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Fuzzy Fractal Brownian Motion: Extensions and Applications

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Abstract: Over the course of this research, we have developed a multifaceted framework for pricing derivatives and modelling asset dynamics that integrates fuzzy set theory, fractional Brownian motion (fBm), jump processes, and classical no-arbitrage principles under a risk-neutral measure. This approach does not merely extend one well-known model. It synthesises multiple advanced elements from mathematical finance and stochastic calculus into a single comprehensive structure. We began by defining an SDE in which fuzzy concepts (upper/lower functions for membership levels) are combined with fractional Brownian motion having Hurst exponent H > ½. Standard fractional finance work typically involves only a fractional Brownian model or a deterministic fuzzy model, but not both simultaneously. Our step of allowing the SDE's drift and diffusion terms to depend on fuzzy processes introduces extra flexibility in capturing uncertainty. This alone is more advanced than most single-parameter fractional or fuzzy finance models. Instead of using a purely classical PDE $\partial_t u + Lu = 0$, we designed a custom PDE with embedded exponential and hyperbolic functions for negative drift. This is an extension of standard PDE–SDE bridging via Feynman–Kac which typically places all operators on the left-hand side, ensuring a known diffusion generator. Our method is a hybrid approach, creating a direct path to solving an enriched PDE that captures both fractional behaviour through exponents involving σt^{2H} and fuzzy membership constraints through special boundary or integral conditions. We then invoke Girsanov's theorem to shift from the "real-world" measure **P** to a "risk-neutral" measure **Q**. This step is typical in modern finance, but we apply it to a fuzzy fractional mixed jump environment. We demonstrate that the drift changes while the diffusion remains unaffected, combined with our no-arbitrage condition $\mu = r - \lambda(J-1)$, it ensures consistency with fundamental asset pricing principles. This bridging of fuzzy fractional assets to classical "risk-neutral" arguments is innovative because standard fuzzy finance seldom performs measure change in this manner. Additionally, we draw on Cheridito's mixed fractional Brownian framework, referencing the condition H>¾ for certain regularity or H> ½ for existence-uniqueness. We then incorporate Poisson i.i.d. jump sequences (V_i) to capture jump risk. This yields an SDE or PDE that unifies continuous fractional noise, discrete jumps, and fuzzy uncertainty. Such synergy between fractional noise, fuzzy membership, and jumps is rarely attempted, making our model significantly more unified than classical jump-diffusions or single-parameter fractional SDEs. Our final expressions for option pricing iterated expectations with respect to $\sigma(\sum Y_i)$, integrals involving $\Phi(d_1)$ and $\Phi(d_2)$, and summations over Poisson counts reflect an extension of the standard Black–Scholes–Merton logic. By combining fuzzy membership, fractional exponents, jump expansions under the risk-neutral measure, we arrived at a unique closed-form payoff integral. In other words, there is only one semimartingale (or fuzzy fractional path) that satisfies our SDE/PDE.

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1. Introduction

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We will now describe the dynamics of a particle, considering the uncertainties involved. The underlying stochastic process is modelled by a fractional Brownian motion (fBm), known for its long-range dependence and self-similarity, which makes it suitable for financial modelling.

The model involves fuzzy processes, denoted by $x_\varepsilon(t)$, which are functions that map the time parameter t to a set of possible values in R with certain degrees of membership. The model also includes a linear coefficient μ and a volatility coefficient σ, both considered constant [22].

To find the closed explicit solution for $\mu \geq 0$, the system of equations is solved to obtain the fuzzy solutions $X_\varepsilon^l(t)$and $X_\varepsilon^u(t)$ for the lower and upper bounds, respectively.

Similarly, the solution is derived for $\mu < 0$ by obtaining the fuzzy solutions $X_\varepsilon^l(t)$and $X_\varepsilon^u(t)$ for the lower and upper bounds, respectively. We will use Feynman-Kac and Itô theorems from the books "Modelling with Itô Stochastic Differential Equations" by Edward Allen Llendeto [24] and "Arbitrage Theory in Continuous Time" by Björk [32] to extend the unique solution to the fuzzy stochastic fractional differential equation (FSFDE) for $\mu \geq 0$. Theorem provides conditions under which a unique solution exists for FSFDE. The statement of the theorem is as follows: suppose that the matrix-valued functions a, b, and σ satisfy the following conditions:

- a is Lipschitz continuous with constant L,
- b is continuous,
- σ is locally Lipschitz continuous, and there exist constants K and p such that $||\sigma(x)|| \leq K(1 + ||x||^p)$ for all x.

Then, for any initial condition X(0) = x(0) with probability 1, there exists a unique solution to the FSFDE in the form: $dX(t) = [aX(t) + b]dt + \sigma(X(t))dW(t)$, where W(t) is a standard d-dimensional Wiener process.

The key idea behind the theorem is to show that when h > ½, the matrix-valued function a, the vector-valued function b, and the matrix-valued function σ satisfy certain regularity conditions that enable us to apply standard techniques from the theory of stochastic differential equations to prove the existence and uniqueness of a solution to the FSFDE. In particular, the Lipschitz continuity condition on 'a' ensures the local boundedness of the solution, while the local Lipschitz continuity condition on σ ensures that the solution is almost surely continuous. The continuity condition on 'b' ensures that the drift term is well-defined and continuous.

2. Applications

We start with the typical partial differential equation used in mathematical finance. $X(t) = X_0 + \int_0^t \mu X(s)ds + \int_0^t \sigma X(s)dB_H(s)$, $X_0 = X(0)$, where X(t) is the value of the financial quantity at time t, X(0) is the initial value of X at time 0, μ and σ are constants representing the drift and volatility of X, respectively, and $B_H(s)$ is the fractional Brownian motion (fBm) process which has long-range dependence and self-similarity properties. We already covered why the fBm process is used in financial modelling. In summary, it can capture the long-term dependencies in financial data, which are not always captured by traditional Brownian motion processes. This makes it suitable for modelling financial quantities that exhibit long-term trends or memory effects. Unlike crisp partial differential equation, the equation involving uncertainties is modelled using fuzzy processes. In the case of linear coefficients, an explicit solution can be obtained. The fractional fuzzy stochastic differential equation (FFSDE) satisfies the assumptions of Yamada-Watanabe-Kunita theorem and the Kushner-Stratanovich equation. The Yamada-Watanabe-Kunita theorem provides conditions for the existence and uniqueness of solutions to certain types of SDEs [33]. Statement of the Theorem - consider the following stochastic differential equation: $dXt = b(Xt)dt + \sigma(Xt)\, dWt$, where:

- X_t is the stochastic process to be solved.
- $b(X_t)$ is a drift function that depends on X_t.
- $\sigma(X_t)$ is a diffusion function that depends on Xt.
- W_t is a standard Wiener process (Brownian motion).

The Yamada-Watanabe-Kunita theorem requires the following assumptions to be satisfied:

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1. The functions b(X) and σ(X) are globally Lipschitz continuous in X with a Lipschitz constant that is independent of time t. In other words, there exists a constant K such that for all x, y in the state space of X, we have:

$$|b(x) - b(y)| \leq K\,|x - y|$$
$$|\sigma(x) - \sigma(y)| \leq K\,|x - y|$$

2. The functions b(X) and σ(X) satisfy a linear growth condition. That is, there exist constants M and L such that for all x in the state space of X, we have:

$$|b(x)| \leq M(1 + |x|)$$
$$|\sigma(x)| \leq L(1 + |x|)$$

SDE is nondegerate meaning the diffusion(covariance) matrix $\sigma\sigma^T$ is positive definite everywhere. Under these assumptions, the Yamada-Watanabe-Kunita theorem guarantees the existence and uniqueness of a strong solution to the stochastic differential equation:

$$dX(t) = bX(t)dt + \sigma X(t)\,dWt$$

The theorem is essential because it establishes the conditions under which solutions to certain stochastic differential equations exist and are unique. The key idea behind the Yamada-Watanabe-Kunita theorem is to show that the nonlinearity of the drift and diffusion coefficients can be controlled by the Lipschitz and growth conditions, respectively, so that the SDE admits a unique solution. The theorem has been extended to FFSDEs, which are SDEs with fuzzy coefficients. In this context, the theorem provides conditions for the existence and uniqueness of solutions to FFSDEs and has important applications in the modelling of systems with uncertain or imprecise information.

The second important condition is the FFSDE model must satisfy the Kushner-Stratonovich equation [34]. It arises in the context of state estimation problems, where the goal is to estimate the unobservable state of a dynamic system based on noisy measurements. The Kushner-Stratonovich equation applies to a class of SDEs that can be written in the following form: $dX_t = f(X_t, t)dt + g(X_t, t)\,dW_t$, where:

X_t is the unobservable state of the system at time t.

$f(X_t, t)$ is a drift function that describes how the state X_t evolves over time. It depends on both the current state X_t and the time t.

$g(X_t, t)$ is a diffusion function that accounts for the effect of random noise on the system. It also depends on the state X_t and time t.

dW_t represents the increment of a Wiener process (Brownian motion), which represents the random noise or uncertainties in the system.

The main challenge in the theory of filtering is to find an optimal estimate of the state X(t) given the available measurements up to time t. This estimate is usually denoted by $\hat{x}_t$ and is known as the filter. Solving the Kushner-Stratonovich equation involves finding the filter $\hat{x}_t$ that minimizes the mean squared error between the true state X(t) and the estimated state $\hat{x}_t$. The key idea behind the equation is to express the SDE in a form that separates the drift and diffusion terms, so that it can be solved using standard techniques from the theory of stochastic calculus. The Kushner-Stratonovich equation takes the following form: $dX(t) = [f(X(t)) + 1/2g(X(t))g'(X(t))]dt + g(X(t))dW(t)$, where g' denotes the derivative of g with respect to its argument. The equation represents an alternative to the Itô formula, which is commonly used to derive solutions to SDEs with linear drift and diffusion coefficients. In contrast to the Itô formula, which involves a stochastic integral with respect to the Wiener process, the Kushner-Stratonovich equation involves a deterministic integral with respect to the derivative of the diffusion coefficient. We have explained two important theorems that were used in this work's derivations.

The equation we are deriving is a stochastic differential equation (SDE) describing a system with two state variables, $X_l^{\varepsilon a}(t)$ and $X_u^{\varepsilon a}(t)$. The SDE has a Wiener process W, which represents the random noise in the system. The SDE can be divided into two cases depending on the value of the drift coefficient μ. When μ ≥ 0, the unique solution to the SDE can be obtained using Feynman-Kac Theorem as in example 4.7 from the book "Modeling with Itô Stochastic Differential Equations" by E. Allen. The solution for $X_l^{\varepsilon a}(t)$ and $X_u^{\varepsilon a}(t)$ can be expressed in a matrix form. The solution involves exponentials of μt, α, and the Wiener process W. When μ < 0, the unique solution to the SDE can also

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be obtained using the same theorem. Again, the solution for $X_l^{\varepsilon a}(t)$ and $X_u^{\varepsilon a}(t)$ can be expressed in a matrix form. The solution involves hyperbolic functions of μt (cosh and sinh), exponentials of α and the Wiener process W. In both cases, the solution involves an integral over the Wiener process W, which is a stochastic integral. The integral is evaluated using the Itô integral, which is a stochastic calculus tool used to integrate stochastic processes with respect to the Wiener process. The integral is approximated using the Stratonovich integral, which is a different stochastic calculus tool that gives a different result than the Itô integral. The result of the Stratonovich integral is denoted by the symbol ⟨ ⟩ in the equation we derive. The final result is an approximation of the solution to the SDE in terms of X(0), $X_l^{\varepsilon 1}(0)$, $X_u^{\varepsilon 1}(0)$, α, σ, μ, and the Wiener process W. The solution is a fuzzy solution because it involves a stochastic process, and the value of $X^{\varepsilon}(t)$ is not deterministic but depends on the realisation of the Wiener process W. Now we will derive it.

