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The Number of All *m*-Independent Sets of *K* Disjoint 3-Regular Graphs with a Given Vertex

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Abstract: Let K be a positive integer and $1 \leq K \leq k$, where k is a given positive integer. Consider a 3-regular graph with 8 vertices and denoted by Q_3. In addition, KQ_3 is composed of K disconnected 3-regular graphs Q_3. In this paper, we first consider the number of all m-independent sets of kQ_3 with $1 \leq m \leq 4k$, denoted by $T_m(kQ_3)$. Then, we consider the number of all m-independent sets of KQ_3 with $1 \leq K \leq k-1$, denoted by $S_m(KQ_3)$.

Keywords: 3-regular graph, m-independent sets

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1. Introduction

A 3-regular graph is a special kind of undirected simple graph in which the degree of each vertex (i.e., the number of edges connected to that vertex) is 3. In other words, in such a graph, each vertex is connected to exactly 3 other vertices. Let $Q_3 = (V, E)$ be a simple undirected 3-regular graph with $|V| = 8$, $|E| = 12$. An independent set is a subset S of V, such that any two vertices in S are not adjacent. A maximum independent set is an independent set of maximal size; its size is denoted by $\alpha(G)$. We say that an m-independent set means that this is an independent set and it has size m.

In 1965, Moon and Moser described in their paper [14] all n-vertex graphs with the maximum number of maximal independent sets. Later, the same problem was also solved for various graph classes, such as trees [2,4,9,13] connected graphs [3,5], forests and graphs with at most one cycle [2], triangle-free graphs [6], graphs with at most r cycles [7,11], and unicyclic connected graph [12]. Graphs with the second [8] and the third [10] largest number of maximal independents sets were also described.

In this paper, we described the graph consisting of K disconnected graph Q_3 denoted as KQ_3, where K can be taken from $\{1, 2, \ldots, k\}$ and k is also a positive integer. Under this set of graphs, we consider its number of certain m-independent sets, where m can be taken from $\{1, 2, \ldots, 4K\}$. However, we will first give formulas to compute the number of m-independent sets of kQ_3, where m can be obtained from $\{1, 2, \ldots, 4k\}$, since $K = k$. We will partition the range of m. In the first case, m is taken from $\{1, 2, \ldots, k\}$; in the second case, m is taken from $\{k+1, \ldots, 4k\}$. We also do the same for the case of KQ_3, K can only be taken from $\{1, 2, \ldots, k-1\}$ obtained.

2. Some Definitions and Notations

Let p is a positive integer, it means that in selecting the independent sets, we obtain points all in these arbitrary p graphs, i.e., take points in these p graphs to m. We know that for graph Q_3, the maximum number of points in its independent set is 4, i.e., $\alpha(Q_3) = 4$. Therefore, for K disjoint graphs Q_3 (see Figure 1), the value of m can be obtained in $[1, 4K]$. When $K = k$, the value of m is obtained in $[1, 4k]$, and when K belongs to some number in $[1, k-1]$, then the value of m can be obtained in $[1, 4K]$. Therefore, in this paper, we are going to consider all the cases.

For kQ_3, we use $T_m(kQ_3)$ to denote the number of a certain m-independent set, where the value of m is taken in [1,4k]. When $1 \leq m \leq k$, the range of values of p is $\left[\lceil \frac{m}{4} \rceil, m\right]$. We need to compute the number of m-independent for some p selected graphs, denoted by $|A_p|$. Thus, $T_m(kQ_3) = \sum_{p=\lceil \frac{m}{4} \rceil}^{m} |A_p|$.

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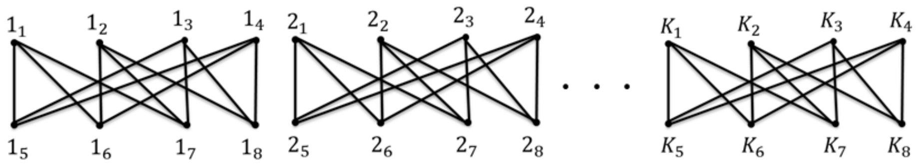

Figure 1. K disjoint 3-regular graphs with 8 vertices

When $k < m \le 4k$, the range of values of p is $\left[\left\lceil \frac{m}{4} \right\rceil, k\right]$, in which case, $T_m(kQ_3) = \sum_{p=\lceil \frac{m}{4} \rceil}^{k} |A_p|$. For KQ_3, we denote by $S_m(KQ_3)$ the number of its certain m-independent sets, and again we consider the range of values of p to obtain the corresponding formula.

3. Main Results

We first consider the number of certain m-independent sets of kQ_3, where $1 \le m \le 4k$.

