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Article

Scalar Field Static Spherically Symmetric Solutions in Teleparallel $F(T)$ Gravity

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Abstract: We investigate in this paper the static radial coordinate dependent spherically symmetric spacetime in teleparallel $F(T)$ gravity for a scalar field source. We begin by setting the static field equations (FEs) to be solved and solve the conservation laws for scalar field potential solutions. We simplify the FEs and then find a general formula for computing the new teleparallel $F(T)$ solutions applicable for any scalar field potential $V(T)$ and coframe ansatz. We compute new non-trivial teleparallel $F(T)$ solutions by using a power-law coframe ansatz for each scalar potential case arising from the conservation laws. We apply this formula to find new exact teleparallel $F(T)$ solutions for several cases of coframe ansatz parameter. The new $F(T)$ solution classes may be relevant for various astrophysical applications inside any dark energy (DE) source described by a fundamental scalar field such as quintessence, phantom energy or quintom system to name only those types.

Keywords: teleparallel $F(T)$ gravity; static scalar field sources; scalar field sources; static spherically symmetric teleparallel spacetimes; dark energy sources; astrophysical systems in dark energy

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1. Introduction

The Teleparallel $F(T)$ gravity is a frame-based theory and constitutes a promising alternative theory to General Relativity (GR) [1–7]. This is defined in terms of spacetime torsion tensor $T^a_{\ bc}$ and torsion scalar T, and its physical quantities are all defined in terms of the coframe $\mathbf{h}^a$ and the spin-connection $\omega^a_{\ bc}$ (and derivatives). We remind that the GR is defined in terms of the metric $g_{\mu\nu}$ and the spacetime curvatures $R^a_{\ b\mu\nu}$, $R_{\mu\nu}$ and R. One feature of teleparallel gravity and its frame-based approach is the new possible spacetime symmetries, especially for non-trivial linear isotropy groups, and its Lorentz-invariant geometries [8–10]. There is thus an approach to determine the symmetries for any independent coframe/spin-connection pairs while treating spacetime curvature and torsion as geometric objects [4,5,11]. From here, any geometry described by a coframe/spin-connection whose curvature and non-metricity are both zero ($R^a_{\ b\mu\nu} = 0$ and $Q_{a\mu\nu} = 0$ conditions) is a teleparallel geometry. Covariantly, this type of geometry is defined as gauge invariant (valid for any g_{ab}). The coframe/spin-connection pairs must satisfy two Lie derivative based relations and we use the Cartan-Karlhede algorithm to solve these two fundamental equations for any teleparallel geometry. For a pure teleparallel $F(T)$ gravity spin-connection solution, we also solve the null Riemann curvature condition leading to a Lorentz transformation based definition of the spin-connection $\omega^a_{\ b\mu}$.

There is a direct equivalent to GR in teleparallel gravity called the Teleparallel Equivalent to GR (TEGR or $F(T) = T$ theory) [1]. The TEGR generalizes to teleparallel $F(T)$ gravity with a function $F(T)$ of the torsion scalar T [12–14]. The Teleparallel $F(T)$-type gravity is locally invariant under the covariant Lorentz definition [7]. In addition to the teleparallel $F(T)$-type gravity, all the previous considerations have been adapted for extended theories as New General Relativity (NGR) (refs. [15–17] and references therein), symmetric teleparallel $F(Q)$-type gravity (refs. [18–21] and references therein) and several extended theories like $F(T,Q)$-type, $F(R,Q)$-type, $F(R,T)$-type, and others (refs. [22–33] and references therein). Therefore, the best approach is to work with the teleparallel $F(T)$ gravity for the current development.

There have been several works and papers on spherically symmetric spacetimes and its solutions in teleparallel $F(T)$ gravity using various approaches and for several types of uses [34–59]. However, the resolution of field equations (FEs) in the orthonormal gauge is done for reasons of coframe/spin-connection pair determinations and for avoiding the extra degrees of freedom (DoF) problem linked to the FEs on proper frames (refs. [35–38] for detailed discussions). However, the symmetric FEs and the $F(T)$ solutions are very similar between different gauges (orthonormal or not); this further confirms the gauge invariance of general teleparallel $F(T)$ gravity FEs. But by opting for the orthonormal gauge, we solve the non-trivial antisymmetric and symmetric parts of FEs covering all DoFs for a non-trivial and orthonormal coframe/spin-connection pair (see refs. [35–39]). In the literature, a lot of papers use tetrads ${e^a}_\mu$ instead of coframes ${h^a}_\mu$, because they are defined in an orthonormal gauge by default [42–55,59]. But the tetrad approach prevents the use of non-orthonormal frames and constitutes the major weakness of the latter approach. To avoid this type of limitation, to respect gauge invariance as well as to work on any type of frame non-necessarily orthonormal, the coframe notation ${h^a}_\mu$ will be used here and has been used in refs. [35–40]. By the last approach, we especially solved the static spherically symmetric (radial coordinate dependent) teleparallel $F(T)$ gravity FEs for any linear and quadratic perfect fluid source [36]. But there are some additional developments on cosmological and time-dependent spherically symmetric teleparallel $F(T)$ and $F(T,B)$ solutions leading to dark energy (DE) source models in cosmic background (see refs. [37–41,60]). For this last reason, it is relevant to consider the static spherically symmetric teleparallel $F(T)$ solutions with a scalar field source to complete the most recent studies on this domain. Beyond the spacetime structures and possible teleparallel solutions, there are further physical justifications and a lot of possible applications.

The most important classes of applications to the static radial coordinate dependent scalar field source teleparallel $F(T)$ solutions are the astrophysical problems (refs. [47–58] for good examples). This especially concerns the black hole (BH), Neutron Star (NS), White Dwarf (WD) and any system containing the previous types of astrophysical objects or any radial centered systems. The last list is so far not exhaustive. The recent literature has showed that any astrophysical system is evolving in a DE dominated universe [61–82]. We have taken into account of this fact in a recent works in teleparallel $F(T)$ gravity by considering a DE perfect fluids dominating the spacetime in ref. [36]. Several recent papers in teleparallel cosmology are done in the scope of DE dominating models [37–40,60,83–86]. We need to take into account of DE perfect fluids and/or scalar field for any astrophysical process in teleparallel gravity in the future. This relevant approach constitutes a more realistic manner to study the astrophysical processes in the universe. The primary aim of this paper is to find new classes of teleparallel $F(T)$ gravity solutions in the scope of treating astrophysical systems evolving in a DE fundamental scalar field. This paper naturally follows the work done and directly complete the analysis done in ref. [36]. But before going to the new possible teleparallel $F(T)$ solutions, we need to further clarify on the types of DE.

The DE and traditionally the scalar field quintessence behavior is usually studied by using the perfect fluid equivalent equation of state (EoS) $P_\phi = \alpha_Q\, \rho_\phi$ where α_Q is the DE index (or quintessence index in some references.). According to this index, the possible DE forms can be summarized as:

1. **Quintessence $-1 < \alpha_Q < -\frac{1}{3}$**: This describes a controlled accelerating universe expansion where energy conditions are always satisfied, i.e., $P_\phi + \rho_\phi > 0$ [61–73].
2. **Phantom energy $\alpha_Q < -1$**: This usually describes an uncontrolled universe expansion accelerating toward a Big Rip event [74–82]. The energy condition is violated, i.e., $P_\phi + \rho_\phi \ngtr 0$.
3. **Cosmological constant $\alpha_Q = -1$**: This is an intermediate limit between the two previous and main types of DE, where $P_\phi + \rho_\phi = 0$. A constant scalar field source $\phi = \phi_0$ will directly lead to this case.
4. **Quintom models:** This is a mixture of previous DE types, usually described by some double scalar field models [87–92]. This type of model is more complete to study and solve in general. Several types of models are in principle possible and these physical processes need further studies in the future.

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The previous DE forms can be fundamentally defined in terms of the same scalar field with specific conditions. However the main interest of the current paper remains the new classes of teleparallel $F(T)$ solutions for static spherically symmetric spacetimes inside a scalar field, which may be useful for astrophysical purposes.

The paper is developed as follows. The section 2 will summarize the teleparallel $F(T)$ gravity theory, the conservation laws, the used coframe/spin-connection pair and the static spherically symmetric FEs. The section 3 will be for solving the unified FE in general and to find the teleparallel $F(T)$ solutions for power-law scalar field sources. The section 4 will focus on exponential, logarithmic and other scalar field source teleparallel $F(T)$ solutions. We will finish by the concluding remarks with future work recommendations in section 5.

2. Summary Of Teleparallel Gravity And Field Equations

2.1. Summary Of Teleparallel Field Equations

The teleparallel action integral is for any gravitational source [1–5,35–40]:

$$S_{F(T)} = \int d^4\,x \left[\frac{h}{2\kappa}\,F(T) + \mathcal{L}_{Source}\right], \tag{1}$$

where κ is the coupling constant and h is the coframe determinant. By applying the least-action principle to the Equation (1), we find the general FEs in terms of coframe and spin-connection [1–5,39]:

$$\kappa\,\Theta_a{}^{\mu} = h^{-1}\,F_T\,\partial_\nu\left(h\,S_a{}^{\mu\nu}\right) + F_{TT}\,S_a{}^{\mu\nu}\,\partial_\mu T + \frac{F}{2}\,h_a{}^{\mu} - F_T\left(T^b{}_{a\nu} + \omega^b{}_{a\nu}\right)S_b{}^{\mu\nu}, \tag{2}$$

where $F = F(T)$, $F_T = \frac{dF(T)}{dT}$, $F_{TT} = \frac{d^2F(T)}{dT^2}$, $\Theta_a{}^{\mu}$ is the energy–momentum, T the torsion scalar, $T^b{}_{a\nu}$ the torsion tensor, $h_a{}^{\mu}$ the coframe (or tetrad for orthonormal frames), $\omega^b{}_{a\nu}$ the spin-connection, and $S_a{}^{\mu\nu}$ the superpotential (torsion dependent). From Equation (2), we find the symmetric and antisymmetric parts of FEs [5,35–40]:

$$\kappa\,\Theta_{(ab)} = F_T\overset{\circ}{G}_{ab} + F_{TT}\,S_{(ab)}{}^{\mu}\,\partial_\mu T + \frac{g_{ab}}{2}\,[F - T\,F_T], \tag{3}$$

$$0 = F_{TT}(T)\,S_{[ab]}{}^{\mu}\,\partial_\mu T, \tag{4}$$

where $\overset{\circ}{G}_{ab}$ the Einstein tensor and g_{ab} the gauge metric. The canonical Energy-Momentum and its conservation laws (same as GR) are obtained from $\mathcal{L}_{Source}$ term of Equation (1) by the least-action principle and defined as [1,2,36–40,60]:

$$\Theta_a{}^{\mu} = \frac{1}{h}\frac{\mathcal{L}_{Source}}{\delta h^a{}_{\mu}}, \quad \Rightarrow \quad \overset{\circ}{\nabla}_{\nu}(\Theta^{\mu\nu}) = 0, \tag{5}$$

where $\overset{\circ}{\nabla}_{\nu}$ is th covariant derivative and $\Theta^{\mu\nu}$ is the conserved energy-momentum tensor. The antisymmetric and symmetric parts of Equation (5) are [5,35–40,60]:

$$\Theta_{[ab]} = 0, \qquad \Theta_{(ab)} = T_{ab}, \tag{6}$$

where T_{ab} is the symmetric part of energy-momentum tensor. In Equation (6), the Θ_{ab} is a symmetric physical quantity. The Equation (6) is valid when the source field interacts with the metric $g_{\mu\nu}$ associated with the coframe $h^a{}_{\mu}$ and the gauge g_{ab}, and is not intricately coupled to the $F(T)$ gravity. This consideration is valid only under the null hypermomentum condition defined by [36,37,39,40,44]:

$$\mathfrak{T}_{ab} = \kappa\Theta_{ab} - F_T\overset{\circ}{G}_{ab} - F_{TT}\,S_{ab}{}^{\mu}\,\partial_\mu T - \frac{g_{ab}}{2}\,[F - T\,F_T] = 0. \tag{7}$$

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In non-zero hypermomentum situations ($\mathfrak{T}^{\mu\nu} \neq 0$ case), we will need to satisfy more complex conservation law equations than Equation (5) as showed in refs. [44,93–95].

