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Article

Algebraic Structures of $2\mathbb{D}$ and $3\mathbb{D}$ Fields of Real Vectors

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Abstract

On the basis of the isomorphic algebraic structures of the field of complex numbers \mathbb{C} and the 2-dimensional *Euclidean* field of real vectors \mathbf{V}_2 , in terms of identical geometric products of elements, this paper brings integral identities for scalar and vector fields in \mathbf{V}_2 , which are vector analogues of the well-known integral identities of complex analysis. Consequently, in this paper, *Theorem 1.*, which is a generalized fundamental theorem of integral calculus in the field \mathbf{V}_2 , is the vector analogue of the *Cauchy* theorem of complex analysis. Therefore, special attention is paid to the vector analogue of *Cauchy's* calculus of residues in the field \mathbf{V}_2 . Finally, at the very end of the paper, the algebraic structure of the $3\mathbb{D}$ field of vectors \mathbf{V}_3 is presented, as well as the corresponding fundamental integral identities.

Keywords: geometric product; the field of vectors

MSC: Primary 14A05; 26B12; Secondary 14A25; 26B20

1. Introduction

A geometric algebra (*Clifford algebra*) is an extension of elementary algebra to work with geometrical objects such as vectors. It is built out of two fundamental operations: addition and geometric product, [2]. The multiplication of vectors alone results in objects called multivectors, among which are bivectors, the name applied in this paper to the objects of the bivector field \mathbb{R}^2 , corresponding to the field of vectors \mathbf{V}_2 . Compared with other formalisms for manipulating geometric objects, geometric algebra supports dividing by a vector. The geometric product was first mentioned by *Grassmann*, who founded the so-called external algebra, [3]. After that, *Clifford* himself greatly expanded upon *Grassmann's* work, to form geometric algebra, named after him *Clifford algebra* [2], by unifying both *Grassmann's* algebra and *Hamilton's* quaternion algebra. In the middle of the 20th century, *Hestenes* repopularized the term geometric algebra [4,5].

On the other hand, although rarely used explicitly, a geometric representation of complex numbers is implicitly based on its structure of the *Euclidean* 2-dimensional vector space. If the binary operation of the product $\bar{w}z$ of two complex numbers \bar{w} and z is considered as the sum of the inner product $w \circ z = (\bar{w}z + w\bar{z})/2 = \mathbf{1} \operatorname{Re}(\bar{w}z)$ and outer product $w \wedge z = (\bar{w}z - w\bar{z})/2 = \mathbf{i} \operatorname{Im}(\bar{w}z)$, where $\mathbf{1} = (1, i0)$ and $\mathbf{i} = (0, i)$, and i is an imaginary unit, it can be said that $\bar{w}z$ is in the form of a geometric product of two *ivectors* (two complex numbers), as two geometric objects belonging to the *ivector* field \mathbb{C} (to the field of complex numbers \mathbb{C}). For any complex number z , its absolute value $|z|$ is its *Euclidean* norm denoted by r , and the argument $\arg z$ is the polar angle φ . Since ordered pairs represent both complex numbers and vectors, the binary operation of the product of two complex numbers (two ordered pairs), in the form of the geometric product wz ($wz = \bar{w} \circ z + \bar{w} \wedge z$), will be the basis for modifying *Grassmann's* geometric product of vectors, which is defined as the sum of the inner (scalar) and outer (bivector) products of two vectors. By this modification, the geometric product of two vectors becomes commutative, similar to the product of complex numbers themselves, which still supports vector division. In this manner, a complete analogy is established between the algebra of complex numbers and the modified *Clifford algebra* in the *Euclidean* 2-dimensional field of vectors

V_2 . On the basis of that analogy, Section 2 of the paper presents the most important vector integral identities, in the *Euclidean* field of real vectors V_2 , which are vector analogies to the well-known integral identities of complex analysis.

In the next section (Section 3), the algebraic structure of the field of vectors V_3 is defined. In addition, this section brings the corresponding integral identities, which are more general than the fundamental integral identities of the *Stokes* theorem and the divergence theorem.

1.1. Realreal Vector Space $V_{\mathbb{C}}$

The ordered pairs $\mathbf{1} = (1, i0)$ and $\mathbf{i} = (0, i)$ are the basis of the 2-dimensional realreal vector space $V_{\mathbb{C}}$ [6], which is the *Cartesian* product $V_{\mathbb{R}} \times iV_{\mathbb{R}}$ of a 1-dimensional real vector space $V_{\mathbb{R}}$ and a 1-dimensional ireal vector space $iV_{\mathbb{R}}$, and as such $V_{\mathbb{C}}$ is an additive *Abelian* (commutative) group of elements $(x, iy) = \mathbf{1}x + \mathbf{i}y$. As both of these 1-dimensional vector spaces are defined over the field of real numbers \mathbb{R} , the real vector space $V_{\mathbb{R}}$ can be said to be a field of real numbers \mathbb{R} , whereas the ireal vector space $iV_{\mathbb{R}}$ cannot be a field, and therefore not a field of imaginary numbers iy . On the other hand, if the vector space $V_{\mathbb{C}}$ is complemented by a binary operation of the product of two elements (a, ib) and (c, id) , which corresponds to the matrix product, in such a manner that

$$\begin{aligned} (a, ib)(c, id) &\doteq \begin{bmatrix} a & ib \\ ib & a \end{bmatrix} \begin{bmatrix} c & id \\ id & c \end{bmatrix} = \\ &= \begin{bmatrix} ac - bd & i(ad + bc) \\ i(ad + bc) & ac - bd \end{bmatrix} \doteq (ac - bd, i(ad + bc)), \end{aligned} \quad (1.1)$$

where both the commutative axiom of multiplication and the associative axiom of multiplication are satisfied, as well as the distributive axiom, and in addition the element $(x, -iy)/(x^2 + y^2)$, which corresponds to the inverse matrix $\begin{bmatrix} x & iy \\ iy & x \end{bmatrix}^{-1}$, is the inverse element of the element $(x, iy) \neq (0, i0)$, then the 2-dimensional realreal vector space $V_{\mathbb{C}}$ can be said to be defined over the field of complex numbers \mathbb{C} , that is, $V_{\mathbb{C}}$ is the field of complex numbers \mathbb{C} . More precisely, on the one hand, the ordered pair $(V_{\mathbb{C}}, \mathbb{R})$ is the vector space $V_{\mathbb{C}}$ over the scalar field of real numbers \mathbb{R} , and on the other hand, after complementation with the binary operation of the product of the elements, the ordered pair $(V_{\mathbb{C}}, \mathbb{C})$ is the vector space over the ivector field \mathbb{C} , that is, $V_{\mathbb{C}}$ is the ivector field \mathbb{C} . Accordingly, complex numbers z can be said to be ivectors $(x, iy) = \mathbf{1}x + \mathbf{i}y$, the elements of the vector space $V_{\mathbb{C}}$, that is, of the ivector field \mathbb{C} and as such can be multiplied either by real numbers as scalars or by complex numbers $z \in \mathbb{C}$ as ivectors. When the ivector (a, ib) is multiplying by the imaginary unit i , then the order of the elements, in the resulting ordered pair, is changed, so the resulting ordered pair is not an element of the ivector field \mathbb{C} . Therefore, from that perspective, multiplying complex numbers $z \in \mathbb{C}$ by imaginary numbers is absolutely unacceptable. A complex number z can be multiplied by the ivector \mathbf{i} , so that $(x, iy)\mathbf{i} = \mathbf{i}(x, iy) = -y\mathbf{1} + x\mathbf{i} = (-y, ix)$ and $\mathbf{i}^2 = \mathbf{i}\mathbf{i} = -\mathbf{1}$.

How the operator $\mathbf{cjs}\cdot = (\cos \cdot, i \sin \cdot) = \mathbf{1} \cos \cdot + \mathbf{i} \sin \cdot$ has the most important properties of an exponential function, since $\mathbf{cjs}\cdot \mathbf{cjs}\cdot = \mathbf{cjs}(\cdot + \cdot)$, $(\mathbf{cjs}\cdot)^{-1} = \mathbf{cjs}(-\cdot)$ and $d(\mathbf{cjs}\cdot) = \mathbf{icjs}\cdot d\cdot$, the operator $\exp(\mathbf{i}\cdot)$ can be said to be the exponential form of the operator $\mathbf{cjs}\cdot$. For above reason, multiplication of the operator $\mathbf{cjs}\cdot = \exp(\mathbf{i}\cdot)$ by imaginary numbers is unacceptable, but multiplication by either a real number or an ivector is acceptable.

1.2. Algebraic Structure of the Field of Real Vectors V_2

The basis of the 2-dimensional real vector space $V_{\mathbb{R}^2}$ (the *Cartesian* square of the 1-dimensional real vector space $V_{\mathbb{R}}$), as an additive *Abelian* group of elements (x, y) , consists of ordered pairs $\mathbf{1} = (1, 0)$ and $\hat{\mathbf{1}} = (0, 1)$, such that $(x, y) = x\mathbf{1} + y\hat{\mathbf{1}}$. Analogous to the binary operation of the product of the elements, which we used to complement the realreal vector space $V_{\mathbb{C}}$, it is also possible to complement

the vector space $V_{\mathbb{R}^2}$, with the binary operation of the product of two elements (a, b) and (c, d) , which corresponds to the matrix product, in such a manner that

$$\begin{aligned} (a, b)(c, d) &\doteq \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \\ &= \begin{bmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{bmatrix} \doteq (ac - bd, ad + bc), \end{aligned} \quad (1.2)$$

where the element $(x/(x^2 + y^2), -y/(x^2 + y^2))$, which corresponds to the inverse matrix $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}^{-1}$, is the inverse element of the vector space $V_{\mathbb{R}^2}^* = V_{\mathbb{R}^2} \setminus \{(0, 0)\}$. In this case both the commutative axiom of multiplication and the associative axiom of multiplication are satisfied, as well as the distributive axiom. Therefore, on the one hand, the ordered pair $(V_{\mathbb{R}^2}, \mathbb{R})$ is the vector space $V_{\mathbb{R}^2}$ over the scalar field of real numbers \mathbb{R} , whereas on the other hand, the ordered pair $(V_{\mathbb{R}^2}, \mathbb{R}^2)$ is a vector space over the bivector field \mathbb{R}^2 , that is, $V_{\mathbb{R}^2}$ is the bivector field \mathbb{R}^2 , whose elements are the bivectors $(x, y) = x\mathbf{1} + y\hat{\mathbf{1}}$. Accordingly, the bireal numbers w are bivectors (x, y) , elements of the vector space $V_{\mathbb{R}^2}$, that is, of the bivector field \mathbb{R}^2 and as such can be multiplied either by real numbers as scalars or by bireal numbers as bivectors.

The field of real vectors, denoted by \mathbf{V}_2 , corresponds to the bivector field \mathbb{R}^2 , in the sense of the correspondence: $\mathbf{1} \doteq \mathbf{e}$ and $\hat{\mathbf{1}} \doteq \hat{\mathbf{e}}$, where the unit vectors \mathbf{e} and $\hat{\mathbf{e}}$ are orthogonal basis vectors of the field \mathbf{V}_2 . The vectors $\mathbf{r} = x\mathbf{e} + y\hat{\mathbf{e}}$, as elements of the field \mathbf{V}_2 , correspond to the bireal numbers $w \in \mathbb{R}^2$. In other words, there is a one to one correspondence between the fields \mathbf{V}_2 and \mathbb{R}^2 . If $(x, -y) \doteq x\mathbf{e} - y\hat{\mathbf{e}} = \bar{\mathbf{r}}$, then $r = \left| \sqrt[2]{|\bar{\mathbf{r}}|} \right| = \|\mathbf{r}\| = \|(x, y)\| = \left| \sqrt[2]{x^2 + y^2} \right|$ is a norm over both fields \mathbf{V}_2 and \mathbb{R}^2 , simultaneously, and $\bar{\mathbf{r}} \doteq (x, -y)(x, y) = r^2\mathbf{1} \doteq r^2\mathbf{e}$. Consequently, $\mathbf{r}^{-1} = \bar{\mathbf{r}}/r^2$. It is quite clear that the inverse element \mathbf{r}^{-1} allows division by a vector in the field \mathbf{V}_2 . The binary operation of the product of two vectors, belonging to the field \mathbf{V}_2 , defined as follows.

