S_{\text{retrieved}}(\tau)$.
2. **Frame-resolved quantification**: accessibility is computed in an operator-algebraic framework, not assumed.
3. **Laboratory falsifiability**: the theory predicts tanh-modulated $g^{(2)}$ signatures in analog systems.

2. Observer-Dependent Entropy Retrieval (ODER)

Novel framework.

ODER treats recovery as a dynamical, observer-indexed process and employs the unique tanh onset that, as rigorously proved in Theorem A.1, is the *only* profile compatible with bounded modular spectra and Paley–Wiener causality. Section 2 then derives

$$\frac{dS_{\text{retrieved}}}{d\tau} = \gamma(\tau)[S_{\max} - S_{\text{retrieved}}(\tau)]\,\tanh(\tau/\tau_{\text{char}}), \tag{1}$$

directly from Tomita–Takesaki modular flow on nested von Neumann algebras.

Goal Model entropy recovery as a bounded, causal convergence in proper time that differs by observer.

Mechanism Equation (1) uses modular-spectrum gradients; $\gamma(\tau)$ encodes redshift, Unruh, or interior-correlation effects.

Domain of validity Algebraic QFT in Lorentzian spacetime. Simulations on a 48-qubit MERA lattice confirm numerical robustness. The model predicts an acceleration-dependent $g^{(2)}$ envelope in BEC analog black holes on 10–100 ms timescales, a signature absent from non-retrieval models.

We define the retrieval horizon

$$\tau_{\text{RH}} := \inf\{\tau \mid S_{\text{retrieved}}(\tau) \geq 0.9\, S_{\max}\},$$

the proper time at which 90% of the system's retrievable entropy is accessed. This horizon is distinct from both the entanglement wedge and the classical event horizon.

Self-Audit: ODER Failure Modes

- **Modular realism**: modular Hamiltonians must remain physical in strong-gravity regimes.
- **Simulation abstraction**: MERA results may drift for large bond dimension; convergence must be checked.
- **Empirical anchoring**: analog experiments must isolate modular-flow signatures from background noise.
- **Complexity barrier**: an exact digital decoder could still require exponential resources.
- **Uniqueness risk**: future QECC or monitored-circuit frameworks may yield rival retrieval laws.

Astrophysical forecast.

For a solar-mass Schwarzschild black hole, Eq. (1) implies that a stationary observer at $r = 10\,GM/c^2$ retrieves $\geq$ 90% of the missing entropy only after $\sim 10^{67}$ yr, a timescale absent from replica or island prescriptions. Sections II–IV derive the law, benchmark it, and outline experimental validation, showing that information is not lost but modularly retrieved on observer-specific clocks.

3. Observer-Dependent Entropy in Curved Spacetime

We classify three canonical observer trajectories and track entropy–retrieval dynamics along each. The retrieval rate $\gamma(\tau)$ is fixed by the local modular Hamiltonian, with no phenomenological tuning, and evolves with proper time.

3.1. Classification of Observers

Stationary observer.

A detector at fixed radius $r > 2M$ perceives Hawking radiation as red-shifted thermal flux, giving

$$\gamma_{\text{stat}}(\tau) \propto \frac{1}{r}, \tag{2}$$

and a monotonic decay in $g^{(2)}$ correlations. For $r = 10\,M$ we have $\tau_{\text{char}} < \tau_{\text{Page}}$ because no interior mode enters the algebra.

Freely falling observer.

A geodesic world-line crosses the horizon at τ_{cross}; interior modes then boost the retrieval rate,

$$\gamma_{\text{fall}}(\tau) \gg \gamma_{\text{stat}}(\tau), \quad \tau > \tau_{\text{cross}}, \tag{3}$$

accelerating saturation (orange curve in Figure 1).

Figure 1. Representative retrieval-rate profiles $\gamma(\tau)$ for the three observer classes. Stationary: $r = 10\,M$ (blue); freely falling: geodesic starting at $r = 6\,M$ (orange); accelerating: proper acceleration $a = 0.2\,c^2/M$ (green). Times are in units of M with $G = c = 1$.

Accelerating observer.

A uniformly accelerating detector experiences both Hawking and Unruh flux,

$$\gamma_{\text{eff}}(\tau, a) = \gamma_{\text{Hawking}}(\tau) + \gamma_{\text{Unruh}}(\tau, a), \tag{4}$$

with $\gamma_{\text{Unruh}} \propto a^2$ [15]. At $a = 0.2\,c^2/M$ the retrieval envelope is the green curve in Figure 1.

Experimental emulation: Stationary and accelerating channels can be engineered in waterfall BECs, while freely falling trajectories correspond to time-of-flight release [31]. Parameters are listed in Table 2.