2.2. Static Spherically Symmetric Coframe And Spin-Connection Components

In the orthonormal gauge $g_{ab} = \eta_{ab} = Diag[-1,1,1,1]$, the relations to satisfy for any teleparallel geometry are [11]:

$$\mathcal{L}_{\mathbf{X}}\mathbf{h}^a = \lambda^a{}_b\mathbf{h}^b \quad \text{and} \quad \mathcal{L}_{\mathbf{X}}\omega^a{}_{bc} = 0, \tag{8}$$

where $\mathbf{h}^a$ is the coframe basis, $\mathcal{L}_{\mathbf{X}}$ is the Lie derivative in terms of Killing Vectors (KV) $\mathbf{X}$, and $\lambda^a{}_b$ is the generator of Lorentz transformations $\Lambda^a{}_b$. For a pure teleparallel geometry, one must also satisfy the zero Riemann curvature requirement (refs. [1–7] and references therein):

$$\begin{aligned} R^a{}_{b\mu\nu} =& \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{e\mu}\omega^e{}_{b\nu} - \omega^a{}_{e\nu}\omega^e{}_{b\mu} = 0, \\ \Rightarrow & \quad \omega^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu \Lambda_b^c. \end{aligned} \tag{9}$$

The solution of Equation (9) leads to the teleparallel spin-connection defined in terms of a Lorentz transformation $\Lambda^a{}_b$. Note also that $\omega^a{}_{b\mu} = 0$ for all proper frames and that $\omega^a{}_{b\mu} \neq 0$ for all non-proper frames.

The teleparallel spherically symmetric spacetimes were defined and discussed in detail with all necessary justifications in refs. [35,36]. The static r-dependent spherically symmetric orthonormal coframe expression is defined by [35,36]:

$$h^a{}_\mu = \text{Diag}[A_1(r),\, A_2(r),\, A_3(r),\, A_3(r)\sin(\theta)]. \tag{10}$$

The Equation (10) is an invariant symmetry frame. The antisymmetric FEs by assuming a spin-connection under the form $\omega_{abc} = \omega_{abc}(\chi(r), \psi(r))$ lead to $\chi = n\,\pi$ and $\psi = 0$ where $n \in \mathbb{Z}$ is an integer and $\cos\chi = \cos(n\,\pi) = \pm 1 = \delta$ [35,36]. We find as non-vanishing static r-dependent non-zero spin-connection components [35,36]:

$$\omega_{233} = \omega_{244} = \frac{\delta}{A_3(r)},\; \omega_{344} = -\frac{\cot(\theta)}{A_3(r)}. \tag{11}$$

The Equation (11) terms are also similar to those obtained in refs. [3,44]. The Equations (10)–(11) recover the five arbitrary functions required for a teleparallel geometry: A_1, A_2, A_3, ψ and χ [35,36]. We note that Equations (10)–(11) are also solutions to Equations (8)–(9), and similar results were been found for a spherically symmetric metric with a non-invariant proper frame approach [51,96–98].

2.3. Static Scalar Field Energy-Momentum Source

The scalar field source Lagrangian density $\mathcal{L}_{Source}$ is defined as [2,39,41,60,99]:

$$\mathcal{L}_{Source} = \frac{h}{2}\overset{\circ}{\nabla}_\nu\phi\,\overset{\circ}{\nabla}{}^\nu\phi - h\,V(\phi), \tag{12}$$

where $V(\phi)$ is the scalar potential. The perfect fluid tensor T_{ab} is defined as [99–102]:

$$T_{ab} = \left(P_\phi + \rho_\phi\right) u_a\, u_b + g_{ab}\, P_\phi, \tag{13}$$

where $u_a = (-1,\,0,\,0,\,0)$ for a stationary fluid, P_ϕ and ρ_ϕ are the pressure and density equivalents in terms of scalar field defined by [99–102]:

$$P_\phi = \frac{\phi'^2}{2} - V(\phi(r)) \quad \text{and} \quad \rho_\phi = \frac{\phi'^2}{2} + V(\phi(r)). \tag{14}$$

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where $\phi' = \partial_r \phi(r)$ and prime will be for derivatives. The DE index (or quintessence index) α_Q as follows [39,61–63,68,71–73]:

$$\alpha_Q = \frac{P_\phi}{\rho_\phi} = \frac{\phi'^2 - 2V(\phi)}{\phi'^2 + 2V(\phi)}, \tag{15}$$

From Equation (5) and by using Equations (14), the static conservation law equation is expressed by [35,36]:

$$\frac{dV}{d\phi} = \phi' \left(\ln A_1\right)' + \phi''. \tag{16}$$

The Equation (16) will be useful to determine the scalar potential expression for any $A_1(r)$ coframe component and scalar field $\phi(r)$ definition.

2.4. Static Scalar Field Source Field Equations

From the FEs components in appendix A, the symmetric FEs and torsion scalar expressions are [35,36]:

$$F_T = F_T(0) \exp\left[\int_{r(T)} dr' \frac{g_1(r')}{k_1(r')}\right], \tag{17}$$

$$\kappa \phi'^2 = 2 F_T \left[g_2(r) - \left(\frac{g_1(r)}{k_1(r)}\right) k_2(r)\right], \tag{18}$$

$$\kappa \left[\phi'^2 + 2V(\phi(r))\right] = -F + 4 F_T \left[g_3(r) - \left(\frac{g_1(r)}{k_1(r)}\right) k_2(r)\right], \tag{19}$$

$$T(r) = -2\left(\frac{\delta}{A_3} + \frac{A_3'}{A_2 A_3}\right)\left(\frac{\delta}{A_3} + \frac{A_3'}{A_2 A_3} + \frac{2 A_1'}{A_1 A_2}\right), \tag{20}$$

where g_i and k_i components are expressed by Equations (A1)–(A4) in appendix A in terms of A_1, A_2, A_3 (and derivatives in r). However, we can simplify and merge the Equations (17)–(19) to obtain a potential dependent equation leading to $F(T)$ solutions in terms of a $r(T)$ relation from Equation (20):

$$F(T) = -2\kappa V(T) + 2 F_T(0) \exp\left[\int_{r(T)} dr' \frac{g_1(r')}{k_1(r')}\right]\left[2 g_3(r(T)) - g_2(r(T)) - \left(\frac{g_1(r(T))}{k_1(r(T))}\right) k_2(r(T))\right], \tag{21}$$

where $V(T) = V(\phi(r(T)))$ is the scalar potential and $r(T)$ can be found by using the Equation (20). From the Equation (21), we will be able to compute any $F(T)$ for any coframe ansatz, a potential $V(T)$ satisfying the Equation (16) and by using the g_i and k_i components.

3. Power-Law Scalar Field Solutions

We will assume a power-law scalar field defined by:

$$\phi(T) = p_0 \left[r(T)\right]^p. \tag{22}$$

For the most of the coming developments, we will use the power-law coframe component ansatz defined as:

$$A_1(r) = a_0\, r^a, \qquad A_2(r) = b_0\, r^b. \tag{23}$$

By assuming a power-law coframe ansatz, we will solve the Equation (16) to find the scalar potential $V(\phi)$ by using Equations (22) and (23) for the following cases:

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1. **General:**

$$\frac{dV}{d\phi} = p\,(a+p-1)\,p_0^{2/p}\,\phi^{1-2/p},$$
$$\Rightarrow\ V(\phi) = V_0 + \frac{p^2\,p_0^{2/p}\,(a+p-1)}{2(p-1)}\,\phi^{2-2/p}. \tag{24}$$

By substituting Equation (22) into Equation (24), we find the $V(T)$ potential:

$$V(T) = V_0 + \frac{p^2\,p_0^2\,(a+p-1)}{2(p-1)}\,(r(T))^{2p-2}, \tag{25}$$

where $r(T)$ is the characteristic equation solution defined from Equation (20).

2. $\mathbf{p = 1}$:

$$\frac{dV}{d\phi} = a\,p_0^2\,\phi^{-1} \qquad \Rightarrow\ V(\phi) = V_0 + a\,p_0^2\,\ln\,\phi. \tag{26}$$

By substituting Equation (22) into Equation (26), we find as $V(T)$:

$$V(T) = V_0 + a\,p_0^2\,\ln\,p_0 + a\,p_0^2\,p\,\ln\,(r(T)), \tag{27}$$

where $r(T)$ is the characteristic equation solution defined from Equation (20).

3. $\mathbf{p \gg 1}$:

$$\frac{dV}{d\phi} = p^2\,\phi \qquad \Rightarrow\ V(\phi) = V_0 + \frac{p^2}{2}\,\phi^2. \tag{28}$$

By substituting Equation (22) into Equation (28), we find as $V(T)$:

$$V(T) = V_0 + \frac{p^2\,p_0^2}{2}\,(r(T))^{2p}, \tag{29}$$

where $r(T)$ is the characteristic equation solution defined from Equation (20). Equation (28) is a simple harmonic oscillator (SHO) potential. This last case is also valid for any coframe ansatz, not only the power-law case.

3.1. Power-Law Ansatz for $A_3 = c_0$

By using Equation (23) and setting $A_3 = c_0$, we find that the characteristic equation from the torsion scalar defined by Equation (20) is [36]:

$$0 = \frac{c_0^2}{2}\,T + 1 + \frac{2\delta\,c_0}{b_0}\,r^{-(1+b)} \qquad \Rightarrow\ r^{-(1+b)}(T) = -\frac{\delta\,b_0}{2c_0}\left(1 + \frac{c_0^2}{2}\,T\right). \tag{30}$$

In terms of Equation (30), we find that:

$$\phi(T) = \phi_0\left(1 + \frac{c_0^2}{2}\,T\right)^{-p/(1+b)}, \tag{31}$$

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where $\phi_0 = p_0 \left(-\frac{2\delta\, c_0}{b_0}\right)^{p/(1+b)}$. By using Equations (A5) in appendix A, the Equation (21) becomes:

$$F(T) = -2\kappa\, V(T) - \frac{2\, F_T(0)}{c_0^2}\left[2 - a\left(1+\frac{c_0^2}{2}\,T\right)\right]^{\frac{(2a-b-1)}{(b+1)}}\left(1+\frac{c_0^2}{2}\,T\right)^{-\frac{a}{(b+1)}}\exp\left[\frac{2}{(b+1)}\left(1+\frac{c_0^2}{2}\,T\right)^{-1}\right]$$
$$\times\left[\frac{\frac{a(1+a+b)}{4}\left(1+\frac{c_0^2}{2}\,T\right)^2 - a\left(1+\frac{c_0^2}{2}\,T\right) - 1}{\left[-\frac{a}{2}\left(1+\frac{c_0^2}{2}\,T\right)+1\right]}\right], \tag{32}$$

where $b \neq -1$. The potential $V(T)$ in Equation (32) from Equations (24)–(28) are for the following situations:

1. **General:** Equation (24) becomes:

$$V(T) = V_0 + \frac{p^2\, p_0^2\,(a+p-1)}{2(p-1)}\left(-\frac{2\delta\, c_0}{b_0}\right)^{2(p-1)/(1+b)}\left(1+\frac{c_0^2}{2}\,T\right)^{-2(p-1)/(1+b)}. \tag{33}$$

2. $\mathbf{p=1}$: Equation (26) becomes:

$$V(T) = \tilde{V}_0 - \frac{a\,p\,p_0^2}{(1+b)}\ln\left(1+\frac{c_0^2}{2}\,T\right), \tag{34}$$

   where $\tilde{V}_0 = V_0 + a\, p_0^2 \ln \phi_0$.