Definition 1. Let $\bar{\mathbf{r}}_1$ and \mathbf{r}_2 be two vectors in \mathbf{V}_2 . Then, the commutative binary operation

$$\begin{aligned} \bar{\mathbf{r}}_1\mathbf{r}_2 &= \frac{1}{2}(\bar{\mathbf{r}}_1\mathbf{r}_2 + \mathbf{r}_1\bar{\mathbf{r}}_2) + \frac{1}{2}(\bar{\mathbf{r}}_1\mathbf{r}_2 - \mathbf{r}_1\bar{\mathbf{r}}_2) := \\ &:= (\bar{\mathbf{r}}_1 \cdot \bar{\mathbf{r}}_2)\mathbf{e} - (\bar{\mathbf{r}}_1 \times \bar{\mathbf{r}}_2) \times \mathbf{e} = (\mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{e} + (\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e}, \end{aligned} \quad (1.3)$$

is the geometric product of $\bar{\mathbf{r}}_1$ and \mathbf{r}_2 .

Obviously, $\mathbf{r}_2\bar{\mathbf{r}}_1 = (\mathbf{r}_2 \cdot \mathbf{r}_1)\mathbf{e} - (\mathbf{r}_2 \times \mathbf{r}_1) \times \mathbf{e} = \bar{\mathbf{r}}_1\mathbf{r}_2$. Here, $\mathbf{r}_1 \cdot \mathbf{r}_2 = \|\mathbf{r}_1\| \|\mathbf{r}_2\| \cos \theta$ and $(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} = \|\mathbf{r}_1\| \|\mathbf{r}_2\| \sin \theta \hat{\mathbf{e}}$, where θ is the angle between \mathbf{r}_1 and \mathbf{r}_2 .

The geometric product of $\bar{\mathbf{r}}_1 = a\mathbf{e} - b\hat{\mathbf{e}}$ and $\mathbf{r}_2 = c\mathbf{e} + d\hat{\mathbf{e}}$ corresponds to the product $(a, -b)(c, d) = (ac + bd, ad - bc)$ of two bivectors $(a, -b)$ and (c, d) , so that

$$\frac{1}{2}(\bar{\mathbf{r}}_1\mathbf{r}_2 + \mathbf{r}_1\bar{\mathbf{r}}_2) := (\mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{e} = \mathbf{r}_1 \circ \mathbf{r}_2 \doteq (a, b) \circ (c, d) \text{ and} \quad (1.4)$$

$$\frac{1}{2}(\bar{\mathbf{r}}_1\mathbf{r}_2 - \mathbf{r}_1\bar{\mathbf{r}}_2) := (\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} = \mathbf{r}_1 \wedge \mathbf{r}_2 \doteq (a, b) \wedge (c, d). \quad (1.5)$$

In addition, $(a, -b)(c, d) = (a, b) \circ (c, d) + (a, b) \wedge (c, d)$ and

$$\begin{aligned} [(a, b) \circ (c, d)]^2 - [(a, b) \wedge (c, d)]^2 &= (a, -b)(c, d)(c, -d)(a, b) = \\ &= (a, -b)(a, b)(c, d)(c, -d) = \|(a, b)\|^2 \|(c, d)\|^2 \mathbf{1}. \end{aligned} \quad (1.6)$$

The concepts of geometric product and bivector, introduced above, are closely related to the same concepts in Clifford algebra [2]. However, there is evidently a crucial difference between Clifford algebra

and the algebra of the field \mathbf{V}_2 , which is reflected in the fact that the geometric product of the elements of the field \mathbf{V}_2 , on the one hand, is the element of the field \mathbf{V}_2 , corresponding to the bivector, the element of the bivector field \mathbb{R}^2 , and on the other hand, it is also commutative, which means that the field \mathbf{V}_2 , in addition to being an additive *Abelian* group, is also a multiplicative *Abelian* group.

If \mathbf{r}_0 is the unit vector of the vector \mathbf{r} ($\mathbf{r} = r\mathbf{r}_0$), then $\mathbf{r}_0^{-1} = \bar{\mathbf{r}}_0$ and

$$\frac{\mathbf{r}_{0_1}}{\mathbf{r}_{0_2}} = \mathbf{r}_{0_1}\bar{\mathbf{r}}_{0_2} = (\mathbf{r}_{0_1} \cdot \mathbf{r}_{0_2})\mathbf{e} - (\mathbf{r}_{0_1} \times \mathbf{r}_{0_2}) \times \mathbf{e}, \quad (1.7)$$

so that

$$\mathbf{r}_{0_1} \circ \mathbf{r}_{0_2} = \frac{1}{2} \left(\frac{\mathbf{r}_{0_2}}{\mathbf{r}_{0_1}} + \frac{\mathbf{r}_{0_1}}{\mathbf{r}_{0_2}} \right) \text{ and } \mathbf{r}_{0_1} \wedge \mathbf{r}_{0_2} = \frac{1}{2} \left(\frac{\mathbf{r}_{0_2}}{\mathbf{r}_{0_1}} - \frac{\mathbf{r}_{0_1}}{\mathbf{r}_{0_2}} \right), \quad (1.8)$$

where $\mathbf{r}_{0_1} \circ \mathbf{r}_{0_2} = (\mathbf{r}_{0_1} \cdot \mathbf{r}_{0_2})\mathbf{e} + \mathbf{r}_{0_1} \wedge \mathbf{r}_{0_2} = (\mathbf{r}_{0_1} \times \mathbf{r}_{0_2}) \times \mathbf{e}$ are the symmetric and antisymmetric parts of the geometric product $\bar{\mathbf{r}}_{0_1}\mathbf{r}_{0_2}$, respectively. Therefore, to emphasize once again, the geometric product of two vectors is a vector in \mathbf{V}_2 . Accordingly, $\mathbf{r}\mathbf{e} = (\bar{\mathbf{r}} \cdot \mathbf{e})\mathbf{e} + (\bar{\mathbf{r}} \times \mathbf{e}) \times \mathbf{e} = x\mathbf{e} + y\hat{\mathbf{e}} = \mathbf{r}$ and $\mathbf{r}\hat{\mathbf{e}} = (\bar{\mathbf{r}} \cdot \hat{\mathbf{e}})\mathbf{e} + (\bar{\mathbf{r}} \times \hat{\mathbf{e}}) \times \mathbf{e} = \mathbf{r}_\perp$. The vector \mathbf{r}_\perp is orthogonal to the vector \mathbf{r} , that is, \mathbf{r}_\perp is the vector obtained by rotating the vector $\mathbf{r} = r\mathbf{r}_0 = r(\mathbf{e} \cos \varphi + \hat{\mathbf{e}} \sin \varphi)$, where φ is the angle between the vector \mathbf{r} and the vector \mathbf{e} , by $\pi/2$ radians in the positive mathematical direction. In addition, the unit vector $\bar{\mathbf{r}}_0 = \cos \varphi \mathbf{e} - \sin \varphi \hat{\mathbf{e}}$ is the inverse vector \mathbf{r}_0^{-1} of the unit vector \mathbf{r}_0 of the vector \mathbf{r} , and therefore also the unit vector of the inverse vector \mathbf{r}^{-1} of the vector \mathbf{r} , so that $\mathbf{r}^{-1} = \bar{\mathbf{r}}_0/r$ and $\bar{\mathbf{r}} = r\bar{\mathbf{r}}_0$.

The main purpose of the next section is to derive vector integral identities, in the field of vectors \mathbf{V}_2 , which are analogous to the known integral identities of complex analysis, on the basis of the analogy of the *ivector* field \mathbb{C} and the bivector field \mathbb{R}^2 , which corresponds to the field of vectors \mathbf{V}_2 .

2. Differential and Integral Calculus in \mathbf{V}_2

As $\hat{\mathbf{c}}\hat{\mathbf{s}} \cdot \hat{\mathbf{c}}\hat{\mathbf{s}} \cdot = \hat{\mathbf{c}}\hat{\mathbf{s}}(\cdot + \cdot)$, $(\hat{\mathbf{c}}\hat{\mathbf{s}} \cdot)^{-1} = \hat{\mathbf{c}}\hat{\mathbf{s}}(\cdot - \cdot) + d(\hat{\mathbf{c}}\hat{\mathbf{s}} \cdot) = \hat{\mathbf{e}}\hat{\mathbf{c}}\hat{\mathbf{s}} \cdot d \cdot$, the bivector operator $\hat{\mathbf{c}}\hat{\mathbf{s}} \cdot = \mathbf{e} \cos \cdot + \hat{\mathbf{e}} \sin \cdot$ also has the properties of an exponential function, similar to the *ivector* operator $\hat{\mathbf{c}}\hat{\mathbf{i}}\hat{\mathbf{s}} \cdot$. The operator $\exp(\hat{\mathbf{e}} \cdot)$ is the exponential form of the operator $\hat{\mathbf{c}}\hat{\mathbf{s}} \cdot$. Since $\mathbf{r}_0 = \hat{\mathbf{c}}\hat{\mathbf{s}}\varphi = \exp(\hat{\mathbf{e}}\varphi)$, one more analogy with complex analysis is the notion of the so-called vector logarithmic function $\log \mathbf{r} = \ln r\mathbf{e} + \varphi\hat{\mathbf{e}}$, where $2\varphi\hat{\mathbf{e}} = 2 \log \mathbf{r}_0 = \log(\mathbf{r}_0/\bar{\mathbf{r}}_0)$. In addition, $\text{Log } \mathbf{r} = \ln r\mathbf{e} + (\varphi \pm 2\pi n)\hat{\mathbf{e}}$, $n \in \mathbb{N}$. Let $\hat{\mathbf{r}}_0 = -\mathbf{r}_{0\perp}^{-1} = \mathbf{e} \sin \varphi + \hat{\mathbf{e}} \cos \varphi = \hat{\mathbf{s}}\hat{\mathbf{e}}\hat{\mathbf{c}}\varphi = \hat{\mathbf{e}}\bar{\mathbf{r}}_0$. The ordered pair of vectors $(\bar{\mathbf{r}}_0, \hat{\mathbf{r}}_0)$ is the inverse orthonormal basis with respect to the orthonormal basis $(\mathbf{r}_0, \mathbf{r}_{0\perp})$ of the field \mathbf{V}_2 . For an arbitrary vector $\% \in \mathbf{V}_2$, the vector $\%\mathbf{r}_0$ is the rotated vector $\%$, in the positive mathematical direction, by the angle φ , and the vector $\%\mathbf{r}_{0\perp}$ by the angle $\pi/2 + \varphi$. The geometric products of the vector $\%$ with the inverse basis vectors $\bar{\mathbf{r}}_0$ and $\hat{\mathbf{r}}_0$ rotate the vector $\%$ by the angles $-\varphi$ and $\pi/2 - \varphi$, respectively, in the positive mathematical direction.

On the basis of the geometric products $\mathbf{e}\mathbf{e} = \mathbf{e}$, $\mathbf{e}\hat{\mathbf{e}} = \hat{\mathbf{e}}$ and $\hat{\mathbf{e}}\hat{\mathbf{e}} = -\mathbf{e}$, as well as,

$$\mathbf{r}_0\mathbf{r}_0 = (\bar{\mathbf{r}}_0 \cdot \mathbf{r}_0)\mathbf{e} + (\bar{\mathbf{r}}_0 \times \mathbf{r}_0) \times \mathbf{e} = (\cos^2 \varphi - \sin^2 \varphi)\mathbf{e} + 2(\cos \varphi \sin \varphi)\hat{\mathbf{e}}, \quad (2.1)$$

$\mathbf{r}_0\bar{\mathbf{r}}_0 = \mathbf{e}$ and $\mathbf{r}_0\hat{\mathbf{r}}_0 = \hat{\mathbf{e}}$ ($\hat{\mathbf{r}} = \bar{\mathbf{r}}\hat{\mathbf{e}}$), all other combinations of geometric products of the basis vectors \mathbf{r}_0 , $\mathbf{r}_{0\perp}$, $\bar{\mathbf{r}}_0$ and $\hat{\mathbf{r}}_0$ can also be obtained.