FFSDE involves a stochastic process X that maps from R+ × Ω to F(R), where F(R) is the space of fuzzy real numbers. $X_l^1(t)$ *and* $X_u^1(t)$ are functions that map from R+ × Ω to R and define the lower and upper bounds of the fuzzy process $[X^1(t)] = X_l^1(t)$ *and* $X_u^1(t)$.

To obtain a closed explicit form of the solution for μ ≥ 0, a system of equations is derived. The solution to the system provides a unique solution, $X_l^{\varepsilon 1}(t)$ *and* $X_u^{\varepsilon 1}(t)$, which is expressed as an explicit formula involving the initial values $X_l^{\varepsilon 1}(0)$ *and* $X_u^{\varepsilon 1}(0)$. This formula involves the exponential function and the fBm process $B_H(s)$. For every $\alpha \in [0, 1]$, a similar procedure is applied to obtain a new system of equations, which is similar to the previous system but involves the fBm process $dB_H^{\varepsilon}(s)$ instead of $dB_H(s)$. The solution to this new system of equations also involves an explicit formula, $X_l^{\varepsilon a}(t)$ and $X_u^{\varepsilon a}(t)$ which is expressed as an integral involving the initial values and the fBm process $dB_H^{\varepsilon}(s)$.

So, the given crisp P.D. equation is $X(t) = X_0 + \int_0^t \mu X(s)ds + \int_0^t \sigma X(s)dB_H(s)$, $X_0 = X(0)$. This is a crisp PDE. To obtain an explicit solution, we first express X(s) in terms of its initial value X_0 and the fBm process $B_H(s)$. $X(s) = X_0 + \int_0^s \mu X(r)dr + \int_0^t \sigma X(r)dB_H(r)$ and apply Itô's formula to the function $f(X(s) = \exp(aX(s))$ for some constant a:

$$df(X(s)) = a\exp(aX(s)ds) + ½\, a^2\, [\exp(aX(s)d(X(s)]^2).$$

Substitute for dx, we get:

$$df(X(s)) = a\exp(aX(s)ds)\,(\mu X(s)ds + \sigma X(s)dB_H(s)) + ½\, a^2 [e^{aX(s)}\sigma^2 X(s)]^2 ds.$$

Integrate both sides from 0 to t, we get:

$$e^{aX(t)} - e^{aX(0)} = \int_0^t \alpha e^{\alpha X(s))}\left(\mu X(s)ds + \sigma X(s)dBH\,(s)\right) + +1/2\,\alpha^2 \int_0^t e^{\alpha X(s)}\sigma^2\, X^2(s)\, ds.$$

Solving for X(t) we obtain: $X = \frac{1}{a}\ln\left(e^{aX(0)} + \int_0^t \alpha e^{\alpha X(s))}\left(\mu X(s)ds + \sigma X(s)dBH\,(s)\right) + 1/2\,\alpha^2 \int_0^t e^{\alpha X(s)}\sigma^2\, X^2(s)\, ds\right)$.

This is the solution to the crisp SDE with a linear coefficient. We are interested in the fuzzy SDE. Consider the FFSDE which satisfies the theorem we proved earlier that there is the existence of a strong and unique solution.

$$\sup_{t\in I}[\, d_\infty^2[(X^{\varepsilon}(t), X_0^{\varepsilon} + \int_0^t f\big(s, X^{\varepsilon}(s)\big)ds + \langle \int_0^t g\big(s, X^{\varepsilon}(s)\big)dB_H^{\varepsilon}(s)\rangle)] = 0$$

Then, the given equation is:

$$\mathrm{X(t)} = X_0 + \int_0^t \mu X(s)ds + \langle \int_0^t \frac{\sigma}{2}(X_l^1(s) + X_u^1(s))dB_H(s)\rangle)$$

To obtain a closed form solution for μ ≥ 0, we obtain the following system of equations for lower and upper bounds.

$$\begin{cases} \mathrm{X}_l^1(\mathrm{t}) = X_l^1(0) + \int_0^t \mu X_l^1(s)ds + \langle \int_0^t \frac{\sigma}{2}(X_l^1(s) + X_u^1(s))dB_H(s)\rangle) \\ \mathrm{X}_u^1(\mathrm{t}) = X_u^1(0) + \int_0^t \mu X_u^1(s)ds + \langle \int_0^t \frac{\sigma}{2}(X_l^1(s) + X_u^1(s))dB_H(s)\rangle) \end{cases}$$

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We add two equations and simplify to get.

$$\mathrm{X}_l^1(\mathrm{t}) + \mathrm{X}_u^1(\mathrm{t}) = X_l^1(0) + X_u^1(0) + \int_0^t \mu(X_l^1(s) + X_u^1(s))ds + \langle \int_0^t \sigma(X_l^1(s) + X_u^1(s))dB_H(s)\rangle)$$

We will now get rid of fBm and replace it with normal Brownian motion. So, we will substitute a derivative of (5) into above equation. Recall (5):

$$B_{H,\varepsilon}(t) = a\int_0^t \varphi^\varepsilon(s)ds + \varepsilon^a B(t)$$

where $\varphi^\varepsilon(s) = \int_0^t (t - s + \varepsilon)^{a-1}]dB(s)$

So, taking derivative $dB_{H,\varepsilon}(t) = \frac{d[a\int_0^t \varphi^\varepsilon(s)ds + \varepsilon^a B(t)]}{dt} = a\varphi^\varepsilon(t) + \varepsilon^a dB(t)$

Substitute these identities into expression for $\mathrm{X}_l^1(\mathrm{t}) + \mathrm{X}_u^1(\mathrm{t})$ will convert this FFSDF into a Wiener process. $\langle\int()\rangle$ this symbol is Stratonovich integral, and this is why we needed the Kushner-Stratonovich theorem so we could go back and fourth between Wiener and fractal processes. The Kushner-Stratonovich equation, which was mentioned earlier in the text, provides a way to relate SDEs driven by a Wiener process to SDEs driven by a fractional Brownian motion process or other fractal processes. This theorem allows for the conversion between the two types of processes, which is necessary in this case to convert the FFSDE involving a fractional Brownian motion into an SDE involving a standard Wiener process. By using the Kushner-Stratonovich theorem and the Stratonovich integral, it becomes possible to work with the more familiar Wiener process while still accounting for the fractal nature of the original process through the appropriate transformations and integrals.

$$\mathrm{X}_l^1(\mathrm{t}) + \mathrm{X}_u^1(\mathrm{t}) = X_l^1(0) + X_u^1(0) + \int_0^t \mu(X_l^1(s) + X_u^1(s))ds + \langle \int_0^t \sigma\big(X_l^1(s) + X_u^1(s)\big)\big(a\varphi^\varepsilon(t) + \varepsilon^a dB(t)\big)\rangle) = X_l^1(0) + X_u^1(0) + \int_0^t \big(\mu + \sigma a\varphi^\varepsilon(s)\big)\big(X_l^1(s) + X_u^1(s)\big)ds + \int_0^t \big(X_l^1(s) + X_u^1(s)\big)\sigma\varepsilon^a dB(t)$$

Now, we can use the solution that we used to solve the crisp stochastic differential equation and substitute back fBm expression from (5).

$$\mathrm{X}_l^{\varepsilon 1}(\mathrm{t}) + \mathrm{X}_u^{\varepsilon 1}(\mathrm{t}) = (X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0))\exp\left(\mu t + \sigma a\int_0^t \varphi^\varepsilon(s)ds - \frac{1}{2}\sigma^2\varepsilon^{2a}t + \sigma\varepsilon^a W(t)\right) = (X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0))\exp\left(\mu t + \sigma B_H^\varepsilon(t) - \frac{1}{2}\sigma^2\varepsilon^{2a}t\right)$$

(27)

Finally, we apply similar approach for every a ∈ [0, 1] to get the system of equations.

$$\begin{cases} \mathrm{X}_l^{\varepsilon a}(\mathrm{t}) = X_l^{\varepsilon a}(0) + \int_0^t \mu X_l^{\varepsilon a}(s)ds + \langle\int_0^t \frac{\sigma}{2}(X_l^{\varepsilon 1}(s) + X_u^{\varepsilon 1}(s))dB_H^\varepsilon(s)\rangle) \\ \mathrm{X}_u^{\varepsilon a}(\mathrm{t}) = X_u^{\varepsilon a}(0) + \int_0^t \mu X_u^{\varepsilon a}(s)ds + \langle\int_0^t \frac{\sigma}{2}(X_l^{\varepsilon 1}(s) + X_u^{\varepsilon 1}(s))dB_H^\varepsilon(s)\rangle) \end{cases}$$

For μ ≥ 0, we apply the solution derived above, namely (27) for upper and lower bounds:

$$\mathrm{X}_l^{\varepsilon a}(\mathrm{t}) = X_l^{\varepsilon a}(0) + \int_0^t \mu X_l^{\varepsilon a}(s)ds + \int_0^t \frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0))\exp\left(\mu s + \sigma B_H^\varepsilon(s) - \frac{1}{2}\sigma^2\varepsilon^{2a}s\right)dB_H^\varepsilon(s),$$

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$$\mathrm{X}_u^{\varepsilon a}(\mathrm{t}) = X_u^{\varepsilon a}(0) + \int_0^t \mu X_u^{\varepsilon a}(s)ds + \int_0^t \frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0))\exp\left(\mu s + \sigma B_H^{\varepsilon}(s) - \frac{1}{2}\sigma^2 \varepsilon^{2a} s\right) dB_H^{\varepsilon}(s).$$

In terms of the Wiener process, we again get rid of fBm and replace it with normal Brownian motion. So, we will substitute a derivative of (5) into above equation same as we did before.

$$\mathrm{X}_l^{\varepsilon a}(\mathrm{t}) = X_l^{\varepsilon a}(0) + \int_0^t \mu X_l^{\varepsilon a}(s)ds + \int_0^t \frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)) \exp\left(\mu s + \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \sigma \varepsilon^a B(t) - \frac{1}{2}\sigma^2 \varepsilon^{2a} s\right) ds\left(a\varphi^{\varepsilon}(t) + \varepsilon^a dB(t)\right) = X_l^{\varepsilon a}(0) + \int_0^t \mu X_l^{\varepsilon a}(s)ds + a\int_0^t \varphi^{\varepsilon}(t)\frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)) \exp\left(\mu s + \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \sigma \varepsilon^a B(t) - \frac{1}{2}\sigma^2 \varepsilon^{2a} s\right) ds + \left(\varepsilon^a \frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)) \int_0^t \exp\left(\mu s + \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \sigma \varepsilon^a B(t) - \frac{1}{2}\sigma^2 \varepsilon^{2a} s\right) dB(t)\right),$$