3. $\mathbf{p \gg 1}$: Equation (28) becomes:

$$V(T) = V_0 + \frac{p^2\, p_0^2}{2}\left(\frac{2\,c_0}{b_0}\right)^{2p/(1+b)}\left(1+\frac{c_0^2}{2}\,T\right)^{-2p/(1+b)}. \tag{35}$$

3.2. *Power-law ansatz for $A_3 = r$*

By using Equation (20) for $A_3 = r$, we find as general characteristic equation [36]:

$$0 = \frac{b_0^2\, T}{2} + b_0^2\, r^{-2} + 2\delta\, b_0\,(1+a)\, r^{-2-b} + (2\,a+1)\, r^{-2-2b}. \tag{36}$$

The Equation (21) in terms of Equation (A6) in appendix A is:

$$F(T) = -2\kappa\, V(T) + \frac{2\, F_T(0)}{b_0^2}\exp\left[\int_{r(T)} dr'\, r'^{b-1}\frac{\left[(2a-a^2+ab+b+1)\, r'^{-2b} - b_0^2\right]}{\left[(a+1)\, r'^{-b} + \delta\, b_0\right]}\right]$$
$$\times\left[(b-3a-2)\, r^{-2b-2}(T) - 2\delta\, b_0\,(a+1)\, r^{-b-2}(T) - \left(\frac{\left[(2a-a^2+ab+b+1)\, r^{-2b}(T) - b_0^2\right]}{\left[(a+1)\, r^{-b}(T) + \delta\, b_0\right]}\right)\right.$$
$$\left.\times\left(r^{-b}(T) + \delta\, b_0\right) r^{-2}(T)\right], \tag{37}$$

The Equation (37) leads to several solutions depending on the a and b values satisfying Equation (36). The analytical cases loeading to new teleparallel $F(T)$ solutions are for the cases (see ref. [36] for result comparisons):

1. $\mathbf{a=b=0}$: Equation (36) becomes:

$$0 = \frac{b_0^2\, T}{2} + (1+\delta\, b_0)^2\, r^{-2} \quad \Rightarrow\ r^{-2}(T) = \frac{b_0^2}{2(1+\delta\, b_0)^2}(-T) = C_{00}\,(-T). \tag{38}$$

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The scalar field $\phi(T)$ will be from Equation (22) definition:

$$\phi(T) = p_0 \left(\frac{\sqrt{2}(1+\delta\, b_0)}{b_0}\right)^p (-T)^{-p/2} = \phi_{00}\, (-T)^{-p/2}. \tag{39}$$

By substituting Equation (38) and (39) into Equation (37), we find that:

$$F(T) = -\, 2\kappa\, V(T) - \frac{2\, F_T(0)}{b_0^2} \frac{(3-\delta\, b_0)}{(1+\delta\, b_0)^{\delta\, b_0}} \left[\frac{b_0}{\sqrt{2}}\right]^{(1+\delta\, b_0)} (-T)^{\frac{(1+\delta\, b_0)}{2}}, \tag{40}$$

where $V(T)$ is for the following cases:

(a) **General:**

$$V(T) = V_0 + \frac{p^2\, p_0^2}{2} \left(\frac{\sqrt{2}(1+\delta\, b_0)}{b_0}\right)^{2p-2} (-T)^{1-p}. \tag{41}$$

(b) $\mathbf{p=1}$: $V(\phi(T)) = V_0 =$ constant. Equation (40) will be a power-law like $F(T)$ solution.

(c) $\mathbf{p \gg 1}$:

$$V(T) = V_0 + \frac{p^2\, p_0^2}{2} \left(\frac{\sqrt{2}(1+\delta\, b_0)}{b_0}\right)^{2p} (-T)^{-p}. \tag{42}$$

If we set $b_0 = \delta$, the Equation (39) simplifies as:

$$F(T) = -\, 2\kappa\, V(T) + F_T(0)\, T, \tag{43}$$

where $V(T)$ is:

(a) **General:**

$$V(\phi(T)) = V_0 + p^2\, p_0^2\, (2)^{3p-4}\, (-T)^{1-p}. \tag{44}$$

(b) $\mathbf{p=1}$: $V(\phi(T)) = V_0 =$ constant. Equation (43) will be a GR (TEGR-like) solution.

(c) $\mathbf{p \gg 1}$:

$$V(T) = V_0 + p^2\, p_0^2\, (2)^{3p-1}\, (-T)^{-p}. \tag{45}$$

2. $\mathbf{a} \neq \{-\mathbf{1}, -\frac{\mathbf{1}}{\mathbf{2}}\}$ and $\mathbf{b=0}$: Equation (22) becomes:

$$0 = \frac{b_0^2\, T}{2} + \left(b_0^2 + 2\delta\, b_0\, (1+a) + 2\, a + 1\right) r^{-2},$$
$$\Rightarrow r^{-2}(T) = \frac{b_0^2}{2\, \left(b_0^2 + 2\delta\, b_0\, (1+a) + 2\, a + 1\right)} (-T) = C_{a0}\, (-T), \tag{46}$$

where $T \leq 0$. The Equation (22) for scalar field is:

$$\phi(T) = p_0 \left(\frac{2^{p/2}\, \left(b_0^2 + 2\delta\, b_0\, (1+a) + 2\, a + 1\right)^{p/2}}{b_0^p}\right) (-T)^{-p/2} = p_0\, \phi_{a0}\, (-T)^{-p/2}. \tag{47}$$

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By substituting Equation (46) and (47) into Equation (37), we find that:

$$F(T) = -\,2\kappa\, V(T) + \frac{2\, F_T(0)}{b_0^2}\, C_{a0}^{\,1-\frac{\left[(2a-a^2+1)-b_0^2\right]}{2\left[(a+1)+\delta\, b_0\right]}} \Bigg[-\,(3a+2) - 2\delta\, b_0\,(a+1) - \left(\frac{\left[(2a-a^2+1)-b_0^2\right]}{\left[(a+1)+\delta\, b_0\right]}\right)(1+\delta\, b_0)\Bigg] (-T)^{1-\frac{\left[(2a-a^2+1)-b_0^2\right]}{2\left[(a+1)+\delta\, b_0\right]}}, \tag{48}$$

where the possible $V(T)$ are:

(a) **General:**

$$V(T) = V_0 + \frac{p^2\, p_0^2\,(a+p-1)}{2(p-1)}\, \phi_{a0}^{2-2/p}\,(-T)^{1-p}. \tag{49}$$

(b) $\mathbf{p = 1}$:

$$V(T) = V_0 + a\, p_0^2\, \ln\left[p_0\, \phi_{a0}\right] - \frac{a\, p\, p_0^2}{2}\, \ln(-T). \tag{50}$$

(c) $\mathbf{p \gg 1}$:

$$V(T) = V_0 + \frac{p^2\, p_0^2}{2}\, \phi_{a0}^2\,(-T)^{-p}. \tag{51}$$

3. $\mathbf{a} \neq \left\{-\mathbf{1}, -\frac{\mathbf{1}}{\mathbf{2}}\right\}$ and $\mathbf{b = 1}$: Equation (36) will be:

$$0 = \frac{b_0^2\, T}{2} + b_0^2\, r^{-2} + 2\delta\, b_0\,(1+a)\, r^{-3} + (2\,a+1)\, r^{-4}, \tag{52}$$

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The solutions are:

$$
\begin{aligned}
& r^{-1}(T) = \\
& \frac{\delta_1\, b_0(a+1)}{2(2a+1)} - \frac{\delta_1}{2}\Bigg[\Bigg(-\frac{2b_0^2}{3(2a+1)} + \frac{b_0^2(a+1)^2}{(2a+1)^2} \\
& + \frac{1}{6\sqrt[3]{2}(2a+1)}\Bigg(\Big(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\, b_0^4\, T \\
& + \sqrt{\Big(\big(16b_0^6 + (432(a+1)^2 - 288(2a+1))\, b_0^4\, T\big)^2 - 4b_0^6\big(4b_0^2 + 24(2a+1)\,T\big)^3\Big)}\Big)^{1/3}\Bigg) \\
& + \frac{2^{4/3} b_0^2\big(b_0^2 + 6\,(2a+1)\,T\big)}{3(2a+1)}\Big(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\, b_0^4\, T \\
& + \sqrt{\Big(\big(16b_0^6 + (432(a+1)^2 - 288(2a+1))\, b_0^4\, T\big)^2 - 4b_0^6\big(4b_0^2 + 24(2a+1)\,T\big)^3\Big)}\Big)^{-1/3}\Bigg)\Bigg]^{1/2} \\
& - \frac{\delta_2}{2}\Bigg[\Bigg(-\frac{4b_0^2}{3(2a+1)} + \frac{2b_0^2(a+1)^2}{(2a+1)^2} \\
& - \frac{1}{6\sqrt[3]{2}(2a+1)}\Bigg(\Big(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\, b_0^4\, T \\
& + \sqrt{\Big(\big(16b_0^6 + (432(a+1)^2 - 288(2a+1))\, b_0^4\, T\big)^2 - 4b_0^6\big(4b_0^2 + 24(2a+1)\,T\big)^3\Big)}\Big)^{1/3}\Bigg) \\
& + \frac{2^{4/3} b_0^2\big(b_0^2 + 6\,(2a+1)\,T\big)}{3(2a+1)}\Big(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\, b_0^4\, T \\
& + \sqrt{\Big(\big(16b_0^6 + (432(a+1)^2 - 288(2a+1))\, b_0^4\, T\big)^2 - 4b_0^6\big(4b_0^2 + 24(2a+1)\,T\big)^3\Big)}\Big)^{-1/3}\Bigg) \\
& + \delta_1\, \frac{8\delta\, b_0^3\, a^2\,(a+1)}{(2a+1)^3}\Bigg[4\Bigg[\Bigg(-\frac{2b_0^2}{3(2a+1)} + \frac{b_0^2(a+1)^2}{(2a+1)^2} \\
& + \frac{1}{6\sqrt[3]{2}(2a+1)}\Bigg(\Big(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\, b_0^4\, T \\
& + \sqrt{\Big(\big(16b_0^6 + (432(a+1)^2 - 288(2a+1))\, b_0^4\, T\big)^2 - 4b_0^6\big(4b_0^2 + 24(2a+1)\,T\big)^3\Big)}\Big)^{1/3}\Bigg) \\
& + \frac{2^{4/3} b_0^2\big(b_0^2 + 6\,(2a+1)\,T\big)}{3(2a+1)}\Big(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\, b_0^4\, T \\
& + \sqrt{\Big(\big(16b_0^6 + (432(a+1)^2 - 288(2a+1))\, b_0^4\, T\big)^2 - 4b_0^6\big(4b_0^2 + 24(2a+1)\,T\big)^3\Big)}\Big)^{-1/3}\Bigg)\Bigg]^{1/2}\Bigg]^{-1}\Bigg]^{1/2}, \qquad (53)
\end{aligned}
$$

where $(\delta_1, \delta_2) = (\pm 1, \pm 1)$. By substituting Equation (22) and (53) into Equation (37), we find that:

$$
\begin{aligned}
F(T) = & -2\kappa\, V(T) - \frac{2\, F_T(0)}{b_0^2}\, \exp[-\delta\, b_0\, r(T)]\, [a + 1 + \delta\, b_0\, r(T)]^{\frac{2a^2-a-1}{a+1}}\, [r(T)]^{\frac{-a^2+3a+2}{a+1}-2} \\
& \times \left[\frac{(3a+1)}{r^2(T)} + \frac{2\delta\, b_0\,(a+1)}{r(T)} + \frac{\left[(3a - a^2 + 2)\, r^{-2}(T) - b_0^2\right]}{\left[(a+1)\, r^{-1}(T) + \delta\, b_0\right]}\left(r^{-1}(T) + \delta\, b_0\right)\right], \qquad (54)
\end{aligned}
$$

where $V(T)$ are described by Equations (25), (27) and (29) with $r(T)$ defined by Equation (53).