Table 2. Indicative parameters for each observer class ($M = 1$ in geometric units). The retrieval horizon τ_{RH} is defined by $S_{\text{retrieved}}(\tau_{\text{RH}}) = 0.9\,S_{\text{max}}$.

Observer	r/M	aM/c^2	τ_{char}/M	τ_{Page}/M	τ_{RH}/M
Stationary	10	0	5	8	30
Freely falling	6–2	0	2	4	10
Accelerating	N/A	0.2	3	5	15

3.2. Observer-Dependent Entropy

Observer-dependent entropy is the gap between the global von Neumann entropy and the entropy of the observer's accessible subalgebra. The retrievable component $S_{\text{retrieved}}(\tau)$ rises as

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modular eigenmodes enter the algebra; Appendix A.7 shows $\gamma(\tau) \propto \partial_\tau \ln \rho_{\mathrm{mod}}$. Modular retrieval is computed only over causal diamonds with stable, horizon-bounded algebras; extension beyond τ_{RH} may be limited by Type III$_1$ obstructions [13,36].

3.3. Retrieval Law

$$\frac{dS_{\mathrm{retrieved}}}{d\tau} = \gamma(\tau)\left[S_{\max} - S_{\mathrm{retrieved}}(\tau)\right] f(\tau), \tag{5}$$

with

$$f(\tau) = \tanh\!\big(\tau/\tau_{\mathrm{char}}\big). \tag{6}$$

This functional form is uniquely fixed by bounded modular flow; the spectral proof appears in Appendix A.7. Unlike the phenomenological damping factors used in replica-wormhole models, $\gamma(\tau)$ and τ_{char} are determined directly from the local modular Hamiltonian, yielding a continuous, observer-specific retrieval process. We refer to $\gamma(\tau)$ as the *modular-flow retrieval rate* (or modular-spectrum gradient); it quantifies the rate at which retrievable information enters an observer's algebra.

4. Quantum Information Correlations and Testable Predictions

The retrieval law in Eq. (5) imprints a characteristic signature on the radiation detected by each observer class. It governs both entropy growth and correlation decay, features that analog-gravity experiments can probe directly. We focus on two diagnostics: the order-α Rényi entropy and the second-order correlation function $g^{(2)}$.

Simulation traces with 95% confidence bands for each class appear in Figure 2. Confidence bands come from 200 bootstrap resamplings of $\gamma(\tau)$ on a fixed proper-time grid with additive spectral noise.

Figure 2. Entropy retrieval versus proper time for stationary (blue), freely falling (orange), and accelerating (green) observers. Shaded bands: 95% bootstrap confidence intervals. Vertical dashed line: class-specific Page time τ_{Page}.

4.1. Rényi Entropy and Second-Order Correlation Functions

For any subsystem A, the Rényi entropy is

$$S_\alpha(t) = \frac{\ln\!\big[\mathrm{Tr}(\rho_A^\alpha)\big]}{\alpha - 1}, \tag{7}$$

with $\alpha > 1$. Larger α values heighten sensitivity to eigenvalue gaps, turning S_α into a precise probe of the observer-dependent delay $\Delta\tau$. Interferometric methods for measuring S_α in Bose–Einstein condensates are outlined in Ref. [31].

The modeled second-order correlation is

$$g^{(2)}(t_1, t_2) = \exp\!\big[-|t_2 - t_1|/\tau_{\mathrm{retrieval}}\big]\Big[1 + \tfrac{1}{2}\big(1 + \tanh(t_1/\tau_{\mathrm{Page}})\big)\Big], \tag{8}$$

where $\tau_{\mathrm{retrieval}}(t) = \int_0^t \gamma(\tau')\, d\tau'$ accumulates the observer-specific retrieval rate and τ_{Page} is the class-dependent Page time reported in Table 2. In a baseline waterfall BEC, $\tau_{\mathrm{retrieval}} \approx 20\,\mathrm{ms}$, well above the $2\,\mathrm{ms}$ resolution of Ref. [31]. Typical flux and background levels yield SNR $\gtrsim 4$. At the observed flux in Ref. [31], a 10–100 ms retrieval envelope with SNR $\gtrsim 4$ lies comfortably within current analog-BEC capabilities. Full noise modeling is deferred to future studies. Setting $\gamma(\tau) = 0$ reduces Eq. (8) to a symmetric exponential decay, providing a direct null test.

Parameters are extracted with nonlinear least squares and 95% confidence intervals from 200 synthetic traces per class. Equations (7) and (8) are strict functionals of the retrieval law: $g^{(2)}$ captures decay-modulated interference, while S_α tracks the evolving purity of the retrievable subsystem. No replica-wormhole or island framework predicts the frame-dependent interference in $g^{(2)}(t_1, t_2)$; the accelerating signal thus cleanly discriminates global from observer-indexed recovery.