Similarly:

$$\mathrm{X}_u^{\varepsilon a}(\mathrm{t}) = X_u^{\varepsilon a}(0) + \int_0^t \mu X_u^{\varepsilon a}(s)ds + \int_0^t \frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)) \exp\left(\mu s + \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \sigma \varepsilon^a B(t) - \frac{1}{2}\sigma^2 \varepsilon^{2a} s\right) ds\left(a\varphi^{\varepsilon}(t) + \varepsilon^a dB(t)\right) = X_u^{\varepsilon a}(0) + \int_0^t \mu X_u^{\varepsilon a}(s)ds + a\int_0^t \varphi^{\varepsilon}(t)\frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)) \exp\left(\mu s + \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \sigma \varepsilon^a B(t) - \frac{1}{2}\sigma^2 \varepsilon^{2a} s\right) ds + \left(\varepsilon^a \frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)) \int_0^t \exp\left(\mu s + \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \sigma \varepsilon^a B(t) - \frac{1}{2}\sigma^2 \varepsilon^{2a} s\right) dB(t)\right)$$

So, to summarise what we have done and give a high-level overview of the steps involved, this is what we did so far. To solve this SDE, we need to find the probability distribution of X(t) given its initial condition X(0). This probability distribution is given by the rough Fokker-Planck equation [35], which is a partial differential equation that describes the time evolution of the probability density function of X(t). However, in the case of our SDE, we can simplify the problem by noticing that $\mathrm{X}_l^{\varepsilon a}(\mathrm{t})$ *and* $\mathrm{X}_u^{\varepsilon a}(\mathrm{t})$ are linear combinations of X, which means that their probability distributions can be obtained from the probability distribution of X(t). We can then use Itô's lemma to transform the SDE into an equivalent SDE for $\mathrm{X}_l^{\varepsilon a}(\mathrm{t})$ *and* $\mathrm{X}_u^{\varepsilon a}(\mathrm{t})$ [24] [27]. Itô's lemma is a rule that allows us to find the SDE satisfied by a function of a stochastic process. The lemma states that if Y(t) is a function of X(t), then the SDE satisfied by Y(t) is given by: $dY(t) = (\partial Y/\partial t)\, dt + (\partial Y/\partial X)\, dX + ½\, (\partial^2 Y/\partial X^2)\, (dX)^2$, where (∂Y/∂t) is the partial derivative of Y with respect to time, (∂Y/∂X) is the partial derivative of Y with respect to X, and (∂2Y/∂X2 is the second partial derivative of Y with respect to X. Using Itô's lemma, we can find the SDEs satisfied by $\mathrm{X}_l^{\varepsilon a}(\mathrm{t})$ *and* $\mathrm{X}_u^{\varepsilon a}(\mathrm{t})$. The SDEs have the same form as the original SDE, but with different drift and diffusion coefficients that depend on α. The final solution involves an integral over the Wiener process W, which is a stochastic integral. The integral is evaluated using the Itô integral or the Stratonovich integral, depending on the convention used. The solution is a fuzzy solution because it involves a fuzzy stochastic process, and the value of $X^{\epsilon}(t)$ is not deterministic but depends on the realisation of the Wiener process W. We will derive a generic solution for stochastic differential equation of the following form which can be applied to each specific case:

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$$\mathrm{X(t)} = X(0) + \int_0^t \mu X(s)ds$$
$$+ \ a \int_0^t \frac{\sigma}{2}(X(0)) \exp\left(\mu s + \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \ \sigma\varepsilon^a B(t)\right.$$
$$\left.- \frac{1}{2}\sigma^2\varepsilon^{2a}s\right) ds$$
$$+ \left(\varepsilon^a \frac{\sigma}{2}(X(0)) \int_0^t \exp \left(\mu s + \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \ \sigma\varepsilon^a B(t)\right.\right.$$
$$\left.\left.- \frac{1}{2}\sigma^2\varepsilon^{2a}s\right) dB(t)\right)$$

Re-writing in terms of derivatives, we need to solve the stochastic differential equation of the form:

$$\mathrm{dX(t)} = \mu X(t)dt + \ a\frac{\sigma}{2}X(0)) \exp\left(\mu t + \sigma a \int_0^t \varphi^{\varepsilon}(u)du + \ \sigma\varepsilon^a B(t)\right.$$
$$\left.- \frac{1}{2}\sigma^2\varepsilon^{2a}s\right)$$
$$+ \left(\varepsilon^a \frac{\sigma}{2}(X(0))\exp \left(\mu t + \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \ \sigma\varepsilon^a B(t)\right.\right.$$
$$\left.\left.- \frac{1}{2}\sigma^2\varepsilon^{2a}s\right) dB(t)\right)$$

with initial condition X(0), we will use Itô's lemma to obtain the solution:

Let Y(t, B(t)) = $e^{\mu t}$X(t). Then, using Itô's lemma, product rule and definition of X to obtain the solution:

$$\mathrm{dY(t, B(t))} = e^{\mu t}dX(t) + e^{\mu t}\frac{\sigma^2}{2}X(t)dt = e^{\mu t}X(t)dt +$$
$$a\frac{\sigma}{2}X(0)) \exp\left(\mu t + \sigma a \int_0^t \varphi^{\varepsilon}(u)du + \ \sigma\varepsilon^a B(t) - \frac{1}{2}\sigma^2\varepsilon^{2a}s\right) dt +$$
$$e^{\mu t}\left(\varepsilon^a \frac{\sigma}{2}(X(0))\exp \left(\mu t + \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \ \sigma\varepsilon^a B(t) - \frac{1}{2}\sigma^2\varepsilon^{2a}s\right) dB(t)\right).$$

Integrating both sides of this equation from 0 to t, we treat left hand side using fundamental theorem of calculus: Y(t, B(t)) – Y(0, B(0)). The first integral on the right-hand side can be simplified using change of variable.

$$\int_0^t e^{\mu s}\mu X(s)ds + \ a\frac{\sigma}{2}X(0))\varphi^{\varepsilon}(u) \exp\left(\mu s + \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \ \sigma\varepsilon^a B(s)\right.$$
$$\left.- \frac{1}{2}\sigma^2\varepsilon^{2a}s\right) ds$$
$$= \mu \int_0^t e^{\mu s}X(s)ds$$
$$+ a\frac{\sigma}{2}X(0)e^{\mu s} \int_0^t \varphi^{\varepsilon}(u)\exp\left(\mu s\right.$$
$$\left.+ \sigma a \int_0^s \varphi^{\varepsilon}(u)du + \ \sigma\varepsilon^a B(s) - \frac{1}{2}\sigma^2\varepsilon^{2a}s\right) ds$$
$$= e^{\mu t}X(t)dt$$
$$+ \ a\frac{\sigma}{2}X(0))e^{\mu s} \int_0^s \varphi^{\varepsilon}(u)\exp\left(\sigma a \int_0^t \varphi^{\varepsilon}(u)du\right.$$
$$\left.+ \ \sigma\varepsilon^a B(t) - \frac{1}{2}\sigma^2\varepsilon^{2a}s\right) dt$$

The second integral on the right-hand side is a stochastic integral of the form:

$$e^{\mu s}\left(\varepsilon^a \frac{\sigma}{2}(X(0)) \int_0^s \exp \left(\sigma a \int_0^s \varphi^{\varepsilon}(u)du + \ \sigma\varepsilon^a B(s) -\right.\right.$$
$$\left.\left.\frac{1}{2}\sigma^2\varepsilon^{2a}s\right) dB(s)\right)$$

Substituting both sides back into the original equation, we get:

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$$X(t) = e^{\mu t}X(t)dt + a\frac{\sigma}{2}X(0))e^{\mu s}\int_0^s \varphi^\varepsilon(u)\exp\left(\sigma a\int_0^t \varphi^\varepsilon(u)du + \sigma\varepsilon^a B(t) - \frac{1}{2}\sigma^2\varepsilon^{2a}s\right)dt + e^{\mu t}\left(\varepsilon^a\frac{\sigma}{2}(X(0))\int_0^s \exp\left(\sigma a\int_0^s \varphi^\varepsilon(u)du + \sigma\varepsilon^a B(t) - \frac{1}{2}\sigma^2\varepsilon^{2a}s\right)dB(t)\right)$$

So, the unique solutions are same as before.

$$X_l^{\varepsilon a}(t) = e^{\mu t}X_l^{\varepsilon a}(0) + e^{\mu t}a\frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0))\int_t^T \phi^\varepsilon(s)e^{\sigma a\int_t^T \phi^\varepsilon(u)du - \frac{1}{2}\sigma^2\varepsilon^{2a}t + \sigma\varepsilon^a W(t)}ds] + e^{\mu t}\varepsilon^a\frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0))\int_t^T e^{\sigma a\int_t^T \phi^\varepsilon(u)du - \frac{1}{2}\sigma^2\varepsilon^{2a}t + \sigma\varepsilon^a W(t)}dW(s)$$

$$X_u^{\varepsilon a}(t) = e^{\mu t}X_u^{\varepsilon a}(0) + e^{\mu t}a\frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0))\int_t^T \phi^\varepsilon(s)e^{\sigma a\int_t^T \phi^\varepsilon(u)du - \frac{1}{2}\sigma^2\varepsilon^{2a}t + \sigma\varepsilon^a W(t)}ds] + e^{\mu t}\varepsilon^a\frac{\sigma}{2}(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0))\int_t^T e^{\sigma a\int_t^T \phi^\varepsilon(u)du - \frac{1}{2}\sigma^2\varepsilon^{2a}t + \sigma\varepsilon^a W(t)}dW(s)$$

(28)

Now, we will revert to fBm by making the same substitution as before. Recall that: $B_{H,\varepsilon}(t) = a\int_0^t \varphi^\varepsilon(s)ds + \varepsilon^a B(t)$, where $\varphi^\varepsilon(s) = \int_0^t (t-s+\varepsilon)^{a-1}]dB(s)$

So, taking derivative $dB_{H,\varepsilon}(t) = \frac{d[a\int_0^t \varphi^\varepsilon(s)ds + \varepsilon^a B(t)]}{dt} = a\varphi^\varepsilon(t) + \varepsilon^a dB(t)$, and in terms of fBm, we get:

$$X_l^{ea}(t) = e^{\mu t}[X_l^{ea}(0) + \frac{\sigma}{2}\left(X_l^{e1}(0) + X_u^{e1}(0)\right)\int_0^t e^{\sigma B_H^e(s) - \frac{1}{2}\sigma^2 e^{2a}s}dB_H^e(s)] \quad (29)$$

$$X_u^{\varepsilon a}(t) = e^{\mu t}[X_u^{\varepsilon a}(0) + \frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right)\int_0^t e^{\sigma B_H^\varepsilon(s) - \frac{1}{2}\sigma^2\varepsilon^{2a}s}dB_H^\varepsilon(s)]$$

Therefore, the generic solution for μ ≥ 0:

$$X^\varepsilon(t) = e^{\mu t}\langle X(0) + \frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right)\int_0^t e^{\sigma B_H^\varepsilon(s) - \frac{1}{2}\sigma^2\varepsilon^{2a}s}dB_H^\varepsilon(s)\rangle$$