Theorem 3.1. *Let m, k, p be positive integers with $1 \le m \le 4k$, $T_m(kQ_3)$ is the number of a certain m-independent set for graph kQ_3.*

(i) If $1 \le m \le k$, $p \in \left[\left\lceil \frac{m}{4} \right\rceil, m\right]$, then

$$T_m(kQ_3) = \sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} |A_p| + |A_m|.$$

(ii) If $k < m \le 4k$, $p \in \left[\left\lceil \frac{m}{4} \right\rceil, k\right]$, then

$$T_m(kQ_3) = \sum_{p=\lceil \frac{m}{4} \rceil}^{k-1} |A_p| + |A_k|.$$

where

$$|A_p| = \binom{k}{p} \left[\sum_{\ell=1 \text{ or } \left\lceil \frac{m-4p}{i} \right\rceil + 4}^{\left\lfloor \frac{m}{p} \right\rfloor} \sum_{i=1 \text{ or } 2p-m}^{\left\lfloor \frac{4p-m}{4-\left\lfloor \frac{m}{p} \right\rfloor} \right\rfloor} \binom{p}{i} \left(2\binom{4}{\ell} + \tau(\ell)4 \right)^{i} \sum_{All\ e_j} \frac{(p-i)!}{\prod_{All\ \delta(e_j)} \delta(e_j)!} \prod_{j=1}^{p-i} \left(2\binom{4}{e_j} + \tau(e_j)4 \right) \right],$$

$$|A_m| = \binom{k}{m} \cdot \left(2\binom{4}{1} \right)^{m},$$

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$$|A_k| = \sum_{i=\lceil \frac{m-k}{3} \rceil}^{m-k} \left\{ \binom{k}{i} \cdot \left(2\binom{4}{1} \right)^{k-i} \cdot \zeta \right\} for\ k < m < 2k,$$

$$|A_k| = \sum_{i=\lceil \frac{m-k}{3} \rceil}^{k-1} \left\{ \binom{k}{i} \cdot \left(2\binom{4}{1} \right)^{k-i} \cdot \zeta \right\} + \left(2\binom{4}{2} + 4 \right)^{k} for\ m = 2k,$$

$$|A_k| = \sum_{i=\lceil \frac{m-k}{3} \rceil}^{k} \left\{ \binom{k}{i} \cdot \left(2\binom{4}{1} \right)^{k-i} \cdot \zeta \right\} for\ 2k < m \leq 4k,$$

$$\zeta = \left[\sum_{All\ \ell_j} \frac{i!}{\prod_{All\ \delta(\ell_j)} \delta(\ell_j)!} \prod_{j=1}^{i} \left(2\binom{4}{\ell_j} + \tau(\ell_j) 4 \right) \right].$$

We will explain the meaning of each symbol in it.

(i) ℓ is a positive integer, which is the same number of points taken for some i of these arbitrary p graphs, i.e.,ℓ, and $1 \leq \ell \leq \left\lfloor \frac{m}{p} \right\rfloor \leq 4$. (Because for each graph, the number of points selected for each graph cannot be more than 4 if the points are guaranteed to be independent, and this is related to $\alpha(Q_3) = 4$).

$$(ii)\ \ \tau(\ell) = \begin{cases} 1, & \text{if } \ell = 2 \\ 0, & \text{otherwise} \end{cases}$$

(iii) e_j is the number of points selected for each of the remaining $p - i$ graphs under the selection of graphs in (i), where $j \in [1, p - i]$, and there are $\ell < e_1 \leq e_2 \leq \cdots \leq e_{p-i} \leq 4$, and $\sum_{j=1}^{p-i} e_j = m - i\ell$. Thus, the summation term in ξ refers to the selection of all different e_j to meet this condition.

$$(iv)\ \ \tau(e_j) = \begin{cases} 1, & \text{if } e_j = 2 \\ 0, & \text{otherwise} \end{cases}$$

(v) $\delta(e_j)$ is the number of times an element is repeated in the $p - i$ elements $\{e_1, e_2, \ldots, e_{p-i}\}$. If some element occurs only once, at this point a certain $\delta(e_j) = 1$. We need to consider all the elements. $eg : \{2, 3, 3, 4, 4, 4\}$. At this point $p - i = 6$, $\prod_{All\ \delta(e_j)} \delta(e_j)! = 1!2!3!$.

Proof. We divide it into two cases and generalize to obtain $|A_m|$, $|A_k|$ and its formula individually.

Case 1. If $1 \leq m \leq k$, then $p \in \left[\lceil \frac{m}{4} \rceil, m \right]$.