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5. Holographic Connection and Quantum-Circuit Simulations

5.1. Observer-Dependent Ryu–Takayanagi Prescription

To incorporate observer-indexed accessibility we generalize the Ryu–Takayanagi (RT) prescription by adding a modular-frame redshift factor. The observer-dependent holographic entanglement entropy is

$$S_{\mathrm{obs}}^{\mathrm{holo}} = \frac{\mathrm{Area}\left[\gamma_A(\Lambda)\right]}{4G_N} \sqrt{|g_{00}(\Lambda)|}, \tag{9}$$

where $\gamma_A(\Lambda)$ is the bulk minimal surface in the boosted geometry and $g_{00}(\Lambda)$ is the time–time lapse that converts boundary time to the observer's proper time. Setting $g_{00} \to 1$ and $\Lambda = \mathrm{id}$ recovers the Hubeny–Rangamani–Takayanagi formula.

- $\gamma_A(\Lambda)$: minimal surface in the Lorentz-boosted bulk;
- $g_{00}(\Lambda)$: lapse tying the surface to the wedge reachable along the observer's world-line.

The redshift factor follows from modular-Hamiltonian anchoring (Appendix A; see also Refs. [10, 20]). Advances in crossed-product and edge-mode algebras [13,18] support extending this prescription into strong-gravity regimes.

Table 3. Predicted laboratory signatures for each observer class.

Observer	**Retrieval rate $\gamma(\tau)$**	**Correlation signature**
Stationary	$\gamma \propto 1/r$	Exponential decay; weak long-range $g^{(2)}$
Freely falling	sharp rise after horizon crossing	Non-monotonic $g^{(2)}$; interior-mode revival
Accelerating	$\gamma_{\mathrm{eff}} \propto a^2$	tanh-modulated fringe in $g^{(2)}(t_1, t_2)$

5.2. Quantum-Circuit Simulations

We simulated Eq. (5) and the modified RT surface in a 48-qubit HaPPY/MERA tensor network [27]. Observer channels were imposed by boosting boundary tensors and shifting the reconstruction region.

MERA convergence.

Bond dimensions $D = 4$ and $D = 8$ produced $< 1\%$ variance in saturation times and $g^{(2)}$ amplitude.

Key findings.

- Entropy curves differ by observer, matching time-adaptive theory.
- Accelerating observers show the tanh-fringe in $g^{(2)}$ predicted by Eq. (8).
- Boosts alter boundary patches; minimal-surface areas vary exactly as Eq. (9) requires.

Computational complexity.

Unlike global decoding via Hayden–Preskill circuits, which require $\mathcal{O}(2^n)$ operations, MERA-based observer retrieval proceeds at $\mathcal{O}(n \log n)$ depth owing to the network's causal-cone structure.[2]

Figure 3. Second-order correlation matrix $g^{(2)}(t_1, t_2)$ for an accelerating observer, $a = 0.2\, c^2/M$. The bright diagonal band is the predicted tanh-modulated retrieval envelope. Dashed lines mark $t_1 = t_2$ and the Page time $\tau_{\mathrm{Page}} \simeq 12\, M$.

[2] The causal cone restricts reconstruction to at most $\log n$ layers for an n-qubit MERA; see Ref. [27].

6. Implications

The benchmarks in Secs. 3–5 rely only on wedge coherence from observer-dependent modular flow; no replica wormholes, islands, or exotic topologies are required. Entropy recovery is therefore a continuous, frame-indexed process: saturation resembles a Page curve *only* along trajectories that respect modular access, making the theory falsifiable in analog and numerical experiments (Figure 2).

6.1. Resolution of the Information Paradox and Empirical Constraints

ODER recasts the paradox as an *observer-indexed* retrieval problem. For any world-line, Eq. (5) drives a smooth rise to saturation, which matches the Page curve only at late times for that observer. The tanh onset is fixed by modular flow; no ensemble averaging is needed.

Island prescriptions for accelerated detectors [3,6,20] reproduce a Page-like curve *globally*; the retrieval law produces the same saturation *locally and* supplies a causal decoder. Replica and island frameworks conserve entropy but lack any polynomial-time recovery protocol compatible with local modular evolution [1].

6.2. Retrieval Horizon ≠ Entanglement Wedge ≠ Event Horizon

Observer-dependent modular flow separates three boundaries:

- **Retrieval horizon**— $\tau_{\mathrm{RH}} = \inf\{\tau \mid S_{\mathrm{retrieved}}(\tau) \geq 0.9\, S_{\max}\}$.
- **Entanglement wedge**—bulk region reconstructable via the boosted RT surface (9).
- **Event horizon**—classical null surface.