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4. $\mathbf{a} \neq \{-\mathbf{1}, -\frac{\mathbf{1}}{\mathbf{2}}\}$ and $\mathbf{b} = -\mathbf{1}$: Equation (36) will be:

$$0 = \left(\frac{b_0^2\,T}{2} + 2\,a + 1\right) + 2\delta\,b_0\,(1+a)\,r^{-1} + b_0^2\,r^{-2},$$
$$\Rightarrow r^{-1}(T) = -\frac{\delta\,(a+1)}{b_0} + \delta_1\sqrt{\frac{a^2}{b_0^2} - \frac{T}{2}}, \tag{55}$$

where $\delta_1 = \pm 1$. The Equation (22) is:

$$\phi(r) = p_0\left(-\frac{\delta\,(a+1)}{b_0} + \delta_1\sqrt{\frac{a^2}{b_0^2} - \frac{T}{2}}\right)^{-p}, \tag{56}$$

By substituting Equation (55) and (56) into Equation (37), we find that:

$$F(T) = -2\kappa\,V(T) - \frac{2\,F_T(0)}{b_0^2}\exp\left[\frac{\delta\,b_0}{r(T)}\right]\frac{[r(T)]^{a+1}}{[(a+1)\,r(T) + \delta\,b_0]^{-\frac{2a^2+a+1}{a+1}}}$$
$$\times\left[3(a+1) + \frac{2\delta\,b_0\,(a+1)}{r(T)} + \frac{\left[a(1-a)\,r^2(T) - b_0^2\right]}{[(a+1)\,r(T) + \delta\,b_0]}\frac{(r(T) + \delta\,b_0)}{r^2(T)}\right], \tag{57}$$

where $r(T)$ is the Equation (55) and $V(T)$ are:

(a) **General:**

$$V(T) = V_0 + \frac{p^2\,p_0^2\,(a+p-1)}{2(p-1)}\left(-\frac{\delta\,(a+1)}{b_0} + \delta_1\sqrt{\frac{a^2}{b_0^2} - \frac{T}{2}}\right)^{2-2p}. \tag{58}$$

(b) $\mathbf{p = 1}$:

$$V(T) = V_0 + a\,p_0^2\,\ln\,p_0 - a\,p_0^2\,p\,\ln\left(-\frac{\delta\,(a+1)}{b_0} + \delta_1\sqrt{\frac{a^2}{b_0^2} - \frac{T}{2}}\right). \tag{59}$$

(c) $\mathbf{p \gg 1}$:

$$V(T) = V_0 + \frac{p^2\,p_0^2}{2}\left(-\frac{\delta\,(a+1)}{b_0} + \delta_1\sqrt{\frac{a^2}{b_0^2} - \frac{T}{2}}\right)^{-2p}. \tag{60}$$

5. $\mathbf{a} \neq \{-\mathbf{1}, -\frac{\mathbf{1}}{\mathbf{2}}\}$ and $\mathbf{b} = -\mathbf{2}$: Equation (36) will be:

$$0 = b_0^2 + \left(\frac{b_0^2\,T}{2} + 2\delta\,b_0\,(1+a)\right)r^2 + (2\,a+1)\,r^4,$$
$$\Rightarrow r(T) = \pm\frac{1}{2\sqrt{2a+1}}\sqrt{\delta_2\,b_0\sqrt{16a^2 + 8\delta\,(a+1)\,b_0\,T + b_0^2\,T^2} - 4\delta\,b_0(a+1) - b_0^2\,T}. \tag{61}$$

Then we will set the positive $r(T)$ case and $\delta_2 = \pm 1$. The scalar field $\phi(T)$ is still described by Equation (22) and then, by substituting Equation (22) and (61) into Equation (37), we find:

$$F(T) = -2\kappa\,V(T) - \frac{2\,F_T(0)}{b_0^2}\exp\left[\frac{\delta\,b_0}{2\,r^2(T)}\right]\frac{[r(T)]^{a+1}}{[(a+1)\,r^2(T) + \delta\,b_0]^{\frac{a^2+a+1}{a+1}}}$$
$$\times\left[(3a+4)\,r^2(T) + 2\delta\,b_0\,(a+1) - \frac{\left[(a^2+1)\,r^4(T) + b_0^2\right]}{[(a+1)\,r^2(T) + \delta\,b_0]}\frac{(r^2(T) + \delta\,b_0)}{r^2(T)}\right], \tag{62}$$

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where $V(T)$ are defined by Equations (25), (27) and (29) forms with $r(T)$ described by Equation (61).

6. $\mathbf{a}=-\mathbf{1}$ and $\mathbf{b}=\mathbf{0}$: Equation (36) becomes:

$$0=\frac{b_0^2\,T}{2}+\left(b_0^2-1\right)r^{-2}\quad\Rightarrow\;r^{-2}(T)=\left(\frac{b_0^2}{2\left(1-b_0^2\right)}\right)T. \tag{63}$$

The scalar field defined by Equation (22) is:

$$\phi(T)=p_0\left(\frac{2^{p/2}\left(1-b_0^2\right)^{p/2}}{b_0^p}\right)T^{-p/2}=p_0\,\phi_{-1,0}\,T^{-p/2}. \tag{64}$$

By substituting Equations (63) and (64) into Equation (37), we find that:

$$F(T)=-\,2\kappa\,V(T)+\frac{2\,F_T(0)}{b_0^3}\left(\frac{b_0^2}{2\left(1-b_0^2\right)}\right)^{1+\frac{\delta}{2b_0}\,\left(2+b_0^2\right)}\left[b_0+\delta\left(2+b_0^2\right)\left(1+\delta\,b_0\right)\right]T^{1+\frac{\delta}{2b_0}\,\left(2+b_0^2\right)}, \tag{65}$$

where $V(T)$ becomes for the cases:

(a) **General:**

$$V(T)=V_0+\frac{p^2\,p_0^2\,(p-2)}{2(p-1)}\,\phi_{-1,0}^{2-2/p}\,T^{1-p}. \tag{66}$$

(b) $\mathbf{p}=\mathbf{1}$:

$$V(T)=V_0-\,p_0^2\,\ln\left(p_0\,\phi_{-1,0}\right)+\frac{p\,p_0^2}{2}\,\ln(T). \tag{67}$$

(c) $\mathbf{p}\gg\mathbf{1}$:

$$V(T)=V_0+\frac{p^2\,p_0^2}{2}\,\phi_{-1,0}^2\,T^{-p}. \tag{68}$$

7. For all $\mathbf{a}=-\mathbf{1}$ and $\mathbf{b}\neq\mathbf{0}$ cases: Equation (37) simplifies as:

$$F(T)=-\,2\kappa\,V(T)+\frac{2\,F_T(0)}{b_0^3}\,\exp\left[\frac{2\delta}{b_0\,b}\,[r(T)]^{-b}-\frac{\delta\,b_0}{b}\,[r(T)]^{b}\right]\Big[(b+3)\,b_0\,r^{-2b}(T)+2\delta\,r^{-3b}(T)$$
$$+\,\delta\,b_0^2\,r^{-b}(T)+b_0^3\Big]\,r^{-2}(T), \tag{69}$$

where the $V(T)$ expressions under the Equation (22) definition:

(a) **General:** Equation (25) becomes:

$$V(T)=V_0+\frac{p^2\,p_0^2\,(p-2)}{2(p-1)}\,[r(T)]^{2p-2}. \tag{70}$$

(b) $\mathbf{p}=\mathbf{1}$: Equation (27) becomes:

$$V(T)=V_0-p_0^2\,\ln\left(p_0\right)-p_0^2\,p\,\ln\left[r(T)\right]. \tag{71}$$

(c) $\mathbf{p}\gg\mathbf{1}$: Equation (29) becomes:

$$V(T)=V_0+\frac{p^2\,p_0^2}{2}\,[r(T)]^{2p}. \tag{72}$$

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8. $\mathbf{a}=-\mathbf{1}$ and $\mathbf{b}=\frac{\mathbf{1}}{\mathbf{2}}$: Equation (36) becomes:

$$0=\frac{b_0^2\,T}{2}+b_0^2\,r^{-2}-r^{-3},$$
$$\Rightarrow\ r^{-1}(T)=\frac{1}{3}\left[b_0^2+\frac{2^{2/3}\,b_0^4}{\left[3^{3/2}\sqrt{27b_0^4\,T^2+8b_0^8\,T}+27b_0^2\,T+4b_0^6\right]^{1/3}}+\frac{\left[3^{3/2}\sqrt{27b_0^4\,T^2+8b_0^8\,T}+27b_0^2\,T+4b_0^6\right]^{1/3}}{2^{2/3}}\right]. \tag{73}$$

The scalar field is described by Equation (22) and then Equation (69) becomes:

$$F(T)=-2\kappa\,V(T)+\frac{2\,F_T(0)}{b_0^3}\,\exp\left[\frac{4\delta}{b_0}\,[r(T)]^{-\frac{1}{2}}-2\delta\,b_0\,[r(T)]^{\frac{1}{2}}\right]\left[\frac{7}{2}\,b_0\,r^{-1}(T)+2\delta\,r^{-\frac{3}{2}}(T)+\delta\,b_0^2\,r^{-\frac{1}{2}}(T)+b_0^3\right]r^{-2}(T), \tag{74}$$

where $V(T)$ are described by Equations (70)–(72) with $r(T)$ defined by the Equation (73).

9. $\mathbf{a}=-\mathbf{1}$ and $\mathbf{b}=-\frac{\mathbf{1}}{\mathbf{2}}$: Equation (36) becomes:

$$0=r^{-2}-\frac{1}{b_0^2}\,r^{-1}+\frac{T}{2}\quad\Rightarrow\ r^{-1}(T)=\frac{1}{2b_0^2}\left[1+\delta_1\sqrt{1-2\,b_0^4\,T}\right], \tag{75}$$

where $\delta_1=\pm 1$. The scalar field is described by Equation (22) and then Equation (69) is:

$$F(T)=-2\kappa\,V(T)+\frac{2\,F_T(0)}{b_0^3}\,\exp\left[-\frac{4\delta}{b_0}\,[r(T)]^{\frac{1}{2}}+2\delta\,b_0\,[r(T)]^{-\frac{1}{2}}\right]\left[\frac{5}{2}\,b_0\,r(T)+2\delta\,r^{\frac{3}{2}}(T)+\delta\,b_0^2\,r^{\frac{1}{2}}(T)+b_0^3\right]r^{-2}(T), \tag{76}$$

where $V(T)$ are described by Equations (70)–(72) with $r(T)$ defined by the Equation (75).

10. $\mathbf{a}=-\mathbf{1}$ and $\mathbf{b}=\mathbf{1}$: Equation (36) becomes:

$$0=r^{-4}-b_0^2\,r^{-2}-\frac{b_0^2\,T}{2}\quad\Rightarrow\ r^{-2}(T)=\frac{b_0^2}{2}\left[1+\delta_1\sqrt{1+\frac{2\,T}{b_0^2}}\right], \tag{77}$$

where $\delta_1=\pm 1$. The scalar field is described by Equation (22) and then Equation (69) is:

$$F(T)=-2\kappa\,V(T)+\frac{2\,F_T(0)}{b_0^3}\,\exp\left[\frac{2\delta}{b_0\,r(T)}-\delta\,b_0\,r(T)\right]\left[\frac{4\,b_0}{r^2(T)}+\frac{2\delta}{r^3(T)}+\frac{\delta\,b_0^2}{r(T)}+b_0^3\right]r^{-2}(T), \tag{78}$$

where $V(T)$ are described by Equations (70)–(72) with $r(T)$ defined by the Equation (77).

