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Article

Cosmology, New Entropy and Thermodynamics of Apparent Horizon

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Abstract: We propose new nonadditive entropy of the apparent horizon $S_K = S_{BH}/(1+\gamma S_{BH}^2)$, where S_{BH} is the Bekenstein–Hawking (BH) entropy and consider the description of new cosmology. When parameter γ vanishes ($\gamma \to 0$) our entropy S_K is converted into BH entropy S_{BH}. By using the holographic principle a new model of holographic dark energy is studied. We obtain the generalised Friedmann's equations for Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime for the barotropic matter fluid with equation of state $p = w\rho$. From the second modified Friedmann's equation we find a dynamical cosmological constant. The dark energy pressure p_D, density energy ρ_D and the deceleration parameter q corresponding to our model are computed. It is shown that at some EoS w and parameter γ there are phases of universe acceleration, deceleration and eternal inflation. Our model, with the help of the holographic principle, can describe the universe inflation and late time of the universe acceleration. We show the current deceleration parameter $q_0 \approx -0.6$ is realized at some model parameters. The generalised entropy of the apparent horizon with the holographic dark energy model may be of interest for new cosmology.

Keywords: entropy; cosmology; holographic principle; dark energy; Friedmann's equations; universe acceleration

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1. Introduction

It is know that black holes can be described by thermodynamics with entropy being proportional to the horizon area [1,2] and temperature is linked with the surface gravity. Thus, gravity is related to ordinary thermodynamics [3–6]. From the first law of apparent horizon thermodynamics Friedmann's equations also can be derived [7–17]. Different entropies were studied in [18–24] which can lead to modified Friedmann's equations. Other holographic dark energy models were considered in Refs. [25–34]. Holographic energy densities, depending on the form of entropy, may describe the dark energy which drives the universe to accelerate [35,36]. The nature of dark energy is unknown and can be described by the LCDM model. We propose here new apparent horizon entropy $S_K = S_{BH}/(1+\gamma S_{BH}^2)$ with S_{BH} being the Bekenstein–Hawking (BH) entropy, that lead to the presence of dark energy so that our model is alternative to the LCDM model. The S_K entropy becomes zero when the BH entropy vanishes and is the monotonically increasing function of the BH entropy S_{BH} and is positive. When parameter γ vanishes we arrive at the BH entropy. It should be noted that the apparent horizon thermodynamics leads to the Friedmann equations, in the framework of Einstein's gravity, only when the matter is a perfect fluid with equation of state (EoS) $p = -\rho$ with p being the matter pressure and ρ is the density energy of matter [16]. Modifying the BH entropy S_{BH} by apparent horizon entropy S_K we study the general case of EoS for barotropic perfect fluid $w = p/\rho$. It is worth noting that the long-range gravitational interactions are described by generalized entropies. It will be shown that S_K entropy leads to modified Friedmann's equations that describe the universe inflation. In our approach the cosmological constant is dynamical and it explains the presents of dark energy.

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2. New Entropy

Let us consider new entropy

$$S_K = -\sum_{i=1}^{W} \frac{p_i \ln p_i}{1+\gamma(\ln(p_i))^2}, \quad (1)$$

with W being a number of states and each state has a probability p_i with the probability, γ is a free parameter. The summation in Equation (1) is performed over all possible system microstates. In the case when $\gamma = 0$ entropy (1) becomes the Gibbs entropy

$$S_G = -\sum_{i=1}^{W} p_i \ln(p_i). \quad (2)$$

If each microstate is populated with equal probability, $1/p_i = W$ $(i = 1, 2, ..., W)$, then Equation (2) is converted into the Boltzmann entropy $S_B = \ln(W)$. By virtue of $1/p_i = W$, we find from Equation (1)

$$S_K = \frac{\ln(W)}{1+\gamma(\ln(W))^2}. \quad (3)$$

The BH entropy is $S_{BH} = \ln(W)$ and from Equation (3) one obtains

$$S_K = \frac{S_{BH}}{1+\gamma S_{BH}^2}. \quad (4)$$

From Equation (4) we find at $\gamma = 0$ the BH entropy S_{BH}. When A and B are two probabilistically independent systems, one has $p_{ij}^{A+B} = p_i^A p_j^B$ and entropy S_K being the nonadditive entropy because $S_K(A+B) \neq S_K(A) + S_K(B)$.