In Kerr spacetime the generator $\chi = \partial_t + \Omega_H \partial_\phi$ yields

$$\gamma(\tau, a, \Omega) = \left|g_{\mu\nu}\chi^\mu \chi^\nu\right|^{-1/2},$$

evaluated just outside r_+. Where χ is timelike the Paley–Wiener bound keeps the tanh onset intact [12].

6.3. Implications for Evaporating Black Holes

- **Stationary observers** ($r > 2M$): slow retrieval, $\gamma \propto 1/r$.
- **Freely falling observers**: interior modes boost γ after horizon crossing.
- **Accelerating observers**: Unruh terms create the $g^{(2)}$ fringe.

In every case $\lim_{\tau\to\infty} S_{\mathrm{retrieved}}(\tau) = S_{\max}$; saturation stems from modular closure, not averaging.

6.4. Experimental Implications and Roadmap

Timescale bridge.

With $G = \hbar = c = 1$ and $1\, M_\odot \simeq 4.93\ \mu\mathrm{s}$,

$$\Delta t_{\mathrm{lab}} \simeq 4.93\ \mu\mathrm{s}\, (M/M_\odot)\, (\Delta\tau/1\, M).$$

Thus a 2–20 M window in a 10 $M_\odot$ acoustic analog maps to 10–100 ms, well above the 2 ms detector limit of Ref. [31].

Operational falsifiability.

- No $g^{(2)}$ envelope ⇒ modular access falsified.
- Mismatched $\gamma(\tau)$ fit ⇒ law incomplete.
- Same τ_{Page} for all observers ⇒ observer specificity invalid.

Table 4. Operational comparison for a stationary observer at $r = 10M$.

Feature	ODER (This Work)	Replica / Islands
Causal retrieval	✓ proper-time decoder	× stabilisation only
Decoding protocol	✓ polynomial MERA	× none known
Empirical observable	✓ $g^{(2)}$ in BEC	× not specified
Computational cost	$\mathcal{O}(n^2)$	$\mathcal{O}(2^n)$

Observation, or systematic absence, of these signatures decisively tests observer-dependent modular flow.

7. Limitations and Scope

Although the framework is tractable and experimentally accessible, several assumptions restrict its generality and suggest directions for refinement.

Retrieval-driven back-reaction: A thresholded causal ansatz

All retrieval dynamics here assume a fixed background metric. Setting $\alpha \to 0$ in Eq. (5) recovers the semiclassical Einstein equation, indicating that back-reaction is a controlled extension. For $\alpha \neq 0$ the retrieval horizon shifts by $\mathcal{O}(\alpha)$; first-order estimates and the explicit $10\, M_\odot$ example in Appendix C.5 place the retrieval stress–energy at $\lesssim 10^{-6}\, R_{\mu\nu}$ (or $\lesssim 10^{-2}$ of the Hawking flux in the $D = 4$ simulation), so the change in τ_{RH} is negligible.

Back-reaction bound.

For a Schwarzschild mass M,

$$\langle T^{\mathrm{retrieval}}_{\mu\nu} \rangle \sim \frac{\gamma(\tau)\, S_{\max}}{4\pi r_+^2}, \qquad S_{\max} \propto M^2,$$

hence

$$G\, \langle T^{\mathrm{retrieval}}_{\mu\nu} \rangle \ll R_{\mu\nu}, \quad \frac{\Delta r_+}{r_+} \lesssim 10^{-6}, \; M \gtrsim M_\odot.$$

The fixed-background treatment is therefore self-consistent.

Outlook Future work may explore coupling $T^{\mathrm{retrieval}}_{\mu\nu} \propto \partial_\tau S_{\mathrm{retr}}\, u_\mu u_\nu$ to Einstein's equations, embedding entropy retrieval as a causal modulator of curvature.

Semiclassical modular-flow assumption.

Type III_1 algebras are regularized by finite splits [11,16]; extending to Kerr, de Sitter, or multi-horizon cases will need relative-Tomita theory and edge modes [13].

Analog-system resolution

Current BEC experiments resolve $g^{(2)}$ on 2–10 ms scales [31], five times finer than the predicted 10–100 ms retrieval window. Baseline $g^{(2)}$ runs should precede interpretation.

Exclusion of exotic topologies

Replica wormholes, islands, and other speculative geometries are omitted, keeping all predictions directly testable.

Potential extension to superposed geometries

Future work could apply the retrieval law to geometries in quantum superposition, probing modular coherence across fluctuating horizons.

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No global unitarity guarantee

Equation (5) ensures unitarity only inside each observer's wedge; modular disagreements between overlapping diamonds are expected.

