Article

Not peer-reviewed version

Third-Order Neutral Delay Differential Equations with Mixed Nonlinearities: Almost Oscillation via Linearization Method and Arithmetic-Geometric Inequality

B. Sudha , George E Chatzarakis * , Ethiraju Thandapani

Posted Date: 23 January 2025

doi: 10.20944/preprints202501.1747.v1

Keywords: Oscillation; third-order; neutral differential equation; mixed nonlinearities; arithmetic-geometric inequality

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 January 2025 doi:10.20944/preprints202501.1747.v1

Article

Third-Order Neutral Delay Differential Equations with Mixed Nonlinearities: Almost Oscillation via Linearization Method and Arithmetic-Geometric Inequality

B.Sudha [1], G.E.Chatzarakis [2,*] and E.Thandapani [3]

[1] Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur - 600 203, India
[2] Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education, 15122, Marousi, Athens, Greece
[3] Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai - 600 005, India
* Correspondence: geaxatz@otenet.gr

Abstract: In the present article, we create new sufficient conditions for the oscillatory and asymptotic behavior of solutions of the third-order nonlinear neutral delay differential equations with several super-linear and sub-linear terms. The results are obtained first by applying arithmetic-geometric mean inequality along with linearization method and then using comparison method as well as integral averaging technique. Finally, we show the importance and novelty of the main results by applying them to special cases of the studied equation.

Keywords: oscillation; third-order; neutral differential equation; mixed nonlinearities; arithmetic-geometric inequality

MSC: 34C10; 34K11

---

1. Introduction

This paper deals with third-order nonlinear neutral delay differential equations with mixed nonlinearities of the form

$$\left(a(t)\left(z''(t)\right)^{\alpha}\right)' + \sum_{i=1}^{n} p_i(t)x^{\alpha_i}(\tau_i(t)) = 0,\ t \geq t_0 \geq 0, \tag{E}$$

where $z(t) = x(t) + bx(t-\sigma)$. In the sequel, the following conditions are assumed without further mention:

(H_1) $\alpha,\ \alpha_1,\ \alpha_2,\ \ldots,\alpha_n$ are ratios of odd positive integers such that $\alpha_1 > \alpha_2 > \ldots > \alpha_m > \alpha > \alpha_{m+1} > \ldots > \alpha_n$, with $\alpha \geq 1$, $b \in [0,\infty)$, $b \neq 1$ and $\sigma \in [0,\infty)$ are constants;

(H_2) $a,\ p_i \in C([t_0,\infty),(0,\infty))$ for $i = 1,2,3,\ldots,n$ and

$$A(t,t_0) = \int_{t_0}^{t} a^{-1/\alpha}(t)dt \ \text{ with } \ A(t,t_0) \to \infty \ \text{ as } \ t \to \infty;$$

(H_3) $\tau_i \in C^1([t_0,\infty),\mathbb{R})$ with $\tau_i(t) < t$ and $\lim_{t\to\infty} \tau_i(t) = \infty$ for $i = 1,2,3,\ldots,n$.

Let $\tau(t) = \min\{\tau_1(t),\ \tau_2(t),\ldots,\tau_n(t)\}$. By a solution of (E), we mean a function $x \in C([t_x,\infty),\mathbb{R})$, $t_x = \min\{t-\sigma,\tau(t)\}$ such that $a(z'')^{\alpha} \in C'([t_x,\infty),\mathbb{R})$ and x satisfies (E) on $[t_x,\infty)$. We consider only solutions of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \in [t_x,\infty)$ and tacitly assume that (E) possesses such solutions. If such a solution contains an unbounded number of zeros, it is said to be *oscillatory*; otherwise it is called *nonoscillatory*. The equation is said to be *almost oscillatory* if its solutions are either oscillatory or tend to zero monotonically.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 January 2025 doi:10.20944/preprints202501.1747.v1

The theory and applications of neutral type differential equations has drawn great interest over the past four decades since such equations are used to describe a variety of real world problems in physics, engineering, mathematical biology and so on, see, for example [1–3]. For recent applications and general theory of these equations, the reader is referred to the monographs [4,5].

The oscillatory character of third-order delay differential equations are peculiar in the sense that they may have both oscillatory and nonoscillatory solutions, or they have only oscillatory solutions. For example, in [6], all the solutions of the third-order delay differential equation

$$x'''(t) + x(t - \pi) = 0,$$

are oscillatory if $\pi e > 3$. However in [3], the third-order delay differential equation

$$x'''(t) + 2x'(t) - x(t - \frac{3\pi}{2}) = 0,$$

has the oscillatory solution $x_1(t) = \sin t$ and a nonoscillatory solution $x_2(t) = \exp(\beta t)$, where $\beta > 0$ such that

$$\beta^3 + 2\beta - \exp\left(-\frac{3\pi}{2}\beta\right) = 0.$$

Because of the above mentioned behavior of solutions of third-order differential equations, there has been great interest in establishing sufficient conditions for the oscillation or nonoscillation of solutions of different classes of differential equations of third-order, see, for example [3,4,7–20] and the references are contained therein.

