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Article

On Rough Parametric Marcinkiewicz Integrals Along Certain Surfaces

Mohammed Ali [1,2,*] and Hussain Al-Qassem [3]

1 College of Integrative Studies, Abdullah Al-Salem University, Firdous Street, Khaldiya, Kuwait
2 Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan
3 Department of Mathematics and Statistics, Qatar University, Doha, Qatar
* Correspondence: myohammed.ali@aasu.edu.kw

Abstract: This work focuses on investigating rough Marcinkiewicz integrals associated to specific surfaces. Whenever the kernel functions belong to $L^q(\mathbb{S}^{m-1})$ space, the L^p boundedness of these Marcinkiewicz integrals is confirmed. This finding along with Yano's extrapolation argument prove the L^p boundedness of the aforementioned integrals under weaker conditions on the kernels. The results in this work improve and generalize various previously known results on Marcinkiewicz integrals.

Keywords: Marcinkiewicz integrals; rough operators; L^p bounds; extrapolation

MSC: 42B20; 42B25; 42B35

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1. Introduction

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with $m \geq 2$, and let $\mathbb{S}^{m-1}$ be the unit sphere in $\mathbb{R}^m$ equipped with the normalized Lebesgue surface measure $d\sigma_m(\cdot)$. Also, let $w' = w/|w|$ for $w \in \mathbb{R}^m \setminus \{0\}$.

Assume that h is a measurable function on $\mathbb{R}^+$ and that Θ is a homogeneous function of degree zero on $\mathbb{R}^m$, integrable over $\mathbb{S}^{m-1}$ and satisfies the condition

$$\int_{\mathbb{S}^{m-1}} \Theta(w')d\sigma_m(w') = 0. \quad (1)$$

For appropriate mappings $\mathcal{P} : \mathbb{R}^m \to \mathbb{R}^d$ and $\phi : \mathbb{R}^+ \to \mathbb{R}$, we consider the Marcinkiewicz integral $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ defined, initially for $f \in C_0^\infty(\mathbb{R}^{d+1})$, by

$$\mathcal{M}_{\Theta,\mathcal{P},\phi,h}(f)(\tilde{x}) = \left(\int_0^\infty \left|\frac{1}{l^\alpha}\int_{|w|\leq l} f(x-\mathcal{P}(w), x_{d+1}-\phi(|w|))\frac{\Theta(w)}{|w|^{m-\alpha}}h(|w|)dw\right|^2 \frac{dl}{l}\right)^{1/2},$$

where $\tilde{x} = (x, x_{d+1}) \in \mathbb{R}^{d+1}$ and $\alpha = \tau + i\kappa$ ($\tau, \kappa \in \mathbb{R}$ with $\tau > 0$).

When $m = d$, $\mathcal{P}(w) = w$, $\phi \equiv 0$, and $h \equiv 1$, we denote $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ by $\mathcal{M}_{\Theta,\alpha}$. Also, when $\alpha = 1$, we denote $\mathcal{M}_{\Theta,\alpha}$ by $\mathfrak{M}_\Theta$ which is basically the classical Marcinkiewicz operator introduced by Stein in [1]. The study of the L^p boundedness of $\mathfrak{M}_\Theta$ has received a large amount of attention by many authors for a long time. For instance, it was proved in [1] that $\mathfrak{M}_\Theta$ is bounded on $L^p(\mathbb{R}^m)$ for $p \in (1,2)$ provided that the kernel function Θ belongs to the space $Lip_\beta(\mathbb{S}^{m-1})$ for some $\beta \in (0,1]$. Later on, the authors of [2] proved the L^p boundedness of $\mathfrak{M}_\Theta$ for all $p \in (1,\infty)$ under the condition $\Theta \in C^1(\mathbb{S}^{m-1})$. Thereafter, Walsh [3] confirmed the $L^2(\mathbb{R}^m)$ boundedness of $\mathfrak{M}_\Theta$ whenever $\Theta \in L(logL)^{1/2}(\mathbb{S}^{m-1})$, and also he found that the assumption $\Theta \in L(\log L)^{1/2}(\mathbb{S}^{m-1})$ is optimal in the sense that if $\Theta \in L(logL)^\epsilon(\mathbb{S}^{m-1})$ for any $\epsilon \in (0,1/2)$, then the operator $\mathfrak{M}_\Theta$ will not be bounded on $L^2(\mathbb{R}^m)$. The result in [3] was

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improved in [4] for the case $p \in (1,\infty)$. On the other hand, the authors of [5] obtained the L^p $(1 < p < \infty)$ boundedness of $\mathfrak{M}_\Theta$ if Θ lies in the space $B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$ for some $q > 1$. Furthermore, they showed that the assumption $\Theta \in B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$ is optimal in the sense that if $\Theta \in B_q^{(0,\epsilon)}(\mathbb{S}^{m-1})$ for any $\epsilon \in (-1,-1/2)$, then $\mathfrak{M}_\Theta$ may not be bounded on $L^2(\mathbb{R}^m)$. Here $B_q^{(0,\epsilon)}(\mathbb{S}^{m-1})$ is referred to the block space that introduced in [6].