- $k = 1, m = 1$,
  $T_1(kQ_3) = \binom{k}{1} \cdot 2\binom{4}{1}$.
- $k = 2, m = 1, m = 2$,
  $T_1(kQ_3) = \binom{k}{1} \cdot 2\binom{4}{1}$;
  $T_2(kQ_3) = \binom{k}{1} \cdot (2\binom{4}{2} + 4) + \binom{k}{2} \cdot \binom{2}{2} \cdot (2\binom{4}{1})^2$.
- $k = 3, m = 1, m = 2, m = 3$,
  $T_1(kQ_3) = \binom{k}{1} \cdot 2\binom{4}{1}$;
  $T_2(kQ_3) = \binom{k}{1} \cdot (2\binom{4}{2} + 4) + \binom{k}{2} \cdot \binom{2}{2} \cdot (2\binom{4}{1})^2$;
  $T_3(kQ_3) = \binom{k}{1} \cdot 2\binom{4}{3} + \binom{k}{2} \cdot \binom{2}{1} \cdot 2\binom{4}{1} \cdot (2\binom{4}{2} + 4) + \binom{k}{3} \cdot (2\binom{4}{1})^3$.
- $k = 4, m = 1, m = 2, m = 3, m = 4$,
  $T_1(kQ_3) = \binom{k}{1} \cdot 2\binom{4}{1}$;
  $T_2(kQ_3) = \binom{k}{1} \cdot (2\binom{4}{2} + 4) + \binom{k}{2} \cdot \binom{2}{2} \cdot (2\binom{4}{1})^2$;
  $T_3(kQ_3) = \binom{k}{1} \cdot 2\binom{4}{3} + \binom{k}{2} \cdot \binom{2}{1} \cdot 2\binom{4}{1} \cdot (2\binom{4}{2} + 4) + \binom{k}{3} \cdot (2\binom{4}{1})^3$;
  $T_4(kQ_3) = \sum_{p=1}^{4} |A_p| = \binom{k}{1} \cdot 2\binom{4}{4} + \binom{k}{2} \cdot \left[\binom{2}{1} \cdot 2\binom{4}{1} \cdot 2\binom{4}{3} + \binom{2}{2} \cdot (2\binom{4}{2} + 4)^2 \right] + \binom{k}{3} \cdot \binom{3}{2} \cdot (2\binom{4}{1})^2 \cdot (2\binom{4}{2} + 4) + \binom{k}{4} \cdot (2\binom{4}{1})^4$.

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Thus, we can obtain $T_m(kQ_3)$ by simply giving the formula for $|A_p|$, where we compute T_m in two subcases.

Subcase 1.1. $1 \le m \le k$, $p \in \left[\left\lceil \frac{m}{4} \right\rceil, m-1\right]$.

By induction, we give the formula for $|A_p|$.

$$|A_p| = \binom{k}{p}\left[\sum_{\ell=1 \text{ or } \left\lceil \frac{m-4p}{i} \right\rceil + 4}^{\left\lfloor \frac{m}{p} \right\rfloor} \sum_{i=1 \text{ or } 2p-m}^{\left\lfloor \frac{4p-m}{4-\left\lfloor \frac{m}{p} \right\rfloor} \right\rfloor} \binom{p}{i}\left(2\binom{4}{\ell} + \tau(\ell)4\right)^i \left(\sum_{All\ e_j} \frac{(p-i)!}{\prod_{All\ \delta(e_j)} \delta(e_j)!} \prod_{j=1}^{p-i}\left(2\binom{4}{e_j} + \tau(e_j)4\right)\right)\right]$$

Let $\xi = \sum_{All\ e_j} \frac{(p-i)!}{\prod_{All\ \delta(e_j)} \delta(e_j)!} \prod_{j=1}^{p-i}\left(2\binom{4}{e_j} + \tau(e_j)4\right)$. Note that we stipulate that $\xi = 1$ when $p - i = 0$.

The case of the range of values of ℓ and i are given below. Since $\ell < e_1 \le e_2 \le \cdots \le e_{p-i} \le 4$, then

$$(p-i)\ell < \sum_{j=1}^{p-i} e_j = m - i\ell \le 4(p-i).$$

By simplifying, we get $\ell < \frac{m}{p}$, i.e. $\ell \le \left\lfloor \frac{m}{p} \right\rfloor$. From $m - i\ell \le 4(p-i)$, we have $i \le \frac{4p-m}{4-\ell}$ and $\ell \ge \frac{m-4p}{i} + 4$. We know that $\ell \le \left\lfloor \frac{m}{p} \right\rfloor$ and i, ℓ are integers, thus $i \le \left\lfloor \frac{4p-m}{4-\left\lfloor \frac{m}{p} \right\rfloor} \right\rfloor$ and $\ell \ge \left\lceil \frac{m-4p}{i} \right\rceil + 4$. Similarly, let $\ell + 1 \le e_1 \le e_2 \le \cdots \le e_{p-i} \le 4$, then

$$(\ell+1)(p-i) \le m - i\ell,$$

it can get

$$i \ge \ell p + p - m \ge 2p - m.$$

Since both p and i are positive integers, the selection of their range of values must this. So, we have $2p - m \ge 1$, then $p \ge \left\lceil \frac{m+1}{2} \right\rceil$. Also, it must have $\frac{m-4p}{i} + 4 \ge 1$, which gives $p \le \left\lfloor \frac{m+3i}{4} \right\rfloor$. In short, if $p \ge \left\lceil \frac{m+1}{2} \right\rceil$, then $i \in \left[2p-m, \left\lfloor \frac{4p-m}{4-\left\lfloor \frac{m}{p} \right\rfloor} \right\rfloor\right]$, otherwise, $i \in \left[1, \left\lfloor \frac{4p-m}{4-\left\lfloor \frac{m}{p} \right\rfloor} \right\rfloor\right]$; when i is determined, for any i, if $p \le \left\lfloor \frac{m+3i}{4} \right\rfloor$, then $\ell \in \left[\left\lceil \frac{m-4p}{i} \right\rceil + 4, \left\lfloor \frac{m}{p} \right\rfloor\right]$, otherwise $\ell \in \left[1, \left\lfloor \frac{m}{p} \right\rfloor\right]$.