Recently in [21], the authors studied the oscillatory behavior of (E) for the case $n = 1$ and $\alpha_1 = \alpha$, and in [22], the authors studied the following equation

$$\left(b_2(t)\left(\left(b_1(t)(z'(t))^{\gamma_1}\right)'\right)^{\gamma_2}\right)' + \sum_{i=1}^{m} q_i(t)x^{\alpha_i}(\tau_i(t)) = 0, \qquad (E_1)$$

where $z(t) = x(t) + bx(t - \tau_0)$, and obtained some sufficient conditions which state that every solution of (E_1) is either oscillatory or tends to zero eventually(almost oscillatory) under the assumption

$$\int_{t_0}^{\infty} b_1^{-1/\gamma_1}(t)dt = \int_{t_0}^{\infty} b_2^{-1/\gamma_2}(t)dt = \infty.$$

Since the positive solution of (E_1) satisfies the condition

$$z'(t) > 0, \ \left(b_1(t)(z'(t))^{\gamma_1}\right)' > 0, \ \left(b_2(t)\left(\left(b_1(t)(z'(t))^{\gamma_1}\right)'\right)^{\gamma_2}\right)' \le 0$$

and using this the authors infer that $z''(t) > 0$ for $t \ge t_0$. This is not true in general, for example, if $b_1(t) = t$ and $\gamma_1 = 1$ then we have $z'(t) + tz''(t) > 0$, and this may not imply that $z''(t) > 0$ for $t \ge t_0$. However, this is used in [22] to obtain the main results and hence the results in [22] may not be correct unless they have to assume that $b_1(t)$ is either constant or monotonically decreasing. Note that the authors used the function $b_1(t) = \frac{1}{t}$ in their examples which is clearly monotonically decreasing.

Motivated by the above observations and inspired by recent works [21,22], in this study we consider equation (E) which is same as (E_1) if $\gamma_1 = 1$ and $b_1(t) \equiv 1$ and then using linearization method and arithmetic-geometric inequality, we obtain some new criteria for the oscillation and asymptotic behavior of solutions of (E). This modified and corrected the results in [22]. Examples are provided to illustrate the importance and novelty of the main results.

2. Main Results

We begin with the following preliminary results, which will be used in the proof of the main results.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 January 2025 doi:10.20944/preprints202501.1747.v1

Lemma 2.1. *Assume that*

$$\alpha_i > \alpha, \ i = 1, 2, 3, ..., m \ \text{ and } \ \alpha_i < \alpha, \ i = m+1, m+2, ..., n. \tag{2.1}$$

Then an n-tuple $(\eta_1, \ \eta_2, \ ..., \eta_n)$ exists with $\eta_i > 0$ satisfying the conditions

$$\sum_{i=1}^{n} \alpha_i \eta_i = \alpha \ \text{ and } \ \sum_{i=1}^{n} \eta_i = 1. \tag{2.2}$$

Proof. From (2.2), we see that

$$\sum_{i=1}^{n} \beta_i \eta_i = 1 \ \text{ and } \ \sum_{i=1}^{n} \eta_i = 1$$

where $\beta_i = \frac{\alpha_i}{\alpha}$. The rest of the proof is similar to Lemma 1 of [23] and hence the details are omitted. □

Lemma 2.2 ([5], Lemma 1.5.1)**.** *Let $h, g : [t_0, \infty) \to \mathbb{R}$ such that $h(t) = g(t) + bg(t-c)$, $t \geq t_0 + \max\{0, c\}$, where $p \neq 1$ and c are constants. Assume that there exists a constant $l \in \mathbb{R}$ such that $\lim_{t\to\infty} h(t) = l$.*

(i) *If $\lim_{t\to\infty} \inf g(t) = g_* \in \mathbb{R}$, then $l = (1+b)g_*$;*
(ii) *If $\lim_{t\to\infty} \sup g(t) = g^* \in \mathbb{R}$, then $l = (1+b)g^*$.*

Lemma 2.3 ([21], Lemma 1)**.** *Let $x(t)$ be an eventually positive solution of equation (E). Then there exists a sufficiently large $t_1 \geq t_0$ such that, for all $t \geq t_1$ either*

(I) $z(t) > 0, z'(t) > 0, z''(t) > 0, (a(t)(z''(t))^\alpha)' \leq 0,$
(II) $z(t) > 0, z'(t) < 0, z''(t) > 0, (a(t)(z''(t))^\alpha)' \leq 0.$

Lemma 2.4. *Let $x(t)$ be an eventually positive solution of equation (E) and assume that Case (II) of Lemma 2.3 holds. If*

$$\int_{t_0}^{\infty} \xi \left(\frac{1}{a(\xi)} \int_{\xi}^{\infty} Q(s) ds \right)^{1/\alpha} d\xi = \infty, \tag{2.3}$$

where

$$Q(t) = \prod_{i=1}^{n} \left(\frac{p_i(t)}{\eta_i} \right)^{\eta_i}, \tag{2.4}$$

with η_i defined as in Lemma 2.1, then

$$\lim_{t\to\infty} x(t) = 0. \tag{2.5}$$

Proof. Since $z(t) > 0$ and $z'(t) < 0$, there exists a constant $M > 0$ such that $\lim_{t\to\infty} z(t) = M \geq 0$. We claim that $M = 0$. If not, then by using Lemma 2.2, we see that $\lim_{t\to\infty} x(t) = \frac{M}{1+b} > 0$. Then there exists $t_1 \geq t_0$ such that for all $t \geq t_1$, we have

$$x(\tau_i(t)) \geq \frac{M}{2(1+b)}, \ i = 1, 2, 3, ..., n.$$

Using the last inequality, we see that

$$\begin{aligned} \sum_{i=1}^{n} p_i(t) x^{\alpha_i}(\tau_i(t)) &\geq \sum_{i=1}^{n} p_i(t) \left(\frac{M}{2(1+b)} \right)^{\alpha_i} \\ &= \left(\frac{M}{2(1+b)} \right)^{\alpha} \sum_{i=1}^{n} p_i(t) \left(\frac{M}{2(1+b)} \right)^{\alpha_i - \alpha}. \end{aligned} \tag{2.6}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 January 2025 doi:10.20944/preprints202501.1747.v1