We point out that the study of the parametric Marcinkiewicz operator $\mathcal{M}_{\Theta,\alpha}$ was initiated in [7] and then continued by many authors. In addition, the study of singular integral operators with rough kernels along surfaces was started in [8], and then continued by many researchers. For instance, the authors of [9] studied the operator $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ when $\phi \equiv 0$, $\Theta \in L(\log L)^{1/2}(\mathbb{S}^{m-1}) \cup B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$, $h \in \nabla_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$, and $\mathcal{P}(w) = (P_1(w), P_2(w), \cdots, P_d(w))$ is a polynomial mapping, where each P_j is a real valued polynomial on $\mathbb{R}^m$. In fact, they established the L^p boundedness of $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ for all $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$. Here, $\Delta_\gamma(\mathbb{R}_+)$ (with $\gamma > 1$) refers to the collection of all functions h that are defined on $\mathbb{R}^+$ and satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+)} = \sup_{j\in\mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(l)|^\gamma \frac{dl}{l} \right)^{1/\gamma} < \infty.$$

The authors of [10] obtained the same results in [9] for the special cases $\mathcal{P}(w) = w$ and for the case $\phi \equiv 0$ is replaced by the condition $\phi \in C^2(\mathbb{R}^+)$ is convex and increasing function with $\phi(0) = 0$. The L^p boundedness of $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ was investigated by many authors under various conditions on Θ, $\mathcal{P}$, ϕ, and h. We refer the readers to consult: For a background information and a sample of past studies relevant to our current study [11–15], for its extensions and developments [16–21] and for recent advances [22–31].

In the light of the results in [9] concerning the operator $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ in the case $\phi \equiv 0$ and of the results concerning the operator $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ in the case $\mathcal{P}(w) = w$, a question arises naturally is whether the boundedness of the operator $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ holds under the same assumptions as in [9] and for certain classes of functions ϕ ?

Our main focus in this paper will be answering the above question in affirmative as described in the following results.

Theorem 1. *Let $\mathcal{P}$ be a polynomial mapping given by $\mathcal{P}(w) = (P_1(w), P_2(w), \cdots, P_d(w))$, where each P_j is a real valued polynomial on $\mathbb{R}^m$, and let ϕ be a function satisfying*

$$\phi(l) = \psi(l) + \varphi(l),$$

where ψ is a polynomial, $\varphi^{(k)}(0) = 0$ for all $1 \leq k \leq M$, $\varphi^{(k)}$ is positive nondecreasing on $\mathbb{R}^+$ for all $1 \leq k \leq M+1$, and $M = \max\{deg(\psi), deg(\mathcal{P})\}$. Assume that $\Theta \in L^q(\mathbb{S}^{m-1})$ for some $q \in (1,2]$ satisfies the condition (1) and that $h \in \nabla_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$. Then, a positive a constant $C_{p,\Theta,h}$ (independent of ϕ and the coefficients of the polynomials P_j and ψ) exists such that

$$\left\| \mathcal{M}_{\Theta,\mathcal{P},\phi,h}(f) \right\|_{L^p(\mathbb{R}^{d+1})} \leq C_{p,\Theta,h} \left(\frac{\gamma}{(q-1)(\gamma-1)} \right)^{1/2} \|f\|_{L^p(\mathbb{R}^{d+1})}$$

for all $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$, where $C_{p,\Theta,h} = C_p \|h\|_{\nabla_\gamma(\mathbb{R}^+)} \|\Theta\|_{L^q(\mathbb{S}^{m-1})}$.

By employing the estimate in Theorem 1 along using an extrapolation argument (see [32,33]), we obtain the following:

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Theorem 2. *Assume that $\mathcal{P}$, ϕ and h are given as in Theorem 1.*
(i) If $\Theta \in L(\log L)^{1/2}(\mathbb{S}^{m-1})$, then

$$\left\|\mathcal{M}_{\Theta,\mathcal{P},\phi,h}(f)\right\|_{L^p(\mathbb{R}^{d+1})} \leq C_p\|h\|_{\nabla_\gamma(\mathbb{R}^+)}\left(\|\Theta\|_{L(\log L)^{1/2}(\mathbb{S}^{m-1})}+1\right)\|f\|_{L^p(\mathbb{R}^{d+1})}$$

for all $|1/p-1/2|<\min\{1/\gamma',1/2\}$;

(ii) If $\Theta \in B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$ for some $q>1$, then

$$\left\|\mathcal{M}_{\Theta,\mathcal{P},\phi,h}(f)\right\|_{L^p(\mathbb{R}^{d+1})} \leq C_p\|h\|_{\nabla_\gamma(\mathbb{R}^+)}\left(\|\Theta\|_{q^{(0,-1/2)}(\mathbb{S}^{s-1})}+1\right)\|f\|_{L^p(\mathbb{R}^{d+1})}$$

for all $|1/p-1/2|<\min\{1/\gamma',1/2\}$.

Remark 1.
(i) We notice that the main result in [9] is attained from Theorem 2 when we take $\phi \equiv 0$. Thus, our result generalizes the result in [9].
(ii) Since $Lip_\beta(\mathbb{S}^{m-1}) \subset L(\log L)^{1/2}(\mathbb{S}^{m-1}) \cup B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$, our results extend the results in [1,2,7].
(iii) our conditions on Θ in Theorem 2 are known to be the best possible in their respective classes for the special cases $m=d$, $\mathcal{P}(w)=w$, $\phi\equiv 0$, $h\equiv 1$, and $\alpha=1$ (see [3,5]).
(iv) For the case $\gamma>2$, our results give the L^p boundedness of $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ for p in the full range $(1,\infty)$.