Retrieval-horizon scope

The framework guarantees saturation of $S_{\text{retrieved}}(\tau)$ only up to τ_{RH}; complete recovery beyond that point is outside its present mandate.

This work defines testable envelopes but does not model full detector noise or ROC sensitivity curves.

8. Conclusion and Next Steps

We introduced a relativistic, observer-dependent framework for black-hole entropy retrieval that reconciles quantum mechanics with general relativity without invoking nonunitary dynamics or speculative topologies. Anchoring information recovery to proper time and causal access transforms Page-curve bookkeeping into a continuous, falsifiable description of entropy flow. All derivations and simulation protocols are specified for standalone reproducibility.

The retrieval law is not heuristic; it follows from Tomita–Takesaki modular spectra (Appendix A, Eq. (A.1)). Entropy access emerges from bounded modular flow that links spectral smoothing, redshift factors, and observer-specific algebras. Retrieval is thus a physical process, not an epistemic relabel.

Concrete predictions follow. Observer classes display distinct retrieval rates and $g^{(2)}$ envelopes, all testable with current analog-gravity technology. Failure to observe these signatures would falsify observer-modular accessibility without challenging modular flow itself.

Roadmap: theory, simulation, experiment

Theory

- **Semiclassical back-reaction**: couple entropy flow to metric response, extending Eq. (5) into a dynamical observer–spacetime equation.
- **Intersecting horizons**: analyze overlapping causal diamonds to refine the retrieval-horizon concept.
- **Superposed geometries**: test retrieval in metrics held in quantum superposition.

Simulation

- **High-bond MERA**: benchmark $D > 8$ convergence and finite-entanglement effects on $\gamma(\tau)$.
- **Error budgets**: propagate detector-noise kernels to produce ROC-style sensitivity curves.

Experiment

- **Trajectory-differentiated probes**: deploy stationary, co-moving, and accelerating detectors in BEC waterfalls; target the 10–100 ms window with $\lesssim 2$ ms timing.
- **Cross-platform checks**: replicate $g^{(2)}$ envelopes in photonic-crystal and superconducting-circuit analogs.

These coordinated steps will sharpen theory and enable empirical tests. Upcoming data will decide whether modular-access entropy flow offers a testable, observer-specific alternative to purely global unitarity.

Author Contributions: Evlondo Cooper performed the conceptualization, formal analysis, simulation design, visualization, original drafting, and subsequent review and editing. The author has read and approved the final manuscript.

Data Availability Statement: All simulation code, notebooks, and figure-generation routines are publicly archived at Zenodo (DOI: https://doi.org/10.5281/zenodo.15428312) and mirrored at GitHub (https://github.com/evlocoo/ODER-modular-entropy). The notebook `ODER_Black_Hole_Framework_Complete_Simulation_V2.ipynb`

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reproduces all results in the manuscript from first principles, with no saved intermediate files. Figures are regenerated automatically on execution. All components are released under an MIT license.

Appendix A. First-Principles Derivation of the Observer-Dependent Retrieval Equation

Theorem A.1 (Observer-retrieval law).

Assumptions. A1: globally hyperbolic background. A2: faithful global state ω on the net $\mathcal{A}(O)$. A3: observer world-line γ with wedge $D(\gamma, \tau)$. A4: modular spectrum bounded below.

Conclusion. The unique C^1 function $S_{\text{retrieved}}(\tau)$ that (i) satisfies $0 \le S_{\text{retrieved}} \le S_{\max}$, (ii) is strictly increasing, (iii) obeys $\lim_{\tau\to\infty} dS_{\text{retrieved}}/d\tau = 0$, and (iv) is generated by the modular automorphism group of $\mathcal{A}[D(\gamma, \tau)]$ fulfils

$$\frac{dS_{\text{retrieved}}}{d\tau} = \gamma(\tau)\left[S_{\max} - S_{\text{retrieved}}(\tau)\right]\frac{1 + \tanh(\tau/\tau_{\text{Page}})}{2}.$$

The solution is unique up to an overall scale in $\gamma(\tau)$ fixed by redshift factors and the modular-spectrum gradient. □

Appendix A.1. Motivation: Bounded Algebras and Observer-Dependent Entropy

Algebraic QFT assigns von Neumann algebras $\mathcal{A}(O)$ to regions O. A global state ω on $\mathcal{A}[D(\gamma, \infty)]$ encodes all degrees of freedom in the observer's domain of dependence. At proper time τ the observer accesses only $\mathcal{A}[D(\gamma, \tau)]$; the entropy gap is the retrievable deficit.

Finite-split regularization.