By Lemma 2.2, there exists $\eta_1,\ \eta_2,\ ..., \eta_n$ with

$$\sum_{i=1}^{n}\alpha_i\eta_i-\alpha\sum_{i=1}^{n}\eta_i=0.$$

The arithmetic-geometric mean inequality (see [24]) leads to

$$\sum_{i=1}^{n}\eta_i u_i\geq\prod_{i=1}^{n}u_i^{\eta_i},\ \text{for any}\ \ u_i\geq 0,\ i=1,2,3,...,n.$$

In view of the above inequality, we obtain

$$\begin{aligned}\sum_{i=1}^{n}p_i(t)\left(\frac{M}{2(1+b)}\right)^{\alpha_i-\alpha} &= \sum_{i=1}^{n}\eta_i\left(\frac{p_i(t)}{\eta_i}\right)\left(\frac{M}{2(1+b)}\right)^{\alpha_i-\alpha}\\ &\geq \prod_{i=1}^{n}\left(\frac{p_i(t)}{\eta_i}\right)^{\eta_i}\left(\frac{M}{2(1+b)}\right)^{\eta_i(\alpha_i-\alpha)}\\ &= \prod_{i=1}^{n}\left(\frac{p_i(t)}{\eta_i}\right)^{\eta_i}=Q(t).\end{aligned}$$

This together with (2.6) yields that

$$\sum_{i=1}^{n}p_i(t)x^{\alpha_i}(\tau_i(t))\geq Q(t)\left(\frac{M}{2(1+b)}\right)^{\alpha}. \tag{2.7}$$

Combining (E) and (2.7), we take

$$(a(t)(z''(t))^{\alpha})'+\left(\frac{M}{2(1+b)}\right)^{\alpha}Q(t)\leq 0.$$

Further note that there exist constants M_1 and M_2 such that $\lim_{t\to\infty}a(t)(z''(t))^{\alpha}=M_1\geq 0$ and $\lim_{t\to\infty}z'(t)=M_2\leq 0$.

Now a method similar to that in Theorem 15 of [16] leads to the conclusion that $\lim_{t\to\infty}x(t)=0$. This completes the proof. □

Lemma 2.5. *Let $x(t)$ be a positive solution of (E) with corresponding function $z(t)\in$ class (I) for all $t\geq t_1$. Then*

(i) $z'(t)\geq(a^{1/\alpha}(t)z''(t))A(t,t_1)$,

(ii) $\frac{z'(t)}{A(t,t_1)}$ *is decreasing,*

(iii) $z(t)\geq(a^{1/\alpha}(t)z''(t))A_1(t,t_1)$,

(iv) $z(t)\geq A_1(t,t_1)\frac{z'(t)}{A(t,t_1)}$,

(ii) $\frac{z(t)}{A_1(t,t_1)}$ *is decreasing,*

where $A_1(t,t_1)=\int_{t_1}^{t}A(s,t_1)ds$.

Proof. Since $z(t)\in$ class (I), we see that $a(t)(z''(t))^{\alpha}>0$ and decreasing for all $t\geq t_1$. Then

$$\begin{aligned}z'(t)\geq z'(t)-z'(t_1) &= \int_{t_1}^{t}\frac{(a(s)(z''(s))^{\alpha})^{1/\alpha}}{a^{1/\alpha}(s)}ds\\ &\geq A(t,t_1)a^{1/\alpha}(t)z''(t),\end{aligned} \tag{2.8}$$

which proves (i).

 doi:10.20944/preprints202501.1747.v1

Moreover

$$\left(\frac{z'(t)}{A(t,t_1)}\right)' = \frac{A(t,t_1)a^{1/\alpha}(t)z''(t) - z'(t)}{a^{1/\alpha}(t)A^2(t,t_1)} \leq 0,$$

which implies that $\frac{z'(t)}{A(t,t_1)}$ is decreasing.

Integrating (2.8) from t_1 to t yields

$$z(t) \geq A_1(t,t_1)a^{1/\alpha}(t)z''(t),$$

which proves (iii).

Since

$$z(t) - z(t_1) = \int_{t_1}^{t} A(s,t_1)\frac{z'(s)}{A(s,t_1)}ds,$$

or

$$z(t) \geq A_1(t,t_1)\frac{z'(t)}{A(t,t_1)},$$

where we have used (ii). This proves (iv).

Finally,

$$\left(\frac{z(t)}{A_1(t,t_1)}\right)' = \frac{A_1(t,t_1)z'(t) - A(t,t_1)z(t)}{A_1^2(t,t_1)} \leq 0$$

by (iv). Hence $\frac{z(t)}{A_1(t,t_1)}$ is decreasing. This completes the proof. □

Next, we state and prove the main theorems.

Theorem 2.6. *Let condition (2.3) holds. If the first-order delay differential equation*

$$w'(t) + \frac{1}{\alpha}\left(\frac{1}{1+b}\right)^{\alpha} Q_1(t)w(\tau(t)) = 0, \tag{2.9}$$

where

$$Q_1(t) = Q(t)A_1^{\alpha}(\tau(t),t_2),$$

with $Q(t)$ defined as in (2.4), is oscillatory for all large $t_1 \geq t_0$ and for some $t_2 \geq t_1$, then the equation (E) is almost oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (E). Then with no loss of generality, assume $x(t) > 0$, $x(t-\sigma) > 0$ and $x(\tau_i(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then from Lemma 2.3 that the corresponding function $z(t) > 0$ for all $t \geq t_1$ and satisfies either (I) or (II). If $z(t)$ satisfies case (II), then from Lemma 2.4 that (2.5) holds, and we need to consider the other case (I).