Throughout the rest of the paper, we assume that the letter C denotes a positive constant whose value is independent of the essential variables and not necessary be the same at each appearance.

2. Some Lemmas

In this section, we give auxiliary lemmas which will play major roles in proving the main results of this work. Let $\mu \geq 2$. For suitable mappings $\mathcal{P}:\mathbb{R}^m \to \mathbb{R}^d$, $\phi:\mathbb{R}^+ \to \mathbb{R}$ and $h:\mathbb{R}^+ \to \mathbb{C}$; we consider the family of measures $\{\mho_{\Theta,\mathcal{P},\phi,h,l} := \mho_l : l \in \mathbb{R}^+\}$ and its related maximal operators $\mho_h^*$ and $M_{h,\mu}$ on $\mathbb{R}^{d+1}$ given by

$$\int_{\mathbb{R}^{d+1}} f d\mho_l = \frac{1}{l^\alpha}\int_{l/2\leq|w|\leq l} f(\mathcal{P}(w),\phi(|w|))\frac{\Theta(y)h(|w|)}{|w|^{m-\alpha}}dw,$$

.

$$\mho_h^* f(\tilde{x}) = \sup_{l\in\mathbb{R}^+}||\mho_l| * f(\tilde{x})|,$$

and

$$M_{h,\mu}f(\tilde{x}) = \sup_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}||\mho_l| * f(\tilde{x})|\frac{dl}{l},$$

where $|\mho_l|$ is defined similar to the definition of $\mho_l$ with replacing Θ by $|\Theta|$ and h by $|h|$.

The following lemma comes from the the results in [34].

Lemma 1. *Let $\mathcal{P}$, ϕ, h, and Θ be given as in Theorem 1. Then there exists a constant $C_p>0$ such that for $f \in L^p(\mathbb{R}^{d+1})$ with $p>\gamma'$, we have*

$$\|\mho_h^*(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C_{p,\Theta,h}\|f\|_{L^p(\mathbb{R}^{d+1})} \tag{2}$$

and

$$\|M_{h,\mu}(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C_{p,\Theta,h}(\ln\mu)\|f\|_{L^p(\mathbb{R}^{d+1})}. \tag{3}$$

Proof. It is easy to check that Hölder's inequality gives that

$$\begin{aligned} &||\mho_l| * f(\tilde{x})| \\ \leq\ & C\|\Theta\|^{1/\gamma'}_{L^1(\mathbb{S}^{m-1})}\|h\|_{\nabla_\gamma(\mathbb{R}_+)}\left(\frac{1}{l}\int_{l/2}^{l}\int_{\mathbb{S}^{m-1}}|\Theta(w)||f(x-\mathcal{P}(tw),x_{d+1}-\phi(t))|^{\gamma'}d\sigma_m(w)dt\right)^{1/\gamma'}. \end{aligned}$$

By Minkowski's inequality we have

$$\|\mho^*_h(f)\|_{L^p(\mathbb{R}^{d+1})} \quad\leq\quad C\|\Theta\|^{1/\gamma'}_{L^1(\mathbb{S}^{m-1})}\|h\|_{\nabla_\gamma(\mathbb{R}^+)}\left(\|\Upsilon^*_l(|f|^{\gamma'})\|_{L^{(p/\gamma')}(\mathbb{R}^{d+1})}\right)^{1/\gamma'}, \tag{4}$$

where

$$\int_{\mathbb{R}^{d+1}} f d\Upsilon_l \quad=\quad \frac{1}{l^\alpha}\int_{1/2l\leq|w|\leq l} f(\mathcal{P}(w),\phi(|w|))\frac{\Theta(w)}{|w|^{m-\alpha}}dw$$

and

$$\Upsilon^*_\Theta(f) \quad=\quad \sup_{l\in\mathbb{R}_+}||\Upsilon_l| * f|.$$

It is clear that

$$\Upsilon^*_\Theta(f)(\tilde{x}) \quad\leq\quad 2W^*(f)(\tilde{x}), \tag{5}$$

where

$$W^*(f)(\tilde{x}) \quad=\quad \left(\sup_{j\in\mathbb{Z}}\int_{2^j\leq|w|\leq 2^{j+1}}|f(x-\mathcal{P}(w),x_{d+1}-\phi(|w|))|\frac{|\Theta(w)|}{|w|^m}dw\right).$$

By Theorem 1.1 in [34], we deduce that

$$\|W^*(f)\|_{L^p(\mathbb{R}^{d+1})} \quad\leq\quad C_p\|\Theta\|_{L^1(\mathbb{S}^{m-1})}\|f\|_{L^p(\mathbb{R}^{d+1})} \tag{6}$$

for all $1 < p < \infty$. Thus, by (4)-(6), we prove the inequality (2) which gives directly (3). The proof of this lemma is complete. □

