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The Least-Norm Solution to a Matrix Equation over the Dual Quaterion Algebra

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Abstract: In this paper, we explore the least-norm solution to the classical matrix equation $AXB = C$ over the dual quaternion algebra. We begin by transforming the definition of the Frobenius norm for dual quaternion matrices into an equivalent form. Using this new expression, we investigate the least-norm solution to the equation $AXB = C$ under solvability conditions. Additionally, we examine the minimum real part of norm solution in cases where a least-norm solution does not exist. Finally, we provide two numerical examples to illustrate the main findings of our study.

Keywords: dual quaternion algebra; matrix equation; the least-norm solution; Frobenius norm

MSC: 15A24, 15B33

1. Introduction

In 1843, Irish mathematician William Hamilton introduced quaternions in [1], defined as

$$\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k \mid i^2 = j^2 = k^2 = -1, ijk = -1, q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where $\mathbb{R}$ denotes the set of real numbers. Nowadays, quaternions and quaternion matrices have captivated mathematicians worldwide, with applications spanning mathematics, signal and color image processing, quantum physics, and computer science (e.g.[2–5]).

In 1873, Clifford introduced dual numbers and dual quaternions in [6]. Since then, these concepts have seen extensive applications in fields such as robotics, computer graphics, and rigid body motions (e.g.[7–11]). For related definitions of dual numbers and dual quaternions, please refer to Section 2.

On the other hand, matrix equations are crucial components of matrix theory. Quaternion and dual quaternion matrix equations have been applied in various fields, such as control theory. Due to the non-commutative nature of quaternion multiplication, research into quaternions and dual quaternion matrix equations is more challenging than research into real and complex matrix equations.

A substantial body of research has explored matrix equations, offering various methods for their resolution (e.g.[12–16]). Especially, the classical matrix equation

$$AXB = C \tag{1}$$

has been extensively studied. Khatri and Mitra [17] discussed the Hermitian and nonnegative definite solution of the linear matrix Equation (1). Peng [18] introduced an iterative method for the least squares symmetric solution of the linear matrix Equation (1). Liang et al. [19] provided an efficient algorithm for the generalized centro-symmetric solution of the linear matrix Equation (1). Chen et al. [20] established the necessary and sufficient conditions for the solvability of the dual quaternion matrix Equation (1). Wang et al. [21] investigated the least squares Hermitian, Persymmetric and Bisymmetric

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solutions of the quaternion matrix Equation (1). Si and Wang [22] considered the general solution to the dual split quaternion matrix Equation (1).

In this paper, we use the following notations: $r(A)$ represents the rank of a quaternion matrix A, A^* denotes the conjugate transpose of A, and $tr(A)$ stands for the trace of A. $I, 0$ refer to the identity matrix and the zero matrix with appropriate dimensions, respectively. The Moore-Penrose (or M-P for short) inverse of a quaternion matrix A is denoted by $A^\dagger$, and it satisfies

$$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (AA^\dagger)^* = AA^\dagger, (A^\dagger A)^* = A^\dagger A.$$

Furthermore, we use $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ to respectively represent two projectors that are induced by $A^\dagger$. As usual, we use $Re[q]$ to denote the real part of a quaternion q.

The remaining contents of this paper are organized as follows. In Section 2, we review definitions of dual numbers, dual quaternions, the Frobenius norm of quaternion matrices and its equivalent expressions, and some important Lemmas for subsequent sections. In Section 3, we explore the least-norm solution to the dual quaternion matrix Equation (1) under consistency conditions and analyze the minimum real part of norm solution when no least-norm solution exists. In Section 4, we present two numerical examples to illustrate the main findings of Section 3. In Section 5, we provide a brief summary.

2. Preliminaries

In this section, we first review definitions related to dual numbers, dual quaternions, and dual quaternion matrices. Then we introduce some results concerning the norm of dual quaternion matrices, the dual quaternion matrix equation, and the equalities related to the M-P inverse of quaternion matrices.

2.1. The Definition of Dual Numbers and Its Total Order Relation

Definition 1 ([23])**.** *The set of dual numbers is represented as follows:*

$$\mathbb{D} = \{x = x_0 + x_1\varepsilon \,|\, \varepsilon^2 = 0 \text{ and } x_0, x_1 \in \mathbb{R}\},$$

where ε is the infinitesimal unit, satisfying $\varepsilon^2 = 0$.

We refer to x_0 as the real part or the standard part of x, and x_1 as the dual part or the infinitesimal part of x. The infinitesimal unit ε commutes with real numbers, complex numbers, and quaternions under multiplication. We call x appreciable if $x_0 \neq 0$, otherwise, we call x infinitesimal.