Since $p \in \left[\left\lceil \frac{m}{4} \right\rceil, m-1\right]$, we can obtain the equation

$$\sum_{p=\left\lceil \frac{m}{4} \right\rceil}^{m-1} |A_p| = \sum_{p=\left\lceil \frac{m}{4} \right\rceil}^{m-1} \left\{\binom{k}{p}\left[\sum_{\ell=1 \text{ or } \left\lceil \frac{m-4p}{i} \right\rceil + 4}^{\left\lfloor \frac{m}{p} \right\rfloor} \sum_{i=1 \text{ or } 2p-m}^{\left\lfloor \frac{4p-m}{4-\left\lfloor \frac{m}{p} \right\rfloor} \right\rfloor} \binom{p}{i} \cdot \left(2\binom{4}{\ell} + \tau(\ell)4\right)^i \cdot \xi\right]\right\}$$

Note that if $4p - m = 0$, i.e., $p = \frac{m}{4}$, then $|A_p| = |A_{\frac{m}{4}}| = \binom{k}{\frac{m}{4}}\left(2\binom{4}{4}\right)^{\frac{m}{4}}$, in which case only $p \in \left[\left\lceil \frac{m}{4} \right\rceil + 1, m-1\right]$ is considered, otherwise $p \in \left[\left\lceil \frac{m}{4} \right\rceil, m-1\right]$.

Subcase 1.2. $1 \le m \le k$, $p = m$.

We can obtain the equation

$$|A_m| = \binom{k}{m} \cdot \left(2\binom{4}{1}\right)^m$$

for $m = k$, we have

$$|A_k| = \left(2\binom{4}{1}\right)^k$$

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Hence, when $1 \le m \le k$, $p \in \left[\lceil \frac{m}{4} \rceil, m\right]$, we obtain the number of certain m-independent sets of kQ_3 from the equation

$$T_m(kQ_3) = \sum_{p=\lceil \frac{m}{4} \rceil}^{m} |A_p| = \sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} |A_p| + |A_m|$$
$$= \sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} \left\{ \binom{k}{p} \left[\sum_{\ell=1 \ or \ \lceil \frac{m-4p}{i} \rceil + 4}^{\lfloor \frac{m}{p} \rfloor} \sum_{i=1 \ or \ 2p-m}^{\left\lfloor \frac{4p-m}{4-\lfloor \frac{m}{p} \rfloor} \right\rfloor} \binom{p}{i} \cdot \left(2\binom{4}{\ell} + \tau(\ell)4 \right)^{i} \cdot \xi \right] \right\}$$
$$+ \binom{k}{m} \cdot \left(2\binom{4}{1} \right)^{m}$$

Case 2. If $k < m \le 4k$, then $p \in \left[\lceil \frac{m}{4} \rceil, k\right]$.

We explain why it is $m \le 4k$, since $\lceil \frac{m}{4} \rceil \le k$, i.e., $k \ge \frac{m}{4}$, so $m \le 4k$. If $m = 4k + 1$,it means that at least one of the independent numbers in graph Q_3 is 5, but $\alpha(Q_3) = 4$, so $m \le 4k$.

Similarly, we only need to compute T_m, which at this point is $T_m(kQ_3) = \sum_{p=\lceil \frac{m}{4} \rceil}^{k-1} |A_p| + |A_k|$, and we also compute them in two subcases.

Subcase 2.1. $k < m \le 4k$, $p \in \left[\lceil \frac{m}{4} \rceil, k-1\right]$.

$$\sum_{p=\lceil \frac{m}{4} \rceil}^{k-1} |A_p| = \sum_{p=\lceil \frac{m}{4} \rceil}^{k-1} \left\{ \binom{k}{p} \left[\sum_{\ell=1 \ or \ \lceil \frac{m-4p}{i} \rceil + 4}^{\lfloor \frac{m}{p} \rfloor} \sum_{i=1 \ or \ 2p-m}^{\left\lfloor \frac{4p-m}{4-\lfloor \frac{m}{p} \rfloor} \right\rfloor} \binom{p}{i} \cdot \left(2\binom{4}{\ell} + \tau(\ell)4 \right)^{i} \cdot \xi \right] \right\}$$

Subcase 2.2. $k < m \le 4k$, $p = k$.

For this case of $p = k$, which is different from $p = m$ in Case 1, then for different values of m, we consider the results of $|A_k|$ by dividing them into three subcases.

Subcase 2.2.1. $k < m < 2k$.

$$|A_k| = \sum_{i=\lceil \frac{m-k}{3} \rceil}^{m-k} \left\{ \binom{k}{i} \cdot \left(2\binom{4}{1} \right)^{k-i} \cdot \left[\sum_{All \ \ell_j} \frac{i!}{\prod_{All \ \delta(\ell_j)} \delta(\ell_j)!} \prod_{j=1}^{i} \left(2\binom{4}{\ell_j} + \tau(\ell_j)4 \right) \right] \right\}$$

We give a brief explanation of the notation in Eq, and give how the range of values of i is obtained.

(i) i is a positive integer which means that for any k graphs of which there are i in the set of points chosen, the number of points chosen for each graph is related to the fact that only one point is chosen for each of the remaining $k - i$ graphs. The number of points selected for each of these i graphs is at least 2, but not more than the maximum number of independents of the graphs. $\binom{k}{i}$ is the fact that for i graphs, there are $\binom{k}{i}$ ways of taking the graphs.