Remark 2. *Let* $0 < m_1 < m_2 < \cdots < m_M$ *be non-negative integers. Then, for any* $w \in \mathbb{R}^m$*, we can write* $\mathcal{P}(w) = \sum_{r=1}^{M}\mathcal{P}^{(r)}(w) + \mathcal{R}^{(r)}(|w|)$*, where* $\mathcal{P}^{(r)}(w) = (P^r_1(w), P^r_2(w), \ldots, P^r_d(w))$*,* $\{P^r_\nu(w) : 1 \leq \nu \leq d, 1 \leq r \leq M\}$ *are real-valued homogeneous polynomials of degree* m_r *with* $|w|^{m_r} \notin span\{P^r_1, \ldots, P^r_d\}$*,* $\mathcal{R}^{(r)}(l) = (\mathcal{R}^{(r)}_1(l), \mathcal{R}^{(r)}_2(l), \ldots, \mathcal{R}^{(r)}_d(l))$*, and* $\{\mathcal{R}^{(r)}_\nu(l) : 1 \leq \nu \leq d, 1 \leq r \leq M\}$ *be polynomials on* $\mathbb{R}$ *of degree less than* m_r*. Let* τ_r *denote the number of elements of* $\{\beta = (\beta_1, \beta_2 \ldots, \beta_m) \in (\mathbb{N} \cup \{0\})^m : |\beta| = m_r\} = \{\beta(1), \beta(2), \ldots, \beta(\tau_r)\}$*. Write* $P^r_k(w) = \sum_{s=1}^{\tau_r} a_{s,k} w^{\beta(s)}$*, and define the linear mapping* $L_r : \mathbb{R}^d \to \mathbb{R}^{\tau_r}$ *by* $L_r(\zeta) = \left(\sum_{k=1}^{d} a^r_{1,k}\zeta_k, \ldots, \sum_{k=1}^{d} a^r_{\tau_r,k}\zeta_k\right)$*. For* $1 \leq r \leq M$*, set* $\mathbf{P}_r(w) = \sum_{k=1}^{r}\mathcal{P}^{(k)}(w) + \mathcal{W}(|w|)$ *and* $\mathbf{P}_0(w) = \mathcal{W}(|w|)$*. Hence, we have* $\mathcal{P}(w) = \mathbf{P}_M(w)$*. For* $1 \leq r \leq M$*, we let* $\mho^{(r)}_l = \mho_{\Theta,\mathbf{P}_r,\phi,h,l}$ *and* $\mho^{(r)*}_h f(\tilde{x}) = \sup_{l\in\mathbb{R}^+}||\mho^{(r)}_l| * f(\tilde{x})|$*.*

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Now we have the following result concerning the measures $\mho_l^{(r)}$:

Lemma 2. *Let $\mathcal{P}$, ϕ, Θ, and h be given as in Theorem 1. Let $\{\mho_l^{(r)} : l \in \mathbb{R}^+, 1 \le r \le M\}$ be a family of Borel measures defined as in Remark 2. Then for $\mu \ge 2$, there exist positive constants δ_r and C such that*

$$\left(\int_{\mu^j}^{\mu^{j+1}} \left| \hat{\mho}_l^{(r)}(\zeta, \zeta_{m+1}) \right|^2 \frac{dl}{l} \right)^{1/2} \le C_{\Theta,h} (\ln \mu)^{1/2}, \tag{7}$$

$$\left(\int_{\mu^j}^{\mu^{j+1}} \left| \hat{\mho}_l^{(r)}(\zeta, \zeta_{m+1}) \right|^2 \frac{dl}{l} \right)^{1/2} \le C_{\Theta,h} (\ln \mu)^{1/2} \left(\mu^{j m_r} |L_r(\zeta)| \right)^{-\frac{1}{4 m_r \ln \mu}}, \tag{8}$$

$$\left(\int_{\mu^j}^{\mu^{j+1}} \left| \hat{\mho}_l^{(r)}(\zeta, \zeta_{m+1}) - \hat{\mho}_l^{(r-1)}(\zeta, \zeta_{m+1}) \right|^2 \frac{dl}{l} \right)^{1/2} \le C_{\Theta,h} (\ln \mu)^{1/2} \left(\mu^{j m_r} |L_r(\zeta)| \right)^{\frac{1}{4 m_r \ln \mu}}, \tag{9}$$

where $C_{\Theta,h} = C \|\Theta\|_{L^q(\mathbb{S}^{m-1})} \|h\|_{\nabla_\gamma(\mathbb{R}^+)}$.

Proof. By the definition of $\mho_l^{(r)}$, it is easy to get (7). In addition, the same arguments as in Proposition 5.1 in [35] lead to (8). By a simple change of variable, we obtain

$$\left(\int_{\mu^j}^{\mu^{j+1}} \left| \hat{\mho}_l^{(r)}(\zeta, \zeta_{m+1}) - \hat{\mho}_l^{(r-1)}(\zeta, \zeta_{m+1}) \right|^2 \frac{dl}{l} \right)^{1/2} \le C_{\Theta,h} (\ln \mu)^{1/2} \left(\mu^{j m_r} |L_r(\zeta)| \right), \tag{10}$$

which when combined with the trivial estimate (7), we conclude that

$$\left(\int_{\mu^j}^{\mu^{j+1}} \left| \hat{\mho}_l^{(r)}(\zeta, \zeta_{m+1}) - \hat{\mho}_l^{(r-1)}(\zeta, \zeta_{m+1}) \right|^2 \frac{dl}{l} \right)^{1/2} \le C_{\Theta,h} (\ln \mu)^{1/2} \left(\mu^{j m_r} |L_r(\zeta)| \right)^{\frac{1}{4 m_r \ln \mu}}. \tag{11}$$