11. $\mathbf{a}=-\mathbf{1}$ and $\mathbf{b}=-\mathbf{1}$: Equation (36) becomes:

$$0=\frac{b_0^2\,T}{2}+b_0^2\,r^{-2}-1\quad\Rightarrow\ r^{-2}(T)=\frac{1}{b_0^2}-\frac{T}{2}, \tag{79}$$

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where $T \leq \frac{2}{b_0^2}$. The scalar field is described by Equation (22) and then Equation (69) is:

$$F(T) = -2\kappa\, V(T) + \frac{2\,F_T(0)}{b_0^3}\, \exp\left[\frac{\delta\, b_0}{r(T)} - \frac{2\delta}{b_0}\, r(T)\right]\left[2b_0\, r^2(T) + 2\delta\, r^3(T) + \delta\, b_0^2\, r(T) + b_0^3\right] \times r^{-2}(T), \tag{80}$$

where $V(T)$ are described by Equations (70)–(72) with $r(T)$ defined by the Equation (79).

12. $\mathbf{a} = -\mathbf{1}$ and $\mathbf{b} = -\frac{\mathbf{3}}{\mathbf{2}}$: Equation (36) becomes:

$$0 = r^{-3} + \frac{T}{2}\, r^{-1} - \frac{1}{b_0^2},$$
$$\Rightarrow r^{-1}(T) = \frac{1}{6^{2/3} b_0^2}\left[\sqrt{6}b_0^4\sqrt{b_0^4\, T^3 + 54} + 18b_0^4\right]^{1/3} - \frac{b_0^2\, T}{6^{1/3}}\left[\sqrt{6}b_0^4\sqrt{b_0^4\, T^3 + 54} + 18b_0^4\right]^{-1/3}. \tag{81}$$

The scalar field is described by Equation (22) and then Equation (69) is:

$$F(T) = -2\kappa\, V(T) + \frac{2\,F_T(0)}{b_0^3}\, \exp\left[\frac{2\delta\, b_0}{3[r(T)]^{\frac{3}{2}}} - \frac{4\delta}{3b_0}[r(T)]^{\frac{3}{2}}\right] \times \left[\frac{3}{2}\, b_0\, r^3(T) + 2\delta\, r^{\frac{9}{2}}(T) + \delta\, b_0^2\, r^{\frac{3}{2}}(T) + b_0^3\right] r^{-2}(T), \tag{82}$$

where $V(T)$ are described by Equations (70)–(72) with $r(T)$ defined by the Equation (81).

13. $\mathbf{a} = -\mathbf{1}$ and $\mathbf{b} = \mathbf{2}$: Equation (36) becomes:

$$0 = \frac{b_0^2\, T}{2} + b_0^2\, r^{-2} - r^{-6},$$
$$\Rightarrow r^{-2}(T) = \left[\frac{2^{2/3}\, b_0^{\frac{4}{3}}}{3^{1/3}}\left[\sqrt{3}\sqrt{27\, T^2 - 16b_0^2} + 9\, T\right]^{-\frac{1}{3}} + \frac{b_0^{\frac{2}{3}}}{6^{2/3}}\left[\sqrt{3}\sqrt{27\, T^2 - 16b_0^2} + 9\, T\right]^{\frac{1}{3}}\right], \tag{83}$$

where $\delta_1 = \pm 1$. The scalar field is described by Equation (22) and then Equation (69) is:

$$F(T) = -2\kappa\, V(T) + \frac{2\,F_T(0)}{b_0^3}\, \exp\left[\frac{\delta}{b_0[r(T)]^2} - \frac{\delta\, b_0}{2}[r(T)]^2\right]\left[\frac{5\, b_0}{r^4(T)} + \frac{2\delta}{r^6(T)} + \frac{\delta\, b_0^2}{r^2(T)} + b_0^3\right] \times r^{-2}(T), \tag{84}$$

where $V(T)$ are described by Equations (70)–(72) with $r(T)$ defined by the Equation (83).

14. $\mathbf{a} = -\mathbf{1}$ and $\mathbf{b} = -\mathbf{2}$: Equation (36) becomes:

$$0 = \frac{b_0^2\, T}{2}\, r^2 + b_0^2 - r^4 \quad \Rightarrow\ r(T) = \frac{\delta_2}{2}\sqrt{b_0^2\, T + \delta_1\, b_0\sqrt{b_0^2\, T^2 + 16}}. \tag{85}$$

The scalar field is described by Equation (22) and then Equation (69) is:

$$F(T) = -2\kappa\, V(T) + \frac{2\,F_T(0)}{b_0^3}\, \exp\left[\frac{\delta\, b_0}{2[r(T)]^2} - \frac{\delta}{b_0}[r(T)]^2\right]\left[b_0\, r^4(T) + 2\delta\, r^6(T) + \delta\, b_0^2\, r^2(T) + b_0^3\right] \times r^{-2}(T), \tag{86}$$

where $V(T)$ are described by Equations (70)–(72) with $r(T)$ defined by the Equation (85).

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15. $\mathbf{a} = -\mathbf{1}$ and $\mathbf{b} = \mathbf{3}$: Equation (36) becomes:

$$0 = \frac{b_0^2\,T}{2} + b_0^2\,r^{-2} - r^{-8},$$
$$\Rightarrow\ r^{-2}(T) =$$
$$\frac{\delta_1}{2}\left[\frac{b_0}{\sqrt[3]{2}\,3^{2/3}}\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{1/3} - 2b_0\sqrt[3]{\frac{2}{3}}\,T\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{-1/3}\right]^{1/2}$$
$$+\frac{\delta_2}{2}\left[2b_0\sqrt[3]{\frac{2}{3}}\,T\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{-1/3} + 2\delta_1\,b_0^2\left[\frac{b_0}{\sqrt[3]{2}\,3^{2/3}}\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{1/3}\right.\right.$$
$$\left.\left. - 2b_0\sqrt[3]{\frac{2}{3}}\,T\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{-1/3}\right]^{-1/2} - \frac{b_0}{\sqrt[3]{2}\,3^{2/3}}\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{1/3}\right]^{1/2}, \tag{87}$$

where $(\delta_1, \delta_2) = (\pm 1, \pm 1)$. The scalar field is described by Equation (22) and then Equation (69) is:

$$F(T) = -2\kappa\,V(T) + \frac{2\,F_T(0)}{b_0^3}\,\exp\left[\frac{2\delta}{3b_0[r(T)]^3} - \frac{\delta\,b_0}{3}[r(T)]^3\right]\left[\frac{6\,b_0}{r^6(T)} + \frac{2\delta}{r^9(T)} + \frac{\delta\,b_0^2}{r^3(T)} + b_0^3\right]$$
$$\times\ r^{-2}(T), \tag{88}$$

where $V(T)$ are described by Equations (70)–(72) with $r(T)$ defined by the Equation (87).

16. $\mathbf{a} = -\frac{\mathbf{1}}{\mathbf{2}}$ and $\mathbf{b} = \mathbf{0}$: Equation (36) becomes:

$$0 = \frac{b_0\,T}{2} + (b_0+\delta)\,r^{-2} \quad\Rightarrow\ r^{-2}(T) = -\frac{b_0}{2(b_0+\delta)}\,T. \tag{89}$$

The Equation (37) becomes by substituting Equation (89):

$$F(T) = -2\kappa\,V(T) - \frac{2\,F_T(0)}{b_0^2}\left[\frac{1}{2} + \delta\,b_0 - \frac{\left(\frac{1}{4}+b_0^2\right)}{\left(\frac{1}{2}+\delta\,b_0\right)}(1+\delta\,b_0)\right]\left[\frac{b_0}{2(b_0+\delta)}\right]^{\frac{\left(\frac{1}{4}+b_0^2\right)}{(1+2\delta\,b_0)}+1}$$
$$\times\ (-T)^{\frac{\left(\frac{1}{4}+b_0^2\right)}{(1+2\delta\,b_0)}+1}. \tag{90}$$

where $V(T)$ are defined in terms of Equation (89) by:

(a) **General:** Equation (25) becomes:

$$V(T) = V_0 + \frac{p^2\,p_0^2\,\left(p-\frac{3}{2}\right)}{2(p-1)}\left(\frac{2(b_0+\delta)}{b_0}\right)^{p-1}(-T)^{1-p}. \tag{91}$$

(b) $\mathbf{p} = \mathbf{1}$: Equation (27) becomes:

$$V(T) = V_0 + a\,p_0^2\,\ln\,p_0 + \frac{p_0^2\,p}{4}\,\ln\left(\frac{b_0}{2(b_0+\delta)}\right) + \frac{p_0^2\,p}{4}\,\ln(-T). \tag{92}$$

(c) $\mathbf{p} \gg \mathbf{1}$: Equation (29) becomes:

$$V(T) = V_0 + \frac{p^2\,p_0^2}{2}\left(\frac{2(b_0+\delta)}{b_0}\right)^{p}(-T)^{-p}. \tag{93}$$

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17. For all $\mathbf{a} = -\frac{1}{2}$ and $\mathbf{b} \neq \mathbf{0}$ cases: Equation (37) will simplify as:

$$F(T) = -2\kappa\, V(T) + \frac{2\,F_T(0)}{b_0^2}\,\exp\left[\int_{r(T)} dr'\, r'^{b-1}\,\frac{\left[\frac{(2b-1)}{4}\,r'^{-2b} - b_0^2\right]}{\left[\frac{1}{2}\,r'^{-b} + \delta\,b_0\right]}\right]\left[\left(b - \frac{1}{2}\right) r^{-2b-2}(T) - \delta\,b_0\,r^{-b-2}(T) - \frac{\left[\frac{(2b-1)}{4}\,r^{-2b}(T) - b_0^2\right]}{\left[\frac{1}{2}\,r^{-b}(T) + \delta\,b_0\right]}\left(r^{-b}(T) + \delta\,b_0\right) r^{-2}(T)\right], \tag{94}$$

where $V(T)$ are defined by:

(a) **General:** Equation (25) becomes:

$$V(T) = V_0 + \frac{p^2\,p_0^2\left(p - \frac{3}{2}\right)}{2(p-1)}\,[r(T)]^{2p-2}. \tag{95}$$

(b) $\mathbf{p = 1}$: Equation (27) becomes:

$$V(T) = V_0 + a\,p_0^2\,\ln\,p_0 - \frac{p_0^2\,p}{2}\,\ln\,[r(T)]. \tag{96}$$

(c) $\mathbf{p \gg 1}$: Equation (29) keeps the same form.

18. $\mathbf{a} = -\frac{1}{2}$ and $\mathbf{b = 1}$: Equation (36) becomes more simple as:

$$\begin{aligned} 0 =& \frac{\delta\,b_0\,T}{2} + \delta\,b_0\,r^{-2} + r^{-3}, \\ \Rightarrow\; r^{-1}(T) =& \frac{1}{3}\left[-\delta\,b_0 + \frac{2^{2/3}\,b_0^2}{\sqrt[3]{-4\delta\,b_0^3 + 3\sqrt{3}\sqrt{27b_0^2\,T^2 + 8b_0^4\,T} - 27\delta\,b_0\,T}}\right. \\ &\left. + \frac{1}{2^{2/3}}\sqrt[3]{-4\delta\,b_0^3 + 3\sqrt{3}\sqrt{27b_0^2\,T^2 + 8b_0^4\,T} - 27\delta\,b_0\,T}\right]. \end{aligned} \tag{97}$$

The Equation (94) becomes by substituting Equation (97):

$$F(T) = -2\kappa\,V(T) - \frac{\delta\,F_T(0)}{b_0}\,[r(T)]^{-\frac{3}{2}}\left(r^{-1}(T) - 2\delta\,b_0\right)\exp[-\delta\,b_0\,r(T)], \tag{98}$$

where $V(T)$ are described by Equations (29), (95) and (96) with $r(T)$ defined by Equation (97).

