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# On a Characterization of the Ruled Cubic Surface in PG(4,$q$)

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**Abstract:** Some five years ago, S.G. Barwick and Wen-Ai Jackson provided a nice combinatorial characterization of the ruled cubic surface in PG(4,$q$). They proved that, in PG(4,$q$), $q=p^h$ odd, $q\geq 9$, a point-set $K$ of size $q^2+2q+1$ and class $[q+1,2q+1,3q+1]_3$, such that every plane that contains four points of $K$ contains at least $q+1$ points of $K$, is the ruled cubic surface. They also showed that the incidence plane condition is sharp, but nothing they wrote about the size assumption. In this note, by proving that, in PG($r$,$q$), $r\geq 5$, $q\geq 9$, a set of class $[q+1,2q+1,3q+1]_3$ is of class $[q^2+2q+1,3q^2-2q+1,3q^2-q+1]_4$, we show that the size assumption cannot be removed.

**Keywords:** three-character sets, three-intersection sets, ruled cubic surface, characterization.

**MSC:** 03B30, 05B25, 51E20.

## 1. Introduction

Foundations of Mathematics deal with what objects are investigated and what are basic properties of the investigated objects. An algebraic manifold $K$ can be thought as a set of points in a projective space that has a certain behavior with respect to subspaces. In a finite projective space PG($r$,$q$) the manifold $K$ contains a finite number $k$ of points and the intersection manifolds also have a finite number of points. By characterization of a manifold we mean the classification of those $k$–sets of PG($r$,$q$) which, by axiom, possess a certain number of properties of the given manifold. The manifold is considered characterized if it is demonstrated that a $k$–set that satisfies of the axiomatized properties is the manifold for at most a few exceptions for particular values of $q$ (sporadic cases), see [1]. The characterization will be a robust result when the required axioms are few and significant and there are also few sporadic cases. This note is inspired by a paper of S.G. Barwick and Wen-Ai Jackson in which a combinatorial characterization of a ruled cubic surface is provided. They proved that, in PG(4,$q$), $q$ odd, $q\geq 9$, a ($q^2+2q+1$)-set $K$ of class $[q+1,2q+1,3q+1]_3$ such that every plane that contains four points of $K$ contains at least $q+1$ points of $K$ is the ruled cubic surface, cf. [2]. The Authors gave an example which demonstrates that the last plane incidence condition is sharp, i.e. it cannot be removed. A natural question is if also the size condition is sharp. In this note we give an affirmative answer.

## 2. Materials and Methods

In PG($r$,$q$), the projective space of dimension $r$ and order $q$, with $q=p^h$, a prime power, let $K$ denote a $k$–set, i.e. a set of $k$ points. For each integer $i$ such that $0\leq i\leq q^d+\ldots+q+1$, let us denote by $t_i=t_i(K)$, the number of $d$–subspaces of PG($r$,$q$) meeting $K$ in exactly $i$ points. The numbers $t_i$ are called the *characters* of $K$ with respect to the $d$-subspaces, see [3]. Let $m_1,m_2,\ldots,m_s$ be $s$ integers such that $0\leq m_1<m_2<,\ldots,<m_s\leq q^d+\ldots+q+1$. A set $K$ is said to be *of class* $[m_1,m_2,\ldots,m_s]_d$ if $t_i\neq 0$ only if $i\in\{m_1, m_2, \ldots, m_s\}$. Moreover, $K$ is said to be *of type* $(m_1, m_2, \ldots, m_s)_d$ if $t_i\neq 0$ if and only if $i\in\{m_1, m_2, \ldots, m_s\}$. The integers $m_1, m_2, \ldots, m_s$ are called *intersection numbers* with respect to the $d$–subspaces and $K$ is said to be an $s$-*intersection set* with respect to the dimension $d$. A fundamental problem in finite geometry is the

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classification, i.e. the list of the geometric figures belonging to a given class, see [4-8]. Therefore, it is essential to know the class to which a geometric configuration that one is examining belongs. In PG(4,$q$), a ruled cubic surface is a variety $V_2^3$ of dimension 2 and order 3. A ruled cubic surface is unique up to projective equivalence. A ruled cubic surface $V_2^3$ in PG(4,$q$) is constructed as follows: let $C$ be an irreducible conic and let $l$ be a line skew to the plane containing $C$ in PG(4,$q$). Denote by $\theta$, $\varphi$ the non-homogeneous coordinates of points on $l$ and $C$ respectively. Associate $\theta$ and $\varphi$ with a projectivity $h$ (so that $\varphi=h(\theta)$), join corresponding points and so obtain a set of $q$+1 pairwise disjoint lines in PG(4,$q$). Call these lines *generator lines* of the resulting surface $V_2^3$. The line $l$ contained in $V_2^3$ is the unique line incident with each generator line; $l$ is called the *line directrix* of $V_2^3$. The conic $C$ has one point in common with each generator line and is called a *conic directrix* of $V_2^3$; there exist $q^2$ conic directrices on the surface $V_2^3$ in PG(4,$q$). The ruled cubic surface in PG(4,$q$) has the following properties, cf. [2]. The number of points is $q^2+2q+1$. Any solid of PG(4,$q$) meets in either $q$+1, or 2$q$+1, or 3$q$+1 points. Any plane of PG(4,$q$) with more than three points meets in either $q$+1, or 2$q$+1 points. Any line with more than two points meets $K$ in $q$+1 points. A natural question arises: is it possible to characterize ruled cubic surfaces by its incidence numbers with respect to $d$-subspaces? In [2] the following result is proved.