(ii) ℓ_j is the number of points selected for each graph for these i graphs and all the points are combined in $(m - (k - i))$ independent sets, where $j \in [1, i]$, and there are $1 < \ell_1 \le \ell_2 \le \cdots \le \ell_i \le 4$, and $\sum_{j=1}^{i} \ell_j = m - (k - i)$. Because for $(k - i)$ graphs, only one point is selected for each graph, while for m independent set, $(m - (k - i))$ independent sets are needed for these i graphs to select the point.

(iii) $\tau(\ell_j) = \begin{cases} 1, & \text{if } \ell_j = 2 \\ 0, & \text{otherwise} \end{cases}$

(iv) Since the definition of $\delta(\ell_j)$ is similar to that of $\delta(e_j)$, we do not redefine it. Similarly the summation term on ℓ_j in Eq. means to pick all different ℓ_j to this condition.

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Since $1 < \ell_1 \le \ell_2 \le \cdots \le \ell_i \le 4$, we have

$$i < \sum_{j=1}^{i} \ell_j = m - k + i \le 4i,$$

which by the inequality that follows gives $i \ge \frac{m-k}{3}$, and i must be a positive integer, hence $i \ge \left\lceil \frac{m-k}{3} \right\rceil$. We let $2 \le \ell_1 \le \ell_2 \le \cdots \le \ell_i \le 4$, in which case we have

$$2i \le m - k + i,$$

so $i \le m - k$. Since $i < k$, hence $m - k < k$, i.e. $m < 2k$.

So this formula holds under the condition that $k < m < 2k$ and the range of values of i is $\left[\left\lceil \frac{m-k}{3} \right\rceil, m-k\right]$.

Subcase 2.2.2. $m = 2k$.

$$|A_k| = \sum_{i=\lceil \frac{m-k}{3} \rceil}^{k-1} \left\{ \binom{k}{i} \cdot \left(2\binom{4}{1}\right)^{k-i} \cdot \left[\sum_{All\ \ell_j} \frac{i!}{\prod_{All\ \delta(\ell_j)} \delta(\ell_j)!} \prod_{j=1}^{i} \left(2\binom{4}{\ell_j} + \tau(\ell_j)4 \right) \right] \right\} + \left(2\binom{4}{2} + 4 \right)^k$$

Subcase 2.2.3. $2k < m \le 4k$.

The range of m at this point is very large and complex to compute, so we consider a simple computation. Since i cannot exceed k, the range of i is $\left[\left\lceil \frac{m-k}{3} \right\rceil, k\right]$. Hence

$$|A_k| = \sum_{i=\lceil \frac{m-k}{3} \rceil}^{k} \left\{ \binom{k}{i} \cdot \left(2\binom{4}{1}\right)^{k-i} \cdot \left[\sum_{All\ \ell_j} \frac{i!}{\prod_{All\ \delta(\ell_j)} \delta(\ell_j)!} \prod_{j=1}^{i} \left(2\binom{4}{\ell_j} + \tau(\ell_j)4 \right) \right] \right\}$$

Set $\zeta = \sum_{All\ \ell_j} \frac{i!}{\prod_{All\ \delta(\ell_j)} \delta(\ell_j)!} \prod_{j=1}^{i} \left(2\binom{4}{\ell_j} + \tau(\ell_j)4 \right)$, the number of points selected for these i graphs goes to the $(m - (k - i))$ independent sets is more difficult, so just simplify this summation formula.

We know that $1 < \ell_1 \le \ell_2 \le \cdots \le \ell_i \le 4$, i.e., $2 \le \ell_1 \le \ell_2 \le \cdots \le \ell_i \le 4$ holds, so the number obtained by ℓ_j is chosen in [2,4], in order to meet

$$\sum_{j=1}^{i} \ell_j = m - k + i,$$

we first consider certain ℓ_j, where $j \in [1, i]$, we set the first n elements are 2, i.e., $\sum_{j=1}^{n} \ell_j = 2n$, at this point, the remaining $i - n$ elements need to be

$$\sum_{j=n+1}^{i} \ell_j = m - k + i - 2n.$$

At this point there are $i - n$ elements that are not 2, and for the selection of n, which has a range, we consider the range of values it takes. Because there are $i - n$ elements that are not 2, then the minimum of these $i - n$ elements is 3, and the maximum cannot be more than 4, so the condition

$$3(i - n) \le m - k + i - 2n \le 4(i - n),$$

is needed. And the inequality $3(i - n) \le m - k + i - 2n$ on the left side gives $n \ge 2i - m + k$; and the inequality $m - k + i - 2n \le 4(i - n)$ on the right side gives $n \le \frac{3i-m+k}{2}$, and because n is a positive integer, so $n \le \left\lfloor \frac{3i-m+k}{2} \right\rfloor$.

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Since n is a positive integer, it stands to reason that $2i - m + k \geq 0$, but if it is less than 0, then n is in the range $\left[0, \left\lfloor \frac{3i-m+k}{2} \right\rfloor\right]$, otherwise $n \in \left[2i - m + k, \left\lfloor \frac{3i-m+k}{2} \right\rfloor\right]$.