19. $\mathbf{a} = -\frac{1}{2}$ and $\mathbf{b = -1}$: Equation (36) becomes:

$$0 = \frac{b_0\,T}{2} + b_0\,r^{-2} + \delta\,r^{-1} \quad\Rightarrow\; r^{-1}(T) = -\frac{\delta}{2b_0} \pm \sqrt{\frac{1}{4b_0^2} - \frac{T}{2}}. \tag{99}$$

The Equation (94) becomes by substituting Equation (99):

$$\begin{aligned} F(T) =& -2\kappa\,V(T) - \frac{2\,F_T(0)}{b_0^2}\,\frac{\exp\left[\frac{\delta\,b_0}{r(T)}\right]}{(r(T) + 2\delta\,b_0)^2} \\ &\times \left[\frac{3}{2}[r(T)]^{1/2} + \frac{\delta\,b_0}{[r(T)]^{1/2}} - \frac{2\left[\frac{3}{4}\,r^2(T) + b_0^2\right]}{[r(T) + 2\delta\,b_0]}\,\frac{(r(T) + \delta\,b_0)}{[r(T)]^{3/2}}\right], \end{aligned} \tag{100}$$

where $V(T)$ are described by Equations (29), (95) and (96) with $r(T)$ defined by Equation (99).

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20. $\mathbf{a} = -\frac{1}{2}$ and $\mathbf{b} = \mathbf{2}$: Equation (36) becomes:

$$0 = \frac{\delta\, b_0\, T}{2} + \delta\, b_0\, r^{-2} + r^{-4} \quad \Rightarrow \quad r^{-2}(T) = -\frac{\delta\, b_0}{2} + \delta_1 \sqrt{\frac{b_0^2}{4} - \frac{\delta\, b_0\, T}{2}}. \tag{101}$$

The Equation (94) becomes by substituting Equation (101):

$$F(T) = -\, 2\kappa\, V(T) + \frac{F_T(0)}{b_0^2} \frac{\exp\left[-\frac{\delta\, b_0}{2} r^2(T)\right]}{(1 + 2\delta\, b_0\, r^2(T))^{\frac{1}{2}}} \\ \times \left[\frac{(3 - 2\delta\, b_0 r^2(T))}{r^{9/2}(T)} - \frac{[3r^{-4}(T) - 4b_0^2]}{[r^{-2}(T) + 2\delta\, b_0]} \frac{(r^{-2}(T) + \delta\, b_0)}{[r(T)]^{1/2}} \right], \tag{102}$$

where $V(T)$ are described by Equations (29), (95) and (96) with $r(T)$ defined by Equation (101).

21. $\mathbf{a} = -\frac{1}{2}$ and $\mathbf{b} = -\mathbf{2}$: Equation (36) will simplify as:

$$0 = b_0 + \left(\frac{b_0\, T}{2} + \delta\right) r^2 \quad \Rightarrow \quad r^2(T) = -\frac{2b_0}{b_0\, T + 2\delta}, \tag{103}$$

where $b_0\, T + 2\delta < 0$. The Equation (94) becomes by substituting Equation (103):

$$F(T) = -\, 2\kappa\, V(T) - \frac{F_T(0)}{b_0^2} \frac{[r(T)]^{1/2} \exp\left[\frac{\delta\, b_0}{2r^2(T)}\right]}{(r^2(T) + 2\delta\, b_0)^{3/2}} \\ \times \left[5r^2(T) + 2\delta\, b_0 - \frac{[5r^4(T) + 4b_0^2]}{[r^2(T) + 2\delta\, b_0]} \frac{(r^2(T) + \delta\, b_0)}{r^2(T)} \right], \tag{104}$$

where $V(T)$ are described by Equations (29), (95) and (96) with $r(T)$ defined by Equation (103).

22. $\mathbf{a} = -\frac{1}{2}$ and $\mathbf{b} = -\mathbf{3}$: Equation (36) becomes:

$$0 = r^3 + \frac{\delta b_0\, T}{2} r^2 + \delta b_0, \\ \Rightarrow \; r(T) = \frac{1}{6} \Bigg[\left[-108\delta\, b_0 - \delta\, b_0^3\, T^3 + 6^{3/2}\, b_0 \sqrt{54 + b_0^2\, T^3} \right]^{1/3} \\ + b_0^2\, T^2 \left[-108\delta\, b_0 - \delta\, b_0^3\, T^3 + 6^{3/2}\, b_0 \sqrt{54 + b_0^2\, T^3} \right]^{-1/3} - \delta\, b_0\, T \Bigg]. \tag{105}$$

The Equation (94) becomes by substituting Equation (105):

$$F(T) = -\, 2\kappa\, V(T) - \frac{F_T(0)}{b_0^2} \frac{r^{\frac{3}{2}}(T) \exp\left[\frac{\delta\, b_0}{3r^3(T)}\right]}{(r^3(T) + 2\delta\, b_0)^{\frac{4}{3}}} \\ \times \left[\left(7r^3(T) + 2\delta\, b_0\right) - \frac{[7r^6(T) + 4b_0^2]}{[r^3(T) + 2\delta\, b_0]} \frac{(r^3(T) + \delta\, b_0)}{r^3(T)} \right], \tag{106}$$

where $V(T)$ are described by Equations (29), (95) and (96) with $r(T)$ defined by Equation (105).

23. $\mathbf{a} = -\frac{1}{2}$ and $\mathbf{b} = \mathbf{4}$: Equation (36) becomes:

$$0 = \frac{b_0\, T}{2} + b_0\, r^{-2} + \delta\, r^{-6}, \\ \Rightarrow \; r^{-2}(T) = \left[\frac{b_0^{\frac{1}{3}}}{6^{\frac{2}{3}}} \left[\sqrt{3}\sqrt{16\delta b_0 + 27\, T^2} - 9\delta\, T \right]^{\frac{1}{3}} - \frac{\delta\, b_0^{\frac{2}{3}}\, 2^{\frac{2}{3}}}{3^{\frac{1}{3}}} \left[\sqrt{3}\sqrt{16\delta b_0 + 27\, T^2} - 9\delta\, T \right]^{-\frac{1}{3}} \right], \tag{107}$$

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where $\delta = \pm 1$. The Equation (94) becomes by substituting Equation (107):

$$F(T) = -2\kappa\, V(T) + \frac{F_T(0)}{b_0^2}\, \frac{\exp\left[-\frac{\delta\, b_0}{4}\, r^4(T)\right]}{[r(T)]^{\frac{13}{2}}\left(1+2\delta\, b_0 r^4(T)\right)^{\frac{3}{4}}} \times \left[\left[7-\delta\, b_0 r^4(T)\right] - \frac{\left[7-4b_0^2 r^8(T)\right]}{\left[1+2\delta\, b_0 r^4(T)\right]}\left[1+\delta\, b_0 r^4(T)\right]\right], \tag{108}$$

where $V(T)$ are described by Equations (29), (95) and (96) with $r(T)$ defined by Equation (107).

24. $\mathbf{a} = -\frac{\mathbf{1}}{\mathbf{2}}$ and $\mathbf{b} = -\mathbf{4}$ case: Equation (36) becomes:

$$0 = r^4 + \frac{\delta\, b_0\, T}{2}\, r^2 + \delta\, b_0 \quad \Rightarrow\ r(T) = \delta_2 \sqrt{-\frac{\delta\, b_0\, T}{4} + \delta_1 \sqrt{\frac{b_0^2\, T^2}{16}} - \delta\, b_0}. \tag{109}$$

The Equation (94) becomes by substituting Equation (109):

$$F(T) = -2\kappa\, V(T) - \frac{2\, F_T(0)}{b_0^2}\, \frac{[r(T)]^{\frac{5}{2}} \exp\left[\frac{\delta\, b_0}{4r^4(T)}\right]}{\left(r^4(T)+2\delta\, b_0\right)^{\frac{5}{4}}} \times \left[\frac{9}{2} r^4(T) + \delta\, b_0 - \frac{\left[\frac{9}{4}\, r^8(T) + b_0^2\right]}{\left[\frac{1}{2}\, r^4(T) + \delta\, b_0\right]}\, \frac{\left(r^4(T)+\delta\, b_0\right)}{r^4(T)}\right], \tag{110}$$

where $V(T)$ are described by Equations (29), (95) and (96) with $r(T)$ defined by Equation (109).

25. $\mathbf{a} = -\frac{\mathbf{1}}{\mathbf{2}}$ and $\mathbf{b} = \mathbf{6}$: Equation (36) becomes:

$$\begin{aligned} 0 =& \frac{b_0\, T}{2} + b_0\, r^{-2} + \delta\, r^{-8}, \\ \Rightarrow r^{-2}(T) =& \frac{\delta_1}{2}\left[2\sqrt[3]{\frac{2}{3}} b_0\, T\left(9\delta\, b_0^2 + \sqrt{3}\, b_0 \sqrt{27 b_0^2 - 32\delta\, b_0\, T^3}\right)^{-1/3} \right. \\ & \left. + \frac{\delta}{\sqrt[3]{2}\, 3^{2/3}}\left(9\delta\, b_0^2 + \sqrt{3}\, b_0\sqrt{27b_0^2 - 32\delta\, b_0\, T^3}\right)^{1/3}\right]^{1/2} \\ & + \frac{\delta_2}{2}\left[-2\sqrt[3]{\frac{2}{3}} b_0\, T\left(9\delta\, b_0^2 + \sqrt{3}\, b_0\sqrt{27b_0^2 - 32\delta\, b_0\, T^3}\right)^{-1/3}\right. \\ & - \frac{\delta\,\delta_2}{2b_0}\left[2\sqrt[3]{\frac{2}{3}} b_0\, T\left(9\delta\, b_0^2 + \sqrt{3}\, b_0\sqrt{27b_0^2 - 32\delta\, b_0\, T^3}\right)^{-1/3}\right. \\ & \left. + \frac{\delta}{\sqrt[3]{2}\, 3^{2/3}}\left(9\delta\, b_0^2 + \sqrt{3}\, b_0\sqrt{27b_0^2 - 32\delta\, b_0\, T^3}\right)^{1/3}\right]^{-1/2} \\ & \left. - \frac{\delta}{\sqrt[3]{2}3^{2/3}}\left(9\delta\, b_0^2 + \sqrt{3}\, b_0\sqrt{27b_0^2 - 32\delta\, b_0\, T^3}\right)^{1/3}\right]^{1/2}, \end{aligned} \tag{111}$$

where the possible solutions are $(\delta_1, \delta_2) = (\pm 1, \pm 1)$. The Equation (94) becomes by substituting Equation (111):

$$F(T) = -2\kappa\, V(T) + \frac{F_T(0)}{b_0^2}\, \frac{\exp\left[-\frac{\delta\, b_0}{6} r^6(T)\right]}{[r(T)]^{\frac{17}{2}}\left(1+2\delta\, b_0 r^6(T)\right)^{\frac{5}{6}}} \times \left[11 - 2\delta\, b_0 r^6(T) - \frac{\left[11-4b_0^2 r^{12}(T)\right]}{\left[1+2\delta\, b_0 r^6(T)\right]}\left(1+\delta\, b_0 r^6(T)\right)\right], \tag{112}$$

where $V(T)$ are described by Equations (29), (95) and (96) with $r(T)$ defined by Equation (85).