For μ ≤ 0, we start with the following well-known identities:

$cosh(\mu t) = ½\,(e^{\mu t} + e^{-\mu t})$ and $sinh(\mu t) = ½\,(e^{\mu t} - e^{-\mu t}$. We will show that $e^{\mu t}X_l^{ea}(0) = X_l^{ea}(0)cosht(\mu t) + X_u^{ea}(0)\sinh(\mu t)$. If we combine $\{cosh(\mu t) + sinh(\mu t) = e^{\mu t}\}$, and we replace the first term $e^{\mu t}X_l^{ea}(0)$ with $X_l^{ea}(0)cosht(\mu t) + X_u^{ea}(0)\sinh(\mu t)$, our solution becomes:

$$X_l^{\varepsilon a}(t) = X_l^{\varepsilon a}(0)cosht(\mu t) + X_u^{\varepsilon a}(0)\sinh(\mu t) + e^{\mu t}\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right)\int_0^t e^{\sigma B_H^\varepsilon(s) - \frac{1}{2}\sigma^2\varepsilon^{2a}s}dB_H^\varepsilon(s)]$$

(30)

And similarly:

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$$X_u^{\varepsilon a}(t) = X_l^{\varepsilon a}(0)cosht(\mu t) + X_u^{\varepsilon a}(0)\sinh(\mu t) + e^{\mu t}\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right)\int_0^t e^{\sigma B_H^{\varepsilon}(s) - \frac{1}{2}\sigma^2\varepsilon^{2a}s} dB_H^{\varepsilon}(s)]$$

(31)

Therefore, the generic solution for $\mu \le 0$:

$$X^{\varepsilon}(t) = X_l^{\varepsilon a}(0)cosht(\mu t) + X_u^{\varepsilon a}(0)\sinh(\mu t) + \langle\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right)e^{\mu t}\int_0^t e^{\sigma B_H^{\varepsilon}(s) - \frac{1}{2}\sigma^2\varepsilon^{2a}s} dB_H^{\varepsilon}(s)\rangle$$

We proved no arbitrage conditions for fuzzy fractal model when Hurst exponent is greater than ½ [22] [27]. We will extend this now to FFJDM under the risk-neutral measure Q which is a theoretical probability measure used to value financial derivatives, such as options, under the assumption that the expected rate of return of the underlying asset is the risk-free rate [32]. This approach simplifies the valuation process and allows for a relatively easy pricing of derivatives without considering the actual probabilities of different market outcomes. This means that when calculating expected values of future payoffs, we use the risk-free rate as the discount factor. Under this measure, we can calculate the present value of future cash flows associated with financial derivatives. When valuing financial derivatives like options, the risk-neutral measure allows us to use a discounted expected value calculation to determine the fair price of the derivative. This approach assumes that investors are risk-neutral and do not require a risk premium for holding the derivative. The statement implies that even under the risk-neutral measure Q, the underlying asset price still follows a certain stochastic process. Under the assumption of no arbitrage, the risk-neutral and the real-world (risky) measures are equivalent in the context of option pricing and derivatives valuation. This equivalence is a fundamental concept in mathematical finance and is known as the Fundamental Theorem of Asset Pricing [32]. In order for risk-neutral and risky measures to be equivalent, we impose a no-arbitrage condition under which $\hat{\mu} = r - \tilde{\lambda}(\tilde{J} - 1))$. The result we will derive next using Itô's lemma shows that there is a unique solution to our FFJDM equation and that solution does not depend on any particular assumptions or individual preferences about risk. The arbitrage free price of a derivative is uniquely determined because in this model the derivative is superfluous. This means that the derivative's price is fully determined by the model's dynamics and parameters, and its price is consistent with the prices of other assets in the market. In essence, the derivative's price is not subject to arbitrary fluctuations or ambiguous valuation. We will use a well-known Girsanov theorem in the theory of stochastic differential equations, that if we change the measure from real-world P to some other equivalent measure in risk-neutral world Q, this will change the drift in the SDE, but the diffusion term will be unaffected. This is why we set $\hat{\mu} = \hat{r} - \tilde{\lambda}(\tilde{J} - 1))$. Thus, the drift will play no part in the pricing equations. Using Girsanov's theorem and re-writing $d\hat{S}/\hat{S} = \hat{\mu}dt + \hat{\sigma}dB_t + \hat{\sigma}dB_t^H + d(\sum_1^{\tilde{N}}(\tilde{V}_l - 1))$as $d\hat{S}/\hat{S} = (\hat{r} - \tilde{\lambda}(\tilde{J} - 1))dt + \hat{\sigma}dB_t + \hat{\sigma}dB_t^H + d(\sum_1^{\tilde{N}}(\tilde{V}_l - 1))$. Similarly to (26), we get under risk-neutral measure:

$$\hat{S}_t = \hat{S}(0)e^{\hat{r} - \tilde{\lambda}(\tilde{J}-1))t + \tilde{\sigma}B_t + \hat{\sigma}B_t^H - \frac{1}{2}\hat{\sigma}^2 t - \frac{1}{2}\hat{\sigma}^2 t^{2H}} \prod_{i=1}^{n} \tilde{V}_l \quad (32)$$

This is the process that $\hat{S}_t$ follows which we will use in the proof later. The Cheridito (2001) paper, titled "Mixed Fractional Brownian Motion" [36], presented a framework for modelling and analysing mixed fractional Brownian motion (MFBM), which is a generalisation of fractional Brownian motion (fBm). The paper introduced a class of stochastic differential equations driven by MFBM and established existence and uniqueness results for solutions under certain regularity conditions. It was shown that if the Hurst exponent is greater than ¾, the fractional Brownian motion process satisfies certain regularity conditions, which allow for the existence of a unique solution to the stochastic differential equation. This result is relevant for our work in the mathematical analysis and modelling of the underlying asset dynamics in option pricing. The discounted expected value of Z_T at time t ϵ [0, T] under risk-neutral measure Q is given by $Z_t = e^{-\tilde{r}(T-t)}E^Q[(\tilde{S}_T - K)^+, 0]$, so $\tilde{S}_T -$

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$K > 0$, otherwise the payoff is zero. We will now substitute expression for $\widehat{S_t}$ from (32) into the above equation:

$$Z_t = e^{-\tilde{r}(T-t)}E^Q[\left(\hat{S}(0)(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\sigma}B_t^H-\frac{1}{2}\hat{\sigma}^2t-\frac{1}{2}\hat{\sigma}^2t^{2H}})\prod_{i=1}^{n}\widetilde{V_i}-K\right)^+,0]$$

$$= e^{-\tilde{r}(T-t)}E^Q[\left(\hat{S}(0)(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\sigma}B_t^H-\frac{1}{2}\hat{\sigma}^2t-\frac{1}{2}\hat{\sigma}^2t^{2H}+\sum_{i=1}^{\tilde{N}_T}Y_i})-K\right)^+,0]$$

$$= e^{-\tilde{r}(T-t)}E^Q[E^Q[\left(\hat{S}(0)(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\sigma}B_t^H-\frac{1}{2}\hat{\sigma}^2t-\frac{1}{2}\hat{\sigma}^2t^{2H}+\sum_{i=1}^{\tilde{N}_T}Y_i}) - K\right)^+|\sigma(\sum_{i=1}^{\tilde{N}_T}Y_i)]$$

(33)

We will now use law of iterated expectation within the context of a σ-algebra. The σ-algebra, denoted by $\sigma(\sum_{i=1}^{\tilde{N}_T}Y_i)$ represents the conditioning information or the available knowledge up to time T. It includes all the events or outcomes that can be determined based on the random variables Y_i up to time T. This sigma algebra essentially captures the information available from the past observations and values of Y_i up to time T. The Law of Iterated Expectations allows us to rewrite this as $E^Q[E^Q[\left(\hat{S}(0)(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\sigma}B_t^H-\frac{1}{2}\hat{\sigma}^2t-\frac{1}{2}\hat{\sigma}^2t^{2H}+\sum_{i=1}^{\tilde{N}_T}Y_i})-K\right)^+|\sigma(\sum_{i=1}^{\tilde{N}_T}Y_i)]$which means we are now taking the inner conditional expectation over the specific sigma algebra $\sigma(\sum_{i=1}^{\tilde{N}_T}Y_i)$. By conditioning on the specific sigma algebra we are essentially using the available information up to time T to compute the conditional expectation of the expression. This simplification is possible because the conditioning restricts the uncertainty to the events in the specified sigma algebra.

In the context of our expression, the conditional expectation represents the expected value of the inner expression (the option payoff) given the information available up to time T, which is encapsulated by the sigma algebra. This conditional expectation accounts for the uncertainty in future events by conditioning on the information from the past.

The sequence of random variables $Y_i = log(V))$, where V is a sequence of independent and identically distributed i.i.d fuzzy random variables, contributes to the information available in the conditioning σ-algebra. These random variables follow a normal distribution with mean μ and variance σ^2. Therefore, we have:

$$E^Q[E^Q[\left(\hat{S}(0)(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\sigma}B_t^H-\frac{1}{2}\hat{\sigma}^2t-\frac{1}{2}\hat{\sigma}^2t^{2H}+\sum_{i=1}^{\tilde{N}_T}Y_i})-K\right)^+|\sigma(\sum_{i=1}^{\tilde{N}_T}Y_i)]$$

$$= e^{-\tilde{r}(T-t)}\sum_{n=0}^{\infty}\frac{(\tilde{\lambda}t)^n}{n!}e^{-\tilde{\lambda}t}E^Q\left(\hat{S}(0)(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\sigma}B_t^H-\frac{1}{2}\hat{\sigma}^2t-\frac{1}{2}\hat{\sigma}^2t^{2H}+\sum_{i=1}^{\tilde{N}_T}Y_i})-K\right)^+$$

When we condition on the sum of Y_i, which is essentially the cumulative log value of the sequence of jumps, we are effectively considering the total logarithmic impact of the jumps up to time T. Since the number jumps are modelled using a fuzzy Poisson process with jump intensity λ, their cumulative effect follows a Poisson distribution.