3. Thermodynamics of Apparent Horizon

We consider here the FLRW flat universe with the metric

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\Omega_2^2). \quad (5)$$

In Equation (5) $a(t)$ is a scale factor and $d\Omega_2^2$ represents the line element of 2-dimensional unit sphere. In the FLRW universe the radius of the apparent horizon $R_h = a(t)r$ is given by

$$R_h = \frac{1}{H}, \quad (6)$$

where $H = \dot{a}(t)/a(t)$ is the Hubble parameter of the universe, and dot over the scale factor being the derivative with respect to the cosmological time t. Inside the space, the total energy is defined as

$$E = \rho V_h = \frac{4\pi}{3}\rho R_h^3, \quad (7)$$

where, ρ is the energy density of matter fields and the first law of apparent horizon thermodynamics reads

$$dE = -T_h dS_h + W dV_h. \quad (8)$$

In cosmology the work density W is given by

$$W = -\frac{1}{2}\mathrm{Tr}(T^{\mu\nu}) = \frac{1}{2}(\rho - p),$$

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and p is the matter pressure. The apparent horizon temperature is represented as

$$T_h = \frac{H}{2\pi}\left|1 + \frac{\dot{H}}{2H^2}\right|. \tag{9}$$

Making us of Equations (6), (7), and (9), we obtain from first law of apparent horizon thermodynamics (8) the equation as follows:

$$\frac{H}{2\pi}\left|1 + \frac{\dot{H}}{2H^2}\right| dS_h = -\frac{4\pi}{3H^3}d\rho + \frac{2\pi(\rho+p)}{H^4}dH. \tag{10}$$

With the aid of the energy momentum conservation (the continuity equation)

$$\dot{\rho} = -3H(\rho+p). \tag{11}$$

we find equation (10) in the form

$$\frac{H}{2\pi}\left|1 + \frac{\dot{H}}{2H^2}\right| \dot{S}_h = -\frac{4\pi\dot{\rho}}{3H^3}\left(1 + \frac{\dot{H}}{2H^2}\right). \tag{12}$$

4. Modified FLRW Equations

Assuming that $\dot{H} \geq 2H^2$) and making use of Equations (11) and (12), we find

$$\frac{H}{2\pi}\dot{S}_h = \frac{4\pi(\rho+p)}{H^2}. \tag{13}$$

By virtue of our entropy (4) ($S_h = S_K$) and BH entropy

$$S_{BH} = \frac{\pi}{GH^2} \tag{14}$$

and utilizing Equations (13) and (14), one obtains the modified Friedmann equation

$$\frac{\dot{H}(1-\gamma\pi^2/(GH^2)^2)}{(1+\gamma\pi^2/(GH^2)^2)^2} = -4\pi G(\rho+p). \tag{15}$$

Equation (15), as $\gamma \to 0$, is converted into the usual Friedmann equation within general relativity for flat universe. Integrating Equation (15) and making use of Equation (12) we find the second modified Friedmann equation

$$H^2 + \frac{bH^2}{b+H^4} - 2\sqrt{b}\arctan\left(\frac{H^2}{\sqrt{b}}\right) = \frac{8\pi G}{3}\rho, \tag{16}$$

where we have defined parameter $b = \pi^2\gamma/G^2$. When $b = 0$ ($\gamma = 0$) Equation (16) becomes the FLRW equation for flat universe in the framework Einstein's gravity. Introducing the effective (a dynamical) cosmological constant

$$\Lambda_{eff} = 6\sqrt{b}\arctan\left(\frac{H^2}{\sqrt{b}}\right) - \frac{3bH^2}{b+H^4}, \tag{17}$$

equation (16) can be put into the usual form of Friedmann's equation

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda_{eff}}{3}. \tag{18}$$

The dynamical cosmological constant Λ_{eff} versus H at some parameters $b = \pi^2\gamma/G^2$ is depicted in Figure 1. In accordance with Figure 1, Λ_{eff} increases as the Habble parameter H increases. As $H \to \infty$ the dynamical cosmological constant becomes $\Lambda_{eff} \to 3\pi\sqrt{b}$. At fixed H, when b increases