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26. $\mathbf{a} = -\frac{1}{2}$ and $\mathbf{b} = -6$: Equation (36) becomes:

$$0 = r^6 + \frac{\delta\, b_0\, T}{2}\, r^2 + \delta\, b_0,$$
$$\Rightarrow\; r^2(T) = \frac{\left[\sqrt{6}\, b_0 \sqrt{\delta\, b_0\, T^3 + 54} - 18\delta\, b_0\right]^{2/3} - \sqrt[3]{6}\,\delta\, b_0\, T}{6^{2/3}\left[\sqrt{6}\, b_0 \sqrt{\delta\, b_0\, T^3 + 54} - 18\delta\, b_0\right]^{1/3}} \tag{113}$$

The Equation (94) becomes by substituting Equation (113):

$$F(T) = -2\kappa\, V(T) - \frac{F_T(0)}{b_0^2}\, \frac{[r(T)]^{\frac{9}{2}} \exp\left[\frac{\delta\, b_0}{6r^6(T)}\right]}{\left(r^6(T) + 2\delta\, b_0\right)^{\frac{7}{6}}}$$
$$\times \left[13r^6(T) + 2\delta\, b_0 - \frac{\left[13r^{12}(T) + 4b_0^2\right]}{\left[r^6(T) + 2\delta\, b_0\right]}\, \frac{\left(r^6(T) + \delta\, b_0\right)}{r^6(T)}\right], \tag{114}$$

where $V(T)$ are described by Equations (29), (95) and (96) with $r(T)$ defined by Equation (113).

27. $\mathbf{a} = -\frac{1}{2}$ and $\mathbf{b} = -8$: Equation (36) becomes:

$$0 = r^8 + \frac{\delta\, b_0\, T}{2}\, r^2 + \delta\, b_0,$$
$$\Rightarrow\; r^2(T) = \frac{\delta_1}{2}\left[\frac{8\delta\, b_0}{\sqrt[3]{3}}\left(\sqrt{3} b_0 \sqrt{27 b_0^2\, T^4 - 4096\delta\, b_0} + 9 b_0^2\, T^2\right)^{-1/3}\right.$$
$$\left. + \frac{1}{2\cdot 3^{2/3}}\left(\sqrt{3} b_0 \sqrt{27 b_0^2\, T^4 - 4096\delta\, b_0} + 9 b_0^2\, T^2\right)^{1/3}\right]^{1/2}$$
$$+ \frac{\delta_2}{2}\left[-\frac{8\delta\, b_0}{\sqrt[3]{3}}\left(\sqrt{3} b_0 \sqrt{27 b_0^2\, T^4 - 4096\delta\, b_0} + 9 b_0^2\, T^2\right)^{-1/3}\right.$$
$$- \delta_1 \delta\, b_0\, T\left[\frac{8\delta\, b_0}{\sqrt[3]{3}}\left(\sqrt{3} b_0 \sqrt{27 b_0^2\, T^4 - 4096\delta\, b_0} + 9 b_0^2\, T^2\right)^{-1/3}\right.$$
$$\left. + \frac{1}{2\cdot 3^{2/3}}\left(\sqrt{3} b_0 \sqrt{27 b_0^2\, T^4 - 4096\delta\, b_0} + 9 b_0^2\, T^2\right)^{1/3}\right]^{-1/2}$$
$$\left. - \frac{1}{2\cdot 3^{2/3}}\left(\sqrt{3} b_0 \sqrt{27 b_0^2\, T^4 - 4096\delta\, b_0} + 9 b_0^2\, T^2\right)^{1/3}\right]^{1/2} \tag{115}$$

The Equation (94) becomes by substituting Equation (115):

$$F(T) = -2\kappa\, V(T) - \frac{F_T(0)}{b_0^2}\, \frac{[r(T)]^{\frac{13}{2}} \exp\left[\frac{\delta\, b_0}{8r^8(T)}\right]}{\left(r^8(T) + 2\delta\, b_0\right)^{\frac{9}{8}}}$$
$$\times \left[17r^8(T) + 2\delta\, b_0 - \frac{\left[17r^{16}(T) + 4b_0^2\right]}{\left[r^8(T) + 2\delta\, b_0\right]}\, \frac{\left(r^8(T) + \delta\, b_0\right)}{r^8(T)}\right], \tag{116}$$

where $V(T)$ are described by Equations (29), (95) and (96) with $r(T)$ defined by Equation (115).

3.3. Other Ansatzes And Possible Comparisons With The Literature

The most of sections 3.1 and 3.2 new teleparallel $F(T)$ solutions have been computed from the general Equation (21) leading to the Equations (32) and (37) for $A_3 = c_0$ and $A_3 = r$ classes of solutions respectively. These new solutions can be easily and directly comparable to the linear perfect fluids teleparallel $F(T)$ solutions found in ref. [36] by a very similar solving approach. The main advantage in favor of this paper concerning the scalar field sources is the general teleparallel $F(T)$ solution computing formula stated by the Equation (21). This general easy-to-use $F(T)$ computation formula can be used for any scalar field and any coframe compoment ansatzes. In a more practical and technical words, we can set in principle any A_1 and A_2 component ansatz for either $A_3 = c_0$ or $A_3 = r$ solutions and generate any new teleparallel $F(T)$ solution depending on ansatz parameter. A very similar approach has been used to find the teleparallel $F(T)$ solutions for general Kantowski-Sachs (KS) spacetimes (pure time-dependent spherically spacetimes) and for Teleparallel Robertson-Walker (TRW) $F(T)$ solutions with scalar field sources in the both cases [39,40]. We have found in the two cases those easy-to-compute teleparallel $F(T)$ solution formula to generate the new solutions applicable to cosmological models, especially the dominating DE universe cases. However, the current paper confirms the relevance of using the scalar field sources, but for focusing on r-coordinate based astrophysical system applications. We used the Equation (23) coframe ansatz to find the most simple analytical new $F(T)$ solutions, but there are other possible ansatzes approaches.

A good example is the partially flat exponential ansatz defined as $A_1(r) = a_0(1 - e^{-ar})$, $A_2(r) = b_0 = 1$ and $A_3 = c_0$ treated in detail in ref. [36]. We set this ansatz in the manner to get a closed form which is leading to a Teleparallel Minkowski spacetime under the $r \to \infty$ limit in the definition of ref. [103]. The closed form coframe ansatz requirement is essential to really obtain new pure analytical teleparallel $F(T)$ solutions. From the last considerations, we studied in detail from the new partially flat exponential ansatz $F(T)$ solutions the possible singularities allowing some new point-like astrophysical spacetime discontinuities. We had suggested that some hidden possible physical processes around the $F(T)$ function singularities can be studied in some future research works. This last aim is beyond the current paper scope, but we can directly use the general Equation (21) by substituting the Equations (A1)–(A4) with the partially flat exponential ansatz for A_1–A_3 components to compute the new $F(T)$ solutions for any scalar field source described by a relevant $V(T)$ potential. A such study may deserve an independent investigation as a good astrophysical application of static spherically symmetric teleparallel spacetime solution.

The last example of non-power-law coframe ansatz solution is not the only possible one. There are a great number of possible ansatzes other than Equation (23) and the partially flat exponential. And there are also other possible scalar field source definitions other than Equation (22). The next section will clarify this last point.

4. Other Scalar Field Source Solutions

The section 3 aim was to find the new teleparallel $F(T)$ solutions for power-law scalar field defined by Equation (22) and the Equations (25), (27) and (29) conservation law solutions for scalar potential $V(T)$. Another conclusion concerning the results obtained in section 3 is that the new $F(T)$ solutions are not only valid for a power-law based scalar field, but also valuable for other scalar field definition and any types of potential $V(T)$ solution of Equation (16). From this point, we are also able to use the same Equation (21) for any type of scalar field source and potential by using the same coframe ansatz and same subcases. We can keep the same teleparallel $F(T)$ solution forms, but we will change the scalar field definition, solve the Equation (16) and use the section (3) $F(T)$ solutions by only change the potential $V(T)$ expressions. This is the aim of this current section: we replace Equations (25), (27) and (29) by different form of scalar field $\phi(T)$ and potential $V(T)$ expressions. We will proceed under this approach for exponential and logarithmic scalar field definitions as relevant case by the same manner as in recent papers for teleparallel cosmological spacetimes [39,40].

4.1. Exponential Scalar Field Solutions

The exponential scalar field is defined by:

$$\phi(T) = p_0 \exp(p\, r(T)). \tag{117}$$

By assuming a power-law coframe ansatz, we will solve the Equation (16) to find the scalar potential $V(\phi)$ for Equations (23) and (117):

$$\frac{dV}{d\phi} = p^2\,\phi + a\,p^2\,\frac{\phi}{\ln(\phi/p_0)},$$
$$\Rightarrow\ V(\phi) = V_0 + \frac{p^2}{2}\,\phi^2 + a\,p^2\,Ei(2\ln(\phi/p_0)). \tag{118}$$

By substituting Equation (117) into Equation (118) and setting $r = r(T)$, we obtain that:

$$V(T) = V_0 + \frac{p^2\,p_0^2}{2}\,\exp(2p\,r(T)) + a\,p^2\,Ei(2p\,r(T)). \tag{119}$$

1. **Power-law ansatz with $A_3 = c_0$**: We find the same $F(T)$ solution form as Equation (32) in section 3.1, but the potential $V(T)$ will be the Equation (119) with $r(T)$ defined by Equation (30). The Equation (119) becomes:

$$V(T) = V_0 + \frac{p^2\,p_0^2}{2}\,\exp\left(\frac{2p\left(-\frac{2\delta\,c_0}{b_0}\right)^{\frac{1}{(1+b)}}}{\left(1+\frac{c_0^2}{2}\,T\right)^{\frac{1}{(1+b)}}}\right) + a\,p^2\,Ei\left(\frac{2p\left(-\frac{2\delta\,c_0}{b_0}\right)^{\frac{1}{(1+b)}}}{\left(1+\frac{c_0^2}{2}\,T\right)^{\frac{1}{(1+b)}}}\right). \tag{120}$$

2. **Power-law ansatz with $A_3 = r$**: We find the same $F(T)$ solution forms as in section 3.2, but only the $V(T)$ potential expressions change by replacing Equations (25), (27) and (29) by the Equation (119) for each subcases treated in this section. The Equation (119) will be for the simplest cases:

   (a) $\mathbf{a = b = 0}$: The potential $V(T)$ in Equation (40) is:

$$V(T) = V_0 + \frac{p^2\,p_0^2}{2}\,\exp\left(\frac{2\sqrt{2}\,p(1+\delta\,b_0)}{b_0\,(-T)^{1/2}}\right). \tag{121}$$

   (b) $\mathbf{a \neq \{-1, -\frac{1}{2}\}}$ and $\mathbf{b = 0}$: The $V(T)$ expression in Equation (48) is:

$$V(T) = V_0 + \frac{p^2\,p_0^2}{2}\,\exp\left(\frac{2\sqrt{2}\,p\left(b_0^2+2\delta\,b_0\,(1+a)+2\,a+1\right)^{1/2}}{b_0(-T)^{1/2}}\right)$$
$$+ a\,p^2\,Ei\left(\frac{2\sqrt{2}\,p\left(b_0^2+2\delta\,b_0\,(1+a)+2\,a+1\right)^{1/2}}{b_0(-T)^{1/2}}\right). \tag{122}$$

   (c) $\mathbf{a \neq \{-1, -\frac{1}{2}\}}$ and $\mathbf{b = -1}$: The $V(T)$ expression in Equation (57) is:

$$V(T) = V_0 + \frac{p^2\,p_0^2}{2}\,\exp\left(2p\left[-\frac{\delta\,(a+1)}{b_0} + \delta_1\sqrt{\frac{a^2}{b_0^2} - \frac{T}{2}}\right]^{-1}\right)$$
$$+ a\,p^2\,Ei\left(2p\left[-\frac{\delta\,(a+1)}{b_0} + \delta_1\sqrt{\frac{a^2}{b_0^2} - \frac{T}{2}}\right]^{-1}\right). \tag{123}$$