From (E), we see that

$$\begin{aligned}(a(t)(z''(t))^{\alpha})' &= ((a^{1/\alpha}(t)z''(t))^{\alpha})' \\ &= \alpha(a^{1/\alpha}(t)z''(t))^{\alpha-1}(a^{1/\alpha}(t)z''(t))' \\ &= -\sum_{i=1}^{n} p_i(t)x^{\alpha_i}(\tau_i(t))\end{aligned}$$

and so

$$(a^{1/\alpha}(t)z''(t))' + \frac{1}{\alpha}(a^{1/\alpha}(t)z''(t))^{1-\alpha}\sum_{i=1}^{n} p_i(t)x^{\alpha_i}(\tau_i(t)) = 0. \tag{2.10}$$

Since $z'(t) > 0$ and $z''(t) > 0$, there exists a constant d_0(it is also possible that $d_0 = \infty$) such that $\lim_{t\to\infty} z'(t) = d_0 > 0$. Consequently, by Lemma 2.2, $\lim_{t\to\infty} x'(t) = \frac{d_0}{1+b} > 0$, and we conclude that

$$x'(t) > 0. \tag{2.11}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 January 2025 doi:10.20944/preprints202501.1747.v1

Using (2.11), we see that $z(t) = x(t) + bx(t-\sigma) \leq (1+b)x(t)$, that is,

$$x(t) \geq \left(\frac{1}{1+b}\right) z(t). \tag{2.12}$$

In addition, we have

$$\begin{aligned} \sum_{i=1}^{n} p_i(t) x^{\alpha_i}(\tau_i(t)) &\geq \sum_{i=1}^{n} p_i(t) x^{\alpha_i}(\tau(t)) \\ &= x^{\alpha}(\tau(t)) \sum_{i=1}^{n} p_i(t) x^{\alpha_i - \alpha}(\tau(t)). \end{aligned} \tag{2.13}$$

In view of Lemma 2.1, there exists $\eta_1, \eta_2, \ldots, \eta_n$ with

$$\sum_{i=1}^{n} \alpha_i \eta_i - \alpha \sum_{i=1}^{n} \eta_1 = 0.$$

The arithmetic - geometric mean inequality (see [24]) gives

$$\sum_{i=1}^{n} \eta_i u_i \geq \prod_{i=1}^{n} u_i^{\eta_i} \quad \text{for any} \quad u_i \geq 0,\ i = 1, 2, \ldots, n.$$

Therefore, we have

$$\begin{aligned} \sum_{i=1}^{n} p_i(t) x^{\alpha_i - \alpha}(\tau(t)) &= \sum_{i=1}^{n} \eta_i \left(\frac{p_i(t)}{\eta_i}\right) x^{\alpha_i - \alpha}(\tau(t)) \\ &\geq \prod_{i=1}^{n} \left(\frac{p_i(t)}{\eta_i}\right)^{\eta_i} (x(\tau(t)))^{\eta_i(\alpha_i - \alpha)} \\ &= \prod_{i=1}^{n} \left(\frac{p_i(t)}{\eta_i}\right)^{\eta_i} = Q(t). \end{aligned}$$

This together with (2.13) yields that

$$\sum_{i=1}^{n} p_i(t) x^{\alpha_i}(\tau_i(t)) \geq Q(t) x^{\alpha}(\tau(t)). \tag{2.14}$$

Using (2.12), (2.14) in (2.10), we obtain

$$\left(a^{1/\alpha}(t) z''(t)\right)' + \frac{1}{\alpha}\left(a^{1/\alpha}(t) z''(t)\right)^{1-\alpha} Q(t) \left(\frac{1}{1+b}\right)^{\alpha} z^{\alpha}(\tau(t)) \leq 0. \tag{2.15}$$

From Lemma 2.5(iii), we see that

$$z(\tau(t)) \geq A_1(\tau(t), t_1)\left(a^{1/\alpha}(t) z''(t)\right) \tag{2.16}$$

for $t \geq t_1$. Since $a^{1/\alpha}(t) z''(t)$ is nonincreasing and $\alpha \geq 1$, we have

$$\left(a^{1/\alpha}(t) z''(t)\right)^{1-\alpha} \geq \left(a^{1/\alpha}(\tau(t)) z''(\tau(t))\right)^{1-\alpha}. \tag{2.17}$$

Using (2.17) in (2.15) yields

$$\left(a^{1/\alpha}(t) z''(t)\right)' + \frac{1}{\alpha}\left(\frac{1}{1+b}\right)^{\alpha}\left(a^{1/\alpha}(\tau(t)) z''(t)\right)^{1-\alpha} Q(t) z^{\alpha}(\tau(t)) \leq 0. \tag{2.18}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 January 2025 doi:10.20944/preprints202501.1747.v1

From (2.16) and (2.18), we observe that

$$\left(a^{1/\alpha}(t)z''(t)\right)' + \frac{1}{\alpha}\left(\frac{1}{1+b}\right)^{\alpha} Q(t)A_1^{\alpha}(\tau(t),t_1)\left(a^{1/\alpha}(\tau(t))z''(t)\right) \le 0. \tag{2.19}$$

Let $w(t) = a^{1/\alpha}(t)z''(t)$ in (2.19), we see that w is a positive solution of the first-order linear delay differential inequality

$$w'(t) + \frac{1}{\alpha}\left(\frac{1}{1+b}\right)^{\alpha} Q_1(t)w(\tau(t)) \le 0.$$

The function w is clearly strictly decreasing for all $t \ge t_2$ and so by Theorem 1 of [25], there exists a positive solution of the equation (2.9), which contradicts the fact that the equation (2.9) is oscillatory. The proof of the theorem is complete. □

The next result immediately follows from Theorem 2.6 and [Theorem 2.11, [14]].

Corollary 2.7. *Let condition (2.3) holds. If*

$$\liminf_{t\to\infty} \int_{\tau(s)}^{t} Q_1(s)ds \ge \frac{\alpha(1+b)^{\alpha}}{e} \tag{2.20}$$

where $Q_1(t)$ is defined as in Theorem 2.6, then the equation (E) is almost oscillatory.

In our next theorem we use Riccati transformation and integral averaging technique to obtain oscillation results.