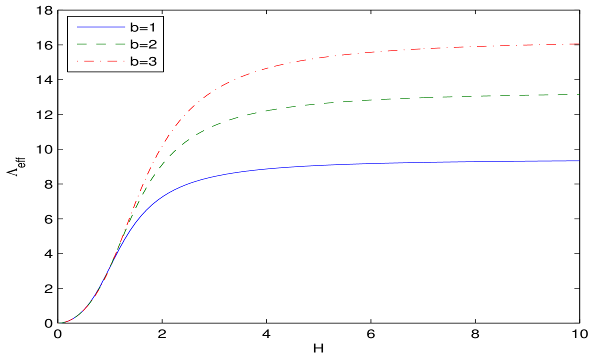

Figure 1. The function Λ_{eff} versus H at $b = \pi^2\gamma/G^2 = 1, 2, 3$. Figure 1 shows that Λ_{eff} increases as b increases. When $H \to \infty$ the dynamical cosmological constant becomes $\Lambda_{eff} \to 3\pi\sqrt{b}$.

the dynamical cosmological constant Λ_{eff} also increases. Making use of Equations (17) and (18) we obtain the dark energy density

$$\rho_D = \frac{3}{8\pi G}\left[2\sqrt{b}\arctan\left(\frac{H^2}{\sqrt{b}}\right) - \frac{bH^2}{b+H^4}\right]. \tag{19}$$

Let us define the normalized density parameters $\Omega_m = \rho/(3M_P^2H^2)$ and $\Omega_D = \rho_D/(3M_P^2H^2)$, where $M_P = 1/\sqrt{8\pi G}$ is the reduced Plank mass. Then from Equations (17), (18) and (19), one finds the equation $\Omega_m + \Omega_D = 1$. By virtue of Equations (17),(18) and (19) we obtain the normalized density of the matter ($w = 0$)

$$\Omega_m = 1 - \frac{2\sqrt{b}}{H^2}\arctan\left(\frac{H^2}{\sqrt{b}}\right) + \frac{b}{b+H^4}. \tag{20}$$

The Ω_m versus H is plotted in Figure 2. For the current era $\Omega_m \approx 0.26$, and in accordance with

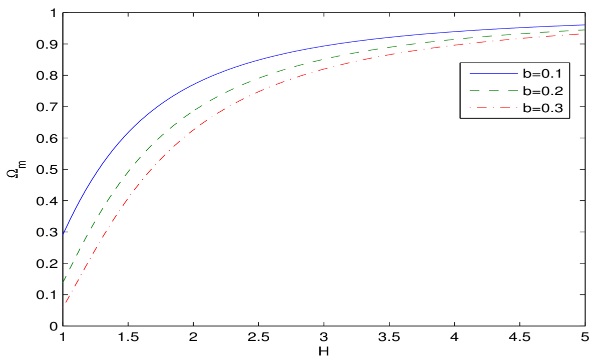

Figure 2. The function Ω_m versus H at $b == 0.1, 0.2, 0.3$. According to Figure 2 Ω_m increases as b decreases at fixed H. When $H \to \infty$ ($R_h \to 0$) one has $\Omega_m \to 1$ and $\Omega_D \to 0$.

Figure 2, one can obtain the corresponding parameters b and H. Assuming that dark substance obeys ordinary conservation law, and there is no mutual interaction between the cosmos components, we find from the continuity equation the dark energy pressure

$$p_D = -\frac{\dot{\rho}_D}{3H} - \rho_D. \tag{21}$$

With the help of Equations (19) and (21) one obtains the pressure

$$p_D = -\frac{b(b+3H^4)\dot{H}}{4\pi G(H^4+b)^2} - \frac{3}{8\pi G}\left[2\sqrt{b}\arctan\left(\frac{H^2}{\sqrt{b}}\right) - \frac{bH^2}{b+H^4}\right]. \tag{22}$$

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Making use of Equations (15), (19) and (22) one finds EoS for the dark energy $w_D = p_D/\rho_D$,

$$w_D = \frac{b(b+3H^4)(1+w)}{H^4(H^4-b)}\left(\frac{H^2}{2\sqrt{b}\arctan(H^2/\sqrt{b}) - bH^2/(H^4+b)} - 1\right) - 1. \tag{23}$$

The w_D versus H is plotted in Figure 3.