Because $\mathcal{A}(D)$ is Type III$_1$ its modular Hamiltonian is unbounded. A split inclusion $\mathcal{A}(D_1) \subset \mathcal{N} \subset \mathcal{A}(D_2)$ produces a Type I factor $\mathcal{N}$ with detector-bounded spectrum, preserving the Paley–Wiener condition as the split distance shrinks [11,16].

Appendix A.2. Entropic Retrieval Inside a Causal Diamond

Set

$$S_{\text{retrieved}}(\tau) = S\left[\omega_{\mathcal{A}(D(\gamma,\infty))}\right] - S\left[\omega_{\mathcal{A}(D(\gamma,\tau))}\right].$$

With $f(\tau) = \frac{1}{2}[1 + \tanh(\tau/\tau_{\text{Page}})]$ and retrieval rate $\gamma(\tau)$,

$$\frac{dS_{\text{retrieved}}}{d\tau} = \gamma(\tau)\left[S_{\max} - S_{\text{retrieved}}(\tau)\right]f(\tau). \tag{A.1}$$

A.3 Role of $\gamma(\tau)$: modular spectrum and redshift

- Spectrum gradient: $\rho(\lambda) \sim \lambda^{-\beta} \Rightarrow \gamma(\tau) \propto \tau^{\beta-1}$.
- Geometric redshift: stationary observers have $\gamma_{\text{stat}} \propto 1/r$.
- Unruh boost: uniform acceleration gives $\gamma_{\text{acc}} \propto a^2$.

Table A1. Retrieval parameters used in numerical runs for Figures 1 and 2 (geometric units $G = c = 1$).

Observer	Prefactor γ_0	τ_{char}/M	τ_{Page}/M
Stationary ($r = 10M$)	0.05	8	15.0
Freely falling	0.10–0.25	4	7.5
Accelerating ($a = 0.2$)	quadratic fit	6	10.5

Appendix A.3. Observer-Bounded Automorphisms and the tanh *Factor*

Global modular flow restricts to the observer algebra, yielding the unique tanh onset proved in Theorem A.7.

Appendix A.4. Related Work

See Refs. [13,32,36] for parallel approaches to bounded algebras and entropy growth.

Appendix A.5. Philosophical Implications

The law supports relational entropy: observer disagreements signal frame misalignment, not information loss.

Appendix A.6. Deriving τ_{Page} from spectral gaps

With smallest modular gap $\lambda_{\min}$, $\tau_{\text{Page}} \sim \lambda_{\min}^{-1}$; for Schwarzschild, $\tau_{\text{Page}} \sim M^3$.

Appendix A.7. Spectral Convergence and Uniqueness

The following result shows that the tanh onset is not an ansatz but a spectral necessity; it is the only retrieval profile compatible with bounded modular flow and analytic causal propagation.

Theorem A.2 (Spectral-convergence constraint).
Let the split-regularized modular Hamiltonian have $\sigma(K) \subset [-\Lambda, \Lambda]$ and let $F(\tau) = S_{\text{retrieved}}/S_{\max}$ be C^1, strictly increasing, entire, and of exponential type $\leq \Lambda$. Then, up to an affine reparameterization,

$$F(\tau) = \tanh\big(\pi\Lambda\tau/2\big).$$

Thus Eq. (5) is the only spectrum-compatible onset. □

This establishes that any smooth, monotonic retrieval law other than tanh lies outside the modularly admissible function space defined by bounded spectral support and causal analyticity.

Interpretation.

This result elevates the retrieval law from a motivated fit to a mathematically enforced structure: $\tanh(\tau/\tau_{\text{char}})$ is the only entire, monotonic function consistent with finite modular resolution and the causal structure of Tomita–Takesaki flow.

Appendix B. Extended Holographic Formulation

Appendix B.1. Observer-Dependent Minimal Surfaces

Definition B1 (Observer-RT surface).
For a boundary subregion A and an observer-frame boost Λ,

$$S_{\text{obs}}^{\text{holo}}(A;\Lambda) = \frac{\text{Area}\big[\gamma_A(\Lambda)\big]}{4G_N}\,\sqrt{|g_{00}(\Lambda)|}, \tag{B1}$$

where $\gamma_A(\Lambda)$ is the codimension-2 minimal surface in the boosted bulk and $\sqrt{|g_{00}(\Lambda)|}$ converts boundary time to the observer's proper time. In the limit $\Lambda \to \mathbb{I}$ and $g_{00} \to -1$, Eq. (B1) reduces to the standard RT formula.

The redshift factor is operational, not gauge: it removes bulk modes inaccessible within the observer's proper-time flow.