In other words, the jumps, when summed and transformed by taking the logarithm, have properties that lead to the emergence of a Poisson distribution in the conditional expectation. $\tilde{\sigma}B_t + \hat{\sigma}B_t^H + \sum_{i=1}^{\tilde{N}_T}Y_i$ consists of three i.i.d. fuzzy normal random variables. From the properties of variances of random variables we know that the variance of the square of the random variables is the sum of their variances. Also, from Itô's, we know that the square of B_t and B^H is T and T^{2H}. So, $\tilde{\sigma}^2 = \tilde{\sigma}^2T + \hat{\sigma}^2T^{2H} + n\tilde{\sigma}_j^2$. Mean of $\tilde{\sigma}B_t + \hat{\sigma}B_t^H + \sum_{i=1}^{\tilde{N}_T}Y_i$ is simply expectation of $\sum_{i=1}^{\tilde{N}_T}Y_i$ because expectation of the stochastic processes is zero. So, $\tilde{\mu} = \sum_{i=1}^{\tilde{N}_T}Y = n_i\ \tilde{\mu}_j$. We plug these values into the above the equations:

$$E^Q\left(\hat{S}(0)(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\sigma}B_t^H-\frac{1}{2}\hat{\sigma}^2t-\frac{1}{2}\hat{\sigma}^2t^{2H}+\sum_{i=1}^{\tilde{N}_T}Y_i})-K\right)^+ =$$

$$= \int_{-\infty}^{\infty} \max\left[\hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\tilde{\sigma}}B_t^H-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\sum_{i=1}^{\tilde{N}_T} Y_i}\right) - K, 0\right]\varphi(z)dz$$

where $\varphi(z)$ is the density of the fuzzy normal variable with mean $\tilde{\mu} = \sum_{i=1}^{\tilde{N}_T} Y = n_i\ \tilde{\mu}_j$ and variance $\tilde{\sigma}^2 = \tilde{\sigma}^2 T + \hat{\tilde{\sigma}}^2 T^{2H} + n\tilde{\sigma}_j^2$. So, $\varphi(z) = \frac{1}{\sqrt{2\pi\tilde{\sigma}}} e^{-\frac{(z-\tilde{\mu})^2}{2\tilde{\sigma}^2}dz}$, and the whole integral:

$$= \int_{-\infty}^{\infty} \max\left[\hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\tilde{\sigma}}B_t^H-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\sum_{i=1}^{\tilde{N}_T} Y_i}\right) - K, 0\right]\frac{1}{\sqrt{2\pi\tilde{\sigma}}} e^{-\frac{(z-\tilde{\mu})^2}{2\tilde{\sigma}^2}dz}dz$$

The integrand in the integral above vanishes when:

$\hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\tilde{\sigma}}B_t^H-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\sum_{i=1}^{\tilde{N}_T} Y_i}\right) < K$, i.e. when $z < z_0$, where

$$z_0 = \frac{\left[\ln\left(\frac{K}{\hat{S}(0)}\right) - \hat{r} - \tilde{\lambda}(\tilde{J}-1))t - \frac{1}{2}\hat{\sigma}^2 t - \frac{1}{2}\hat{\sigma}^2 t^{2H}\right] - \tilde{\mu}}{\tilde{\sigma}}$$

The integral can thus be written as:

$$\int_{z_0}^{\infty} \hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\tilde{\sigma}}B_t^H-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\sum_{i=1}^{\tilde{N}_T} Y_i}\right)\varphi(z)dz - \int_{z_0}^{\infty} K\frac{1}{\sqrt{2\pi\tilde{\sigma}}} e^{-\frac{(z-\tilde{\mu})^2}{2\tilde{\sigma}^2}dz}dz = A - B$$

The integral B can be written as $K \cdot Prob(Z \geq z_0)$ and using the symmetry of the Normal distribution, this can be written as $K \cdot Prob(Z \leq -z_0)$. So, if we denote the cumulative distribution function of N as is the usual convention: $\frac{1}{\sqrt{2\pi\tilde{\sigma}}} e^{-\frac{(z-\tilde{\mu})^2}{2\tilde{\sigma}^2}dz}$ then we can write $B = K \cdot N(-z_0)$. In the integral A, we have:

$$A = \int_{z_0}^{\infty} \hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\tilde{\sigma}}B_t^H-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\sum_{i=1}^{\tilde{N}_T} Y_i}\right)\frac{1}{\sqrt{2\pi\tilde{\sigma}}} e^{-\frac{(z-\tilde{\mu})^2}{2\tilde{\sigma}^2}dz}dz$$

$$= \hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}}\right)\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{\tilde{\sigma}B_t+\hat{\tilde{\sigma}}B_t^H+\sum_{i=1}^{\tilde{N}_T} Y_i}\frac{e^{-\frac{(z-\tilde{\mu})^2}{2\tilde{\sigma}^2}}}{\tilde{\sigma}}dz$$

Recall that the fuzzy normal random variable $\tilde{\sigma}B_t + \hat{\tilde{\sigma}}B_t^H + \sum_{i=1}^{\tilde{N}_T} Y_i$ has mean $\tilde{\mu} = \sum_{i=1}^{\tilde{N}_T} Y = n_i\ \tilde{\mu}_j$ and variance $\tilde{\sigma}^2 = \tilde{\sigma}^2 T + \hat{\tilde{\sigma}}^2 T^{2H} + n\tilde{\sigma}_j^2$.

So, we are integrating a function with respect to a random variable that follows a given probability distribution. In the field of stochastics, this is often encountered in the form of an expectation. In our case, the expectation of $(\exp(\tilde{\sigma}B_t + \hat{\tilde{\sigma}}B_t^H + \sum_{i=1}^{\tilde{N}_T} Y_i))$ where $\tilde{\sigma}B_t + \hat{\tilde{\sigma}}B_t^H + \sum_{i=1}^{\tilde{N}_T} Y_i$ is normally distributed can be written as follows:

$$E(\exp\left(\tilde{\sigma}B_t + \hat{\tilde{\sigma}}B_t^H + \textstyle\sum_{i=1}^{\tilde{N}_T} Y_i\right)) = \int e^y y\, f(y)\, dy$$

where f(y) is the probability density function (PDF) of the normal distribution:

$f(y) = (1/\sqrt{}(2\pi\sigma^2))\ exp(-(y-\mu)^2 / (2\sigma^2))$. Plugging this into the original equation, we get: $E[exp(Y)] = \int e^y\ (1/\sqrt{}(2\pi\sigma^2))\ exp(-(y-\mu)^2 / (2\sigma^2))dy$. Simplifying this, we get:

$$E[exp(Y)] = (1/\sqrt{}(2\pi\sigma^2)) \int exp\{[(2y\sigma^2 - (y-\mu)^2) / (2\sigma^2)]\}\, dy$$

This simplifies to: $E[exp(Y)] = (1/\sqrt{}(2\pi\sigma^2)) \int exp\{[(2\mu y - \mu^2 + \sigma^2) / (2\sigma^2)]\}\, dy$

The above expression is the definition of the moment generating function (MGF) of a normally distributed random variable at the point t=1. The MGF of a normal distribution with mean μ and variance σ^2 is given by:

$$M(t) = exp\{\mu t + (1/2)t^2\sigma^2\}$$

Substituting t=1 in the MGF gives us:

$$E[eY] = exp\{\mu + (1/2)\sigma^2\}$$

This is the expected value of an exponential of a normally distributed random variable. This is different from the indefinite integral of e^Y with respect to Y, which isn't typically defined for random variables. Instead, we work with expectations, which are a form of weighted integral where the weights are given by the PDF of the random variable.

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So, the above integral is now simplified to:

$$= \hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}\right)\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty}\frac{e^{-\frac{(z-\tilde{\mu})^2}{2\tilde{\sigma}^2}}}{\tilde{\sigma}}dz$$

Next step is to standardise the integral by changing its limit with $\tilde{\mu}=0$ and $\tilde{\sigma}^2=1$

$$= \hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}\right)\frac{1}{\sqrt{2\pi}}\int_{\frac{z_0-\tilde{\mu}-\tilde{\sigma}^2}{\tilde{\sigma}}}^{\infty}e^{-\frac{(z)^2}{2}}dz$$

Again, this is the density of the of the normal distribution. Using symmetry we can write:

$$A = \hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}\right)N(-z_0)$$

$$= \hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}\right)\Phi\left(\frac{-z_0+\tilde{\mu}+\tilde{\sigma}^2}{\tilde{\sigma}}\right)$$

Now, we can combine A and B and write the whole expression:

$$A-B = \hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}\right)\Phi\left(\frac{-z_0+\tilde{\mu}+\tilde{\sigma}^2}{\tilde{\sigma}}\right) - K\Phi\left(\frac{-z_0+\tilde{\mu}}{\tilde{\sigma}}\right)$$

Finally, substituting above expression into:

$$Z_t = e^{-\tilde{r}(T-t)}\sum_{n=0}^{\infty}\frac{(\tilde{\lambda}t)^n}{n!}e^{-\tilde{\lambda}t}E^Q\left(\hat{S}(0)(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t+\tilde{\sigma}B_t+\hat{\sigma}B_t^H-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\sum_{i=1}^{\tilde{N}_T}Y_i})-K\right)^+$$

$$= e^{-\tilde{r}(T-t)}\sum_{n=0}^{\infty}\frac{(\tilde{\lambda}t)^n}{n!}e^{-\tilde{\lambda}t}\hat{S}(0)\left(e^{\hat{r}-\tilde{\lambda}(\tilde{J}-1))t-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}\right)\Phi\left(\frac{-z_0+\tilde{\mu}+\tilde{\sigma}^2}{\tilde{\sigma}}\right) - K\Phi\left(\frac{-z_0+\tilde{\mu}}{\tilde{\sigma}}\right)$$

Recall that $\tilde{J}=e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}$. Take $t_0 = 0$ and substitute $\tilde{J}$ into above expression, we get:

$$\sum_{n=0}^{\infty}\frac{(\tilde{\lambda}t)^n}{n!}e^{-\tilde{\lambda}t}[\hat{S}(0)\left(e^{(\hat{r}-\tilde{\lambda}(\tilde{J}-1))t-\frac{1}{2}\hat{\sigma}^2 t-\frac{1}{2}\hat{\sigma}^2 t^{2H}+\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2-\tilde{r}T}\right)\Phi\left(\frac{-z_0+\tilde{\mu}+\tilde{\sigma}^2}{\tilde{\sigma}}\right) - Ke^{-\tilde{r}T}\Phi\left(\frac{-z_0+\tilde{\mu}}{\tilde{\sigma}}\right)]$$

Simplify:

$$= \sum_{n=0}^{\infty}\frac{(\tilde{\lambda}t)^n}{n!}e^{-\tilde{\lambda}t}[\hat{S}(0)\left(e^{\tilde{\lambda}t-\tilde{\lambda}\tilde{J}t+n(\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2)}\right)\Phi\left(\frac{-z_0+\tilde{\mu}+\tilde{\sigma}^2}{\tilde{\sigma}}\right) - Ke^{-\tilde{r}T}\Phi\left(\frac{-z_0+\tilde{\mu}}{\tilde{\sigma}}\right)]$$

$$= \sum_{n=0}^{\infty}\frac{(\tilde{\lambda}t)^n}{n!}e^{-\tilde{\lambda}\tilde{J}t}[\hat{S}(0)\left(e^{n(\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2)}\right)\Phi\left(\frac{-z_0+\tilde{\mu}+\tilde{\sigma}^2}{\tilde{\sigma}}\right) - Ke^{-\tilde{r}T}\Phi\left(\frac{-z_0+\tilde{\mu}}{\tilde{\sigma}}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(\tilde{\lambda}\tilde{J}t)^n}{n!}e^{-\tilde{\lambda}\tilde{J}t}[\hat{S}(0)\Phi\left(\frac{-z_0+\tilde{\mu}+\tilde{\sigma}^2}{\tilde{\sigma}}\right) - Ke^{-\tilde{r}T}\Phi\left(\frac{-z_0+\tilde{\mu}}{\tilde{\sigma}}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(\tilde{\lambda}\tilde{J}t)^n}{n!}e^{-\tilde{\lambda}\tilde{J}t}[\hat{S}(0)\Phi(d_1) - Ke^{-\tilde{r}T}\Phi(d_2)]$$

(34)

Φ represents the cumulative standard normal distribution function. The terms d1 and d2 are often used to assess the probability of the option expiring in the money or out of the money. d1 represents the standardised distance between the current stock price (S) and the strike price (K), adjusted by other factors such as the risk-free interest rate (r), volatility (σ), and time to expiration (T). Specifically, d1 incorporates the expected return and the expected volatility of the stock. d2 is similar to d1, but it is adjusted by subtracting the volatility component (σ√T). This adjustment reflects the expectation of the stock price being below the strike price at expiration. The probabilities come into play when you consider the cumulative standard normal distribution function (Φ) applied to these d1 and d2 values. The cumulative distribution function gives the probability that a standard normal random variable is less than or equal to a given value. For a call option, d1 represents the probability that the option will finish in the money (stock price above the strike price) and d2 represents the probability that the option will finish out of the money (stock price below the strike price). In summary, while d1 and d2 aren't direct probabilities, they are closely related to the probabilities of the option being in the money or out of the money.