If we introduce the differential operator $d = dr\partial_r + d\varphi\partial_\varphi$, then $d\mathbf{r} = dr\mathbf{r}_0 + d\varphi\mathbf{r}_\perp$. Hence, $d\mathbf{r}_\perp = \hat{\mathbf{e}}d\mathbf{r} = dr\mathbf{r}_{0\perp} - d\varphi\mathbf{r}$ and $d\hat{\mathbf{r}} = \hat{\mathbf{e}}d\bar{\mathbf{r}} = \hat{\mathbf{e}}(d\bar{\mathbf{r}}_0 - d\varphi\hat{\mathbf{r}}) = d\bar{\mathbf{r}}_0 + d\varphi\bar{\mathbf{r}}$. Since $2r \cos \varphi \mathbf{e} = \mathbf{r} + \bar{\mathbf{r}}$ and $2r \sin \varphi \hat{\mathbf{e}} = \mathbf{r} - \bar{\mathbf{r}}$, the vector operators of partial derivatives are introduced, as a vector analogue of the *Virtinger* operators [16],

$$\bar{\partial}_r = \partial_r r \partial_r + \partial_r \varphi \partial_\varphi = \frac{1}{2} (\bar{\mathbf{r}}_0 \partial_r - \frac{\hat{\mathbf{r}}_0}{r} \partial_\varphi) \text{ and } \bar{\partial}_{\bar{r}} = \bar{\partial}_r = \frac{1}{2} (\mathbf{r}_0 \partial_r + \frac{\mathbf{r}_{0\perp}}{r} \partial_\varphi). \quad (2.2)$$

Here,

$$\partial_r r = \frac{1}{2r} \partial_r r^2 = \frac{\bar{\mathbf{r}}_0}{2} \text{ and } \partial_r \varphi = \cos^2 \varphi \partial_r \tan \varphi = \left(\frac{\mathbf{r} + \bar{\mathbf{r}}}{2r} \right)^2 \frac{-2\hat{\mathbf{e}}\bar{\mathbf{r}}}{(\mathbf{r} + \bar{\mathbf{r}})^2} = -\frac{\hat{\mathbf{r}}_0}{2r}. \quad (2.3)$$

It is important to emphasize that when geometric products and geometric quotients are differentiated, the same rules apply as when ordinary products and quotients are differentiated, so that $d(\mathbf{r}/\bar{\mathbf{r}}) = (\bar{\mathbf{r}}d\mathbf{r} - \mathbf{r}d\bar{\mathbf{r}})/\bar{\mathbf{r}}^2$. Let's prove this,

$$\begin{aligned} d\frac{\mathbf{r}}{\bar{\mathbf{r}}} &= (d\frac{1}{r^2})\mathbf{r}^2 + \frac{1}{r^2}d\mathbf{r}^2 = -2(\frac{\mathbf{r}^2}{r^3}dr - \frac{1}{r^2}\mathbf{r}d\mathbf{r}) = \\ &= -2[\frac{\mathbf{r}^2}{2r^4}(\bar{\mathbf{r}}d\mathbf{r} + \mathbf{r}d\bar{\mathbf{r}}) - \frac{1}{r^2}\mathbf{r}d\mathbf{r}] = -2[\frac{1}{2\bar{\mathbf{r}}^2}(\bar{\mathbf{r}}d\mathbf{r} + \mathbf{r}d\bar{\mathbf{r}}) - \frac{1}{\bar{\mathbf{r}}^2}\bar{\mathbf{r}}d\mathbf{r}] = \frac{\bar{\mathbf{r}}d\mathbf{r} - \mathbf{r}d\bar{\mathbf{r}}}{\bar{\mathbf{r}}^2}. \end{aligned} \quad (2.4)$$

Definition 2. The geometric product $d\mathbf{r}\bar{\partial}_{\mathbf{r}}$ is an operator of a $1\mathbb{D}$ differential form.

The vector operator

$$\mathbf{d} = \mathbf{r}_0(d\mathbf{r}\bar{\partial}_{\mathbf{r}} + d\bar{\mathbf{r}}\bar{\partial}_{\bar{\mathbf{r}}}) = 2\mathbf{r}_0(d\mathbf{r} \circ \bar{\partial}_{\bar{\mathbf{r}}}) = \mathbf{r}_0(d\mathbf{r}\partial_r + d\varphi\partial_\varphi) = \mathbf{r}_0d, \quad (2.5)$$

is a radial vector differential operator. The vector operator

$$\mathbf{d}_\perp = \mathbf{r}_0(d\mathbf{r}\bar{\partial}_{\mathbf{r}} - d\bar{\mathbf{r}}\bar{\partial}_{\bar{\mathbf{r}}}) = -2\mathbf{r}_0(d\mathbf{r} \wedge \bar{\partial}_{\bar{\mathbf{r}}}) = \mathbf{r}_{0\perp}(rd\varphi\partial_r - \frac{1}{r}dr\partial_\varphi) = \mathbf{r}_{0\perp}\hat{d} \quad (2.6)$$

is a transverse vector differential operator.

It is obvious that $2\bar{\partial}_{\mathbf{r}}$ is a gradient operator. The symmetric part of the geometric product $2\bar{\partial}_{\mathbf{r}}F$ is the divergence vector (div) of the vector field $F = F\mathbf{r}_0 + F_\perp\mathbf{r}_{0\perp}$, and the antisymmetric part is the curl of F , since

$$\begin{aligned} 2\bar{\partial}_{\mathbf{r}}F &= \frac{\bar{\mathbf{r}}_0\mathbf{r}_0}{r}[\partial_r(rF) + \partial_\varphi F_\perp] + \frac{\bar{\mathbf{r}}_0\mathbf{r}_{0\perp}}{r}[\partial_r(rF_\perp) - \partial_\varphi F] = \\ &= \bar{\mathbf{r}}_0\mathbf{r}_0 \operatorname{div} F + \operatorname{curl} F \times \bar{\mathbf{r}}_0\mathbf{r}_0. \end{aligned} \quad (2.7)$$

On the other hand,

$$\bar{\mathbf{r}}_0\bar{\partial}_{\bar{\mathbf{r}}} \circ \bar{\mathbf{r}}_0F + \bar{\mathbf{r}}_0\bar{\partial}_{\bar{\mathbf{r}}} \wedge \bar{\mathbf{r}}_0F = \mathbf{r}_0\bar{\partial}_{\bar{\mathbf{r}}}(\bar{\mathbf{r}}_0F) = -\frac{\bar{\mathbf{r}}_0}{2r}F + \bar{\partial}_{\mathbf{r}}F. \quad (2.8)$$

Consequently,

$$2\bar{\partial}_{\mathbf{r}}F = \frac{2}{r^2}[(\bar{\mathbf{r}}\bar{\partial}_{\bar{\mathbf{r}}} \circ \bar{\mathbf{r}}F) + (\bar{\mathbf{r}}\bar{\partial}_{\bar{\mathbf{r}}} \wedge \bar{\mathbf{r}}F)], \quad (2.9)$$

so that $\operatorname{div} F = 2(\bar{\mathbf{r}}\bar{\partial}_{\bar{\mathbf{r}}} \cdot \bar{\mathbf{r}}F)/r^2$ and $\operatorname{curl} F = 2(\bar{\mathbf{r}}\bar{\partial}_{\bar{\mathbf{r}}} \times \bar{\mathbf{r}}F)/r^2$. Similarly,

$$\mathbf{r}_0\bar{\partial}_{\mathbf{r}} \circ \bar{\mathbf{r}}_0F + \mathbf{r}_0\bar{\partial}_{\mathbf{r}} \wedge \bar{\mathbf{r}}_0F = \bar{\mathbf{r}}_0\bar{\partial}_{\bar{\mathbf{r}}}(\bar{\mathbf{r}}_0F) = \bar{\mathbf{r}}_0(\frac{1}{2r}F + \bar{\partial}_{\mathbf{r}}F) \text{ and} \quad (2.10)$$

$$2\bar{\partial}_{\bar{\mathbf{r}}}F = \frac{\mathbf{r}_0}{r}[\mathbf{r}_0(r\partial_rF - \partial_\varphi F_\perp) + \mathbf{r}_{0\perp}(r\partial_rF_\perp + \partial_\varphi F) - F] = \operatorname{grad} F. \quad (2.11)$$

Further, since $\mathbf{r} = r\mathbf{c}\hat{\mathbf{s}}\varphi = r \exp(\varphi\hat{\mathbf{e}})$, it follows that $2\varphi\hat{\mathbf{e}} = \log(\mathbf{r}/\bar{\mathbf{r}})$,

$$\mathbf{d}\varphi = \mathbf{r}_0(\bar{\partial}_{\mathbf{r}}\varphi d\mathbf{r} + \bar{\partial}_{\bar{\mathbf{r}}}\varphi d\bar{\mathbf{r}}) = \mathbf{r}_0d\varphi = -\frac{\mathbf{r}_{0\perp}}{2}d\log(\frac{\mathbf{r}}{\bar{\mathbf{r}}}) \text{ and} \quad (2.12)$$

$$d\mathbf{r} = \mathbf{r}_0(\bar{\partial}_{\mathbf{r}}rd\mathbf{r} + \bar{\partial}_{\bar{\mathbf{r}}}rd\bar{\mathbf{r}}) = \mathbf{r}_0d\mathbf{r} = \mathbf{r}_0d(\mathbf{r}\bar{\mathbf{r}})^{\frac{1}{2}} = \frac{\mathbf{r}}{2}d\log(\mathbf{r}\bar{\mathbf{r}}). \quad (2.13)$$

Therefore,

$$\hat{\mathbf{r}}_0\mathbf{d}\mathbf{r}d\varphi = \mathbf{r}_{0\perp}d\mathbf{r}d\varphi = \frac{\mathbf{r}}{4}d\log(\mathbf{r}\bar{\mathbf{r}})d\log(\frac{\mathbf{r}}{\bar{\mathbf{r}}}). \quad (2.14)$$

In addition,

$$\frac{\mathbf{r}}{4}d\log(\mathbf{r}\bar{\mathbf{r}})d\log(\frac{\mathbf{r}}{\bar{\mathbf{r}}}) = \frac{\mathbf{r}}{4r^4}(\bar{\mathbf{r}}d\mathbf{r} + \mathbf{r}d\bar{\mathbf{r}})(\bar{\mathbf{r}}d\mathbf{r} - \mathbf{r}d\bar{\mathbf{r}}) = \frac{\mathbf{r}}{4r^2}[(\bar{\mathbf{r}}_0d\mathbf{r})^2 - (\mathbf{r}_0d\bar{\mathbf{r}})^2]. \quad (2.15)$$

In accordance with above,

$$\frac{r_0}{2}(\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) = \frac{r_0}{4}(\bar{\mathbf{d}}\mathbf{r}\bar{\mathbf{d}}\mathbf{r} - \mathbf{d}\bar{\mathbf{r}}\mathbf{d}\bar{\mathbf{r}}) = \hat{\mathbf{r}}d\mathbf{r}d\varphi = \mathbf{r}_\perp d\mathbf{r}d\varphi, \tag{2.16}$$

since $\bar{\mathbf{d}}\bar{\mathbf{r}} = \bar{\mathbf{r}}_0 d\bar{\mathbf{r}} = \bar{\mathbf{d}}\mathbf{r}$. The vector identity just derived can be obtained explicitly, if we introduce the determinant of the *Jacobi* matrix (*Jacobian*) of the bijective mapping $\mathbf{V}_2 \rightarrow \mathbf{V}_2$, defined by the system of vector equations $2 \ln r\mathbf{e} = \log(\mathbf{r}\bar{\mathbf{r}})$ and $2\varphi\hat{\mathbf{e}} = \log(\mathbf{r}/\bar{\mathbf{r}})$, as follows

$$\mathbf{J} = \begin{vmatrix} \partial_r \log(\mathbf{r}\bar{\mathbf{r}}) & \partial_\varphi \log(\mathbf{r}\bar{\mathbf{r}}) \\ \partial_r \log(\mathbf{r}/\bar{\mathbf{r}}) & \partial_\varphi \log(\mathbf{r}/\bar{\mathbf{r}}) \end{vmatrix} = \begin{vmatrix} 2r^{-1}\mathbf{e} & \mathbf{0} \\ \mathbf{0} & 2\hat{\mathbf{e}} \end{vmatrix} = \frac{4}{r}\hat{\mathbf{e}}. \tag{2.17}$$

In this case, $4\mathbf{r}_{0\perp}d\mathbf{r}d\varphi = \mathbf{r}\mathbf{J}d\mathbf{r}d\varphi = \mathbf{r}d\log(\mathbf{r}\bar{\mathbf{r}})d\log(\mathbf{r}/\bar{\mathbf{r}})$, which leads to (2.14). The vector $d\mathbf{S} = (\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r})/2 = r d\mathbf{r}d\varphi \mathbf{r}_{0\perp} \bar{\mathbf{r}}_0 = d\mathbf{S}\hat{\mathbf{e}}$, corresponding to the bivector $r d\mathbf{r}d\varphi \hat{\mathbf{1}}$ of the bivector field \mathbb{R}^2 , is the *Lebesgue* measure of the infinitesimal surface of the field of vectors \mathbf{V}_2 .

Definition 3. Let $\bar{\partial}_{\bar{\mathbf{r}}}^2 = \bar{\partial}_{\bar{\mathbf{r}}}\bar{\partial}_{\bar{\mathbf{r}}}$. Then, the geometric product $d\mathbf{S}\bar{\partial}_{\bar{\mathbf{r}}}^2$ is an operator of a $2\mathbb{D}$ differential form.