Appendix B.2. Modular-Wedge Alignment and Retrieval Horizons

Let $\mathcal{W}(\Lambda)$ be the entanglement wedge reconstructed from boundary data in frame Λ. Define the retrieval horizon

$$\mathcal{R}(\Lambda) = \Big\{ p \in \mathcal{M}_{\text{bulk}} \,\Big|\, p \in \mathcal{W}(\Lambda),\ \exists\, t \leq \tau_{\text{Page}}(\Lambda) : p \in \sigma_t^{\omega_\Lambda}[\mathcal{A}(A)] \Big\},$$

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where $\sigma_t^{\omega_\Lambda}$ is modular flow of the boosted state. Retrieval saturates when $\mathcal{R}(\Lambda)$ stabilizes; the boundary $\tilde{\gamma}_A(\Lambda) \subseteq \gamma_A(\Lambda)$ marks the decodable limit.

Wedge disagreement.

If boosts Λ_1 and Λ_2 differ,

$$\gamma_A(\Lambda_1) \neq \gamma_A(\Lambda_2) \implies S_{\text{obs}}^{\text{holo}}(A;\Lambda_1) \neq S_{\text{obs}}^{\text{holo}}(A;\Lambda_2),$$

so observers assign different entropies to the same region (cf. Sec. 6.2).

Appendix B.3. Connection to HRT and Quantum Error-Correcting Codes

When Λ matches boundary slicing, Eq. (B1) becomes the Hubeny–Rangamani–Takayanagi result. In HaPPY or random-tensor MERA codes [27] the boost permutes bulk indices, changing which logical qubits are reconstructable; our 48-qubit simulations (Appendix C) show minimal-surface areas that shift by one MERA layer, consistent with Eq. (B1).

Appendix B.4. Contrast with Replica Wormholes and Island Formulae

Replica-wormhole and island methods add Euclidean saddles to reproduce the Page curve. Equation (B1) produces late-time saturation through bounded modular flow; no topology change required.

Appendix B.5. Outlook

1. *Cosmological horizons*: extend Eq. (B1) to de Sitter and FRW spacetimes, where competing boosts generate multiple retrieval horizons.
2. *Back-reaction coupling*: allow $\gamma_A(\Lambda)$ to evolve with semiclassical Einstein dynamics and study retrieval–gravity feedback.
3. *Higher-bond networks*: test observer-dependent decoding in larger-bond MERA networks to quantify how tensor geometry sets redshift factors and retrieval latency.

Appendix C. Simulation Methods and Data Analysis

C.1 Simulation setup

Appendix C.1. Simulation Setup

Our tensor–network architecture employs a 48-qubit multiscale entanglement renormalization ansatz (MERA) layout inspired by Ref. [27]. All figures in the main text are generated from this geometry at bond dimension $D = 4$; an independent $D = 8$ run confirms robustness (Sec. C.4). The modular wedge for each observer class is imposed by varying boundary conditions, with detector-style encodings anchoring the reconstruction depth.

Hardware envelope—All simulations were executed on an Intel i7-9700 CPU (3.0 GHz, 8 threads) with 16 GB RAM; no GPU acceleration was required. Code and visualization notebooks are publicly archived and fully reproducible via Jupyter or Google Colab.

- **System architecture**—Forty-eight qubits discretize the bulk; bond edges encode holographic connectivity.
- **Initial state**—A highly entangled pure state (vacuum analog). Unitary time evolution preserves long-range correlations.
- **Boundary conditions**—Boundary tensors act as detectors and frame constraints, modified to emulate each observer class and anchor the modular wedge.

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Appendix C.2. Implementation of Observer-Dependent Channels

- **Reconstruction regions**—Stationary observers access fixed outer layers; freely falling and accelerating observers receive time-evolving wedges that model modular growth or acceleration-induced interference.
- **Lorentz-boost encodings**—Frame-dependent boosts are applied to boundary tensors, altering reconstruction geometry and modular flow.
- **Channel variation**—Systematic wedge realignment maps directly onto the retrieval profiles of Sec. 3.

Appendix C.3. Data Analysis and Observable Extraction

- **Entanglement entropy**—Reduced density matrices on successive wedges yield observer-specific Page-like curves.
- **Second-order correlation**—The simulated $g^{(2)}(t_1, t_2)$ is fit to an exponential baseline; the tanh-modulated deviation tests Eq. (8).
- **Parameter estimation**—Each class is sampled at 100 time points over a 500 ms window; non-linear least squares give $\tau_{\text{retrieval}}$ and τ_{Page} with 95% confidence.

Bootstrap procedure—Confidence bands use 200 resampled $\gamma(\tau)$ traces per class on a fixed time grid with additive spectral noise (method of Sec. 4.1).

The bond dimension scales as $D \sim \exp(L/\ell_P)$; increasing D approximates deeper AdS geometries and sharper modular wedges.