Assume that $x = x_0 + x_1\varepsilon, y = y_0 + y_1\varepsilon \in \mathbb{D}, k \in \mathbb{R}$, then we have:

$$\begin{aligned} x + y &= x_0 + y_0 + (x_1 + y_1)\,\varepsilon, \\ xy &= x_0y_0 + (x_0y_1 + x_1y_0)\,\varepsilon, \\ kx &= kx_0 + kx_1\varepsilon. \end{aligned}$$

Then we define a total order relation over dual numbers. Assume $x = x_0 + x_1\varepsilon, y = y_0 + y_1\varepsilon \in \mathbb{D}$. We say $x = y$ if $x_0 = y_0$ and $x_1 = y_1$. We say $x < y$ if $x_0 < y_0$, or $x_0 = y_0$ and $x_1 < y_1$. Therefore, we call x a positive dual number if $x > 0$, we call x a nonnegative dual number if $x \geq 0$.

Next, we discuss the inverse element and square root of dual numbers. It is straightforward to observe that $1 = 1 + 0\,\varepsilon$ is the unit element in $\mathbb{D}$. If x is appreciable ($x_0 \neq 0$), then x is invertible and

$$x^{-1} = x_0^{-1} + x_0^{-1}x_1x_0^{-1}\varepsilon, \tag{2}$$

satisfying $xx^{-1} = x^{-1}x = 1$. If x is infinitesimal, then x is not invertible.

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If x is positive and appreciable, then the square root of x can be expressed as

$$\sqrt{x} = \sqrt{x_0} + \frac{x_1}{2\sqrt{x_0}}\varepsilon. \tag{3}$$

Specifically, if $x = 0$, then $\sqrt{x} = 0$.

2.2. The Definition of Dual Quaternions and Dual Quaternion Matrices

Definition 2 ([23]). *The set of dual quaternions is represented as follows:*

$$\mathbb{DQ} = \{q = q_0 + q_1\varepsilon \,|\, \varepsilon^2 = 0 \text{ and } q_0, q_1 \in \mathbb{H}\},$$

where ε is the infinitesimal unit.

We refer to q_0 as the real part or the standard part of q, and q_1 as the dual part or the infinitesimal part of q. We call q appreciable if $q_0 \neq 0$, otherwise, we call q infinitesimal. The conjugate of q is defined as $q^* = q_0^* + q_1^*\varepsilon$.

Qi et al. presented the following Lemma in [23], showing that a mixed product sum of two quaternions is a real number. We supplement the conclusion of this Lemma and provide a proof.

Lemma 1. *Suppose that $p = p_0 + p_1 i + p_2 j + p_3 k, q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{Q}$, where $p_0, p_1, p_2, p_3, q_0, q_1, q_2, q_3 \in \mathbb{R}$. Then*

$$\begin{aligned} pq^* + qp^* &= p^*q + q^*p = 2p_0q_0 + 2p_1q_1 + 2p_2q_2 + 2p_3q_3 \\ &= 2Re[pq^*] = 2Re[qp^*] = 2Re[p^*q] = 2Re[q^*p], \end{aligned}$$

which is a real number.

Proof. Since $qp^* = (pq^*)^*$ and $q^*p = (p^*q)^*$, we obtain:

$$\begin{aligned} pq^* + qp^* &= 2Re[pq^*] = 2Re[qp^*] = 2p_0q_0 + 2p_1q_1 + 2p_2q_2 + 2p_3q_3, \\ p^*q + q^*p &= 2Re[p^*q] = 2Re[q^*p] = 2p_0q_0 + 2p_1q_1 + 2p_2q_2 + 2p_3q_3. \end{aligned}$$

Thus,

$$\begin{aligned} pq^* + qp^* &= p^*q + q^*p = 2p_0q_0 + 2p_1q_1 + 2p_2q_2 + 2p_3q_3 \\ &= 2Re[pq^*] = 2Re[qp^*] = 2Re[p^*q] = 2Re[q^*p]. \quad \square \end{aligned}$$

According to the above Lemma, we can define the magnitude of $q \in \mathbb{DQ}$ as follows:

$$|q| := \begin{cases} |q_0| + \frac{q_0q_1^* + q_1q_0^*}{2|q_0|}\varepsilon, \text{if } q_0 \neq 0, \\ |q_1|\,\varepsilon, \text{otherwise}, \end{cases} \tag{4}$$

which is a dual number.

Next, we will introduce dual quaternion matrices and discuss their norm.

Definition 3 ([23]). *The set of dual quaternion matrices is represented as follows:*

$$\mathbb{DQ}^{m\times n} = \{X = X_0 + X_1\varepsilon \,|\, \varepsilon^2 = 0 \text{ and } X_0, X_1 \in \mathbb{H}^{m\times n}\},$$

where ε is the infinitesimal unit.

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Assume that $X = X_0 + X_1\varepsilon, Y = Y_0 + Y_1\varepsilon \in \mathbb{DQ}^{m\times n}$, then we have:

$$\begin{aligned} X + Y &= X_0 + Y_0 + (X_1 + Y_1)\,\varepsilon, \\ XY &= X_0Y_0 + (X_0Y_1 + X_1Y_0)\,\varepsilon, \\ X^* &= X_0^* + X_1^*\varepsilon. \end{aligned}$$

Before defining the norm of dual quaternion matrices, we will first introduce the norm of quaternion matrices.