This ends the proof of the lemma. □

By employing similar arguments as that employed in [35], we get the following:

Lemma 3. *Let $\mathcal{P}$ and ϕ be given as in Theorem 1, and let $\mu \ge 2$, $h \in \nabla_\gamma(\mathbb{R}^+)$ with $\gamma > 1$ and $\Theta \in L^q(\mathbb{S}^{\alpha-1})$ with $1 < q \le 2$. Then there exists $C_{p,\Theta,h} > 0$ such that*

$$\left\| \left(\sum_{j \in \mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} |\mho_l * \mathcal{T}_j|^2 \frac{dl}{l} \right)^{1/2} \right\|_{L^p(\mathbb{R}^{d+1})} \le C_{p,\Theta,h} (\ln \mu)^{1/2} \left\| \left(\sum_{j \in \mathbb{Z}} |\mathcal{T}_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{d+1})} \tag{12}$$

for all $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$, where $\{\mathcal{T}_j(\cdot,\cdot),\ j \in \mathbb{Z}\}$ is any set of functions on $\mathbb{R}^{d+1}$.

Proof. Since $\nabla_\gamma(\mathbb{R}^+) \subseteq \nabla_2(\mathbb{R}^+)$ for any $\gamma \ge 2$, it suffices to prove this lemma only for the case $\gamma \in (1, 2]$. In this case, we have $|1/p - 1/2| < 1/\gamma'$, which means that $p \in (\frac{2\gamma}{3\gamma-2}, \frac{2\gamma}{2-\gamma})$. First, if $p \in [2, \frac{2\gamma}{2-\gamma})$, then by duality there exists a function $\mathcal{G} \in L^{(p/2)'}(\mathbb{R}^{d+1})$ such that $\|\mathcal{G}\|_{L^{(p/2)'}(\mathbb{R}^{d+1})} \le 1$ and

$$\left\| \left(\sum_{j \in \mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} |\mho_l * \mathcal{T}_j|^2 \frac{dl}{l} \right)^{1/2} \right\|_{L^p(\mathbb{R}^{d+1})}^2 = \int_{\mathbb{R}^{d+1}} \sum_{j \in \mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} |\mho_l * \mathcal{T}_j(\tilde{x})|^2 \frac{dl}{l} |\mathcal{G}(\tilde{x})| d\tilde{x}.$$

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Employing Schwartz's inequality, we deduce that

$$\begin{aligned}\left|\mho_l * \mathcal{T}_j(\tilde{w})\right|^2 &\leq C\|\Theta\|_{L^q(\mathbb{S}^{m-1})}\|h\|^{\gamma}_{\nabla_\gamma(\mathbb{R}^+)}\int_{\frac{1}{2}l}^{l}\int_{\mathbb{S}^{m-1}}|\Theta(w)| \\ &\times \left|\mathcal{T}_j(x-\mathcal{P}(tw),x_{d+1}-\phi(t))\right|^2|h(t)|^{2-\gamma}d\sigma_m(w)\frac{dt}{t}.\end{aligned}$$

Thanks to Hölder's inequality and Lemma 1, we obtain

$$\begin{aligned}&\left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\left|\mho_l * \mathcal{T}_j\right|^2\frac{dl}{l}\right)^{1/2}\right\|^2_{L^p(\mathbb{R}^{d+1})} \\ &\leq C\|\Theta\|_{L^q(\mathbb{S}^{m-1})}\|h\|^{\gamma}_{\nabla_\gamma(\mathbb{R}^+)}\left\|\sum_{j\in\mathbb{Z}}|\mathcal{T}_j|^2\right\|_{L^{(p/2)}(\mathbb{R}^{d+1})}\left\|M_{|h|^{2-\gamma},\mu}(\overline{\mathcal{G}})\right\|_{L^{(p/2)'}(\mathbb{R}^{d+1})} \\ &\leq C(\ln\mu)\|\Theta\|_{L^q(\mathbb{S}^{m-1})}\|h\|^{\gamma}_{\nabla_\gamma(\mathbb{R}_+)}\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{T}_j|^2\right)^{1/2}\right\|^2_{L^p(\mathbb{R}^{d+1})}\left\|\mho^*_{|h|^{2-\gamma}}(\overline{\mathcal{G}})\right\|_{L^{(p/2)'}(\mathbb{R}^{d+1})} \\ &\leq C^2_{p,\Theta,h}(\ln\mu)\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{T}_j|^2\right)^{1/2}\right\|^2_{L^p(\mathbb{R}^{d+1})}, \qquad (13)\end{aligned}$$

where $\overline{\mathcal{G}}(\tilde{x}) = \mathcal{G}(-\tilde{x})$.