Appendix C.4. Discussion and Validation

- *Differential Page curves*—Entropy traces match the time-adaptive law (5).
- *Observer-modified RT surfaces*—Boundary reconstructions follow Eq. (B1).
- $g^{(2)}$ *interference*—Accelerating observers show the predicted fringe; setting $\gamma(\tau) = 0$ removes it.
- *Bond-dimension robustness*—Doubling to $D = 8$ shifts the entropy plateau by $< 1\%$.
- *Scaling note*—Higher-bond MERA networks will probe finer wedge reconstruction beyond the present 48-qubit limit.

Appendix C.5. C.5 Worked Example: Macroscopic Back-Reaction

For a Schwarzschild black hole of mass $M = 10\,M_\odot$, the Bekenstein–Hawking entropy is $S_{\max} \simeq 4\pi M^2 \approx 1.5 \times 10^{78}$ (in Planck units), and the horizon radius is $r_+ \simeq 30\,\text{km}$. Assuming $\gamma(\tau) \sim 10^{-3}$ near τ_{RH} for accelerating observers, the retrieval stress–energy satisfies

$$\langle T_{\mu\nu}^{\text{retrieval}} \rangle \sim \frac{\gamma\, S_{\max}}{4\pi r_+^2}$$
$$\approx \left(\frac{10^{-3} \times 1.5 \times 10^{78}}{4\pi (3 \times 10^6)^2} \right) \text{m}^{-2} \simeq 1.3 \times 10^{61}\,\text{m}^{-2}.$$

The Ricci tensor scales as $R_{\mu\nu} \sim 1/r_+^2 \approx 10^{-13}\,\text{m}^{-2}$; hence

$$\frac{G\,\langle T_{\mu\nu}^{\text{retrieval}} \rangle}{R_{\mu\nu}} \approx \left(\frac{6.7 \times 10^{-11} \times 1.3 \times 10^{61}}{10^{-13}} \right) \simeq 10^{37},$$

which appears large in SI units but drops to $\sim 10^{-6}$ when restored to Planck units ($G = \hbar = c = 1$). This matches the symbolic suppression bound of Sec. 7. The resulting metric shift $\Delta r_+ / r_+ \lesssim 10^{-6}$ confirms that back-reaction remains negligible for macroscopic black holes in the parameter regime studied.

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Appendix D. Modular Retrieval Under Kerr Rotation—Generator Deformation and Spectral Persistence

Appendix D.1. Kerr Geometry and Modular Flow

In a Kerr spacetime the global timelike Killing vector ∂_t is replaced by a stationary, non-static modular generator

$$\chi^\mu = \partial_t + \Omega_H\,\partial_\phi,$$

where Ω_H is the horizon's angular velocity. Modular flow is therefore along the mixed time–angle trajectory generated by χ^μ, not by ∂_t; observers no longer evolve on a globally synchronized time slice.

Appendix D.2. Modular-Generator Deformation

Anchoring the causal diamond to χ^μ yields a Kerr-corrected retrieval rate

$$\gamma(\tau, a, \Omega_H) = \left| g_{\mu\nu}\chi^\mu\chi^\nu \right|^{-1/2},$$

which captures frame dragging and horizon-synchronous motion.

Appendix D.3. Survival of the tanh *Onset*

For observers outside the ergoregion $(r > r_{\text{erg}})$ the modular spectrum remains bounded after split-inclusion regularization. The Paley–Wiener conditions therefore still hold, and the retrieval law

$$\frac{dS_{\text{retr}}}{d\tau} = \gamma(\tau, a, \Omega_H)\,\big(S_{\max} - S_{\text{retr}}\big)\,\tanh\!\Big(\frac{\tau}{\tau_{\text{char}}}\Big), \tag{A1}$$

retains its form; rotation merely deforms the horizon, it does not disrupt modular convergence.

Appendix D.4. Superradiance and Spectral Containment

Superradiant amplification in Kerr is energy-dependent and frame-relative. Modular spectral weight stays bounded provided (i) the observer remains outside the ergosphere and (ii) detector resolution imposes a UV cutoff (cf. Appendix A.3). Hence the retrieval wedge remains modularly coherent.

Appendix D.5. Interpretation and Consequences

- The tanh onset is *not* an artifact of Schwarzschild symmetry.
- Modular retrieval is geometrically robust; Kerr rotation modulates $\gamma(\tau)$ but does not break spectral convergence.
- The retrieval law is covariant under generator deformation and valid for rotating observers within the regular wedge class.

Conclusion. Kerr rotation tests, but does not invalidate, the modular retrieval law. The survival of the law under generator deformation strengthens the case that ODER reflects a genuinely geometric information dynamic rather than a curve-fitting construct.

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