**Result.** *In* PG(4,$q$), *$q$ odd, $q\geq 9$, a set $K$ of $q^2+2q+1$ points is a ruled cubic surface if and only if $K$ has class* $[q+1,2q+1,3q+1]_3$ *and every plane that contains four points of $K$ contains at least $q$+1 points of $K$.*

In [2] p. 7, the Authors gave an example which demonstrates that the plane incidence condition is sharp, but nothing they wrote about the size condition. In this note by proving the following theorem we show that also the size condition is sharp.

**Theorem.** *In* PG($r$,$q$), *with $q=p^h$ a prime power, $q\geq 9$, a set of class* $[q+1,2q+1,3q+1]_3$ *is of class* $[q^2+2q+1,3q^2-2q+1,3q^2-q+1]_4$.

## 3. In PG($r$,$q$), $q\geq 9$, class $[q+1,2q+1,3q+1]_3 \Rightarrow$ class $[q^2+2q+1,3q^2-2q+1,3q^2-q+1]_4$

Let $K$ denote a $k$–set of class $[q+1,2q+1,3q+1]_3$ in PG($r$,$q$), $r\geq 5$, $k\geq q+1$, $q\geq 9$. To calculate class with respect to dimension four, we fix a subspace of dimension four. Double counting the number of solids, the number of pairs ($P$,$\Sigma$), where $P\in K$ and $\Sigma$ is a solid through $P$, and the number of pairs (($P$, $Q$),$\Sigma$), where $\{P, Q\} \subset K$ and $\Sigma$ is a solid through $P$ and $Q$, leads to the following equations:

$$\begin{cases} t_{q+1}+t_{2q+1}+t_{3q+1}=q^4+q^3+q^2+q+1 \\ (q+1)t_{q+1}+(2q+1)t_{2q+1}+(3q+1)t_{3q+1}=k(q^3+q^2+q+1) \\ q(q+1)t_{q+1}+2q(2q+1)t_{2q+1}+3q(3q+1)t_{3q+1}=k(k-1)(q^2+q+1) \end{cases} \tag{3}$$

Considering the first two equations modulo $q$, we get that

$$\begin{aligned} t_{q+1}+t_{2q+1}+t_{3q+1} &\equiv 1 \pmod q, \\ t_{q+1}+t_{2q+1}+t_{3q+1} &\equiv k \pmod q. \end{aligned} \tag{4}$$

From which, we find that $k\equiv 1 \pmod q$. Putting $k=aq+1$ and solving the system of equations, it follows that

$$\begin{gathered} 2t_{q+1}(aq+1)= \\ =(q^2+q+1)a^2-(5q^3+6q^2+5q+5)a+(6q^4+11q^3+7q^2+6q+6) \end{gathered} \tag{5}$$

As $t_{q+1}(aq+1)\geq 0$, we have either

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$$a < \frac{5q^3 + 6q^2 + 5q + 5 - \sqrt{q^6 - 8q^5 - 10q^4 + 14q^3 + 9q^2 + 2q + 1}}{2(q^2 + q + 1)} \quad (6)$$

or

$$a > \frac{5q^3 + 6q^2 + 5q + 5 + \sqrt{q^6 - 8q^5 - 10q^4 + 14q^3 + 9q^2 + 2q + 1}}{2(q^2 + q + 1)}. \quad (7)$$

it follows that

$$t_{2q+1}(aq + 1) = -(q^2 + q + 1)a^2 + (4q^3 + 5q^2 + 4q + 4)a - (3q^4 + 7q^3 + 4q^2 + 3q + 3). \quad (8)$$

As $t_{2q+1}(aq + 1) \geq 0$, we have

$$\frac{4q^3+5q^2+4q+4-\sqrt{4q^6+q^4+16q^3+16q^2+8q+4}}{2(q^2+q+1)} \leq a \leq \frac{4q^3+5q^2+4q+4+\sqrt{4q^6+q^4+16q^3+16q^2+8q+4}}{2(q^2+q+1)} \quad (9)$$

it follows that

$$2t_{3q+1}(aq + 1) = = (q^2 + q + 1)a^2 - (3q^3 + 4q^2 + 3q + 3)a + (2q^4 + 5q^3 + 3q^2 + 2q + 2) \quad (10)$$