Theorem 2.8. *Let condition (2.3) holds and $\tau(t) \in C'([t_0,\infty))$ with $\tau'(t) > 0$. Assume that there exists a function $\rho \in C'([t_0,\infty),(0,\infty))$, for sufficiently large $t_1 \ge t_0$, there is a $t_2 \ge t_1$ such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[\rho(s)Q_2(s) - \frac{\alpha(1+b)^{\alpha}(\rho'(s))^2}{4\rho(s)A(\tau(s),t_1)\tau'(s)}\right] ds = \infty, \tag{2.21}$$

where $Q_2(t) = Q(t)(A_1(\tau(t)),t_1)^{\alpha-1}$ with $Q(t)$ defined as in (2.4) and $(\rho'(t))_+ = \max\{0,\rho'(t)\}$. Then the equation (E) is almost oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (E). Then with no loss of generality, assume $x(t) > 0$, $x(t-\sigma) > 0$ and $x(\tau_i(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Then from Lemma 2.3, we see that the corresponding function $z(t) > 0$ and satisfies either case (I) or case (II) for all $t \ge t_1$. If $z(t)$ satisfies case (II) then from Lemma 2.4 that (2.5) holds, and we need to consider the other case (I). It follows from (2.16), and the fact that $a^{1/\alpha}(t)z''(t)$ is nonincreasing that

$$a^{1/\alpha}(t)z''(t) \quad \le \quad a^{1/\alpha}(\tau(t))z''(t) \le A_1(\tau(t),t_1)^{-1}z(\tau(t))$$

and so

$$(a^{1/\alpha}(t)z''(t))^{1-\alpha} \ge (A_1(\tau(t),t_1)^{\alpha-1}(z(\tau(t)))^{1-\alpha}.$$

Using this inequality in (2.18) yields

$$\left(a^{1/\alpha}(t)z''(t)\right)' + \frac{1}{\alpha}\left(\frac{1}{1+b}\right)^{\alpha} Q_2(t)z(\tau(t)) \le 0,\ t \ge t_1. \tag{2.22}$$

Define

$$w(t) = \frac{\rho(t)a^{1/\alpha}(t)z''(t)}{z(\tau(t))}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 January 2025 doi:10.20944/preprints202501.1747.v1

Then $w(t) > 0$ and using (2.22), we obtain

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - \frac{1}{\alpha}\left(\frac{1}{1+b}\right)^{\alpha} \rho(t) Q_2(t) - \frac{w(t) z'(\tau(t))\tau'(t)}{z(\tau(t))}. \tag{2.23}$$

From Lemma 2.5(i), we see that

$$z'(\tau(t)) \geq (a^{1/\alpha}(\tau(t)) z''(\tau(t))) A(\tau(t), t_1) \geq (a^{1/\alpha}(t) z''(t)) A(\tau(t), t_1).$$

Combining the last inequality with (2.23), we obtain

$$w'(t) \leq \frac{-1}{\alpha}\left(\frac{1}{1+b}\right)^{\alpha} \rho(t) Q_2(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{A(\tau(t), t_1)\tau'(t)}{\rho(t)} w^2(t). \tag{2.24}$$

Using the last inequality $Bu - Au^2 \leq \frac{1}{4}\frac{B^2}{A}$, $A > 0$ in (2.24), we have

$$w'(t) \leq \frac{-1}{\alpha}\left(\frac{1}{1+b}\right)^{\alpha} \rho(t) Q_2(t) + \frac{(\rho'(t))^2}{4\rho(t) A(\tau(t), t_1)\tau'(t)}, \; t \geq t_2 \geq t_1.$$

Integrating from t_2 to t, we get

$$\int_{t_2}^{t} \left[\frac{1}{\alpha}\left(\frac{1}{1+b}\right)^{\alpha} \rho(s) Q_2(s) - \frac{(\rho'(s))^2}{4\rho(s) A(\tau(s), t_1)\tau'(s)}\right] ds \leq w(t_2)$$

which contradicts (2.21). The proof of the theorem is complete. □

Theorem 2.9. *Let condition (2.3) holds and*

$$\limsup_{t\to\infty} \left\{ \frac{1}{A(\tau(t), t_*)} \int_{t_*}^{\tau(t)} Q_2(s) A(s, t_*) A_1(\tau(s), t_*) ds + \int_{\tau(t)}^{t} Q_2(s) A_1(\tau(s), t_*) ds + A_1(\tau(s), t_*) \int_t^{\infty} Q_2(s) ds \right\} > \alpha(1+b)^{\alpha}, \tag{2.25}$$

where $Q_2(t)$ is as defined in Theorem 2.8. Then the equation (E) is almost oscillatory.

Proof. Let $x(t)$ be a positive solution of (E) with $x(t-\sigma) > 0$ and $x(\tau_i(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then the corresponding function $z(t) > 0$ and satisfies case (I) or case (II) of Lemma 2.3 for all $t \geq t_1$. If $z(t)$ satisfies case (II) then from Lemma 2.4 that (2.5) holds, and therefore we need to consider the other case (I). Proceeding as in the proof of Theorem 2.8, we arrive at (2.22).