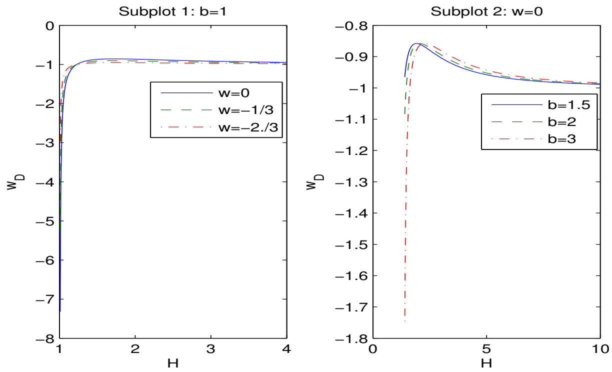

Figure 3. Left panel: The function w_D versus H at $b = 1$, $w = 0, -1/3, -2/3$, $G = \rho_0 = a_0 = 1$. According to Figure 3 when EoS parameter for the matter w increases at fixed H the w_D also increases (at $H > 1.5$). At large H EoS parameter for dark energy w_D approaches to -1. **Right panel**: In accordance with figure when parameter b increases at fixed H (at $H > 1.5$), EoS parameter for dark energy w_D also increases and $\lim_{H\to\infty} w_D = -1$.

In accordance with left panel of Figure 3 at $b = 1$ and $w = 0, -1/3, -2/3$ when w increases, EoS parameter for dark energy w_D also increases (at $H > 1.5$). According to right panel of Figure 3 at $w = 1$ and $b = 2, 3, 4$ when b increases, w_D also increases (at $H > 1.5$). From Equation (23), it follows that $\lim_{H\to\infty} w_D = -1$ so that the dynamical cosmological constant leads to EoS of dark energy $w_D = -1$ at large Habble parameter H (small apparent horizon radius R_h). As a result, universe inflation is due to dynamical cosmological constant. When $H \to 0$ ($R_h \to \infty$) the dynamical cosmological constant vanishes ($\Lambda_{eff} \to 0$). Thus, after Big Bang ($R_h \approx 0$) we have the de Sitter space, $p_D + \rho_D = 0$.

According to the second law of apparent horizon thermodynamics we have the requirement $\dot{S}_K \geq 0$ and from Equation (4) one obtains $(1-\gamma S_{BH}^2)\dot{S}_{BH}/(1+\gamma S_{BH}^2)^2 \geq 0$ or $(1-\gamma S_{BH}^2) \geq 0$ and $\dot{S}_{BH} = -2\pi\dot{H}/(GH^3) \geq 0$. As a result, these requirements, for positive Hubble parameter, lead to $\dot{H} \leq 0$ and $(1-\gamma S_{BH}^2) \geq 0$. Then from Equation (15) we find that at $1-\gamma\pi^2/(GH^2)^2) \geq 0$ one has $\rho + p \geq 0$ and for the positive energy density we obtain for EoS parameter the requirement $w \geq -1$.

Now we explore the redshift $z = a_0/a(t) - 1$, where a_0 corresponds to a scale factor at the current time. From the continuity equation (11) and EoS $p = w\rho$ we obtain the density energy of matter

$$\rho = \rho_0\left(\frac{1+z}{a_0}\right)^{3(1+w)}, \tag{24}$$

where ρ_0 is the density energy of matter at the present time. With the aid of Equations (16) and (24) one finds equation as follows:

$$H^2 + \frac{bH^2}{b+H^4} - 2\sqrt{b}\arctan\left(\frac{H^2}{\sqrt{b}}\right) = \frac{8\pi G}{3}\rho_0\left(\frac{1+z}{a_0}\right)^{3(1+w)}. \tag{25}$$

making use of Equations (24) and (25) we obtain the redshift

$$z = a_0\left(\frac{3}{8\pi\rho_0 G}\left(H^2 + \frac{bH^2}{H^4+b} - 2\sqrt{b}\arctan\left(\frac{H^2}{\sqrt{b}}\right)\right)\right)^{1/(3(1+w))} - 1. \tag{26}$$

We plotted the function of Habble parameter versus redshift z in Figure 4 for $G = \rho_0 = a_0 = 1$. Figure 4 shows that sas redshift z increases the Habble parameter H also increases. According to left panel