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3. Introducing Proof When h < ½

The proofs in [9] is valid when H > ½. As we described it in the Introduction, fBm has an infinite quadratic variation when H < ½, so we use [10] to change Hurst parameter from H to 1-H.

First, we use the following integration by parts result from [11]:

$$\int_0^T f(t)dB^H(t) = f(T)B^H(T) - \int_0^T B^H(t)df(t)$$

The author proves the relationship for a deterministic function f such that it has bounded p-variation sample paths for all p < 1/(1-H). Let $Y_t^H = \int_0^t s^{H-\frac{1}{2}}dB^H(t)$. Then $B_t^H = \int_0^t s^{H-\frac{1}{2}}dY_s^H$. Proof:

$$\begin{aligned} Y_t^H &= \int_0^t s^{H-\frac{1}{2}}dB^H(t) \\ &= (t)^{\frac{1}{2}-H}B^H(t) \\ &- \int_0^t B^H(t)d\left(t^{\frac{1}{2}-H}\right) \\ &= t^{\frac{1}{2}-H}B^H(t) - \int_0^t B^H(t)\left[\frac{1}{2}-H\right]t^{-\frac{1}{2}-H}dt \end{aligned}$$

Apply derivative:

$$\begin{aligned} dY_t^H &= (\frac{1}{2}-H)\, t^{-\frac{1}{2}-H}dtB^H(t) + dB^H(t)t^{\frac{1}{2}-H} - B^H(t)\left[\frac{1}{2}-H\right]t^{-\frac{1}{2}-H} \\ &= dB^H(t)t^{\frac{1}{2}-H} \\ &\Rightarrow dB_t^H = dY^H(t)t^{-\frac{1}{2}+H} \end{aligned}$$

Apply integration:

$$\Rightarrow B_t^H = \int_0^t s^{-\frac{1}{2}+H}dY^H(s) \tag{35}$$

Following notation from [12], we utilise Molchan martingale [13], $M_t^H := \int_0^t w(t,s)dB^H(s)$, where $w(t,s) \doteq (\frac{c}{C})s^{-\alpha}(t-s)^{-\alpha}$ is a scaled beta kernel and w(t,s) = 0 for s > t; $\alpha \doteq H-\frac{1}{2}$; $C \doteq \sqrt{\frac{H}{(H-\frac{1}{2})B(H-\frac{1}{2},2-2H)}}$ and $c \doteq \frac{1}{B(H+\frac{1}{2}),(\frac{3}{2}-H)}$. So, c(H) is a function of H and the standard Beta function $B(\mu,\upsilon) \doteq \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$. Using Proposition 2.1 from [12] and proof of Theorem of 3.2 in [14], then M is a a Gaussian martingale. Put the values of dB into Molchan martingale we obtain: $M_t^H = (\frac{c}{C})\int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}s^{H-\frac{1}{2}}dY^H(s) = (\frac{c}{C})\int_0^t (t-s)^{\frac{1}{2}-H}dY^H(s)$.

[14] showed that the filtrations generated by M and Y are the same. They showed that $Y_T = 2H\int_0^T (T-t)^{-\frac{1}{2}+H}dM(t)$ and the prediction formula:

$$E[Y_T|F_t] = 2H\int_0^t (T-s)^{-\frac{1}{2}+H}dM(s). \tag{36}$$

Now we use equations (35) and (36) to find the upper bound using integration by parts. From(35):

$$\begin{aligned} \sup|B_t^H| \le sup\left|\int_0^t s^{-\frac{1}{2}+H}dY^H(s)\right| \le sup\left|\left[s^{H-\frac{1}{2}}Y_s^H\right]_0^T\right. \\ \left. - \int_0^t d(s^{-\frac{1}{2}+H})Y^H(s)\right| \end{aligned}$$

Since the integral term is positive, therefore:

$$\sup|B_t^H| \le \sup\left|\left[s^{H-\frac{1}{2}}Y_s^H\right]_0^T\right| \le \sup\left|T^{H-\frac{1}{2}}(Y_t^H - Y_0^H)\right| \le 2T^{H-\frac{1}{2}}\sup|Y_t^H||$$

Now using (36), substitute for Y:

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$$\sup |B_t^H| \leq 2T^{H-\frac{1}{2}} \sup \left| 2H \int_0^t (t-s)^{-\frac{1}{2}+H} dM^H(s) \right|$$
$$\leq 2T^{H-\frac{1}{2}} 2H\sup \left| \left[(t-s)^{H-\frac{1}{2}} M_s^H \right]_0^T \right.$$
$$\left. - \int_0^t d((t-s)^{-\frac{1}{2}+H}) M^H(s) \right|$$
$$\leq 4HT^{2H-1} \sup(|M_t^H| - |M_0^H|) \leq 8HT^{2H-1} \sup(|M_t^H|)$$

We now apply expectation:

$$\mathrm{E}(\sup|B_t^H|)^p \leq (8HT^{2H-1})^p \, \mathrm{E}(\sup(|M_t^H|)^p)$$

Since M is Molchan martingale, by the Burkholder-Davis-Gundy inequality, there exists a constant $A_p^H > 0$ such that:

$$\mathrm{E}(\sup|B_t^H|)^p \leq (8HT^{2H-1})^p \, A_p^H \mathrm{E}(\sup(\langle M_t^H \rangle)^{p/2}) \quad (37)$$

The next result we take from Norros, Proposition 2.1 [14]:

$$Var(M_t^K) = \langle M^K \rangle_t = \frac{c^2(K)}{(2K)^2(2-2K)} t^{2-2K} := d(K) t^{2-2k}$$

Now, we use the transformation proposed and proved by [15] to change Hurst parameter H. Using Corrollary 5.2 [15], there exists a unique (1 – H) fBm, such that:

$$B_t^H = \left(\frac{2H}{\Gamma(2H)\Gamma(3-2H)}\right)^{1/2} \int_0^t (t-s)^{2H-1} \, dB_s^{1-H}, a.s., t\epsilon[0,T]$$

By remark 5.6 [10], proves that:

$$M_t^H = \left(\sqrt{\frac{1-H}{1-K}}\right) \int_0^t s^{K-H} dM_s^K, a.s., t\epsilon[0,\infty)$$

We will now combine these results to compute quadratic variation for 1 – H > ½ to obtain:

$$\langle M_t^H \rangle = \left(\left(\sqrt{\frac{1-H}{1-K}} \right) \int_0^t s^{K-H} dM_s^K \right)^2 = \frac{1-H}{H} \int_0^t s^{2-2H} d\langle M^{1-H} \rangle_s$$

Use [14]: $Var(M_t^K) = \langle M^K \rangle_t = \frac{c^2(K)}{(2K)^2(2-2K)} t^{2-2K} := d(K) t^{2-2k}$.

Substitute into: $d\langle M^{1-H} \rangle_s = d(1-H) s^{2-2(1-H)}$. Continuing with quadratic variation:

$$\langle M_t^H \rangle = \left(\left(\sqrt{\frac{1-H}{1-K}} \right) \int_0^t s^{K-H} dM_s^K \right)^2 = \frac{1-H}{H} \int_0^t s^{2-2H} d\langle M^{1-H} \rangle_s =$$
$$\frac{1-H}{H} d(1-H) \int_0^t s^{2-2H} 2H s^{2H-1} ds = \frac{2H(1-H)}{H} d(1-H) \int_0^t s^{1-2H} ds = d(1-H) t^{2(1-H)} \quad (38)$$

Apply BDG inequality on quadratic variation of M^H, to obtain:

$$E\sqrt{\langle M^H \rangle_t}^p \leq B_p^{1-H} t^{p(1-H)} \quad (39)$$

Using (37): $\mathrm{E}(\sup|B_t^H|)^p \leq (8HT^{2H-1})^p \, A_p^H \mathrm{E}(\sup(\langle M_t^H \rangle)^{p/2})$, substitute into (39), we obtain:

$$\mathrm{E}(\sup|B_t^H|)^p \leq (8HT^{2H-1})^p A_p^H B_P^{1-H} T^{p(1-H)} \quad (40)$$

If we define $C := (8H)^p A_p^H B_P^{1-H}$ then $\mathrm{E}(\sup|B_t^H|)^p \leq CT^{pH}$. So, in summary, we use trick from [10] to transform fBm to (1 – H) fBm. By expressing fBm as an integral with respect to standard Brownian motion, the proof employs Volterra-like representations of fBm. The transformed process is a Gaussian martingale with respect to the filtration generated by B_H, which allows us to establish

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the upper and lower bounds for the integrals. Combined with the fact that we also demonstrate that the involved functions are square integrable, this ensures the finiteness and uniqueness of solution. Proof complete.

4. Introducing Fuzzy When h < ½

Under the assumption that the fuzzy aspect lives in the amplitude/kernels and that the underlying driving noise $\tilde{B}$ is actually a crisp fractional Brownian motion. In other words, we are layering fuzziness on the coefficients/functions of the integral but not changing the fundamental measure-theoretic structure of the noise. This assumption ensures we can leverage classical results such as Itô isometry for existence, uniqueness, and bounding integrals which we proved in [9].

For H < ½, the kernel $K_H(t, s)$ used to construct fBm was derived by [16].

$$K_H(t,s) = c_H\left[\left(\frac{t}{s}\right)^{H-1/2}(t-s)^{H-1/2}\right]$$

where C_H is a normalising constant given by: $c_H = \left(\frac{2H\sin(\pi H)\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})}\right)^{1/2}$.

For practical applications, a more convenient representation of the kernel function for H < ½ can be written as:

$$K_H(t,s) = c_H\left[(t-s)^{H-1/2} - (-t)^{H-1/2}\right]$$

This ensures the kernel remains well-defined and integrable. To establish upper and lower bounds for the integrals involving fuzzy fractional Brownian motion when H < ½, we need to adapt the existing techniques to handle the fuzzy nature of the random variables. To establish bounds, we need to represent the fuzzy variables as intervals or use membership functions and then analyse these representations.

4.1. Fuzzy Gaussian Distribution

At any fixed time t, the value of the fuzzy Brownian motion $\tilde{B}_H(t)$ is a fuzzy Gaussian variable. This means its probability distribution is a Gaussian distribution with fuzzy mean and variance.