2.1. Generalization of the Fundamental Theorem of Integral Calculus in \mathbf{V}_2

Let γ be a closed smooth *Jordan* curve, which is the boundary ∂G of an arbitrary region G in \mathbf{V}_2 , and r_i ($i = 1, 2, \dots, n_\gamma$) be isolated points on the curve γ , surrounded by circles $c(r_i, \varepsilon)$ with centers at r_i and arbitrarily small radius ε , which intersect the curve γ at points r_{1i} and r_{2i} and do not intersect each other. If ρ_j ($j = 1, 2, \dots, n_G$) are isolated points in G , surrounded by circles $c(\rho_j, \varepsilon)$, which do not intersect each other, then it is possible to form a simply connected region \mathcal{R}_ε , inside which all points r_i and ρ_j are, by connecting the circles $c(\rho_j, \varepsilon)$, successively, the first with the second, the second with the third, etc., using parallel straight line segments l_{j1} and l_{j2} , at a mutual distance $\delta_j \ll \varepsilon$, as well as by connecting the circles $c(r_i, \varepsilon)$ on the curve γ with the circles $c(\rho_j, \varepsilon)$, one-to-one, using the parallel straight line segments l_{i1} and l_{i2} , at a mutual distance $\delta_i \ll \varepsilon$. The boundary $\partial\mathcal{R}_\varepsilon$ of \mathcal{R}_ε (blue region in Figure 1), inside G , divides the region G into n_γ subregions G_i .

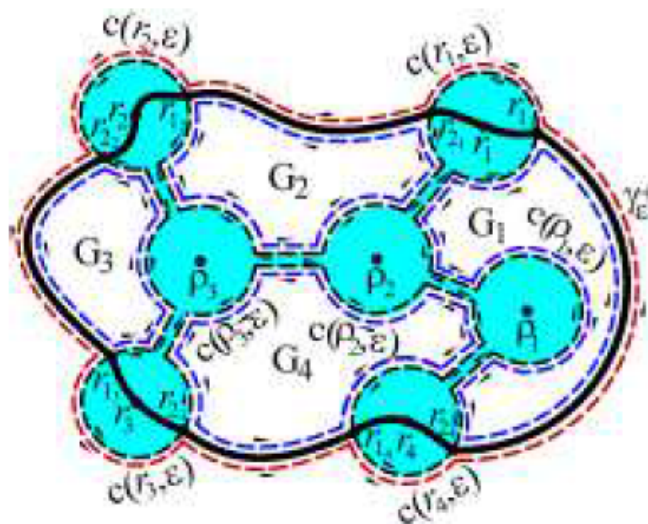


Figure 1

Accordingly, the vector integral operator over γ is introduced, as follows

$$vt \int_{\gamma^+}^{\circ} \bar{\mathbf{r}}_0(\mathbf{d} + \mathbf{d}_\perp) = 2vt \int_{\gamma^+}^{\circ} d\mathbf{r}\bar{\partial}_{\bar{\mathbf{r}}} = vt \int_{\gamma^+}^{\circ} [(d\mathbf{r} \circ \text{grad}) - (d\mathbf{r} \wedge \text{grad})], \tag{2.18}$$

where vt denotes the total value of an improper integral [9–14], such that

$$\begin{aligned} vt \int_{\gamma^+}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}} &= \lim_{\varepsilon \rightarrow 0^+} \int_{\gamma_{\varepsilon}^+}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}} = \lim_{\varepsilon \rightarrow 0^+} \left(\int_{\partial\mathcal{R}_{\varepsilon}}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}} + \sum_{i=1}^{n_{\gamma}} \int_{\partial G_i}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}} \right) \\ &= vp \int_{\gamma}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}} + \lim_{\varepsilon \rightarrow 0^+} \sum_{i=1}^{n_{\gamma}} \int_{\partial c(r_i, \varepsilon)}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}} = vt \int_{\gamma^-}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}} + \lim_{\varepsilon \rightarrow 0^+} \sum_{i=1}^{n_{\gamma}} \int_{\partial c(r_i, \varepsilon)}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}}, \end{aligned} \quad (2.19)$$

and vp denotes the *Cauchy* principal value. The vector integral operator

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{\partial\mathcal{R}_{\varepsilon}}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}} &= 2\pi\hat{\mathbf{e}} \sum_{i=1}^{n_{\gamma}} \text{Res}(\bar{\partial}_{\mathbf{r}}, \mathbf{r}_i) + \sum_{i=1}^{n_{\gamma}} \left(\int_{l_i^+} d\mathbf{r}\bar{\partial}_{\mathbf{r}} - \int_{l_i^-} d\mathbf{r}\bar{\partial}_{\mathbf{r}} \right) + \\ &+ 2\pi\hat{\mathbf{e}} \sum_{j=1}^{n_G} \text{Res}(\bar{\partial}_{\mathbf{r}}, \%_j) + \sum_{j=1}^{n_G-1} \left(\int_{l_j^+} d\mathbf{r}\bar{\partial}_{\mathbf{r}} - \int_{l_j^-} d\mathbf{r}\bar{\partial}_{\mathbf{r}} \right), \end{aligned} \quad (2.20)$$

where $\int_{l_i^+} d\mathbf{r}\bar{\partial}_{\mathbf{r}} = \lim_{\varepsilon \rightarrow 0^+} \int_{l_{i1}} d\mathbf{r}\bar{\partial}_{\mathbf{r}}$, that is, $\int_{l_i^-} d\mathbf{r}\bar{\partial}_{\mathbf{r}} = \lim_{\varepsilon \rightarrow 0^+} \int_{l_{i2}} d\mathbf{r}\bar{\partial}_{\mathbf{r}}$, as well as $\lim_{\varepsilon \rightarrow 0^+} \int_{c(r_i, \varepsilon)}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}} = 2\pi\hat{\mathbf{e}}\text{Res}(\bar{\partial}_{\mathbf{r}}, \mathbf{r}_i)$ and $\lim_{\varepsilon \rightarrow 0^+} \int_{c(\%_j, \varepsilon)}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}} = 2\pi\hat{\mathbf{e}}\text{Res}(\bar{\partial}_{\mathbf{r}}, \%_j)$, is the residue operator in G .

The vector integral operator over G is introduced, as follows

$$2vt \int_{G^+} (\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) \bar{\partial}_{\bar{\mathbf{r}}}^2 = 2vp \int_G (\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) \bar{\partial}_{\bar{\mathbf{r}}}^2 + \lim_{\varepsilon \rightarrow 0^+} \int_{\partial\mathcal{R}_{\varepsilon}}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}}. \quad (2.21)$$

Here, $vp \int_G (\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) \bar{\partial}_{\bar{\mathbf{r}}}^2 = \lim_{\varepsilon \rightarrow 0^+} \sum_{i=1}^n \int_{G_i} (\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) \bar{\partial}_{\bar{\mathbf{r}}}^2$, $(\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) \bar{\mathbf{r}}_0 = 2\hat{\mathbf{r}}drd\varphi$ and

$$vt \int_{G^+} (\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) \bar{\partial}_{\bar{\mathbf{r}}}^2 - vt \int_{G^-} (\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) \bar{\partial}_{\bar{\mathbf{r}}}^2 = 2\pi\hat{\mathbf{e}} \sum_{i=1}^{n_{\gamma}} \text{Res}(\bar{\partial}_{\mathbf{r}}, \mathbf{r}_i), \quad (2.22)$$

so that

$$\begin{aligned} 2vt \int_{G^+} (\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) \bar{\partial}_{\bar{\mathbf{r}}}^2 &= vp \int_G r dr d\varphi (\hat{\mathbf{r}}_0 \mathbf{r}_0 \text{div grad} + \hat{\mathbf{r}}_0 \mathbf{r}_0 \times \text{curl grad}) + \\ &+ \lim_{\varepsilon \rightarrow 0^+} \int_{\partial\mathcal{R}_{\varepsilon}}^{\circ} d\mathbf{r}\bar{\partial}_{\mathbf{r}}, \end{aligned} \quad (2.23)$$

since

$$\begin{aligned} \bar{\partial}_{\bar{\mathbf{r}}}^2 &= \bar{\partial}_{\bar{\mathbf{r}}}\bar{\partial}_{\bar{\mathbf{r}}} = \frac{\bar{\mathbf{r}}_0}{4r^2} [\mathbf{r}_0 [r\partial_r(r\partial_r) + \partial_{\varphi^2}^2] + \mathbf{r}_{\perp} (\partial_{r\varphi}^2 - \partial_{\varphi r}^2)] = \\ &= \frac{\bar{\mathbf{r}}_0}{4} (\mathbf{r}_0 \text{div grad} + \mathbf{r}_0 \times \text{curl grad}). \end{aligned} \quad (2.24)$$

Obviously,

$$\text{div grad} = \frac{1}{r^2} [r\partial_r(r\partial_r) + \partial_{\varphi^2}^2] = \frac{4}{r^2} (\bar{\mathbf{r}}_0 \bar{\partial}_{\bar{\mathbf{r}}} \cdot \bar{\mathbf{r}}_0 \bar{\partial}_{\bar{\mathbf{r}}}) \text{ and} \quad (2.25)$$

$$\text{curl grad} = \frac{\mathbf{r}_0 \times \mathbf{r}_{\perp}}{r^2} (\partial_{\varphi r}^2 - \partial_{r\varphi}^2) = \frac{4}{r^2} (\bar{\mathbf{r}}_0 \bar{\partial}_{\bar{\mathbf{r}}} \times \bar{\mathbf{r}}_0 \bar{\partial}_{\bar{\mathbf{r}}}) = \mathbf{0}. \quad (2.26)$$

Definition 4. Let $!_{\mathbf{r}}$ denote the vector differential form obtained by applying $d\mathbf{r}\bar{\partial}_{\mathbf{r}}$ to some scalar or vector field in \mathbf{V}_2 , which has continuous partial derivatives at every point of G . Then, $!_{\mathbf{r}}$ is regular in G .

Definition 5. Let $!_{\mathbf{r}}$ be regular almost everywhere in G (everywhere except on the finite set $S \subset G$ of singular points $\mathbf{r}_i \in \partial G$ and $\%_j \in G \setminus \partial G$). Then, $!_{\mathbf{r}}$ is integrally summable on the set S if and only if

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\partial\mathcal{R}_{\varepsilon}}^{\circ} !_{\mathbf{r}} \in \mathbf{V}_2. \quad (2.27)$$

The proof of the fundamental theorem that follows, as can be seen, is an explicit consequence of *Green's* theorem [6].

Theorem 1. Let $!_r$ and $!_S$ be regular almost everywhere in G (everywhere except on the finite set $S \subset G$ of singular points $r_i \in \partial G$ and $\%_j \in G \setminus \partial G$). Then,

$$vt \int_{\gamma^+}^{\circ} !_r = vt \int_{G^+} d!_r, \quad (2.28)$$

where $d!_r = 2!_S$.

Proof. Since

$$\begin{aligned} d\mathbf{r}\bar{\partial}_r &= d\mathbf{r} \circ \text{grad} - d\mathbf{r} \wedge \text{grad} = \\ &= \frac{\bar{\mathbf{r}}_0}{2} (\mathbf{d} + \mathbf{d}_\perp) = \frac{\bar{\mathbf{r}}_0}{2} [\mathbf{r}_0 (d\mathbf{r}\partial_r + d\varphi\partial_\varphi) + \mathbf{r}_{0\perp} (rd\varphi\partial_r - \frac{1}{r}dr\partial_\varphi)], \end{aligned}$$

it follows from Green's theorem and the vector relation (2.24) that

$$\int_{\partial G_i}^{\circ} !_r = \int_{G_i} !_S, \text{ for every } i = 1, 2, \dots, n_\gamma.$$

Therefore, according to (2.19) and (2.22),

$$\begin{aligned} vt \int_{\gamma^+}^{\circ} !_r &= \lim_{\varepsilon \rightarrow 0^+} \left(\int_{\partial \mathcal{R}_\varepsilon}^{\circ} !_r + \sum_{i=1}^{n_\gamma} \int_{\partial G_i}^{\circ} !_r \right) = \\ &= \lim_{\varepsilon \rightarrow 0^+} \left(\int_{\partial \mathcal{R}_\varepsilon}^{\circ} !_r + \sum_{i=1}^{n_\gamma} \int_{G_i} !_S \right) = 2vt \int_{G^+} !_S. \end{aligned}$$

□

Remark 1. If the set of singular points, either on the contour γ or in the region G , is an empty set, then the choice of a representative point (either \mathbf{r} on the contour γ or $\%$ in the region G , respectively) is arbitrary. If, in addition, the field is uniform [7], then $\int_{l_i^+} !_r = \int_{l_i^-} !_r$, so in that case, the choice of representative points is not necessary. On the contrary, if there is a limit $\lim_{\varepsilon \rightarrow 0^+} \int_{\gamma_\varepsilon^+}^{\circ} !_r$, and $\int_{\partial \mathcal{R}_\varepsilon}^{\circ} !_r$ tends to infinity as ε tends to zero, then the limit $\lim_{\varepsilon \rightarrow 0^+} \sum_{i=1}^n \int_{\partial G_i}^{\circ} !_r = vp \int_G !_S$ is also infinite. Obviously, the indeterminate form of the difference of two infinities, in this emphasized case, leads to the finite limit $\lim_{\varepsilon \rightarrow 0^+} \int_{\gamma_\varepsilon^+}^{\circ} !_r$.