Again, n is not more than k, so it can obtain

$$\frac{3i - m + k}{2} \leq \frac{3k - m + k}{2} < k$$

from $i \leq k$, and the result is $m > 2k$. Thus the range of m is $(2k, 4k]$, and as m increases, the maximum value of n is not more than k.

Hence, when $k < m < 2k$, $p \in [\lceil \frac{m}{4} \rceil, k]$, we obtain the number of certain m-independent sets of kQ_3 from the equation

$$\begin{aligned} T_m(kQ_3) &= \sum_{p=\lceil \frac{m}{4} \rceil}^{k} |A_p| = \sum_{p=\lceil \frac{m}{4} \rceil}^{k-1} |A_p| + |A_k| \\ &= \sum_{p=\lceil \frac{m}{4} \rceil}^{k-1} \left\{ \binom{k}{p} \left[\sum_{\ell=1 \text{ or } \lceil \frac{m-4p}{i} \rceil + 4}^{\lfloor \frac{m}{p} \rfloor} \sum_{i=1 \text{ or } 2p-m}^{\left\lfloor \frac{4p-m}{4-\lfloor \frac{m}{p} \rfloor} \right\rfloor} \binom{p}{i} \cdot \left(2\binom{4}{\ell} + \tau(\ell)4 \right)^i \cdot \xi \right] \right\} \\ &+ \sum_{i=\lceil \frac{m-k}{3} \rceil}^{m-k} \left\{ \binom{k}{i} \cdot \left(2\binom{4}{1} \right)^{k-i} \cdot \zeta \right\}. \end{aligned}$$

when $m = 2k$, $p \in [\lceil \frac{m}{4} \rceil, k]$,

$$\begin{aligned} T_m(kQ_3) &= \sum_{p=\lceil \frac{m}{4} \rceil}^{k} |A_p| = \sum_{p=\lceil \frac{m}{4} \rceil}^{k-1} |A_p| + |A_k| \\ &= \sum_{p=\lceil \frac{m}{4} \rceil}^{k-1} \left\{ \binom{k}{p} \left[\sum_{\ell=1 \text{ or } \lceil \frac{m-4p}{i} \rceil + 4}^{\lfloor \frac{m}{p} \rfloor} \sum_{i=1 \text{ or } 2p-m}^{\left\lfloor \frac{4p-m}{4-\lfloor \frac{m}{p} \rfloor} \right\rfloor} \binom{p}{i} \cdot \left(2\binom{4}{\ell} + \tau(\ell)4 \right)^i \cdot \xi \right] \right\} \\ &+ \sum_{i=\lceil \frac{m-k}{3} \rceil}^{k-1} \left\{ \binom{k}{i} \cdot \left(2\binom{4}{1} \right)^{k-i} \cdot \zeta \right\} + \left(2\binom{4}{2} + 4 \right)^k. \end{aligned}$$

when $2k < m \leq 4k$, $p \in [\lceil \frac{m}{4} \rceil, k]$,

$$\begin{aligned} T_m(kQ_3) &= \sum_{p=\lceil \frac{m}{4} \rceil}^{k} |A_p| = \sum_{p=\lceil \frac{m}{4} \rceil}^{k-1} |A_p| + |A_k| \\ &= \sum_{p=\lceil \frac{m}{4} \rceil}^{k-1} \left\{ \binom{k}{p} \left[\sum_{\ell=1 \text{ or } \lceil \frac{m-4p}{i} \rceil + 4}^{\lfloor \frac{m}{p} \rfloor} \sum_{i=1 \text{ or } 2p-m}^{\left\lfloor \frac{4p-m}{4-\lfloor \frac{m}{p} \rfloor} \right\rfloor} \binom{p}{i} \cdot \left(2\binom{4}{\ell} + \tau(\ell)4 \right)^i \cdot \xi \right] \right\} \\ &+ \sum_{i=\lceil \frac{m-k}{3} \rceil}^{k} \left\{ \binom{k}{i} \cdot \left(2\binom{4}{1} \right)^{k-i} \cdot \zeta \right\}. \end{aligned}$$

While a portion of the graphs select only one point, or only two points as part of their independent sets in the process of selecting points, the remaining graphs can only select 3 or 4 points for the whole m independent sets, which is still complicated and therefore still to be solved.

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In fact, for the case of $n = 0$, let both ends of the equation of n are 0, that is,

$$2i - m + k = 0, \quad \frac{3i - m + k}{2} = 0.$$

the extremes of both sides of n is actually for i graphs are all selected 3 points (the left side of the equation), or all selected 4 points (the right side of the equation) to m-independence. Therefore, in the process of calculation, if i satisfies the condition of the equation, the selection of ℓ_j becomes more straightforward and favorable to get the result.

Next, we consider the number of certain m-independent sets in KQ_3, where K can be obtained from $[1, k-1]$, and we explore its connection with the number of corresponding m-independent sets of kQ_3, for what deformation of the previously discussed formulas yields the new needed formulae.

Before the discussion of the situation, we need to compare how p changes for KQ_3 when m is taken from a different range than when it is taken from the same m for kQ_3. Since our previous formulas have been based on the final quantity based on the value of p taken, we consider the quantity of the m-independent set of KQ_3, i.e., $S_m(KQ_3)$, in the same way. Let's make a distinction here by setting the previous p to p_k and the p here to p_K.