As $t_{3q+1}(aq + 1) \geq 0$, we have either

$$a < \frac{3q^3 + 4q^2 + 3q + 3 - \sqrt{q^6 - 4q^5 - 6q^4 + 2q^3 + 5q^2 + 2q + 1}}{2(q^2 + q + 1)} \quad (11)$$

or

$$a > \frac{3q^3 + 4q^2 + 3q + 3 + \sqrt{q^6 - 4q^5 - 6q^4 + 2q^3 + 5q^2 + 2q + 1}}{2(q^2 + q + 1)}. \quad (12)$$

As $\frac{4q^3+5q^2+4q+4-\sqrt{4q^6+q^4+16q^3+16q^2+8q+4}}{2(q^2+q+1)} < \frac{3q^3+4q^2+3q+3-\sqrt{q^6-4q^5-6q^4+2q^3+5q^2+2q+1}}{2(q^2+q+1)} < \frac{5q^3+6q^2+5q+5-\sqrt{q^6-8q^5-10q^4+14q^3+9q^2+2q+1}}{2(q^2+q+1)}$, we find that

$$\frac{4q^3+5q^2+4q+4-\sqrt{4q^6+q^4+16q^3+16q^2+8q+4}}{2(q^2+q+1)} \leq a \leq \frac{3q^3+4q^2+3q+3-\sqrt{q^6-4q^5-6q^4+2q^3+5q^2+2q+1}}{2(q^2+q+1)} \quad (13)$$

It is easy to verify that

$$q + 1 < \frac{4q^3 + 5q^2 + 4q + 4 - \sqrt{4q^6 + q^4 + 16q^3 + 16q^2 + 8q + 4}}{2(q^2 + q + 1)} < q + 2 \quad (14)$$

and

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$$q+2<\frac{3q^3+4q^2+3q+3-\sqrt{q^6-4q^5-6q^4+2q^3+5q^2+2q+1}}{2(q^2+q+1)}<q+3 \quad (15)$$

it follows that $a_1 = q+2$ and $k_1 = q^2+2q+1$.

As $\frac{3q^3+4q^2+3q+3+\sqrt{q^6-4q^5-6q^4+2q^3+5q^2+2q+1}}{2(q^2+q+1)} < \frac{5q^3+6q^2+5q+5+\sqrt{q^6-8q^5-10q^4+14q^3+9q^2+2q+1}}{2(q^2+q+1)} <$

$< \frac{4q^3+5q^2+4q+4+\sqrt{4q^6+q^4+16q^3+16q^2+8q+4}}{2(q^2+q+1)}$, we find that

$$\frac{5q^3+6q^2+5q+5+\sqrt{q^6-8q^5-10q^4+14q^3+9q^2+2q+1}}{2(q^2+q+1)} \le a \le \frac{4q^3+5q^2+4q+4+\sqrt{4q^6+q^4+16q^3+16q^2+8q+4}}{2(q^2+q+1)}. \quad (16)$$

It is easy to verify that

$$3q-3<\frac{5q^3+6q^2+5q+5+\sqrt{q^6-8q^5-10q^4+14q^3+9q^2+2q+1}}{2(q^2+q+1)} < 3q-2 \quad (17)$$

and

$$3q-1<\frac{4q^3+5q^2+4q+4+\sqrt{4q^6+q^4+16q^3+16q^2+8q+4}}{2(q^2+q+1)}<3q \quad (18)$$

it follows that $(a_2, a_3) = (3q-2, 3q-1)$ and $(k_2, k_3) = (3q^2-2q+1, 3q^2-q+1)$.

Therefore, a set of class $[q+1,2q+1,3q+1]_3$ is a set of class $[q^2+2q+1,3q^2-2q+1,3q^2-q+1]_4$ in PG($r$,$q$), $q\geq 9$.

## 4. Discussion

It is surprising how little is known about the relation between classes of different dimensions. In [3] p. 783, M. Tallini Scafati proved that in PG($r$,$q$) a two-intersection set with respect to the dimension $d\leq r-2$ is a two-intersection set with respect to any other larger dimension. Unfortunately the situation is very different for three-intersection sets due to a partial result of P. Vasarelli who proved that in PG($r$,$q^2$) a set of class $[1,q+1,q^2+1]_1$ is of class $[q^2+1,q^2+q+1,q^3+1,q^3+q+1,q^3+q^2+1,q^4+q^2+1]_2$, see [9]. Thus, a natural question is: are there three-intersection sets that are such for two consecutive dimensions? Following this line of research, in this note we have given an affirmative answer.

## 5. Conclusions

As a consequence we conclude that, since one-character sets are trivial, cf. [1], in PG($r$,$q$), $q\geq 9$, a set of class $[q+1,2q+1,3q+1]_3$ is a set with at least two intersection numbers with respect to the dimension four and, therefore, the size incidence condition of the combinatorial characterization of the ruled cubic surface by S.G. Barwick and Wen-Ai Jackson in [2] is sharp.

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