Now, if $p \in (\frac{2\gamma}{3\gamma-2}, 2)$, by duality there exists a class of functions $\{g_j(\tilde{x},l)\}$ on $\mathbb{R}^{d+1}\times\mathbb{R}^+$ such that

$$\left\|\left\|\|g_j\|_{L^2([\mu^j,\mu^{j+1}],\frac{dl}{l})}\right\|_{l^2}\right\|_{L^{p'}(\mathbb{R}^{d+1})} \leq 1$$

and

$$\begin{aligned}\left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\left|\mho_l * \mathcal{T}_j\right|^2\frac{dl}{l}\right)^{1/2}\right\|_{L^p(\mathbb{R}^{d+1})} &= \int_{\mathbb{R}^{d+1}}\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}(\mho_l * \mathcal{T}_j(\tilde{x}))g_j(\tilde{x},l)\frac{dl}{l}d\tilde{x} \\ \leq C_p(\ln\mu)^{1/2}\|\mathcal{H}(g_j)\|^{1/2}_{L^{(p'/2)}(\mathbb{R}^{d+1})}&\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{T}_j|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{d+1})}, \qquad (14)\end{aligned}$$

where

$$\mathcal{H}(g_j)(\tilde{x}) = \sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\left|\mho_l * g_j(\tilde{x},l)\right|^2\frac{dl}{l}.$$

Since $p < 2$ we have $p' > 2$. Hence by duality, there exists a function $\mathcal{V} \in L^{(p'/2)'}(\mathbb{R}^{d+1})$ satisfying $\|\mathcal{V}\|_{L^{(p'/2)'}(\mathbb{R}^{d+1})} \leq 1$ and

$$\begin{aligned}\|\mathcal{H}(g_j)\|_{L^{(p'/2)}(\mathbb{R}^{d+1})} &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{d+1}} \int_{\mu^j}^{\mu^{j+1}} |\mho_l * g_j(\tilde{x},l)|^2 \frac{dl}{l} \mathcal{V}(\tilde{x}) d\tilde{x} \\ &\leq C\|\Theta\|_{L^q(\mathbb{S}^{m-1})} \left\| \left(\sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} |g_j(\tilde{x},l)|^2 \frac{dl}{l} \right) \right\|_{L^{(p'/2)}(\mathbb{R}^{d+1})} \\ &\times \|h\|^{\gamma}_{\nabla_\gamma(\mathbb{R}^+)} \left\| \mho^*_{|g|^{2-\gamma}}(\mathcal{V}) \right\|_{L^{(p'/2)'}(\mathbb{R}^{d+1})} \leq C^2_{p,\Theta,h}. \end{aligned} \tag{15}$$

By the inequalities (14) and (15), we get (12) if $p \in (\frac{2\gamma}{3\gamma-2}, 2)$, which in turn along with (13) ends the proof of the lemma. □

3. Proof of Theorem 1

Let $\Theta \in L^q(\mathbb{S}^{m-1})$ for some $1 < q \leq 2$ and $h \in \nabla_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$. Set $\mu = 2^{\gamma' q'}$. By Minkowski's inequality, we get

$$\begin{aligned}\mathcal{M}_{\Theta,\mathcal{P},\phi,h}(f)(\tilde{x}) &\leq \sum_{j=0}^{\infty} \left(\int_{\mathbb{R}^+} \left| \frac{1}{l^\alpha} \int_{2^{-j-1}l<|w|\leq 2^{-j}l} f(x-\mathcal{P}(w), x_{d+1}-\phi(|w|)) \frac{\Theta(w)}{|w|^{m-\alpha}} h(|w|) dw \right|^2 \frac{dt}{t} \right)^{1/2} \\ &\leq \frac{2^\tau}{(2^\tau-1)} \left(\int_{\mathbb{R}^+} |\mho_l * f(\tilde{x})|^2 \frac{dt}{t} \right)^{1/2} = C \left(\int_{\mathbb{R}^+} \left| \mho_l^{(M)} * f(\tilde{x}) \right|^2 \frac{dt}{t} \right)^{1/2}. \end{aligned} \tag{16}$$

For $j \in \mathbb{Z}$, let $\{\mathcal{A}_j\}$ be a collection of $C^\infty((0,\infty))$ functions satisfying the following:

$$\begin{aligned} 0 &\leq \mathcal{A}_j \leq 1, \ \sum_{j\in\mathbb{Z}} \mathcal{A}_j(l) = 1, \\ \operatorname{supp}(\mathcal{A}_j) &\subseteq [\mu^{-j-1}, \mu^{-j+1}], \ \textit{and} \ \left| \frac{d^n \mathcal{A}_j(l)}{dl^n} \right| \leq \frac{C_n}{l^n}, \end{aligned}$$

where C_n is independent of $\{\mu^j; j \in \mathbb{Z}\}$. Define the operator $\widehat{T_j(f)}(\zeta,\zeta_{d+1}) = \mathcal{A}_j(|L_M(\zeta)|)\, \hat{f}(\zeta,\zeta_{d+1})$. Thus, for any $f \in C_0^\infty(\mathbb{R}^{d+1})$, Minkowski's inequality yields

$$\left(\int_{\mathbb{R}^+} \left| \mho_l^{(M)} * f(\tilde{x}) \right|^2 \frac{dt}{t} \right)^{1/2} \leq C \sum_{s\in\mathbb{Z}} \mathcal{F}_s(f)(\tilde{x}), \tag{17}$$

where

$$\begin{aligned} \mathcal{F}_s(f)(\tilde{x}) &= \left(\int_{\mathbb{R}^+} |\mathcal{J}_s(f)(\tilde{x},t)|^2 \frac{dt}{t} \right)^{1/2}, \\ \mathcal{J}_s(f)(\tilde{x},t) &= \sum_{j\in\mathbb{Z}} \mho_l^{(M)} * T_{j+s} * f(\tilde{x}) \chi_{[\mu^j,\mu^{j+1})}(t). \end{aligned}$$

Thus, to prove Theorem 1, it suffices to show that

$$\|\mathcal{F}_s(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C_{p,\Theta,h} (\ln \mu)^{1/2}\, 2^{-\frac{\varepsilon|s|}{2}} \|f\|_{L^p(\mathbb{R}^{d+1})}. \tag{18}$$

for all $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$ and for some $\varepsilon > 0$.