Now we have

$$\begin{cases} 1 \leq m \leq k, & p \in \left[\lceil \frac{m}{4} \rceil, m\right] \\ k < m \leq 4k, & p_k \in \left[\lceil \frac{m}{4} \rceil, k\right] \end{cases} \tag{1}$$

similarly,

$$\begin{cases} 1 \leq m \leq K, & p \in \left[\lceil \frac{m}{4} \rceil, m\right] \\ K < m \leq 4K, & p_K \in \left[\lceil \frac{m}{4} \rceil, K\right] \end{cases} \tag{2}$$

where K can be obtained from $[1, k-1]$. Since K is smaller than k, this shows that $[1, K] \subseteq [1, k]$, then for KQ_3, when picking m, the same m is picked as in kQ_3, and their p belong to the same range, i.e., $\left[\lceil \frac{m}{4} \rceil, m\right]$. Therefore when calculating the number of m-independent sets in KQ_3, where $1 \leq m \leq K$, we only need to compare it with the same m in kQ_3, which has the number of m-independent sets, so as to be able to obtain the relationship between them.

For $K < m \leq 4K$, we can divide it into

$$\begin{cases} K < m_1 \leq k, & p_K \in \left[\lceil \frac{m_1}{4} \rceil, K\right] \\ k < m_2 \leq 4K, & p_K \in \left[\lceil \frac{m_2}{4} \rceil, K\right] \end{cases} \tag{3}$$

we can see that the range of values of m_2 at this point has an inclusive relationship with the range of values of m in the second case of kQ_3, including the range of values of p, i.e. $(k, 4K] \subseteq (k, 4k]$ and $\left[\lceil \frac{m_2}{4} \rceil, K\right] \subseteq \left[\lceil \frac{m}{4} \rceil, k\right]$, where $m = m_2$. Thus in calculating the number of m_2-independent sets in KQ_3, where $k < m_2 \leq 4K$. Again we only need to compare the same m in kQ_3 with the formula for solving for the number of m-independent sets in it.

Theorem 3.2. *Let m, k, K, p be positive integers with $1 \leq m \leq 4K$, where $1 \leq K \leq k-1$. $S_m(kQ_3)$ is the number of a certain m-independent set for graph KQ_3.*

(i) If $1 \leq m \leq K$, $p \in \left[\lceil \frac{m}{4} \rceil, m\right]$, then

$$\begin{aligned} S_m(KQ_3) &= \sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} |A_{p_K}| + |A_{m_K}| \\ &= \sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} \frac{(k-p)!K!}{(K-p)!k!} |A_p| + \frac{(k-m)!K!}{(K-m)!k!} |A_m|. \end{aligned}$$

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(ii) If $K < m \leq 4K$, $p \in \left[\lceil \frac{m}{4} \rceil, K\right]$, then

$$S_m(KQ_3) = \sum_{p_K=\lceil \frac{m}{4} \rceil}^{K} |A_{p_K}| = \sum_{p_K=\lceil \frac{m}{4} \rceil}^{K} \frac{(k-p)!K!}{(K-p)!k!} |A_p|.$$

where $|A_p|$ and $|A_m|$ are the corresponding formulas of Theorem 3.1.

Proof. We still split into two cases according to (3.2) and consider the following case. □

Case 1. If $1 \leq m \leq K$, where $K \in [1, k-1]$, then $p \in \left[\lceil \frac{m}{4} \rceil, m\right]$.

At this point there is the formula

$$S_m(KQ_3) = \sum_{p=\lceil \frac{m}{4} \rceil}^{m} |A_p| = \sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} |A_p| + |A_m|$$

To distinguish this from the previous formula, we let $\sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} |A_p| = \sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} |A_{p_K}|$ and $|A_m| = |A_{m_K}|$, thus

$$S_m(KQ_3) = \sum_{p=\lceil \frac{m}{4} \rceil}^{m} |A_p| = \sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} |A_{p_K}| + |A_{m_K}|$$

Subcase 1.1. $1 \leq m \leq K$, $p \in \left[\lceil \frac{m}{4} \rceil, m-1\right]$.

Since

$$|A_{p_K}| = \binom{K}{p} \left[\sum_{\ell=1 \text{ or } \lceil \frac{m-4p}{i} \rceil + 4}^{\lfloor \frac{m}{p} \rfloor} \sum_{i=1 \text{ or } 2p-m}^{\left\lfloor \frac{4p-m}{4-\lfloor \frac{m}{p} \rfloor} \right\rfloor} \binom{p}{i} \cdot \left(2\binom{4}{\ell} + \tau(\ell)4\right)^i \cdot \zeta \right].$$

Set $\zeta = \sum_{All\ e_j} \frac{(p-i)!}{\prod_{All\ \delta(e_j)} \delta(e_j)!} \prod_{j=1}^{p-i} \left(2\binom{4}{e_j} + \tau(e_j)4\right)$. For finding the number of m-independent sets of kQ_3, the $|A_p|$ we are seeking is compared to it, then we will obtain

$$\frac{|A_p|}{|A_{p_K}|} = \frac{\binom{k}{p}}{\binom{K}{p}} = \frac{(K-p)!k!}{(k-p)!K!},$$

thus

$$|A_{p_K}| = \frac{(k-p)!K!}{(K-p)!k!} |A_p|.$$

Then we can get

$$\sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} |A_{p_K}| = \sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} \frac{(k-p)!K!}{(K-p)!k!} |A_p|.$$

Subcase 1.2. $1 \leq m \leq K$, $p = m$.