According to (2.28), since

$$(\bar{\mathbf{d}}\mathbf{r} \wedge \mathbf{d}\bar{\mathbf{r}})(\bar{\partial}_{\mathbf{r}\mathbf{r}}^2 - \bar{\partial}_{\mathbf{r}\mathbf{r}}^2) = -\hat{\mathbf{r}}_0 \mathbf{r}_{0\perp} r dr d\varphi \|\text{curl grad}\| = \bar{\mathbf{r}}_0 \mathbf{r}_0 dr d\varphi (\partial_{r\varphi}^2 - \partial_{\varphi r}^2), \quad (2.29)$$

it follows that

$$\begin{aligned} \bar{\mathbf{r}}_0 \mathbf{r}_0 vt \int_{\gamma^+}^{\circ} d &= vt \int_{\gamma^+}^{\circ} (d\mathbf{r}\bar{\partial}_r + d\bar{\mathbf{r}}\bar{\partial}_{\mathbf{r}}) = vt \int_{G^+} (\bar{\mathbf{d}}\mathbf{r} \wedge \mathbf{d}\bar{\mathbf{r}})(\bar{\partial}_{\mathbf{r}\mathbf{r}}^2 - \bar{\partial}_{\mathbf{r}\mathbf{r}}^2) = \\ &= \bar{\mathbf{r}}_0 \mathbf{r}_0 vt \int_{G^+} dr d\varphi (\partial_{r\varphi}^2 - \partial_{\varphi r}^2). \end{aligned} \quad (2.30)$$

If $r\partial_r(r\partial_r) = -\partial_{\varphi^2}^2$, then $\bar{\partial}_{\mathbf{r}\mathbf{r}}^2 = \bar{\partial}_{\mathbf{r}\mathbf{r}}^2 = \mathbf{0}$ and

$$\bar{\partial}_{\mathbf{r}^2}^2 = \frac{1}{2} \frac{\bar{\mathbf{r}}_0}{r} [\bar{\mathbf{r}}_0 \partial_{r^2}^2 - \hat{\mathbf{r}}_0 (\partial_{\varphi r}^2 - \frac{1}{r} \partial_\varphi)] = \bar{\mathbf{r}}_0 \partial_r (\bar{\partial}_{\mathbf{r}}), \quad (2.31)$$

since

$$\begin{aligned} \bar{\partial}_{\mathbf{r}^2}^2 &= \bar{\partial}_{\mathbf{r}} \bar{\partial}_{\mathbf{r}} = \frac{1}{4} (\bar{\mathbf{r}}_0 \partial_r - \hat{\mathbf{r}}_0 \frac{1}{r} \partial_\varphi) (\bar{\mathbf{r}}_0 \partial_r - \hat{\mathbf{r}}_0 \frac{1}{r} \partial_\varphi) = \\ &= \frac{\bar{\mathbf{r}}_0}{4r^2} [\bar{\mathbf{r}}_0 [r\partial_r(r\partial_r) - \partial_{\varphi^2}^2 - 2r\partial_r] - \hat{\mathbf{r}}_0 (\partial_{\varphi r}^2 + \partial_{r\varphi}^2 - \frac{2}{r} \partial_\varphi)]. \end{aligned} \quad (2.32)$$

Clearly, $\bar{\partial}_{\mathbf{r}^2}^2 = \bar{\partial}_{\mathbf{r}^2}^2$.

2.2. Integrals of Scalar and Vector Fields

The vector differential of a scalar field $F : \mathbf{V}_2 \rightarrow \mathbb{R}$ is as follows

$$dF = \mathbf{r}_0(\partial_{\mathbf{r}}F d\mathbf{r} + \partial_{\bar{\mathbf{r}}}F d\bar{\mathbf{r}}) = \mathbf{r}_0(f d\mathbf{r} + \bar{f} d\bar{\mathbf{r}}), \quad (2.33)$$

where $f = \partial_{\mathbf{r}}F$. The second vector partial derivative of F is the first vector partial derivative of the vector field f , so that

$$\begin{aligned} \partial_{\mathbf{r}}f = \partial_{\mathbf{r}^2}^2 F = \frac{\bar{\mathbf{r}}_0}{4r^2} [r\partial_r(r\partial_r F) - \partial_{\varphi^2}^2 F - 2r\partial_r F] \bar{\mathbf{r}}_0 - \\ - [(\partial_{\varphi r}^2 F + \partial_{r\varphi}^2 F) - \frac{2}{r}\partial_{\varphi} F] \hat{\mathbf{r}} \text{ and} \end{aligned} \quad (2.34)$$

$$\partial_{\bar{\mathbf{r}}}f = \partial_{\bar{\mathbf{r}}^2}^2 F = \frac{\bar{\mathbf{r}}_0 \mathbf{r}_0}{4r^2} [r\partial_r(r\partial_r F) + \partial_{\varphi^2}^2 F]. \quad (2.35)$$

If $f = \partial_{\mathbf{r}}F$ and $\bar{f} = \partial_{\bar{\mathbf{r}}}F$ are uniform vector fields, then by applying the vector integral operator (2.30) to the scalar field F , a vector integral identity is obtained

$$\begin{aligned} \bar{\mathbf{r}}_0 \mathbf{r}_0 vt \int_{\gamma^+}^{\circ} dF = vt \int_{\gamma^+}^{\circ} (\partial_{\mathbf{r}}F d\mathbf{r} + \partial_{\bar{\mathbf{r}}}F d\bar{\mathbf{r}}) = vt \int_{\gamma^+}^{\circ} (f d\mathbf{r} + \bar{f} d\bar{\mathbf{r}}) = \\ = vt \int_{G^+} (\partial_{\mathbf{r}}^2 F - \partial_{\bar{\mathbf{r}}}^2 F)(d\bar{\mathbf{r}} \wedge d\mathbf{r}) = \bar{\mathbf{r}}_0 \mathbf{r}_0 vt \int_{G^+} (\partial_{\varphi r}^2 F - \partial_{r\varphi}^2 F) dr d\varphi = \\ = (2\pi\hat{\mathbf{e}}) \left[\sum_{i=1}^{n_{\gamma}} \text{Res}(f, \mathbf{r}_i) + \sum_{j=1}^{n_G} \text{Res}(f, \%_j) + \sum_{i=1}^{n_{\gamma}} \overline{\text{Res}(\bar{f}, \mathbf{r}_i)} + \sum_{j=1}^{n_G} \overline{\text{Res}(\bar{f}, \%_j)} \right], \end{aligned} \quad (2.36)$$

where $\lim_{\varepsilon \rightarrow 0^+} \int_{\partial\mathcal{R}_{\varepsilon}}^{\circ} \bar{f} d\bar{\mathbf{r}} = (2\pi\hat{\mathbf{e}}) [\sum_{i=1}^{n_{\gamma}} \overline{\text{Res}(\bar{f}, \mathbf{r}_i)} + \sum_{j=1}^{n_G} \overline{\text{Res}(\bar{f}, \%_j)}]$. An integral identity of complex analysis, which is an analogue of the vector integral identity (2.36), is the integral identity of *Cauchy's integral theorem* [7].

As $\partial_{\varphi r}^2 F = \partial_{r\varphi}^2 F$ and if, in addition, $r\partial_r(r\partial_r F) = -\partial_{\varphi^2}^2 F$, then $\partial_{\mathbf{r}}\bar{f} = \partial_{\bar{\mathbf{r}}}f = \mathbf{0}$, that is,

$$\mathbf{r}\partial_r f = -\mathbf{r}_{0\perp} \partial_{\varphi} f \text{ and} \quad (2.37)$$

$$\mathbf{r}\partial_{\bar{\mathbf{r}}}f = \mathbf{r}\partial_{\mathbf{r}^2}^2 F = r\partial_r(\partial_{\mathbf{r}}F) = r\partial_r f. \quad (2.38)$$

A vector field f , satisfying the *Cauchy-Riemann* condition $\mathbf{r}\partial_r f = -\mathbf{r}_{0\perp} \partial_{\varphi} f$, is said, analogous to complex analytic functions, to be an analytic vector field. Hence, an analytic vector field is a vector derivative of the *Laplace* scalar field F . Clearly, the coordinate components of the analytic vector field f are also *Laplace* scalar fields.

Assume that the analytic vector field f , as the vector derivative of the *Laplace* scalar field F , is not defined at the point $\% \in \text{int}.G$, where G is a region in the field of vectors \mathbf{V}_2 , bounded by a closed smooth *Jordan* curve γ , as well as at point r on curve γ . The vector integral identity

$$\begin{aligned} \bar{\mathbf{r}}_0 \mathbf{r}_0 vt \int_{\gamma^+}^{\circ} dF = vt \int_{\gamma^+}^{\circ} \partial_{\mathbf{r}}F d\mathbf{r} = vt \int_{\gamma^+}^{\circ} f d\mathbf{r} = \\ = (2\pi\hat{\mathbf{e}}) [\text{Res}(f, \mathbf{r}) + \text{Res}(f, \%)] + \int_{I^+} f d\mathbf{r} - \int_{I^-} f d\mathbf{r}, \end{aligned} \quad (2.39)$$

is a vector analogue of the integral identity of *Cauchy's integral theorem*, which is slightly generalized, since in this emphasized case

$$\begin{aligned} vt \int_{G^+} \partial_{\bar{\mathbf{r}}}^2 F (d\bar{\mathbf{r}} \wedge d\mathbf{r}) = \lim_{\varepsilon \rightarrow 0^+} \int_{\partial\mathcal{R}_{\varepsilon}}^{\circ} f d\mathbf{r} = \\ = (2\pi\hat{\mathbf{e}}) [\text{Res}(f, \mathbf{r}) + \text{Res}(f, \%)] + \int_{I^+} f d\mathbf{r} - \int_{I^-} f d\mathbf{r}. \end{aligned} \quad (2.40)$$

Definition 6. If a vector field F is differentiable (regular), but not analytic, in an arbitrary region G of the field \mathbf{V}_2 , bounded by a closed smooth Jordan curve γ , then the gradient of F , as the surface (spatial) derivative of F , is defined by

$$\lim_{G \rightarrow \%_k} \frac{1}{\mathbf{S}_G} \int_{\gamma} F d\mathbf{r} = 2\bar{\partial}_{\mathbf{r}} F(\%_k) = \text{grad } F(\%_k), \quad (2.41)$$

where $2\mathbf{S}_G = \int_{\gamma} \bar{\mathbf{r}} d\mathbf{r} = \int_{\gamma} \mathbf{r} \wedge d\mathbf{r}$.

Remark 2. Obviously, some vector field F is regular if and only if the differential form $\bar{\mathbf{r}} = \bar{\partial}_{\mathbf{r}} F$ is regular. The gradient defined above is the vector analogue in \mathbf{V}_2 of the Pompeiu areolar derivative of a complex function, [8].

Similarly, based on the vector identity (2.9), the so-called cumulative surface (spatial) derivative of F can be defined as follows.