First, we estimate $\|\mathcal{F}_s(f)\|_{L^2(\mathbb{R}^{d+1})}$ as follows: By Plancherel's Theorem, Fubini's Theorem and Lemma 2, we obtain

$$\|\mathcal{F}_s(f)\|^2_{L^2(\mathbb{R}^{d+1})} \leq \sum_{j\in\mathbb{Z}} \int_{\mathcal{O}_{j+s}} \left(\int_{\mu^j}^{\mu^{j+1}} \left|\hat{\mho}_l^{(M)}(\zeta,\zeta_{d+1})\right|^2 \frac{dt}{t} \right) \left|\hat{f}(\zeta,\zeta_{d+1})\right|^2 d\zeta d\zeta_{d+1}$$

$$\leq C^2_{p,\Theta,h}(\ln\mu) \sum_{j\in\mathbb{Z}} \int_{\mathcal{O}_{j+s}} \left|\mu^{jm_M} L_M(\zeta)\right|^{\pm\frac{\epsilon}{q'\gamma'}} \left|\hat{f}(\zeta,\zeta_{d+1})\right|^2 d\zeta d\zeta_{d+1}$$

$$\leq C^2_{p,\Theta,h}(\ln\mu)\, 2^{-\varepsilon|s|} \sum_{j\in\mathbb{Z}} \int_{\mathcal{O}_{j+s}} \left|\hat{f}(\zeta,\zeta_{d+1})\right|^2 d\zeta d\zeta_{d+1}$$

$$\leq C^2_{p,\Theta,h}(\ln\mu)\, 2^{-\varepsilon|s|} \|f\|^2_{L^2(\mathbb{R}^{d+1})}, \tag{19}$$

where $\mathcal{O}_j = \left\{(\zeta,\zeta_{d+1}) \in \mathbb{R}^d \times \mathbb{R} : |L_M(\zeta)| \in [\mu^{-j-1}, \mu^{-j+1}]\right\}$ and $\varepsilon \in (0,1)$.

Now, let us estimate $\|\mathcal{F}_s(f)\|_{L^p(\mathbb{R}^{d+1})}$. By utilizing Lemma 3 and Littlewood–Paley theory, we deduce

$$\|\mathcal{F}_s(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C \left\| \left(\sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \left(\left|\mho_l^{(M)} * T_{j+s} * f\right| \right)^2 \frac{dl}{l} \right)^{1/2} \right\|_{L^p(\mathbb{R}^{d+1})}$$

$$\leq C_{p,\Theta,h}(\ln\mu)^{1/2} \left\| \left(\sum_{j\in\mathbb{Z}} |T_{j+s} * f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{d+1})}$$

$$\leq C_{p,\Theta,h} \left(\frac{\gamma}{(\gamma-1)(q-1)} \right)^{1/2} \|f\|_{L^p(\mathbb{R}^{d+1})}. \tag{20}$$

Consequently, by interpolating between (19) with (20), we get (18), which in turn along with (16)-(17) completes the proof of Theorem 1.

4. Conclusions

In this paper, we obtained sharp L^p bounds for parametric Marcinkiewicz integrals $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ whenever their kernel functions belong to the space $L^q(\mathbb{S}^{m-1})$ for some $q > 1$. These estimates allow us to employ Yano's extrapolation argument to prove the L^p boundedness of $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ whenever the singular kernels functions Θ are either in the space $L(\log L)^{1/2}(\mathbb{S}^{m-1})$ or in the space $B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$ for some $q > 1$. Our results improve or extend several known results as those in [1–5,7–10,13].

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References

1. E. Stein, E. On the functions of Littlewood-Paley, Lusin and Marcinkiewicz. *Trans. Amer. Math. Soc.* **1958**, 88, 430–466.
2. Benedek, A.; Calderon, A.; Panzone, R. Convolution operators on Banach space valued functions. *Proc. Nat. Acad. Sci. U.S.A.* **1962**, 48, 356–365.
3. Walsh, T. On the function of Marcinkiewicz. *Studia Math.* **1972**, 44, 203–217.
4. Al-Salman, A.; Al-Qassem, H.; Cheng, L.; Pan, Y. L^p bounds for the function of Marcinkiewicz. *Math. Res. Lett.* **2002**, 9, 697–700.