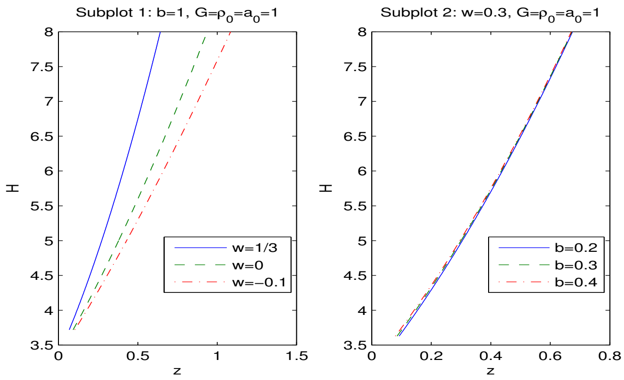

Figure 4. **Left panel**: The function H versus z at $b = 1$, $w = 1/3, 0, -0.1$, $G = \rho_0 = a_0 = 1$. In accordance with Figure 4 when z increases H also increases. When EoS parameter w increases, at fixed z, the H also increases. **Right panel**: $b = 0.2, 0.3, 0.4$, $w = 0.3$, $G = \rho_0 = a_0 = 1$. When parameter b increases at fixed z the Habble parameter also increases.

of Figure 4, when EoS parameter w increases, at fixed z, the H also increases. Right panel of Figure 4 shows that when parameter b increases at fixed z the Habble parameter also increases.

Let us investigate the phases of acceleration and deceleration of the universe. The deceleration parameter is given by

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2}. \qquad (27)$$

If $q < 0$ we have the acceleration phase but when $q > 0$ the phase of the universe deceleration takes place. By virtue of Equations (15), (24) and (27) we obtain the deceleration parameter as a function of redshift z

$$q = \frac{4\pi G\rho_0(1+w)(H^4+b)^2}{H^6(H^4-b)}\left(\frac{1+z}{a_0}\right)^{3(1+w)} - 1. \qquad (28)$$

With the help of Equations (25) and (28) one finds the deceleration parameter q in the form

$$q = \frac{3(1+w)(H^4+b)^2}{2H^6(H^4-b)}\left(H^2 + \frac{bH^2}{b+H^4} - 2\sqrt{b}\arctan\left(\frac{H^2}{\sqrt{b}}\right)\right) - 1. \qquad (29)$$

In Figure 5 we depicted the deceleration parameter q versus the Hubble parameter H. For some values

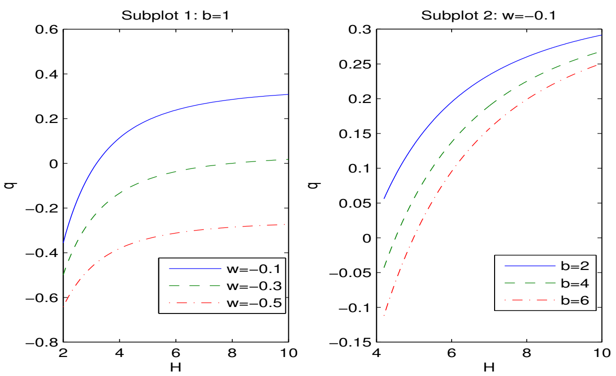

Figure 5. Figure 5 shows that q increases as H increases. **Left panel**: The function q versus H at $b = 1$, $w = -0.1, -0.3$, -0.5. When EoS parameter w increases at fixed b and H, the deceleration parameter q also increases. At $w = -0.1$ and -0.3 there are two phases, acceleration $q < 0$ and deceleration $q > 0$ and at $w = -0.5$ we have only the acceleration phase (the eternal inflation). **Right panel**: According to figure, when parameter b (and γ) increases at fixed w and H the deceleration parameter q decreases. Here, at $b = 4$ and $b = 6$ we have two phases: acceleration and deceleration.

of w and γ there are two phases: inflation ($q < 0$) and deceleration ($q > 0$) but at some w and γ we

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have only eternal university acceleration (inflation), $q < 0$. In accordance with figure, when redshift z increases the deceleration parameter q also increases. According to left panel of Figure 5, when EoS parameter w increases the deceleration parameter q also increases at fixed γ. At $w = -0.1$ and -0.3 there are two phases, acceleration $q < 0$ and deceleration $q > 0$ but at $w = -0.5$ we have the acceleration phase (the eternal inflation). Accordance to right panel of Figure 5, when parameter b (and γ) increases, at fixed w, the deceleration parameter q decreases. At $b = 4$ and $b = 6$ we have two phases: acceleration and deceleration.