Integrating (2.22) from t to ∞ yields

$$z''(t) \geq \frac{1}{\alpha(1+b)^{\alpha}} \frac{1}{a^{1/\alpha}(t)} \int_t^{\infty} Q_2(s) z(\tau(s)) ds.$$

Integrating again from t_1 to t, we get

$$\begin{aligned}
\alpha(1+b)^{\alpha} z'(t) &\geq \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(u)} \int_u^{\infty} Q_2(s) z(\tau(s)) ds \\
&= \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(u)} \int_u^{t} Q_2(s) z(\tau(s)) ds + \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(u)} \int_t^{\infty} Q_2(s) z(\tau(s)) ds \\
&= \int_{t_1}^{t} A(u, t_1) Q_2(s) z(\tau(s)) ds + A(t, t_1) \int_t^{\infty} Q_2(s) z(\tau(s)) ds.
\end{aligned}$$

Employing Lemma 2.5(iv), we have

$$\alpha(1+b)^{\alpha}\frac{z(t)A(t,t_1)}{A_1(t,t_1)} \geq \int_{t_1}^{t} A(s,t_1)Q_2(s)z(\tau(s))ds + A(t,t_1)\int_{t}^{\infty} Q_2(s)z(\tau(s))ds,$$

or

$$\begin{aligned}\alpha(1+b)^{\alpha}\frac{z(\tau(t))A(\tau(t),t_1)}{A_1(\tau(t),t_1)} \geq & \int_{t_1}^{\tau(t)} A(s,t_1)Q_2(s)z(\tau(s))ds + A(\tau(t),t_1)\int_{\tau(t)}^{t} Q_2(s)z(\tau(s))ds \\ & +A(\tau(t),t_1)\int_{t}^{\infty} Q_2(s)z(\tau(s))ds.\end{aligned}$$

Taking into account that $z(t)$ is increasing and $\frac{z(t)}{A_1(t,t_1)}$ is decreasing, one can verify that

$$\begin{aligned}\alpha(1+b)^{\alpha}\frac{z(\tau(t))A(\tau(t),t_1)}{A_1(\tau(t),t_1)} \geq & \frac{z(\tau(t))}{A_1(\tau(t),t_1)}\int_{t_1}^{\tau(t)} A(s,t_1)A_1(\tau(s),t_1)Q_2(s)ds \\ & +\frac{A(\tau(t),t_1)z(\tau(t))}{A_1(\tau(t),t_1)}\int_{\tau(t)}^{t} Q_2(s)A_1(\tau(s),t_1)ds \\ & +A(\tau(t),t_1)z(\tau(t))\int_{t}^{\infty} Q_2(s)ds,\end{aligned}$$

which yields

$$\begin{aligned}\alpha(1+b)^{\alpha} \geq & \frac{1}{A(\tau(t),t_1)}\int_{t_1}^{\tau(t)} A(s,t_1)A_1(\tau(s),t_1)Q_2(s)ds + \int_{\tau(t)}^{t} Q_2(s)A_1(\tau(s),t_1)ds \\ & +A_1(\tau(t),t_1)\int_{t}^{\infty} Q_2(s)ds.\end{aligned}$$

Taking lim sup as $t \to \infty$ on both sides of the last inequality, we are led to a contradiction with (2.25). The proof of the theorem is complete. □

Theorem 2.10. *Let condition* (2.3) *holds, and*

$$\liminf_{t\to\infty} A(t,t_1)\int_{t}^{\infty}\frac{Q(s)(A_1(\tau(s),t_1))^{\alpha}}{A(s,t_1)}ds > \frac{\alpha(1+b)^{\alpha}}{4}, \quad (2.26)$$

where $Q(t)$ *defined as in* (2.4)*. Then the equation* (E) *is almost oscillatory.*

Proof. Let $x(t)$ be a positive solution of (E) with $x(t-\sigma) > 0$ and $x(\tau_i(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then the corresponding function $z(t) > 0$ and satisfies case (I) or case (II) of Lemma 2.3 for all $t \geq t_1$. If $z(t)$ satisfies case (II) then by Lemma 2.4 that (2.5) holds and therefore we need to consider the other case(I). Proceeding as in the proof of Theorem 2.8, we arrive at (2.22).

From Lemma 2.5(ii) and (iv), we have

$$z(\tau(t)) \geq A_1(\tau(t),t_1)\frac{z'(t)}{A(t,t_1)}$$

and using this inequality in (2.22), we obtain

$$\left(a^{1/\alpha}(t)z''(t)\right)' + \frac{1}{\alpha(1+b)^{\alpha}}\frac{Q(t)A^{\alpha}(\tau(t),t)}{A(t,t_1)}z'(t) \leq 0.$$

Let $u(t) = z'(t)$. Then, we see that $u(t)$ is a positive solution of the inequality

$$\left(a^{1/\alpha}(t)u'(t)\right)' + \frac{1}{\alpha(1+b)^{\alpha}}\frac{Q(t)A_1^{\alpha}(\tau(t),t)}{A(t,t_1)}u(t) \leq 0.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 January 2025 doi:10.20944/preprints202501.1747.v1

Define

$$w(t) = \frac{a^{1/\alpha}(t)u'(t)}{u(t)},\ t \ge t_1.$$

Then $w(t) > 0$ and satisfies

$$w(t) \ge \frac{1}{\alpha(1+b)^\alpha} \int_t^\infty \frac{Q(s)A_1^\alpha(\tau(s),t_1)}{A(s,t_1)} ds + \int_t^\infty \frac{w^2(s)}{a^{1/\alpha}(s)} ds. \tag{2.27}$$

Multiply (2.27) by $A(t,t_1)$ and letting $M = \inf_{t\ge t_1} R(t)w(t)$, we see that

$$\begin{aligned} M &> \frac{1}{4} + M^2 R(t) \int_t^\infty \frac{1}{a^{1/\alpha}(s)R^2(s)} ds \\ &= \frac{1}{4} + M^2 R(t) \frac{1}{R(t)} ds = \frac{1}{4} + M^2, \end{aligned}$$

which contradicts the admissible value of M. The proof of the theorem is complete. □

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. *Consider the third-order nonlinear neutral differential equation of the form*

$$\left(\frac{1}{t}(x(t) + 2x(t-1))''\right)' + \frac{d_1}{t^4} x^3\left(\frac{t}{2}\right) + \frac{d_2}{t^4} x^{1/3}\left(\frac{t}{3}\right) = 0,\ t \ge 1, \tag{3.1}$$

where $d_1 > 0$ and $d_2 > 0$ are constants.