5. Al-Qassem, H.; Al-Salman, A. A note on Marcinkiewicz integral operators. *J. Math. Anal. Appl.* **2003**, 282, 698–710.
6. Jiang, Y.; Lu, S. A class of singular integral operators with rough kernel on product domains. *Hokkaido Math. J.* **1995** 24, 1–7.
7. Hörmander, L. Estimates for translation invariant operators in L^p space. *Acta Math.* **1960**, 104, 93–139.
8. Kim, W.; Wainger, S.; Wright, J.; Ziesler, S. Singular Integrals and Maximal Functions Associated to Surfaces of Revolution. *Bull. London Math. Soc.* **1996**, 28, 291–296.
9. Al-Qassem, H.; Pan, Y. On certain estimates for Marcinkiewicz integrals and extrapolation. *Collect. Math.* **2009**, 60, 123–145.
10. Al-Salman, A.; Al-Qassem, H. Flat Marcinkiewicz integral operators. *Turkish J. Math.* **2002**, 26, 329–338.
11. Ding, Y. On Marcinkiewicz integral, Proc. of the conference Singular integrals and related topics III, Oska, Japan , **2001**.
12. Ding, Y.; Fan, D.; Pan, Y. On the L^p boundedness of Marcinkiewicz integrals. *Michigan Math. J.* **2002**, 50, 17–26.
13. Ding, Y.; Lu,S.; Yabuta, K. *A problem on rough parametric Marcinkiewicz functions*. *Austral. Math. Soc.* **2002**, 72, 13–21.
14. Sakamota, M.; Yabuta, K. Boundedness of Marcinkiewicz functions. *Studia Math.* **1999**, 135, 103–142.
15. Torchinsky, A.; Wang, S. A note on the Marcinkiewicz integral. *Colloq. Math.* **1990**, 61, 235–243.
16. Al-Salman, A. A note on parabolic Marcinkiewicz integrals along surfaces. *Trans. A Razmadze Math. Inst.* **2010**, 154, 21–36.
17. Ali, M. L^p Estimates for Marcinkiewicz Integral Operators and Extrapolation. *J. Ineq. Appl.*, **2014**, 1, art. 269.
18. Chen, Y.; Ding, Y. L^p bounds for the parabolic Marcinkiewicz integral with rough kernels. *J. Korean Math. Soc.* **2007**, 44, 733–745.
19. Liu, F.; Fu, Z.; Jhang, S. Boundedness and continuity of Marcinkiewicz integrals associated to homogeneous mappings on Triebel-Lizorkin spaces. *Front. Math. China.* **2019**, 14, 95–122.
20. Wang, F.; Chen, Y.; Yu, W.L^p bounds for the parabolic Littlewood-Paley operator associated to surfaces of revolution. *Bull. Korean Math. Soc.* **2012**, 29, 787–797.
21. Xue, Q.; Ding, Y.; Yabuta, K. Parabolic Littlewood-Paley g-function with rough kernels. *Acta Math. Sin. (Engl. Ser.)* **2008**, 24, 2049–2060.
22. Al-Qassem, H.; Ali, M. A note on generalized parabolic Marcinkiewicz integrals with Grafakos-Stefanov kernels. *symmetry*. **2025**, art. 429.
23. Ali, M.; Katatbeh, Q. Generalized parabolic Marcinkiewicz integrals associated with polynomial compound curves with rough kernels. *Demon. Math.*, **2020**, 53, 44-–57
24. Ali, M.; Katatbeh, Q.; Al-Refai, O.; Al-Shutnawi, B. Estimates for functions of generalized Marcinkiewicz operators related to surfaces of revolution. *AIMS Mathematic.* **2024**, 9, 22287-22300.
25. Gao, W.; Tang, L. BLO estimates for Marcinkiewicz integrals associated with Schrödinger operators. *Proc. Indian Acad. Sci. Math. Sci.* **2019**, 129, 1–10
26. Gürbüz, F. A note concerning Marcinkiewicz integral with rough kernel. *Infinite Dim. Anal. Quan. Prob Related Topics*. **2021**, 24, art. 53.
27. Jia, H.; Yang, D.; Yuan, W.; Zhang, Y. Estimates for Littlewood–Paley operators on ball Campanato-type function spaces. *Results in Mathematics.* **2023**, 78, 1–37.
28. Khalil, O.; Tao, S.; Bechi, A. Boundedness of Marcinkiewicz integral with rough kernel and their commutator on weighted Herz space with variable exponent. *Acta Math. Univ. Comenianae.* **2023**, 2, 145–163.
29. Liu, F. A note on Marcinkiewicz integral associated to surfaces of revolution. *J. Aust. Math. Soc.* **2018**, 104, 380–402.
30. Zhang, J.; He, Q.; Xue, Q. On weighted boundedness and compactness of commutators of Marcinkiewicz integral associated with Schrödinger operators. *Ann. Func. Anal.* **2023**, 14, art. 59.
31. Liu, F.; Mao, S. L^p bounds for nonisotropic Marcinkiewicz integrals associated to surfaces. *J. Aust. Math. Soc.* **2015**, 99, 380–398.
32. Sato, S. Estimates for singular integrals and extrapolation. *Studia Math.* **2009**, 192, 219–233.
33. Yano, S. Notes on Fourier analysis. XXIX. An extrapolation theorem. *J. Math. Soc. Japan*. **1951**, 3, 296–305.

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34. Al-Salman, A.; Al-Jarrah, A. L^p Estimates of Rough Maximal Functions Along Surfaces with Applications. *Acta Math. Sinica, English Series*. **2016**, 32, 925–942.
35. Fan, D.; Pan, Y. Singular integral operators with rough kernels supported by subvarieties. *Amer J. Math.* **1997**, 199, 799–839.