At this point, we consider $|A_{m_K}|$, it has

$$|A_{m_K}| = \binom{K}{m} \cdot \left(2\binom{4}{1}\right)^m,$$

similarly, we know

$$|A_m| = \binom{k}{m} \cdot \left(2\binom{4}{1}\right)^m,$$

they are compared to get

$$\frac{|A_m|}{|A_{m_K}|} = \frac{\binom{k}{m}}{\binom{K}{m}} = \frac{(K-m)!k!}{(k-m)!K!},$$

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then we can get

$$|A_{m_K}| = \frac{(k-m)!K!}{(K-m)!k!}\,|A_m|,$$

hence,

$$S_m(KQ_3) = \sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} |A_{p_K}| + |A_{m_K}|$$
$$= \sum_{p=\lceil \frac{m}{4} \rceil}^{m-1} \frac{(k-p)!K!}{(K-p)!k!}\,|A_p| + \frac{(k-m)!K!}{(K-m)!k!}\,|A_m|.$$

Case 2. If $K < m \leq 4K$, where $K \in [1, k-1]$, then $p_K \in \left[\lceil \frac{m}{4} \rceil, K\right]$.

We divide the range of values of m.

Subcase 2.1. $K < m_1 \leq k$, $p_K \in \left[\lceil \frac{m_1}{4} \rceil, K\right]$.

We know that the range of values of m_1 at this point is actually a subset of the range of values of m of Case 1. in kQ_3, at this point we have $(K, k] \subseteq (1, k]$, and we find that when we take the same number, the ranges of values of their corresponding p are also a containment relation, which means that when we calculate the number of a certain m_1-independent set, we only need to calculate a certain part of p about the same m of kQ_3. In fact, p is taken up to K.

Since

$$S_{m_1}(KQ_3) = \sum_{p_K=\lceil \frac{m_1}{4} \rceil}^{K} |A_{p_K}|,$$

we know

$$|A_{p_K}| = \frac{(k-p)!K!}{(K-p)!k!}\,|A_p|,$$

hence,

$$S_{m_1}(KQ_3) = \sum_{p_K=\lceil \frac{m_1}{4} \rceil}^{K} |A_{p_K}| = \sum_{p_K=\lceil \frac{m_1}{4} \rceil}^{K} \frac{(k-p)!K!}{(K-p)!k!}\,|A_p|.$$

Subcase 2.2. $k < m_2 \leq 4K$, $p_K \in \left[\lceil \frac{m_2}{4} \rceil, K\right]$.

For the range of values of m_2 , we can easily interrelate with the range of values of m of case 2. in the previous kQ_3, since at this point we have $(k, 4K] \subseteq (k, 4k]$. And, for them to take the same m, the range of p_K in KQ_3 happens to be a subset of the range of p in kQ_3, and the range of p_K happens to be in the range of p up to the element K. Therefore, we do not need to consider the case where $p = k$, but only subcase2.1.

Similarly,

$$S_{m_2}(KQ_3) = \sum_{p_K=\lceil \frac{m_2}{4} \rceil}^{K} |A_{p_K}|.$$

we know

$$|A_{p_K}| = \frac{(k-p)!K!}{(K-p)!k!}\,|A_p|.$$

hence,

$$S_{m_2}(KQ_3) = \sum_{p_K=\lceil \frac{m_2}{4} \rceil}^{K} |A_{p_K}| = \sum_{p_K=\lceil \frac{m_2}{4} \rceil}^{K} \frac{(k-p)!K!}{(K-p)!k!}\,|A_p|.$$

This shows that for Case 2., regardless of the value of m, we have the formula

$$S_m(KQ_3) = \sum_{p_K=\lceil \frac{m}{4} \rceil}^{K} |A_{p_K}| = \sum_{p_K=\lceil \frac{m}{4} \rceil}^{K} \frac{(k-p)!K!}{(K-p)!k!}\,|A_p|.$$

holds as long as $K < m \leq 4K$, where $K \in [1, k-1]$.

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4. Conclusions

In this paper, we obtain for K disjoint regular graphs Q_3 with 8 vertices, defined as KQ_3, the formula for the number of certain m-independent sets about it, where $1 \leq K \leq k$ and $1 \leq m \leq 4K$. We first explore the number of certain m-independent sets about kQ_3, where $1 \leq m \leq 4k$. By taking different values of m, we make a case-by-case discussion and obtain the formula for its correlation. Under this condition, we again considered the case of less than k disjoint graphs Q_3, i.e., KQ_3, where $1 \leq K \leq k-1$, and obtained formulas for the number of certain m-independent sets of them. We obtained the simple formula for KQ_3 by finding the relation between kQ_3 and the number of the same m-independent set of KQ_3.

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