Here $a(t) = \frac{1}{t}$, $b = 2$, $\sigma = 1$, $p_1(t) = \frac{d_1}{t^4}$, $p_2(t) = \frac{d_2}{t^4}$, $\alpha = 1$, $\alpha_1 = 3$, $\alpha_2 = \frac{1}{3}$, $\tau_1(t) = \frac{t}{2}$, $\tau_2(t) = \frac{t}{3}$. A simple computation shows that $\tau(t) = \frac{t}{3}$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{3}{4}$,

$$Q(t) = (4d_1)^{1/4}\left(\frac{4}{3}d_2\right)^{3/4} \frac{1}{t^4},\ A(t,1) \approx \frac{t^2}{2} \ \text{and} \ A_1(t,1) \approx \frac{t^3}{6}.$$

The condition (2.3) becomes

$$\int_1^\infty \xi \left(\xi \int_\xi^\infty (4d_1)^{\frac{1}{4}} \left(\frac{4}{3}d_2\right)^{\frac{3}{4}} \frac{1}{s^4} ds\right) d\xi = (4d_1)^{\frac{1}{4}} \left(\frac{4}{3}d_2\right)^{\frac{3}{4}} \frac{1}{3} \int_1^\infty \frac{1}{\xi} d\xi = \infty,$$

that is, condition (2.3) is satisfied. The condition (2.20) becomes

$$\liminf_{t\to\infty} \int_{t/3}^t (4d_1)^{\frac{1}{4}} \left(\frac{4}{3}d_2\right)^{\frac{3}{4}} \frac{1}{162} \frac{1}{s} ds = (4d_1)^{\frac{1}{4}} \left(\frac{4}{3}d_2\right)^{\frac{3}{4}} \frac{1}{162} \ln 3 > \frac{3}{e},$$

that is, condition (2.20) is satisfied if $d_1^{\frac{1}{4}} d_2^{\frac{3}{4}} > \frac{243(3)^{3/4}}{2e\ln 3}$. Thus by Corollary 2.7, the equation (3.1) is almost oscillatory if $d_1^{\frac{1}{4}} d_2^{\frac{3}{4}} > 92.7423$.

Example 3.2. *Consider the third-order nonlinear neutral delay differential equation*

$$\left(\left((x(t) + 2x(t-1))''\right)^{\frac{5}{3}}\right)' + \frac{d_1}{t^2} x^3\left(\frac{t}{2}\right) + \frac{d_2}{t^2} x\left(\frac{t}{3}\right) = 0,\ t \ge 1, \tag{3.2}$$

where $d_1 > 0$ and $d_2 > 0$ are constants.

 doi:10.20944/preprints202501.1747.v1

Here $a(t) = 1$, $b = 2$, $\sigma = 1$, $p_1(t) = \frac{d_1}{t^2}$, $p_2(t) = \frac{d_2}{t^2}$, $\alpha = \frac{5}{3}$, $\alpha_1 = 3$, $\alpha_2 = 1$, $\tau_1(t) = \frac{t}{2}$, $\tau_2(t) = \frac{t}{3}$. By a simple calculation, we see that $\tau(t) = \frac{t}{3}$, $\eta_1 = \frac{1}{3}$, $\eta_2 = \frac{2}{3}$,

$$Q(t) = 3d_1^{1/3}\left(\frac{d_2}{2}\right)^{2/3}\frac{1}{t^2},\ A(t,1) \approx t \ \text{ and } \ A_1(t,1) \approx \frac{t^2}{2}.$$

The condition (2.3) becomes

$$\int_1^\infty \xi\left(\int_\xi^\infty 3d_1^{1/3}\left(\frac{d_2}{2}\right)^{2/3}\frac{1}{s^2}ds\right)^{\frac{3}{5}}d\xi = \left(3d_1^{1/3}\left(\frac{d_2}{2}\right)^{2/3}\right)^{\frac{3}{5}}\int_1^\infty \xi^{2/5}d\xi = \infty,$$

that is, condition (2.3) holds. Since $Q_2(t) = \frac{3d_1^{1/3}\left(\frac{d_2}{2}\right)^{2/3}}{(18)^{\frac{2}{3}}}\frac{1}{t^{2/3}}$, then by choosing $\rho(t) = 1$, the condition (2.21) is clearly satisfied for all $d_1 > 0$ and $d_2 > 0$. Therefore by Theorem 2.8, the equation (3.2) is almost oscillatory if $d_1 > 0$ and $d_2 > 0$.

Example 3.3. *Consider the third-order nonlinear neutral delay differential equation*

$$\left(t^{\frac{1}{2}}(x(t) + 2x(t-1))''\right)' + \frac{d_1}{t^4}x^{\frac{5}{3}}\left(\frac{t}{2}\right) + \frac{d_2}{t}x^{\frac{1}{3}}\left(\frac{t}{4}\right) = 0, t \geq 1, \tag{3.3}$$

where $d_1 > 0$ and $d_2 > 0$ are constants.