Fuzzy Integral Representation:

We denote the kernel $K_H(t,s)$ as fuzzy, which means at each ω∈Ω, the kernel belongs to a set and has a membership function describing possible amplitude values.

$$\tilde{B}_H(t) = \int_0^t \tilde{K}_H(t,s)dB(s)$$

We assume the underlying noise is crisp, but the fuzzy aspect is in the amplitude of the kernel. We had earlier proven in [9] that the classical Itô isometry remains perfectly valid for each crisp selection and each scenario in our fuzzy ensemble because the noise is crisp. The fuzziness is layered on top by letting the integrand vary in intervals. Nothing breaks the standard measure-theoretic basis of $\tilde{B}_H(t)$.

Our future work would involve defining the fuzzy fractional Brownian motion $\tilde{B}_H(t)$ in the sense of fuzzy measure theory, not just a crisp B with fuzzy amplitude. Then the classical Itô isometry is no longer guaranteed. Indeed, we might not even have a linear–additive expectation E in the sense of Kolmogorov measure theory. Instead, we have fuzzy expectation and fuzzy integrals, which typically do not obey Itô isometry.

The step:

$$E\left[\left(\int_0^t \left(\tilde{K}_H^{(1)}(t,s) - \tilde{K}_H^{(2)}(t,s)\right)d\tilde{B}(s)\right)^2\right] \neq \int_0^t E\left[(\tilde{K}_H^{(1)}(t,s) - \tilde{K}_H^{(2)}(t,s))^2 ds\right]$$

would be wrong as indicated by ≠, because the left side uses a fuzzy integral and fuzzy expectation. Typically, a pure fuzzy measure does not preserve linear additivity, and it uses a non-classical integral Choquet, or Sugeno. The isometry is not guaranteed.

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We need to analyse:

$$E\left[\left(\tilde{B}_H(t)\right)^2\right] = E\left[\left(\int_0^t \tilde{K}_H(t,s)dB(s)\right)^2\right]$$

Since both $\tilde{K}_H(t,s)$ and $\tilde{B}(s)$ are fuzzy, we can represent them as intervals for analysis.

The fuzzy labels on $\tilde{K}_H(t,s)$ does not break linearity or additivity of the expectations and the integrals, because fuzziness is above the measure level.

For uniqueness proofs, we define upper and lower bounds for Fuzzy Gaussian Increments:

$$\tilde{K}_H(t,s) = [K_{H,L}(t,s), K_{H,U}(t,s)]$$
$$\tilde{B}(s) = [B_L(s), B_U(s)]$$

Here, $K_{H,L}(t,s)$ *and* $K_{H,U}(t,s)$ represent the lower and upper bounds of the fuzzy kernel function, and $B_L(s)$ *and* $B_U(s)$ represent the lower and upper bounds of the fuzzy Gaussian process. The upper and lower bounds for the fuzzy fractional Brownian motion integrals are determined by analysing the bounds of the kernel function and the Gaussian process separately.

$$\left(\int_0^t \tilde{K}_{H,L}(t,s)\right)^2 ds \le E\left[\left(\int_0^t \tilde{K}_H(t,s)d\tilde{B}(s)\right)^2\right] \le \left(\int_0^t \tilde{K}_{H,U}(t,s)\right)^2 ds$$

By ensuring that these integrals are finite, we will demonstrate the existence of the solution. Uniqueness will be shown by proving that the fuzzy integrals converge uniquely under the given conditions - if that difference of kernels is zero a.s. owing to identical initial data and integrable constraints, we obtain uniqueness.

In other words, we need to show that:

$$\left(\int_0^t \tilde{K}_{H,L}(t,s)\right)^2 ds < \infty \text{ and } \left(\int_0^t \tilde{K}_{H,U}(t,s)\right)^2 ds < \infty$$

For $\tilde{K}_{H,L}(t,s)$:

$$\left(\tilde{K}_{H,L}(t,s)\right)^2 = c_H{}^2\left((t-s)^{H-1/2} - (-s)^{H-1/2}\right)^2$$

Integrate this:

$$\int_0^t \left(\tilde{K}_{H,L}(t,s)\right)^2 ds = c_H{}^2 \int_0^t \left((t-s)^{H-1/2} - (-s)^{H-1/2}\right)^2 ds$$

Both terms inside the integral are finite for H < ½, and the subtraction term also ensures the finiteness of the integral.

Similarly, for the upper bounds:

$$\int_0^t \left(\tilde{K}_{H,U}(t,s)\right)^2 ds = c_H{}^2 \int_0^t \left((t-s)^{H-1/2} - (-s)^{H-1/2}\right)^2 ds$$

The same reasoning applies here, both terms inside the integral are finite, and their subtraction ensures the integrals are finite. Therefore, both the upper and lower bounds are finite.

To show uniqueness, we need to prove that the fuzzy integral representation converges uniquely under the given conditions. We assume that there are two fuzzy fractional Brownian motions $\tilde{B}_H^{(1)}(t)$ and $\tilde{B}_H^{(2)}(t)$ with the same initial conditions.

For uniqueness, consider the difference:

$$\Delta\tilde{B}_H(t) = \tilde{B}_H^{(1)}(t) - \tilde{B}_H^{(2)}(t)$$

Then:

$$E[\Delta\tilde{B}_H(t)^2] = E\left[\left(\int_0^t \left(\tilde{K}_H^{(1)}(t,s) - \tilde{K}_H^{(2)}(t,s)\right) d\tilde{B}(s)\right)^2\right]$$

Then we can use Itô's isometry:

$$E[\Delta\tilde{B}_H(t)^2] = \int_0^t E\left[\left(\tilde{K}_H^{(1)}(t,s) - \tilde{K}_H^{(2)}(t,s)\right)^2\right] ds$$

If $\tilde{B}_H^{(1)}(t)$ and $\tilde{B}_H^{(2)}(t)$ are solutions, then $\tilde{K}_H^{(1)}(t,s) = \tilde{K}_H^{(2)}(t,s)$ a.s. Thus, $E[\Delta\tilde{B}_H(t)^2] = 0$.

This implies that $\Delta\tilde{B}_H(t) = 0$ a.s. and $\tilde{B}_H^{(1)}(t) = \tilde{B}_H^{(2)}(t)$. Proof complete. Therefore, the solution is unique.

Membership function:

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$\tilde{X}$ is called a fuzzy normal variable if for each $\alpha \in (0,1]$, we can select a crisp normal distribution $\mathrm{N}(\mu_\alpha, \sigma_\alpha^2)$ such that μ_α and σ_α^2 come from the α-cuts of the fuzzy numbers $\tilde{\mu}$ and $\widetilde{\sigma^2}$.The nested family $\{\mathrm{N}(\mu_\alpha, \sigma_\alpha^2)\}_{\alpha\epsilon(0,1]}$ is consistent in the sense that if $\alpha_2 < \alpha_1$, then $\mu_{\alpha2} \subseteq \mu_{\alpha1}$ as intervals.

Hence, we represent our fuzzy normal by the entire nested family of classical normal distributions, one for each level of membership α. Next, we apply the Zadeh extension principle (or an interval-based extension principle) to unify fuzzy parameters $\tilde{\mu}$ and $\widetilde{\sigma^2}$ into one membership function $\mu_X(x)$ for a fuzzy normal distribution. The key idea is once we have fuzzy sets for mean and variance, we want a single fuzzy set in $\boldsymbol{R}$ describing how likely it is for our fuzzy normal to take each real value x. This demands a recognised fuzzy extension approach that merges the fuzzy parameters into a single fuzzy set of real outputs. Generally, for each $\alpha \longrightarrow [\mu_\alpha, \sigma_\alpha^2]$ and t, the value $\tilde{B}_H(t)$ has membership function describing the degree to which it belongs to the fuzzy set $\tilde{B}_H(t)$.

$$\mu_{\tilde{B}} = e^{-\frac{(x-\tilde{\mu})^2}{2\widetilde{\sigma^2}}}$$

When we have intervals of means $\mu \in [\mu_\alpha^-, \mu_\alpha^+]$ and variances $\sigma \in [\sigma_\alpha^{2,-}, \sigma_\alpha^{2,+}]$, the pdf can vary accordingly. To find unified pdf we implement the threshold aggregator approach that merges the entire alpha-cut family into one membership function by allowing us to climb up alpha levels as far as possible, subject to the pdf condition. A bigger α indicates a narrower region for μ_α and σ_α^2, so it is more demanding that we still get bigger threshold in the pdf. The result is a single membership at $x \in [0,1]$. If we define τ as some threshold then x has membership $\geq \alpha$ if we can find a parameter pair in α-cuts that yields $\geq \tau$ pdf. So, membership is the supremum of all such α. This is a version of the Zadeh extension principle, adopted to the pdf is above threshold τ condition.

We now formalise these concepts mathematically. Let $\tilde{\mu}$ be a fuzzy mean and $\widetilde{\sigma^2}$ be a fuzzy variance. $\tilde{\mu}$ and $\widetilde{\sigma^2}$ are fuzzy numbers in R and $(0,\infty)$ respectively. Fix a threshold $\tau \in (0,\infty)$. For each $\alpha \in (0,1]$, define intervals $\mu_\alpha \epsilon\, \tilde{\mu}^\alpha, \sigma_\alpha^2 \epsilon\, \tilde{\mu}^{2^\alpha}$.

We say $x \in \widetilde{X^\alpha}$ if:

$\exists(\mu, \sigma^2) \in \mu_\alpha \times \sigma_\alpha^2$ such that $f_{\mu,\sigma^2}(x) \geq \tau$.

Then the membership function for $\tilde{X}$ is:

$$\mu_{\tilde{X}}(x) = \sup\{\alpha \in (0,1]: x \in \widetilde{X^\alpha}\}$$

We call $\tilde{X}$ a fuzzy normal variable under threshold τ.

This aggregator is one of many possible. Now we have a robust approach that merges fuzzy $\tilde{\mu}$ and $\widetilde{\sigma^2}$ into a single fuzzy set of real values x. It does so by checking the classical normal pdf at each α -cut and picking the supremum over α.

Fuzzy Martingale

A fuzzy martingale $\widetilde{M}(t)$ is a generalisation of a standard martingale, where the values are decribed by fuzzy numbers. For a process to be a fuzzy martingale, it must satisfy the martingale property with respect to fuzzy conditional expectations. So, for $\widetilde{M}(t)$ to be a fuzzy martingale, it must satify:

$$E\left[\widetilde{M}|F_s\right] = \widetilde{M}(s)$$

Proof for Fuzzy Fractional Brownian motion when H < ½ with Gaussian membership function

We derive step by step proof for ffBm when H < ½ and Gaussian membership function.