Definition 7. If a vector field F is differentiable (regular), but not analytic, in an arbitrary region G of the field \mathbf{V}_2 , bounded by a closed smooth Jordan curve γ , then the cumulative surface (spatial) derivative of F is defined by

$$\lim_{G \rightarrow \%_k} \frac{1}{\mathbf{S}_G} \int_{\gamma} F d\bar{\mathbf{r}} = -2\bar{\partial}_{\mathbf{r}} F(\%_k) = \bar{\mathbf{r}}_0 \mathbf{r}_0 \times \text{curl } F(\%_k) - \bar{\mathbf{r}}_0 \mathbf{r}_0 \text{div } F(\%_k). \quad (2.42)$$

By (2.41), if F is a regular and uniform vector field in the ε -neighborhood $C_{\varepsilon}^0 \setminus \{\mathbf{0}\}$ of its singular point $\% = \mathbf{0}$ and $\lim_{\mathbf{r} \rightarrow \mathbf{0}} \bar{\partial}_{\mathbf{r}}(r^2 F) = \%_0 \in \mathbf{V}_2$, then

$$\begin{aligned} \text{Res}(F, \mathbf{0}) &= (2\pi \hat{\mathbf{e}})^{-1} \lim_{\varepsilon \rightarrow 0^+} \int_{C_{\varepsilon}^0} F d\mathbf{r} = - \lim_{\varepsilon \rightarrow 0^+} \frac{\hat{\mathbf{e}}}{2\pi \varepsilon^2} \int_{C_{\varepsilon}^0} r^2 F d\mathbf{r} = \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\mathbf{S}_{C_{\varepsilon}^0}} \int_{C_{\varepsilon}^0} \bar{\partial}_{\mathbf{r}}(r^2 F) d\mathbf{S} = \lim_{\mathbf{r} \rightarrow \mathbf{0}} \bar{\partial}_{\mathbf{r}}(r^2 F) = \%_0. \end{aligned} \quad (2.43)$$

If $\bar{\partial}_{\mathbf{r}} F = \mathbf{0}$, then $\text{Res}(F, \mathbf{0}) = \lim_{\mathbf{r} \rightarrow \mathbf{0}} \mathbf{r} F = \%_0$, which is another vector analogy to the well-known result of complex analysis. Let F be an analytic vector field, such that $\lim_{r \rightarrow 0^+} (F / \ln r)$ leads to the determinate form only after the application of *L'Hospital's rule* n times. Then, the vector formula for $\text{Res}(f, \mathbf{0})$, being analogous to the complex analysis formula, can be obtained via the vector identity $\mathbf{r} f = r \partial_r F$, see (2.38), where $f = \bar{\partial}_{\mathbf{r}} F$. Namely, since the same vector identity applies to the analytic vector field $\mathbf{r}^n f = r^n \partial_r F \underbrace{\mathbf{r}_0 \dots \mathbf{r}_0}_{n-1}$, it follows that

$$\bar{\partial}_{\mathbf{r}}(\mathbf{r}^n f) = (nr^{n-1} \partial_r F + r^n \partial_r^2 F) \underbrace{\mathbf{r}_0 \dots \mathbf{r}_0}_{n-2} \quad (2.44)$$

$$\bar{\partial}_{\mathbf{r}}^2(\mathbf{r}^n f) = [n(n-1)r^{n-2} \partial_r F + 2nr^{n-1} \partial_r^2 F + r^n \partial_r^3 F] \underbrace{\mathbf{r}_0 \dots \mathbf{r}_0}_{n-3}. \quad (2.45)$$

Accordingly, applying *L'Hospital's rule*,

$$\begin{aligned} \frac{1}{(n-1)!} \lim_{\mathbf{r} \rightarrow \mathbf{0}} \bar{\partial}_{\mathbf{r}}^{n-1}(\mathbf{r}^n f) &= \frac{n!}{(n-1)!} \lim_{r \rightarrow 0^+} \sum_{k=1}^n \binom{n-1}{n-k} \frac{r^k \partial_r^k F}{k!} = \\ &= \left[\sum_{k=1}^n (-1)^{n-k} \binom{n}{n-k} \right] \frac{\lim_{r \rightarrow 0^+} r^n \partial_r^n F}{(n-1)!} = \frac{(-1)^{n-1}}{(n-1)!} \lim_{r \rightarrow 0^+} r^n \partial_r^n F. \end{aligned} \quad (2.46)$$

Further, since $f = \bar{\mathbf{r}}_0 \partial_r F$ is an analytic vector field, it follows that

$$\bar{\partial}_{\mathbf{r}}^{n-1} f = \left(\frac{r}{\mathbf{r}}\right)^n \partial_r^n F \quad \text{and} \quad (2.47)$$

$$-\frac{1}{(n-1)!} \lim_{\mathbf{r} \rightarrow \mathbf{0}} (-\mathbf{r})^n \bar{\partial}_{\mathbf{r}}^{n-1} f = \frac{(-1)^{n-1}}{(n-1)!} \lim_{r \rightarrow 0^+} r^n \partial_r^n F = \lim_{\mathbf{r} \rightarrow \mathbf{0}} \mathbf{r} f. \quad (2.48)$$

This means that *L'Hospital's* rule can be explicitly applied to the vector field $\mathbf{r}f$.

If some analytic vector field f is regular in an arbitrary region G bounded by a closed smooth *Jordan* curve γ , then for the vector field

$$f(\mathbf{r} + \%_A) - [f(\%_A) + \sum_{k=1}^{n-1} \partial_{\mathbf{r}^k}^k f(\%_A) \frac{\mathbf{r}^k}{k!}],$$

where $\{\mathbf{0}, \%_A\} \subset \text{int}.G$, according to (2.39), (2.46) and (2.48), the following is true

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} vt \int_{c_\varepsilon^0} \{f(\mathbf{r} + \%_A) - [f(\%_A) + \sum_{k=1}^{n-1} \partial_{\mathbf{r}^k}^k f(\%_A) \frac{\mathbf{r}^k}{k!}]\} \frac{d\mathbf{r}}{\mathbf{r}^{n+1}} &= \\ = 2\pi \hat{\mathbf{e}} \lim_{\mathbf{r} \rightarrow \mathbf{0}} \frac{1}{\mathbf{r}^n} \{f(\mathbf{r} + \%_A) - [f(\%_A) + \sum_{k=1}^{n-1} \partial_{\mathbf{r}^k}^k f(\%_A) \frac{\mathbf{r}^k}{k!}]\} &= 2\pi \hat{\mathbf{e}} \frac{\partial_{\mathbf{r}^n}^n f(\%_A)}{n!}. \end{aligned} \quad (2.49)$$

Hence

$$\frac{n!}{2\pi} vt \int_{\gamma} \frac{f(\mathbf{r} + \%_A)}{\mathbf{r}^{n+1}} d\mathbf{r} = \hat{\mathbf{e}} \partial_{\mathbf{r}^n}^n f(\%_A), \quad (2.50)$$

since $vt \int_{\gamma} \mathbf{r}^{-n} d\mathbf{r} = \mathbf{0}$, whenever $n \geq 2$. This is the vector analogue of the well-known *Cauchy's* integral formula.

If some vector field $F = F\mathbf{r}_0 + F_{\perp}\mathbf{r}_{0\perp}$ is such that the scalar fields F and F_{\perp} have continuous first partial derivatives in region G , bounded by the closed smooth *Jordan* curve γ , almost everywhere (everywhere except on the singular set $S \subset G$), then by applying the vector integral operator (2.30) to the vector field F , one comes to the following vector integral identity

$$\begin{aligned} vt \int_{\gamma^+} dF &= vt \int_{\gamma^+} (\partial_{\mathbf{r}} F d\mathbf{r} + \partial_{\mathbf{r}} F d\bar{\mathbf{r}}) = vt \int_{G^+} (\partial_{\mathbf{r}\mathbf{r}}^2 F - \partial_{\mathbf{r}\mathbf{r}}^2 F)(d\bar{\mathbf{r}} \wedge d\mathbf{r}) = \\ &= (2\pi \hat{\mathbf{e}}) \left[\sum_{i=1}^{n_{\gamma}} \text{Res}(\partial_{\mathbf{r}} F, r_i) + \sum_{j=1}^{n_G} \text{Res}(\partial_{\mathbf{r}} F, \%_j) + \sum_{i=1}^{n_{\gamma}} \overline{\text{Res}(\partial_{\mathbf{r}} F, r_i)} + \sum_{j=1}^{n_G} \overline{\text{Res}(\partial_{\mathbf{r}} F, \%_j)} \right], \end{aligned} \quad (2.51)$$

since

$$\partial_{\varphi r}^2 F - \partial_{r\varphi}^2 F = \mathbf{r}_0(\partial_{\varphi r}^2 F - \partial_{r\varphi}^2 F) + \mathbf{r}_{0\perp}(\partial_{\varphi r}^2 F_{\perp} - \partial_{r\varphi}^2 F_{\perp}) = \mathbf{0}. \quad (2.52)$$

Consequently,

$$\begin{aligned} \partial_{\mathbf{r}\mathbf{r}}^2 F &= \partial_{\mathbf{r}\mathbf{r}}^2 F = \frac{1}{2} \partial_{\mathbf{r}}(\bar{\mathbf{r}}_0 \mathbf{r}_0 \text{div} F + \text{curl} F \times \bar{\mathbf{r}}_0 \mathbf{r}_0) = \\ &= \frac{r\partial_r(r\partial_r F) + \partial_{\varphi^2}^2 F}{4r^2} = \frac{\text{div grad} F}{4}. \end{aligned} \quad (2.53)$$

Clearly, in the general case, $\text{curl grad} F = 4(\bar{\mathbf{r}}\partial_{\mathbf{r}} \times \bar{\mathbf{r}}\partial_{\mathbf{r}})F/r^2 = \mathbf{0}$ is not the same as $\text{curl}(\bar{\mathbf{r}}_0 \text{grad} F) = 4(\bar{\mathbf{r}}\partial_{\mathbf{r}} \times \bar{\mathbf{r}}_0 \partial_{\mathbf{r}} F)/r^2$. Namely,

$$\frac{\mathbf{r}_0}{r^2} (\bar{\mathbf{r}}\partial_{\mathbf{r}} \circ \bar{\mathbf{r}}_0 \text{grad} F) = \frac{\mathbf{r}_0}{2} [\partial_{r^2}^2 F + \frac{1}{r^2} (\partial_{\varphi^2}^2 F - \partial_{\varphi} F_{\perp})] \text{ and} \quad (2.54)$$

$$\frac{\mathbf{r}_{0\perp}}{r^2} (\bar{\mathbf{r}}\partial_{\mathbf{r}} \wedge \bar{\mathbf{r}}_0 \text{grad} F) = \frac{\mathbf{r}_{0\perp}}{2} [\partial_{r^2}^2 F_{\perp} + \frac{1}{r^2} (\partial_{\varphi^2}^2 F_{\perp} + \partial_{\varphi} F)]. \quad (2.55)$$

So, $\text{div grad} F = 4(\bar{\mathbf{r}}\partial_{\mathbf{r}} \cdot \bar{\mathbf{r}}\partial_{\mathbf{r}})F/r^2$ differs from $\text{div}(\bar{\mathbf{r}}_0 \text{grad} F) = 4(\bar{\mathbf{r}}\partial_{\mathbf{r}} \cdot \bar{\mathbf{r}}_0 \partial_{\mathbf{r}} F)/r^2$. Accordingly,

$$\begin{aligned} 4\partial_{\mathbf{r}\mathbf{r}}^2 F &= \text{div grad} F = \frac{2\mathbf{r}_0}{r^2} (\bar{\mathbf{r}}\partial_{\mathbf{r}} \circ \bar{\mathbf{r}}_0 \text{grad} F + \bar{\mathbf{r}}\partial_{\mathbf{r}} \wedge \bar{\mathbf{r}}_0 \text{grad} F) + \\ + \frac{\bar{\mathbf{r}}_0}{r} \text{grad} F &= \mathbf{r}_0 \text{div}(\bar{\mathbf{r}}_0 \text{grad} F) + \text{curl}(\bar{\mathbf{r}}_0 \text{grad} F) \times \mathbf{r}_0 + \frac{\bar{\mathbf{r}}_0}{r} \text{grad} F, \end{aligned} \quad (2.56)$$

since

$$\begin{aligned} \mathbf{r}_0(\bar{\mathbf{r}}_0\bar{\partial}_{\bar{\mathbf{r}}} \circ \bar{\mathbf{r}}_0\bar{\mathbf{r}}_0 \text{grad } F + \bar{\mathbf{r}}_0\bar{\partial}_{\bar{\mathbf{r}}} \wedge \bar{\mathbf{r}}_0\bar{\mathbf{r}}_0 \text{grad } F) &= \mathbf{r}_0[\mathbf{r}_0\bar{\partial}_{\bar{\mathbf{r}}}(\bar{\mathbf{r}}_0\bar{\mathbf{r}}_0 \text{grad } F)] = \\ &= -\frac{\bar{\mathbf{r}}_0}{r} \text{grad } F + \bar{\partial}_{\bar{\mathbf{r}}} \text{grad } F \text{ and} \end{aligned} \quad (2.57)$$

$$2\bar{\partial}_{\bar{\mathbf{r}}}(\bar{\mathbf{r}}_0 \text{grad } F) = \bar{\mathbf{r}}_0\mathbf{r}_0 \text{div}(\bar{\mathbf{r}}_0 \text{grad } F) + \text{curl}(\bar{\mathbf{r}}_0 \text{grad } F) \times \bar{\mathbf{r}}_0\mathbf{r}_0, \quad (2.58)$$

which can be explicitly obtained if in (2.9) F is formally replaced by $\bar{\mathbf{r}}_0 \text{grad } F$. Therefore, the two identities 5. and 6., on page 85., in Section 3.16., Chapter 3., in [15], should be replaced by: 5. $\text{curl grad } F \equiv \mathbf{0}$ and 6. $\text{div grad } F = \mathbf{0}$ if F is either an analytic vector field ($\bar{\partial}_{\bar{\mathbf{r}}}F = \mathbf{0}$) or a Laplace vector field ($\bar{\partial}_{\bar{\mathbf{r}}}F = \mathbf{0}$). In both of these cases, the vector field F satisfies Laplace's equation $r\partial_r(r\partial_r F) = -\partial_{\varphi^2}^2 F$.