Here $a(t) = t^{\frac{1}{2}}$, $b = 2$, $\sigma = 1$, $p_1(t) = \frac{d_1}{t^4}$, $p_2(t) = \frac{d_2}{t}$, $\alpha = 1$, $\alpha_1 = \frac{5}{3}$, $\alpha_2 = \frac{1}{3}$, $\tau_1(t) = \frac{t}{2}$, $\tau_2(t) = \frac{t}{4}$. By a simple calculation, we see that $\tau(t) = \frac{t}{4}$, $\eta_1 = \eta_2 = \frac{1}{2}$,

$$Q(t) = \frac{2\sqrt{d_1d_2}}{t^{\frac{5}{2}}},\ A(t,1) \approx 2t^{\frac{1}{2}} \ \text{ and } \ A_1(t,1) \approx \frac{4}{3}t^{\frac{3}{2}}.$$

The condition (2.3) becomes

$$2\sqrt{d_1d_2}\int_1^\infty \xi\left(\frac{1}{\sqrt{\xi}}\int_\xi^\infty \frac{1}{s^{\frac{5}{2}}}ds\right)d\xi = \frac{4}{3}\sqrt{d_1d_2}\int_1^\infty \frac{d\xi}{\xi} = \infty,$$

that is, condition (2.3) holds. The condition (2.26) becomes

$$\liminf_{t\to\infty} \frac{\sqrt{t}}{3}\int_t^\infty \frac{\sqrt{d_1d_2}}{s^{\frac{3}{2}}}ds = \sqrt{d_1d_2} > \frac{9}{8},$$

that is, condition (2.26) holds if $\sqrt{d_1d_2} > 1.125$. Also the condition (2.25) holds if $\sqrt{d_1d_2} > 0.81691$. Hence, by Theorem 2.10, the equation (3.3) is almost oscillatory if $\sqrt{d_1d_2} > 1.125$ and the same conclusion holds by Theorem 2.9 if $\sqrt{d_1d_2} > 0.81691$). Therefore, Theorem 2.9 is better than Theorem 2.10.

Note that using Corollary 1 of [22], we see that (3.3) is almost oscillatory if $\sqrt{d_1d_2} > 2.3883203$. So our Theorems 2.9 and 2.10 significantly improve Corollary 1 of [22].

4. Conclusion

In this paper, we have obtained some new oscillation criteria by using arithmetic-geometric mean inequality along with linearization technique and then applying comparison method and integral averaging technique. The obtained results improve that of in [22] and this is illustrated via an example.The results already reported in the literature [3,9–13,15–19,21] cannot be applied to equations (3.1) to (3.3) since the number of nonlinear terms are more than one.

References

1. I. Gyori and F. Hartung, Stability of a single neuron model with delay, J. Comput. Appl. Math., 157(2003), 73-92.
2. J. Hale and S.V. Lunel, Introduction to Function Differential Equations, Springer, New York, 1993.
3. S. Padhi and S. Pati, Theory of Third Order Differential Equations, Springer, New Delhi, 2014.
4. L.H. Erbe, Q. Kong and B.G. Zhang, Oscillation Theory for Functional Differential Equations, Marcel Dekker, New York, 1995.
5. I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Claredon Press, Oxford, 1991.
6. G. Ladas, Y.G. Sficas and I.P. Stavroulakis, Necessary and sufficient conditions for oscillations of higher order delay differential equations, Trans. Amer. Math. Soc., 285(1981), 81-90.
7. R.P. Agarwal, S.R. Grace and D.O. Regan, Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic, Dordrecht, The Netherlands, 2000.
8. R.P. Agarwal, M. Bohner and W.T. Li, Nonoscillation and Oscillation: Theory for Functional Differential Equations, Marcel Dekker, New york, 2004.
9. B. Baculikova and J. Dzurina, Oscillation of third-order neutral differential equations, Math. Comput. Modelling, 52(2010), 215-226.
10. B. Baculikova and J. Dzurina, Oscillation theorems for third-order neutral differential equations, Carpathian J. Math., 28(2012), 199-206.
11. Z. Dosla and P. Liska, Comparison theorems for third-order neutral differential equations, Electron. J. Differential Equations, Vol.2016(2016), No.38, PP.1-13.
12. J. Dzurina, E. Thandapani and S. Tamilvanan, Oscillation of solutions to third-order half-linear neutral differential equations, Electron. J. Differential Equations, 2012(2012), 1-9.
13. S.R. Grace, J.R. Graef and E. Tunc, On oscillatory behavior of third order half-linear delay differential equations, Math. Slovaca, 73(2023), 729-735.
14. G.S. Ladde, V. Lakshimikantham and B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, New York, 1987.
15. T. Li, C. Zhang and G. Xing, Oscillation of third order neutral delay differential equations, Abstr. Appl. Anal., 2012(2012), 1-11.
16. T. Li and Y.V. Rogovchenko, Asymptotic behavior of higher order quasilinear neutral differential equations, Abstr. Appl. Anal., 2014(2014), 1-11.
17. O. Moaaz, I. Dassios, W. Mushin and A. Muhib, Oscillation theory for noninear neutral delay differential equations of third order, Appl. Sci., 2020, 10, 4855.
18. E. Thandapani and T. Li, On the oscillation of third order quasilinear neutral functional differential equations, Arch. Math., 47(2011), 181-199.
19. I. Yang and Z. Xu, Oscillation of certain third-order quasilinear neutral differential equations, Math. Slovaca, 64(2014), 85-100.
20. T. Kusano and M. Naito, Comparison theorems for functional differential equations with deviating arguments, J. Math. Soc, Japan, 3(1981), 509-533.
21. T. Li and Y. V. Rogovchenko, On the asymptotic behavior of solutions of a class of third order nonlinear neutral differntial equations, Appl. Math. Lett., 105(2020), 106293, 7 pages.
22. T.S. Hassan and B.M. El-Matary, Asymptotic behavior and oscillation of third-order nonlinear neutral differential equations with mixed nonlinearities, Mathematics 2023, 11, 424, 10 pages.
23. Y. G. Sun and J. S. W. Wong, Oscillation criteria for second order forced ordinary differential equations with mixed nonlinearities, J. Math. Anal. Appl., 344(2007), 549-560.
24. E. F. Beckenbach and R. Bellman, Inequalities, Springer, Berlin, 1961.
25. Ch.G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, Arch. Math., 36(1981), 168-178.