   (d) $\mathbf{a = -1}$ and $\mathbf{b = 0}$: The $V(T)$ expression in Equation (65) is:

$$V(T) = V_0 + \frac{p^2\,p_0^2}{2}\,\exp\left(\frac{2\sqrt{2}\,p(1-b_0^2)^{1/2}}{b_0\,T^{1/2}}\right) - p^2\,Ei\left(\frac{2\sqrt{2}\,p(1-b_0^2)^{1/2}}{b_0\,T^{1/2}}\right). \tag{124}$$

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(e) **a** $= -$**1** and **b** $= -$**1**: The $V(T)$ expression in Equation (80) is:

$$V(T) = V_0 + \frac{p^2\,p_0^2}{2}\,\exp\left(2p\left[\frac{1}{b_0^2} - \frac{T}{2}\right]^{-1/2}\right) - p^2\,Ei\left(2p\left[\frac{1}{b_0^2} - \frac{T}{2}\right]^{-1/2}\right). \tag{125}$$

(f) **a** $= -\frac{1}{2}$ and **b** $=$ **0**: The $V(T)$ expression in Equation (90) is:

$$V(T) = V_0 + \frac{p^2\,p_0^2}{2}\,\exp\left(2p\left[-\frac{2(b_0+\delta)}{b_0\,T}\right]^{1/2}\right) - \frac{p^2}{2}\,Ei\left(2p\left[-\frac{2(b_0+\delta)}{b_0\,T}\right]^{1/2}\right). \tag{126}$$

(g) **a** $= -\frac{1}{2}$ and **b** $= -$**1**: The $V(T)$ expression in Equation (100) is:

$$\begin{aligned} V(T) =& V_0 + \frac{p^2\,p_0^2}{2}\,\exp\left(2p\left[-\frac{\delta}{2b_0} \pm \sqrt{\frac{1}{4b_0^2} - \frac{T}{2}}\right]^{-1}\right) \\ & - \frac{p^2}{2}\,Ei\left(2p\left[-\frac{\delta}{2b_0} \pm \sqrt{\frac{1}{4b_0^2} - \frac{T}{2}}\right]^{-1}\right). \end{aligned} \tag{127}$$

The other subcases of section 3.2 can be computed by the same manner as in the previous simple examples.

4.2. Logarithmic Scalar Field Solutions

The exponential scalar field is defined by:

$$\phi(T) = p_0\,\ln(p\,r(T)). \tag{128}$$

By assuming a coframe power-law ansatz, we will solve the Equation (16) to find the scalar potential $V(\phi)$ for Equations (23) and (128):

$$\begin{aligned} \frac{dV}{d\phi} &= (a-1)\,p_0\,p^2\,\exp(-2\phi/p_0), \\ \Rightarrow\ V(\phi) &= V_0 + \frac{(1-a)}{2}\,p_0^2\,p^2\,\exp(-2\phi/p_0). \end{aligned} \tag{129}$$

By substituting Equation (128) into Equation (129) and setting $r = r(T)$, we obtain that:

$$V(T) = V_0 + \frac{(1-a)}{2}\,p_0^2\,r^{-2}(T). \tag{130}$$

1. **Power-law ansatz with** $A_3 = c_0$: As in section 4.1, the $F(T)$ solution is under the same form as Equation (32) in section 3.1. The potential $V(T)$ will be the Equation (130) with $r(T)$ defined by Equation (30) as:

$$V(T) = V_0 + \frac{(1-a)}{2}\,p_0^2\left(\frac{b_0}{2c_0}\right)^{\frac{2}{(1+b)}}\left(1 + \frac{c_0^2}{2}\,T\right)^{\frac{2}{(1+b)}}. \tag{131}$$

2. **Power-law ansatz with** $A_3 = r$: As in section 4.1, we find the same $F(T)$ solution forms than section 3.2, but only the $V(T)$ potential expressions change for the Equation (130) form for each subcases treated in this section. The Equation (130) will be for the simplest cases:

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(a) **a** $=$ **b** $=$ **0**: The potential $V(T)$ in Equation (40) is:

$$V(T)=V_0+\frac{p_0^2\,b_0^2}{4(1+\delta\,b_0)^2}(-T). \tag{132}$$

(b) **a** $\neq\{$**−1**, $-\frac{1}{2}\}$ and **b** $=$ **0**: The $V(T)$ expression in Equation (48) is:

$$V(T)=V_0+\frac{p_0^2\,b_0^2\,(1-a)}{4\left(b_0^2+2\delta\,b_0\,(1+a)+2\,a+1\right)}\,(-T). \tag{133}$$

(c) **a** $\neq\{$**−1**, $-\frac{1}{2}\}$ and **b** $=$ **−1**: The $V(T)$ expression in Equation (57) is:

$$V(T)=V_0+\frac{(1-a)}{2}\,p_0^2\left[-\frac{\delta\,(a+1)}{b_0}+\delta_1\sqrt{\frac{a^2}{b_0^2}-\frac{T}{2}}\right]^2. \tag{134}$$

(d) **a** $=$ **−1** and **b** $=$ **0**: The $V(T)$ expression in Equation (65) is:

$$V(T)=V_0+\frac{p_0^2\,b_0^2}{2(1-b_0^2)}\,T. \tag{135}$$

(e) **a** $=$ **−1** and **b** $=$ **−1**: The $V(T)$ expression in Equation (80) is:

$$V(T)=V_0+p_0^2\left[\frac{1}{b_0^2}-\frac{T}{2}\right]. \tag{136}$$

(f) **a** $=-\frac{1}{2}$ and **b** $=$ **0**: The $V(T)$ expression in Equation (90) is:

$$V(T)=V_0-\frac{3p_0^2\,b_0}{8(b_0+\delta)}\,T. \tag{137}$$

(g) **a** $=-\frac{1}{2}$ and **b** $=$ **1**: The $V(T)$ expression in Equation (100) is:

$$V(T)=V_0+\frac{3}{4}\,p_0^2\left[-\frac{\delta}{2b_0}\pm\sqrt{\frac{1}{4b_0^2}-\frac{T}{2}}\right]^2. \tag{138}$$

As in section 4.1, the other subcases of section 3.2 can be computed by the same manner as in the current section examples.

5. Concluding Remarks

We have treated, simplified and solved the static teleparallel $F(T)$ FEs by summarizing everything to the single Equation (21) to obtain the teleparallel $F(T)$ solutions. We have obtained solutions for subcases similar to those treated in ref. [36] by replacing the astrophysical perfect fluid by a scalar field as energy-momentum source with only the radial dependence (r-coordinate). This scalar field source and then the conservation law solutions (described by Equation (16) for the potential $V(T)$) can represent a local source of DE where an astrophysical system evolves. We obtain teleparallel $F(T)$ solutions with several common points with those of the linear perfect fluids found in ref. [36]. More concretely, Equation (21) allows to generate all $F(T)$ solutions for any type of ansatz, scalar field $\phi(T)$ and any type of scalar potential $V(T)$ solutions of Equation (16). By this approach, the general forms of the $F(T)$ solutions obtained in section 3 remain intact, regardless of the potential $V(T)$. This constitutes a major strength favoring the approach based on Equation (21). Section 4 shows that one can also replace the power-law scalar field $\phi(T)$ used in section 3 by exponential and logarithmic $\phi(T)$

(sections 4.1 and 4.2 respectively) while keeping the same forms of $F(T)$ solutions as in section 3 and respecting Equation (16). In short, this work simultaneously and logically completes the advances made in ref. [36].

All this will make easier the use of teleparallel $F(T)$ solutions and scalar field sources for various future astrophysical applications involving systems immersed and evolving in a DE dominating universe that can fundamentally be described by a scalar field. We should note that a relatively similar approach had been used in several recent works in teleparallel $F(T)$ gravity, but for KS and TRW spacetimes depending only on the time-coordinate [39,40]. These latest advances have at the same time laid the foundations for this present paper by clarifying some points on the DE. However, the present contribution with its new Teleparallel $F(T)$ solutions will allow to deal much more easily and realistically with the various types of possible astrophysical systems in the DE, harmonizing better with cosmological models. This has become possible with the powerful tools of Teleparallel Gravity.

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Abbreviations

The following abbreviations are used in this manuscript:

AL	Alexandre Landry	BH	Black Hole
DE	Dark Energy	DoF	Degree of Freedom
EoS	Equation of State	FE	Field Equation
GR	General Relativity	KS	Kantowski-Sachs
KV	Killing Vector	NGR	New General Relativity
NS	Neutron star	TEGR	Teleparallel Equivalent of General Relativity
TRW	Teleparallel Robertson-Walker	WD	White Dwarf

Appendix A. Field Equation Components

This appendix presents the exact FE components found and used in refs. [35,36]. There are general, constant A_3 and $A_3 = r$ power-law ansatz FE components useful for the paper purposes.

Appendix A.1. General components

$$\frac{g_1}{k_1} = \frac{\left[-A_2\,A_3^2\,A_1'' - A_1\,A_2\,A_3\,A_3'' + A_1\,A_2\,A_3'^2 + (A_1\,A_2)'\,A_3\,A_3' + A_3^2\,A_1'\,A_2' - A_1\,A_2^3\right]}{\left[A_1\,A_2\,A_3\,A_3' + A_2\,A_3^2\,A_1' + \delta\,A_1\,A_2^2\,A_3\right]} \tag{A1}$$

$$g_2 = \frac{1}{A_1\,A_2^3\,A_3}\left[-A_1\,A_2\,A_3'' + (A_1\,A_2)'\,A_3'\right] \tag{A2}$$

$$g_3 = \frac{1}{A_1\,A_2^3\,A_3^2}\left[-A_1\,A_2\,A_3\,A_3'' - A_1\,A_2\,A_3'^2 - A_2\,A_3\,A_3'\,A_1' - \delta\,A_1\,A_2^2\,A_3' + A_1\,A_3\,A_2'\,A_3' - \delta\,A_2^2\,A_3\,A_1'\right], \tag{A3}$$

$$k_2 = \frac{1}{A_2^2\,A_3}\left[A_3' + \delta\,A_2\right]. \tag{A4}$$

Appendix A.2. $A_3 = c_0 =$ constant power-law components

The Equations (A1)–(A4) with the Equations (23) power-law ansatz are:

$$\frac{g_1}{k_1} = \frac{\left[(a(1-a+b))\, r^{-2b-2} - \left(\frac{b_0}{c_0}\right)^2\right]}{\left[a\, r^{-b-1} + \left(\frac{\delta\, b_0}{c_0}\right)\right] r^{-b}}, \quad g_2 = 0, \quad g_3 = -\left(\frac{\delta\, a}{b_0\, c_0}\right) r^{-b-1}, \quad k_2 = \left(\frac{\delta}{b_0\, c_0}\right) r^{-b}. \tag{A5}$$

Appendix A.3. $A_3 = r$ power-law components

The Equations (A1)–(A4) with Equations (23) power-law ansatz are:

$$\frac{g_1}{k_1} = \frac{\left[\left(2a - a^2 + ab + b + 1\right) r^{-2b} - b_0^2\right]}{\left[(a+1)\, r^{2(1-b)-1} + \delta\, b_0\, r^{(1-b)}\right]}, \qquad g_2 = \frac{(a+b)}{b_0^2}\, r^{-2b-2},$$
$$g_3 = \frac{1}{b_0^2}\left[(-a+b-1)\, r^{-2b-2} - \delta\, b_0\, (a+1)\, r^{-b-2}\right], \qquad k_2 = \frac{1}{b_0^2}\left[r^{-2b-1} + \delta\, b_0\, r^{-b-1}\right]. \tag{A6}$$

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