Making use of Equation (29) one obtains the asymptotic

$$\lim_{H\to\infty} q = \frac{3w+1}{2}. \tag{30}$$

Equation (30) shows that the asymptotic of the deceleration parameter depends only on the entropy parameter γ ($b = \pi^2\gamma/G^2$). We obtain from Equation (29 at $b = 0$ ($\gamma = 0$) that $q = (3w+1)/2$. The approximate real and positive solutions to Equation (29) for H at $q = 0$, $G = 1$, $w = -0.1$ are given in Table 1 for some parameters γ. When $q = 0$ the transition redshifts z_t, we obtained from Equation (26) at $G = a_0 = \rho_0 = 1$. According to Table I shows that when the entropy parameter γ increases

Table 1. The approximate solutions to Equation (29) for H at $q = 0$, $G = a_0 = \rho_0 = 1$, $w = -0.1$.

γ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
H	3.777	4.180	4.492	4.749	4.971	5.166	5.341	5.501	5.648
z_t	-3.513	-3.625	-3.714	-3.790	-3.856	-3.915	-1.606	-1.621	-1.634

the Hubble parameter H also increases. For a divided point $q = 0$ between two pases, universe acceleration and deceleration, the transition reshift z_t is negative and decreases. From Equation (26) we obtain, for the current era when $z = 0$, $w = -2/3$, approximate solutions for the Habble parameter H and the deceleration parameter q from Equation (29) for different γ, presented in Table II. In Table

Table 2. The approximate solutions to Equations (3.13) and (3.15) for the current era $z = 0$ at $G = 1$, $a_0 = \rho_0 = 1$, $w = -2/3$.

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
H	3.352	3.509	3.621	3.711	3.786	3.853	3.912	3.965	4.015	4.060
q	-0.618	-0.646	-0.664	-0.676	-0.686	-0.694	-0.701	-0.707	-0.712	-0.717

II, negative values of the deceleration parameter q show the acceleration phase of the universe at the current time. The deceleration parameter at the current time is $q_0 \approx -0.6$ [38]. In accordance with Table II there is entropy parameter $\gamma \approx 0.1$ which can give that result.

5. Summary

In conclusion, we have proposed new entropy $S_K = S_{BH}/(1+\gamma S_{BH}^2)$ which possesses similar property as the Bekenstein–Hawking entropy S_{BH}; it becomes zero when the apparent horizon radius R_h vanishes. The S_K monotonically increases when the apparent horizon radius R_h increases and S_K is positive. We have studied the barotropic perfect fluid with flat FLRW universe. By exploring the first law of apparent horizon thermodynamics we obtained the modified Friedmann's equations. We have the addition term in the second Friedmann's equation which is a dynamical cosmological constant. We have showed that holographic dark energy is the source of the universe inflation. It is worth mentioning that Barrow's and Tsallis's entropies also lead to Einsten's equations with the dynamical cosmological constant [37]. We have found that for some parameters our model have phases of universe inflation and deceleration and eternal inflation. The transition redshifts when

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$q = 0$, presented in Table I were calculated for some EoS parameter w and for entropy parameter γ. According to Table II, at $\gamma \approx 0.1$ and $w = -2/3$ the current deceleration parameter $q_0 \approx -0.6$ is realised. It was shown that dynamical cosmological constant gives EoS of dark energy $w_D = -1$ at large Habble parameter H (small apparent horizon radius R_h). So, after Big Bang the de Sitter space takes place ($p_D + \rho_D = 0$) and universe inflation is due to dynamical cosmological constant. We have showed that when $R_h \to \infty$ ($H \to 0$) dynamical cosmological constant vanishes ($\Lambda_{eff} \to 0$). It is worth noting that similar results were discussed in other models [39,40]. Thus,cosmology based on the modified Friedmann equations obtained may be of interest for a description of inflation and late time universe acceleration.

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