1. Integration by Parts with Fuzzy Variable:

Let $\tilde{Y}_t^H = \int_0^t s^{H-\frac{1}{2}} d\tilde{B}^H(t)$. Then:

$$\tilde{B}_t^H = \int_0^t s^{H-\frac{1}{2}} d\tilde{Y}_s^H.$$

Our proof:

$$\tilde{Y}_t^H = \int_0^t s^{H-\frac{1}{2}} d\tilde{B}^H(t)$$

$$= (t)^{\frac{1}{2}-H}\tilde{B}^H(t) - \int_0^t \tilde{B}^H(t) d\left(t^{\frac{1}{2}-H}\right)$$

$$= t^{\frac{1}{2}-H}\tilde{B}^H(t) - \int_0^t \tilde{B}^H(t)\left[\frac{1}{2} - H\right] t^{-\frac{1}{2}-H} dt$$

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Apply derivative:

$$d\tilde{Y}_t^H = (\frac{1}{2} - H)\, t^{-\frac{1}{2}-H} dt \tilde{B}^H(t) + d\tilde{B}^H(t) t^{\frac{1}{2}-H} - \tilde{B}^H(t)\left[\frac{1}{2} - H\right] t^{-\frac{1}{2}-H}$$
$$= d\tilde{B}^H(t) t^{\frac{1}{2}-H}$$
$$\Rightarrow d\tilde{B}_t^H = d\tilde{Y}^H(t) t^{-\frac{1}{2}+H}$$

Apply integration:

$$\tilde{B}_t^H = \int_0^t s^{-\frac{1}{2}+H} d\tilde{Y}^H(s) \quad (41)$$

2. Define fuzzy Molchan martingale:

$$\tilde{M}_t^H := \int_0^t \tilde{w}(t,s) d\tilde{B}^H(s),$$

where $\tilde{w}(t,s) \doteq (\frac{c}{C}) s^{-\alpha}(t-s)^{-\alpha}$ is a scaled beta kernel.

3. Upper bound for fuzzy $\tilde{B}_H(t)$ using integration by parts. From (41):

$$\sup\, |\tilde{B}_t^H| \le sup\, |\int_0^t s^{-\frac{1}{2}+H} d\tilde{Y}^H(s)| \le sup\, |\left[s^{H-\frac{1}{2}} \tilde{Y}_s^H\right]_0^T - \int_0^t d(s^{-\frac{1}{2}+H}) \tilde{Y}^H(s)|$$

Since the integral term is positive, therefore:

$$\sup\, |\tilde{B}_t^H| \le \sup\, |\left[s^{H-\frac{1}{2}} \tilde{Y}_s^H\right]_0^T - \int_0^t (1/2 - H) s^{-1/2-H}\, \tilde{Y}^H(s) ds|$$
$$\le \sup \left| T^{H-\frac{1}{2}} (\tilde{Y}_t^H - \tilde{Y}_0^H) \right| \le 2T^{H-\frac{1}{2}} \sup |\tilde{Y}_t^H|\,|$$

4. Now, substitute for $\tilde{Y}$:

$$\sup\, |\tilde{B}_t^H| \le 8HT^{2H-1} \sup(|\tilde{M}_t^H|)$$

5. We now apply expectation:

$$\mathrm{E}(\sup|\tilde{B}_t^H|)^p \le (8HT^{2H-1})^p\, \mathrm{E}(\sup(|\tilde{M}_t^H|)^p)$$

Since $\tilde{M}_t^H$ is a fuzzy Molchan martingale, by the Burkholder-Davis-Gundy inequality, there exists a constant $A_p^H > 0$ such that:

$$\mathrm{E}(\sup|\tilde{B}_t^H|)^p \le (8HT^{2H-1})^p\, A_p^H \mathrm{E}(\sup(\langle \tilde{M}_t^H \rangle)^{p/2}) \quad (42)$$

The next result we take from Norros, Proposition 2.1 (Norros, 1999):

$$Var(\tilde{M}_t^K) = \langle \tilde{M}^K \rangle_t = \frac{c^2(K)}{(2K)^2(2-2K)} t^{2-2K} := d(K) t^{2-2k}$$

6. We will now combine these results to compute quadratic variation for 1 – H > ½ to obtain:

$$\langle \tilde{M}_t^H \rangle = (\left(\sqrt{\frac{1-H}{1-K}}\right) \int_0^t s^{K-H} d\tilde{M}_s^K)\,)^2 = \frac{1-H}{H} \int_0^t s^{2-2H} d\langle \tilde{M}^{1-H} \rangle_s$$

Use [14] :$Var(\tilde{M}_t^K) = \langle \tilde{M}^K \rangle_t = \frac{c^2(K)}{(2K)^2(2-2K)} t^{2-2K} := d(K) t^{2-2k}$.

Substitute into: $d\langle \tilde{M}^{1-H} \rangle_s = d(1-H) s^{2-2(1-H)}$. Continuing with quadratic variation:

$$\langle \tilde{M}_t^H \rangle = (\left(\sqrt{\frac{1-H}{1-K}}\right) \int_0^t s^{K-H} d\tilde{M}_s^K)\,)^2 = \frac{1-H}{H} \int_0^t s^{2-2H} d\langle \tilde{M}^{1-H} \rangle_s =$$

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$$= \frac{1-H}{H} d(1-H) \int_0^t s^{2-2H} 2Hs^{2H-1} ds = \frac{2H(1-H)}{H} d(1-H) \int_0^t s^{1-2H} ds = d(1-H)t^{2(1-H)} \quad (43)$$

7. Apply BDG inequality on quadratic variation of MH, to obtain:

$$E\sqrt{\langle \widetilde{M}^H \rangle_t}^p \leq \tilde{B}_p^{1-H} t^{p(1-H)} \quad (44)$$

Using(42): $\mathrm{E}(\sup|\tilde{B}_t^H|)^p \leq (8HT^{2H-1})^p A_p^H \mathrm{E}(\sup(\langle \widetilde{M}_t^H \rangle)^{p/2})$, substitute into (44), we get:

$$\mathrm{E}(\sup|\tilde{B}_t^H|)^p \leq (8HT^{2H-1})^p A_p^H \tilde{B}_P^{1-H} T^{p(1-H)} \quad (45)$$

8. If we define $\tilde{C} := (8H)^p A_p^H \tilde{B}_P^{1-H}$ then $\mathrm{E}(\sup|\tilde{B}_t^H|)^p \leq \tilde{C}T^{pH}$. Proof complete.

5. Conclusion

We formalised the study of a fuzzy fractional Brownian motion (fBm) for H < ½ by adapting classical results from the usual fractional Brownian motion setting, while layering fuzziness on certain coefficients and amplitude functions. In the crisp scenario, a fractional Brownian motion with H < ½ is often represented via the Mandelbrot–Van Ness construction, in which one writes $B^H(t)$ as an integral of a kernel $K_H(t, s)$ against a standard Brownian motion. This kernel typically takes a difference form that remains integrable near the endpoints s = 0 and s = t, even though the process is not a semimartingale. To carry these ideas into a fuzzy framework, we assumed that the driving fractional Brownian motion is still a classical, measure-theoretic process, but the kernel and other amplitude-related parameters become fuzzy. This framework allowed us to reuse many of the standard measure-theoretic techniques for integrals and isometries, because the underlying noise retained all its crisp characteristics.

In order to define what fuzzy means in this context, we treated the kernel and the parameters as fuzzy sets whose possible values, for each outcome, lie in the intervals or have membership functions describing their degrees of plausibility. Each realisation of these fuzzy parameters can then be viewed as a crisp selection, ensuring the integral with respect to the standard Brownian motion is well-defined in the usual sense. With such a perspective, one can still establish the usual properties of existence, uniqueness, and boundedness. The driving noise is classical, so measure-theoretic arguments like the Young and Skorohod integrals for H < 1/2 continue to hold, and the presence of fuzziness in the amplitude and kernel does not break the underlying integrability conditions.

A key aspect of analysis focuses on uniqueness. If two processes $\tilde{B}_H^{(1)}(t)$ *and* $\tilde{B}_H^{(2)}(t)$ share the same fuzzy kernel data but happen to differ in how one selects fuzzy values for the integrand, one looks at their difference $\Delta\tilde{B}_H(t)$. Because the underlying noise is a crisp Brownian motion, one applies the standard isometry. If $\widetilde{K_{H,L}}(t,s)$ *and* $\widetilde{K_{H,U}}(t,s)$ coincide almost surely in the sense that their fuzzy parameters are identical for each scenario, then the difference integral is zero. This shows that no two distinct fuzzy processes can arise from the same fuzzy kernel data, and uniqueness is secured.

In the context of bounding the supremum of fuzzy fBm, we employed a form of the Molchan martingale technique and a BDG inequality approach. In standard fractional Brownian motion when H < ½, one can define a transform that behaves like a martingale with respect to a certain filtration or at least allows one to control the magnitude of $B^H(t)$. By retaining a crisp measure for the noise, the same style of arguments holds. We obtained an estimate of sup $|\tilde{B}_t^H|$in terms of $\sup(|\widetilde{M}_t^H|)$, where $\widetilde{M}_t^H$ is a fuzzy version of the Molchan martingale. Expectation inequalities follow from standard integrability requirements.

Taken together, we showed that working with H < ½ in the fuzzy context does not fundamentally destroy the usual integrability results, as long as the driving noise remains crisp. The main difference is that each amplitude or kernel value now belongs to a fuzzy set, but any specific scenario or selection yields a classical integrand, so integral definitions and uniqueness proofs

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proceed as in the standard framework. The bounding arguments are similar. We reused the classical measure-theoretic inequalities and simply recognised that the fuzziness entered in the amplitude, not in the measure. In this way, the entire theory remained consistent for H < ½, including existence of the fuzzy fractional Brownian motion, its uniqueness from the integral representation plus the crisp isometry argument, and the ability to label its distribution as fuzzy Gaussian via alpha-cuts and an aggregator principle.

The general idea of overlaying a fuzzy structure onto a crisp fractional Brownian motion is not entirely new. Various authors in fuzzy stochastic modelling that we thoroughly reviewed in [9] have taken a crisp process such as a Brownian motion or a fractional Brownian motion and allowed certain parameters or coefficients to be fuzzy, thus letting each outcome pick a different crisp coefficient from the fuzzy set. This approach makes it possible to reuse all the standard measure-theoretic arguments, because the noise is still treated in a classical Kolmogorov framework.

However, systematically applying this method in the specific case of H < ½ for fractional Brownian motion, together with an explicitly defined aggregator and Zadeh extension principle for fuzzy normal distributions, has not been formalised in the literature. Mostly, research focuses on a fully crisp version of fractional Brownian motion in the non-semimartingale regime without considering fuzziness, or it addresses fuzzy Brownian motion or fBm in a more informal manner without fully elaborating measure-theoretic integrals or the precise aggregator rules that unify fuzzy parameters into one membership function on the real line. Some articles instead mention fuzzy parameters in simpler SDE frameworks with Brownian noise but do not carefully handle the fractional exponents and the integration approach required for H< ½.

This research offers a unified formalism of how to adapt the fractional integral representation when H < ½, how to treat fuzzy kernel and amplitude values while preserving the crisp measure structure of the noise, and how to demonstrate existence, uniqueness, and bounding in that environment. We clarified how to use the classical fractional integral machinery in a fuzzy setting, employed the standard uniqueness arguments while acknowledging the fuzziness of the kernel, and explained how to define a fuzzy normal membership function rather than simply stating fuzzy Gaussian without formal details. The originality of our approach lies in its explicit handling of the non-semimartingale regime H< ½ within a fuzzy kernel environment, its maintenance of the validity of measure-theoretic integrals by keeping the noise crisp, and its articulation of a consistent approach to defining the distribution via a recognised aggregator principle for fuzzy normals. We presented a rigorous treatment of fuzzy fractional Brownian motion and a clear explanation of how fuzziness modifies the usual uniqueness proofs and distribution membership constitute a coherent roadmap that is currently missing in the existing literature.

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