Consequently,

$$\begin{aligned} vt \int_{\gamma^+}^{\circ} \frac{\bar{\mathbf{r}}_0}{2} (\mathbf{d}F + \mathbf{d}_{\perp}F) &= vt \int_{\gamma^+}^{\circ} \bar{\partial}_{\bar{\mathbf{r}}}F d\mathbf{r} = \\ &= vt \int_{G^+} \bar{\partial}_{\bar{\mathbf{r}}}^2 F (\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) = \frac{\bar{\mathbf{r}}_0\mathbf{r}_{0\perp}}{2} vt \int_{G^+} r \text{div grad } F d\mathbf{r} d\varphi. \end{aligned} \quad (2.59)$$

On the other hand, let $F = F_{\mathbf{r}_0} + F_{\perp}\mathbf{r}_{0\perp}$ be continuous in an arbitrary region G bounded by a closed smooth Jordan curve γ , in which the partial derivatives $\partial_r F$, $\partial_{\varphi} F$, $\partial_r F_{\perp}$ and $\partial_{\varphi} F_{\perp}$ exist and satisfy the Cauchy-Riemann equations

$$\partial_r F = \frac{1}{r} \partial_{\varphi} F_{\perp} \text{ and } \partial_r F_{\perp} = -\frac{1}{r} \partial_{\varphi} F. \quad (2.60)$$

Then, according to the Looman-Menchoff theorem [1], both the analytic vector field $\bar{\mathbf{r}}_0 F$ and the Laplace vector field $\mathbf{r}_0 \bar{F}$ can be said to be regular (holomorphic) vector fields in G . Therefore, on the basis of (2.50),

$$(2\pi\hat{\mathbf{e}})^{-1} \int_{\gamma}^{\circ} \frac{\bar{\mathbf{r}}_0 F}{\mathbf{r}^{n+1}} d\mathbf{r} = \text{Res}\left(\frac{\bar{\mathbf{r}}_0 F}{\mathbf{r}^{n+1}}, \mathbf{0}\right) = \frac{\lim_{\mathbf{r} \rightarrow \mathbf{0}} \bar{\partial}_{\bar{\mathbf{r}}}^n(\bar{\mathbf{r}}_0 F)}{n!}. \quad (2.61)$$

In addition,

$$(2\pi\hat{\mathbf{e}})^{-1} \left[\int_{\gamma}^{\circ} \frac{F}{\mathbf{r}} d\mathbf{r} - vp \int_G \frac{\bar{\mathbf{r}}_0 \text{grad } F}{r} d\mathbf{S} \right] = F(\mathbf{0}) \text{ and} \quad (2.62)$$

$$(2\pi\hat{\mathbf{e}})^{-1} \left[\int_{\gamma}^{\circ} \frac{\bar{F}}{\bar{\mathbf{r}}} d\bar{\mathbf{r}} + vp \int_G \frac{\mathbf{r}_0 \text{div } \bar{F} + \text{curl } \bar{F} \times \mathbf{r}_0}{r} d\mathbf{S} \right] = -\bar{F}(\mathbf{0}), \quad (2.63)$$

where $\bar{\mathbf{r}} \text{grad } F = -F$ and $\mathbf{r} \text{div } \bar{F} - \mathbf{r} \times \text{curl } \bar{F} = -\bar{F}$. These vector integral formulas are analogous to the Cauchy-Pompeiu integral formula of complex analysis [17].

On the basis of the previous results one can say that there is a complete analogy between complex analysis in \mathbb{C} and real vector analysis in \mathbf{V}_2 , thus all the results of complex analysis are applicable to scalar and vector fields in \mathbf{V}_2 and vice versa. In doing so, z is formally replaced by \mathbf{r} , and the imaginary unit i , more precisely the i -vector \hat{i} , is replaced by the vector $\hat{\mathbf{e}}$ and vice versa ($z \rightleftharpoons \mathbf{r}$ and $\hat{i} \rightleftharpoons \hat{\mathbf{e}}$). This conclusion can be even more obvious if a formally analogous method of deriving previously obtained vector identities is applied to the field of complex vectors $\mathbf{V}_{\mathbb{C}}$, which corresponds to the i -vector field (field of complex numbers) \mathbb{C} , in the sense of the correspondence: $\mathbf{1} \equiv \mathbf{e}$ and $\hat{i} \equiv \hat{\mathbf{e}}$, where the unit vector \mathbf{e} and the pseudo-unit vector $\hat{\mathbf{e}}$ ($\hat{\mathbf{e}} \cdot \hat{\mathbf{e}} = -1$) form an orthogonal basis of the field of complex vectors $\mathbf{V}_{\mathbb{C}}$, whose algebraic structure is based on the geometric product of two complex vectors $\mathbf{r}_1 = a\mathbf{e} + b\hat{\mathbf{e}}$ and $\mathbf{r}_2 = c\mathbf{e} + d\hat{\mathbf{e}}$ as follows [9]

$$\begin{aligned} \bar{\mathbf{r}}_1 \mathbf{r}_2 &= \frac{1}{2}(\bar{\mathbf{r}}_1 \mathbf{r}_2 + \mathbf{r}_1 \bar{\mathbf{r}}_2) + \frac{1}{2}(\bar{\mathbf{r}}_1 \mathbf{r}_2 - \mathbf{r}_1 \bar{\mathbf{r}}_2) := (\mathbf{r}_1 \cdot \bar{\mathbf{r}}_2) \mathbf{e} - (\bar{\mathbf{r}}_1 \times \bar{\mathbf{r}}_2) \times \mathbf{e} = \\ &= (\bar{\mathbf{r}}_1 \cdot \mathbf{r}_2) \mathbf{e} + (\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} = [(a\mathbf{e} - b\hat{\mathbf{e}}) \cdot (c\mathbf{e} + d\hat{\mathbf{e}})] \mathbf{e} + [(a\mathbf{e} + b\hat{\mathbf{e}}) \times (c\mathbf{e} + d\hat{\mathbf{e}})] \times \mathbf{e}. \end{aligned} \quad (2.64)$$

3. Algebraic Structure of the Field of Vectors \mathbf{V}_3

The *Euclidean* space \mathbb{R}^3 consists of three *Euclidean* spaces \mathbb{R}^2 , which means that the field of vectors \mathbf{V}_3 , isomorphic to it, consists of three fields of vectors \mathbf{V}_2 , with base vectors \mathbf{e}_i and $\hat{\mathbf{e}}_i$ ($i = 1, 2, 3$), such that $\mathbf{e}_1 = \hat{\mathbf{e}}_3, \mathbf{e}_2 = \hat{\mathbf{e}}_1$ and $\mathbf{e}_3 = \hat{\mathbf{e}}_2$. Hence, the three plane vectors $\langle \mathbf{a} \rangle_i = \langle \mathbf{a}\mathbf{e} + \hat{\mathbf{a}}\hat{\mathbf{e}} \rangle_i$, where $a_i = \mathbf{a} \cdot \mathbf{e}_i$ and $\hat{a}_i = \mathbf{a} \cdot \hat{\mathbf{e}}_i$, are component vectors of an arbitrary spatial vector $\mathbf{a} \in \mathbf{V}_3$, such that $2\mathbf{a} = \langle \mathbf{a} \rangle_i \mathbf{e}^i$. An index repeated as subscript and superscript in a product represents summation over the range of the index, by the *Einstein* summation convention.

Definition 8. Let $\langle \mathbf{a} \rangle_i = \langle \mathbf{a}\mathbf{e} + \hat{\mathbf{a}}\hat{\mathbf{e}} \rangle_i$ and $\langle \mathbf{b} \rangle_i = \langle \mathbf{b}\mathbf{e} + \hat{\mathbf{b}}\hat{\mathbf{e}} \rangle_i$ be component vectors of two vectors \mathbf{a} and \mathbf{b} in \mathbf{V}_3 , respectively, such that $\bar{\mathbf{a}} = \langle \bar{\mathbf{a}} \rangle_i \mathbf{e}^i = \langle \mathbf{a}\mathbf{e} - \hat{\mathbf{a}}\hat{\mathbf{e}} \rangle_i \mathbf{e}^i = \mathbf{0}$. The commutative geometric product $\bar{\mathbf{a}}\mathbf{b}$ in \mathbf{V}_3 is defined as the sum of the geometric products of the component vectors $\langle \bar{\mathbf{a}} \rangle_i$ and $\langle \mathbf{b} \rangle_i$, as follows

$$\bar{\mathbf{a}}\mathbf{b} = \mathbf{a} \circ \mathbf{b} + \mathbf{a} \wedge \mathbf{b} = \frac{\langle \bar{\mathbf{a}}\mathbf{b} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}} = [\langle \mathbf{a} \circ \mathbf{b} \rangle_i + \langle \mathbf{a} \wedge \mathbf{b} \rangle_i] \frac{\mathbf{e}^i}{\sqrt[2]{3}}. \quad (3.1)$$

Since

$$\mathbf{a}\bar{\mathbf{b}} = \frac{\langle \mathbf{a}\bar{\mathbf{b}} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}} = \frac{\langle \bar{\mathbf{b}}\mathbf{a} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}} = [\langle \mathbf{b} \circ \mathbf{a} \rangle_i + \langle \mathbf{b} \wedge \mathbf{a} \rangle_i] \frac{\mathbf{e}^i}{\sqrt[2]{3}}, \quad (3.2)$$

it follows that $\mathbf{a} \circ \mathbf{b} = (\bar{\mathbf{a}}\mathbf{b} + \mathbf{a}\bar{\mathbf{b}})/2$ and $\mathbf{a} \wedge \mathbf{b} = (\bar{\mathbf{a}}\mathbf{b} - \mathbf{a}\bar{\mathbf{b}})/2$.

On the other hand, the vector

$$\mathbf{a}^{-1} = \frac{1}{\sqrt[2]{3}} \langle \mathbf{a}^{-1} \rangle_i \mathbf{e}^i, \quad (3.3)$$

where $\langle \mathbf{a}^{-1} \rangle_i = \langle \bar{\mathbf{a}} / \|\mathbf{a}\|^2 \rangle_i$ and $\|\mathbf{a}\|_i^2 = (a^2 + \hat{a}^2)_i$, such that

$$\mathbf{a}^{-1}\mathbf{a} = \frac{\langle \mathbf{a}^{-1}\mathbf{a} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}} = \frac{\mathbf{e}_i \mathbf{e}^i}{\sqrt[2]{3}}, \quad (3.4)$$

is the inverse vector of the spatial vector \mathbf{a} , which allows division by the vector in \mathbf{V}_3 . If $\langle \hat{a}\hat{b} - \hat{a}b \rangle$ is denoted by the bracket $[a, \hat{b}]$, then

$$\mathbf{a} \wedge \mathbf{b} = \frac{\langle \mathbf{a} \wedge \mathbf{b} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}} = \frac{\langle \mathbf{a} \times \mathbf{b} \rangle_i \times \mathbf{e}^i}{\sqrt[2]{3}} = \frac{[a, \hat{b}]_i \hat{\mathbf{e}}^i}{\sqrt[2]{3}} \text{ and} \quad (3.5)$$

$$\mathbf{a} \circ \mathbf{b} = \frac{\langle \mathbf{a} \circ \mathbf{b} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}} = \frac{\langle \mathbf{a} \cdot \mathbf{b} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}} = \frac{\langle ab + \hat{a}\hat{b} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}}. \quad (3.6)$$

3.1. Integral Identities in \mathbf{V}_3

Let $\mathbf{F} = \langle \mathbf{F} \rangle_i \mathbf{e}^i = \langle F\mathbf{r}_0 + F_\perp \mathbf{r}_{0\perp} \rangle_i \mathbf{e}^i$ be a spatial vector field, such that

$$\langle \mathbf{F}d\bar{\mathbf{r}} \rangle_i = \langle (Fdr + rF_\perp d\varphi)\mathbf{e} + (F_\perp dr - rFd\varphi)\hat{\mathbf{e}} \rangle_i. \quad (3.7)$$

For the $1\mathbb{D}$ vector differential forms in \mathbf{V}_3

$$\mathfrak{!}_{\bar{\mathbf{r}}} = \mathbf{F}d\bar{\mathbf{r}} = \frac{\langle \mathbf{F}d\bar{\mathbf{r}} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}} \text{ and } \mathfrak{!}_{\mathbf{r}} = \bar{\mathbf{F}}d\mathbf{r} = \frac{\langle \bar{\mathbf{F}}d\mathbf{r} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}}, \quad (3.8)$$

where $2d\mathbf{r} = \langle d\mathbf{r} \rangle_i \mathbf{e}^i$, the $2\mathbb{D}$ vector differential forms in \mathbf{V}_3 are as follows

$$d\mathfrak{!}_{\bar{\mathbf{r}}} = -2\bar{\partial}_{\mathbf{r}}F d\hat{\mathbf{S}} = -2\frac{\langle \bar{\partial}_{\mathbf{r}}F d\hat{\mathbf{S}} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}} \text{ and} \quad (3.9)$$

$$d\mathfrak{!}_{\mathbf{r}} = 2\bar{\partial}_{\mathbf{r}}\bar{F} d\hat{\mathbf{S}} = 2\frac{\langle \bar{\partial}_{\mathbf{r}}\bar{F} d\hat{\mathbf{S}} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}}, \quad (3.10)$$

where $2d\hat{\mathbf{S}}_i = \langle d\bar{\mathbf{r}} \wedge \bar{d}\mathbf{r} \rangle_i$.

According to *Theorem 1*,

$$\langle vt \int_{\partial\gamma^+} \mathbf{F} \circ d\mathbf{r} \rangle_i \mathbf{e}^i = -2 \langle vt \int_{G^+} (\bar{\partial}_{\bar{\mathbf{r}}} \wedge \mathbf{F}) d\hat{\mathbf{S}} \rangle_i \mathbf{e}^i \text{ and} \quad (3.11)$$

$$\langle vt \int_{\partial\gamma^+} \mathbf{F} \wedge d\mathbf{r} \rangle_i \mathbf{e}^i = 2 \langle vt \int_{G^+} (\bar{\partial}_{\bar{\mathbf{r}}} \circ \mathbf{F}) d\hat{\mathbf{S}} \rangle_i \mathbf{e}^i, \quad (3.12)$$

where the regions G_i , bounded by closed smooth *Jordan* curves γ_i , are the normal projections of the smooth surface \mathcal{S} in \mathbf{V}_3 onto \mathbf{V}_2^i , so that

$$vt \int_{\partial S^+} \bar{\mathbf{l}}_{\bar{\mathbf{r}}} = vt \int_{\partial S^+} \mathbf{F} d\bar{\mathbf{r}} = \frac{1}{\sqrt[2]{3}} \langle vt \int_{\gamma^+} \mathbf{F} d\bar{\mathbf{r}} \rangle_i \mathbf{e}^i = \quad (3.13)$$

$$= -\frac{2}{\sqrt[2]{3}} \langle vt \int_{G^+} \bar{\partial}_{\bar{\mathbf{r}}} \mathbf{F} d\hat{\mathbf{S}} \rangle_i \mathbf{e}^i = -2 \int_{S^+} \bar{\partial}_{\bar{\mathbf{r}}} \mathbf{F} d\hat{\mathbf{S}} = \int_{S^+} d\bar{\mathbf{l}}_{\bar{\mathbf{r}}} \text{ and}$$

$$vt \int_{\partial S^+} \bar{\mathbf{l}}_{\mathbf{r}} = vt \int_{\partial S^+} \bar{\mathbf{F}} d\mathbf{r} = \frac{1}{\sqrt[2]{3}} \langle vt \int_{\gamma^+} \bar{\mathbf{F}} d\mathbf{r} \rangle_i \mathbf{e}^i = \quad (3.14)$$

$$= \frac{2}{\sqrt[2]{3}} \langle vt \int_{G^+} \bar{\partial}_{\bar{\mathbf{r}}} \bar{\mathbf{F}} d\hat{\mathbf{S}} \rangle_i \mathbf{e}^i = 2 \int_{S^+} \bar{\partial}_{\bar{\mathbf{r}}} \bar{\mathbf{F}} d\hat{\mathbf{S}} = \int_{S^+} d\bar{\mathbf{l}}_{\mathbf{r}}.$$

Furthermore, since, see (2.9),

$$\langle (\bar{\partial}_{\bar{\mathbf{r}}} \bar{\mathbf{F}} - \bar{\partial}_{\bar{\mathbf{r}}} \mathbf{F}) d\hat{\mathbf{S}} \rangle_i \cdot \mathbf{e}^i = \langle (\text{curl } \mathbf{F} \times \mathbf{e}) \circ d\hat{\mathbf{S}} \rangle_i \cdot \mathbf{e}^i = \quad (3.15)$$

$$= \langle \text{curl } \mathbf{F} \rangle_i \cdot \langle \mathbf{e} \times d\hat{\mathbf{S}} \rangle^i = \langle \text{curl } \mathbf{F} \rangle^i \cdot d\mathbf{S}_i = \text{curl } \mathbf{F} \cdot d\mathbf{S} \text{ and}$$

$$\mathbf{e}^i \times \langle (\bar{\partial}_{\bar{\mathbf{r}}} \bar{\mathbf{F}} + \bar{\partial}_{\bar{\mathbf{r}}} \mathbf{F}) d\hat{\mathbf{S}} \rangle_i = \mathbf{e}^i \times \langle (\text{div } \mathbf{F} \mathbf{e} \times d\hat{\mathbf{S}}) \times \mathbf{e} \rangle_i = \langle \text{div } \mathbf{F} \mathbf{e} \rangle^i \times d\hat{\mathbf{S}}_i, \quad (3.16)$$

the previous two vector identities leads to the generalized *Stokes* integral identity

$$vt \int_{\partial S^+} \mathbf{F} \cdot d\mathbf{r} = 2vt \int_{S^+} (\bar{\partial}_{\bar{\mathbf{r}}} \wedge \mathbf{F}) \cdot d\mathbf{S}, \quad (3.17)$$

where $\mathbf{F} \cdot d\mathbf{r} = \langle \mathbf{F} \circ d\mathbf{r} \rangle_i \cdot \mathbf{e}^i$ and $d\mathbf{S} = \mathbf{e}^i \times d\hat{\mathbf{S}}_i$, as well as to the vector integral identity

$$vt \int_{\partial S^+} \mathbf{F} \times d\mathbf{r} = \frac{2}{\sqrt[2]{3}} \langle vt \int_{S^+} \bar{\partial}_{\bar{\mathbf{r}}} \circ \mathbf{F} \rangle^i \times d\hat{\mathbf{S}}_i, \quad (3.18)$$

where $\mathbf{F} \times d\mathbf{r} = \mathbf{e}^i \times \langle \mathbf{F} \wedge d\mathbf{r} \rangle_i$. Obviously, the fundamental theorem in \mathbf{V}_3 is as follows

Theorem 2. Let $\bar{\mathbf{l}}_{\bar{\mathbf{r}}}$ and $d\bar{\mathbf{l}}_{\bar{\mathbf{r}}}$ be regular almost everywhere in $\mathcal{S} \subset \mathbf{V}_3$ (everywhere except on the finite set $S \subset \mathcal{S}$ of singular points). Then,

$$vt \int_{\partial S^+} \bar{\mathbf{l}}_{\bar{\mathbf{r}}} = vt \int_{S^+} d\bar{\mathbf{l}}_{\bar{\mathbf{r}}}. \quad (3.19)$$

On the other hand, for an arbitrary region V in the field \mathbf{V}_3 , bounded by a closed smooth surface ∂V , if an arbitrary vector field $\mathbf{F} = F\mathbf{e} + \hat{F}\hat{\mathbf{e}}$ satisfies the conditions of the divergence (*Gauss-Ostrogradsky*) theorem, then a procedure, similar to that for obtaining the integral identity (2.27), leads to the following integral identities

$$vt \int_{\partial V} \mathbf{F}_i (\mathbf{n}^i \cdot d\mathbf{S}) = vt \int_{\partial V} \langle \mathbf{F} d\mathbf{S} \rangle_i \mathbf{e}^i = 2vt \int_V \langle \bar{\partial}_{\bar{\mathbf{r}}} \circ F\mathbf{e} + \bar{\partial}_{\bar{\mathbf{r}}} \wedge \hat{F}\hat{\mathbf{e}} \rangle_i \mathbf{e}^i dV \text{ and} \quad (3.20)$$

$$vt \int_{\partial V} \langle \mathbf{F} \times \mathbf{n} \rangle_i (\mathbf{n}^i \cdot d\mathbf{S}) = -vt \int_{\partial V} \langle \mathbf{F} d\hat{\mathbf{S}} \rangle_i \mathbf{e}^i = 2vt \int_V \langle \bar{\partial}_{\bar{\mathbf{r}}} \circ \hat{F}\hat{\mathbf{e}} + \bar{\partial}_{\bar{\mathbf{r}}} \wedge F\mathbf{e} \rangle_i \mathbf{e}^i dV, \quad (3.21)$$

where $\mathbf{n}_i = \langle \mathbf{e} \times \hat{\mathbf{e}} \rangle_i$ and $d\mathbf{S}_i = \langle \hat{\mathbf{e}} \times (\mathbf{e} \times d\hat{\mathbf{S}}) \rangle_i$. Therefore, since $d\bar{\mathbf{S}}_i = \langle d\mathbf{S} - d\hat{\mathbf{S}} \rangle_i$ it follows that

$$vt \int_{\partial V}^{\circ} \mathbf{F} d\bar{\mathbf{S}} = 2vt \int_V \bar{\partial}_r \mathbf{F} dV. \quad (3.22)$$

In addition, if $2\langle \bar{\partial}_r \circ \mathbf{F} \rangle_i \cdot \mathbf{e}^i = \operatorname{div} \mathbf{F}$ and $\mathbf{e}^i \times 2\langle \bar{\partial}_r \wedge \mathbf{F} \rangle_i = \operatorname{curl} \mathbf{F}$, then

$$vt \int_{\partial V^+}^{\circ} \mathbf{F} \cdot d\mathbf{S} = vt \int_{V^+} \operatorname{div} \mathbf{F} dV \text{ and} \quad (3.23)$$

$$vt \int_{\partial V^+}^{\circ} d\mathbf{S} \times \mathbf{F} = vt \int_{V^+} \operatorname{curl} \mathbf{F} dV, \quad (3.24)$$

which is a generalization of the well-known integral formulas in field theory. In this acute case, $\mathbf{F} d\bar{\mathbf{S}}$ is the vector differential form $!_{\bar{\mathbf{S}}}$, such that

$$d!_{\bar{\mathbf{S}}} = 2\bar{\partial}_r \mathbf{F} dV = 2 \frac{\langle \bar{\partial}_r \mathbf{F} \rangle_i \mathbf{e}^i}{\sqrt[2]{3}} dV. \quad (3.25)$$

Consequently, another fundamental theorem in \mathbf{V}_3 can be formulated as follows

Theorem 3. Let $!_{\bar{\mathbf{S}}}$ and $d!_{\bar{\mathbf{S}}}$ be regular almost everywhere in $V \subset \mathbf{V}_3$ (everywhere except on the finite set $S \subset V$ of singular points). Then,

$$vt \int_{\partial V^+}^{\circ} !_{\bar{\mathbf{S}}} = vt \int_{V^+} d!_{\bar{\mathbf{S}}}. \quad (3.26)$$

4. Conclusion

Based on the integral identities obtained in the previous sections, all fundamental integral identities, from *Cauchy's* integral identity of complex analysis, through the *Kelvin-Stokes (Green's)* integral identity and the *Stokes* integral identity, as well as *Gauss-Ostrogradsky's* integral identity, all the way to the *Newton-Leibniz* formula [13], can be expressed by one vector integral identity

$$vt \int_{\partial \Omega^+}^{\circ} ! = vt \int_{\Omega^+} d!, \quad (4.1)$$

where $\partial \Omega$ is the boundary of the corresponding compact region Ω in \mathbf{V}_3 . For the *Newton-Leibniz* formula, the vector differential form $!$ is the vector field $\mathbf{F} = F\mathbf{e}$, such that

$$\Delta \mathbf{F}(I) = [F(b^+) - F(a^-)] \mathbf{e} = vt \int_{\partial I^+}^{\circ} ! \quad (4.2)$$

is an interval vector field and $d!(\%) = 2\bar{\partial}_r \mathbf{F}(\%) d\mathbf{r} = [\lim_{I \rightarrow \%} (\Delta \mathbf{F} / \tau_I)] d\mathbf{r}$, where I is some compact interval $[a, b]$ of the real line \mathbb{R} and $\tau_I = \mu_I \mathbf{e}$ is the vector *Lebesgue* measure of I .

So, in (4.1)

$$d!(\%) = \lim_{\Omega \rightarrow \%} \int_{\partial \Omega}^{\circ} !. \quad (4.3)$$

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