Definition 4 ([24])**.** *Let $X = (x_{ij}) \in \mathbb{H}^{m\times n}$, the Frobenius norm of X is defined as*

$$||X|| := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|x_{ij}|^2} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}^*x_{ij}} = \sqrt{\mathrm{tr}X^*X}. \tag{5}$$

Next, we introduce the definition of the norm of dual quaternion matrices.

Definition 5 ([24])**.** *Assume $X = X_0 + X_1\varepsilon = (x_{ij}) \in \mathbb{DQ}^{m\times n}$, the Frobenius norm of X is defined as*

$$||X|| := \begin{cases} \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|x_{ij}|^2}, \text{if } X_0 \neq 0, \\ ||X_1||\,\varepsilon, \text{otherwise.} \end{cases} \tag{6}$$

In order to facilitate the discussion of the least-norm solution to the matrix Equation (1) in the following text, we have equivalently revised the definitions provided above.

Proposition 1. *Suppose $X = X_0 + X_1\varepsilon = (x_{ij}) \in \mathbb{DQ}^{m\times n}$, then we have*

$$||X|| = \begin{cases} ||X_0|| + \frac{\mathrm{tr}(X_1^*X_0 + X_0^*X_1)}{2||x_0||}\varepsilon, \text{if } X_0 \neq 0, \\ ||X_1||\,\varepsilon, \text{otherwise.} \end{cases} \tag{7}$$

$$= \begin{cases} \sqrt{\mathrm{tr}X_0^*X_0} + \frac{\mathrm{Re}[\mathrm{tr}X_1^*X_0]}{\sqrt{\mathrm{tr}X_0^*X_0}}\varepsilon, \text{if } X_0 \neq 0, \\ \sqrt{\mathrm{tr}X_1^*X_1}\,\varepsilon, \text{otherwise.} \end{cases} \tag{8}$$

Proof. If $X_0 \neq 0$, in accordance with (3) and Lemma 1, we have

$$\begin{aligned} ||X|| &= \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|x_{ij}|^2} \\ &= \sqrt{\sum_{(x_{ij})_0\neq 0}\left(|(x_{ij})_0| + \frac{(x_{ij})_0(x_{ij})_1^* + (x_{ij})_1(x_{ij})_0^*}{2|(x_{ij})_0|}\varepsilon\right)^2 + \sum_{(x_{ij})_0=0}(|(x_{ij})_1|\varepsilon)^2} \\ &= \sqrt{\sum_{(x_{ij})_0\neq 0}(|(x_{ij})_0|^2 + ((x_{ij})_0(x_{ij})_1^* + (x_{ij})_1(x_{ij})_0^*)\varepsilon)} \\ &= \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(|(x_{ij})_0|^2 + ((x_{ij})_0(x_{ij})_1^* + (x_{ij})_1(x_{ij})_0^*)\varepsilon)} \\ &= \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|(x_{ij})_0|^2 + (\sum_{i=1}^{m}\sum_{j=1}^{n}((x_{ij})_0(x_{ij})_1^* + (x_{ij})_1(x_{ij})_0^*))\varepsilon} \end{aligned}$$

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$$
\begin{aligned}
&= \sqrt{||X_0||^2 + (\sum_{i=1}^{m}\sum_{j=1}^{n}((x_{ij})_0(x_{ij})_1^* + (x_{ij})_1(x_{ij})_0^*))\varepsilon} \\
&= ||X_0|| + \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}((x_{ij})_0(x_{ij})_1^* + (x_{ij})_1(x_{ij})_0^*)}{2||X_0||}\varepsilon \\
&= ||X_0|| + \frac{\mathrm{tr}(X_1^*X_0 + X_0^*X_1)}{2||X_0||}\varepsilon \\
&= \sqrt{\mathrm{tr}X_0^*X_0} + \frac{\mathrm{Re}[\mathrm{tr}X_1^*X_0]}{\sqrt{\mathrm{tr}X_0^*X_0}}\varepsilon = \sqrt{\mathrm{tr}X_0^*X_0} + \frac{\mathrm{Re}[\mathrm{tr}X_0^*X_1]}{\sqrt{\mathrm{tr}X_0^*X_0}}\varepsilon.
\end{aligned}
$$

If $X_0 = 0$, since $||X_1|| = \sqrt{\mathrm{tr}X_1^*X_1}$, we have $||X|| = ||X_1||\varepsilon = \sqrt{\mathrm{tr}X_1^*X_1}\varepsilon$. □

2.3. Some Important Lemmas

In this part, we present several key Lemmas that are crucial for the subsequent proofs.

Lemma 2 ([20]). *Let $A = A_0 + A_1\varepsilon \in \mathbb{DQ}^{m\times n}, B = B_0 + B_1\varepsilon \in \mathbb{DQ}^{k\times l}, C = C_0 + C_1\varepsilon \in \mathbb{DQ}^{m\times l}$ be known. Put*

$$A_2 = A_1L_{A_0}, B_2 = R_{B_0}B_1, C_2 = C_1 - A_0A_0^\dagger C_0B_0^\dagger B_1 - A_1A_0^\dagger C_0B_0^\dagger B_0, \tag{9}$$
$$A_3 = R_{A_0}A_2, C_{21} = R_{A_0}C_2, B_3 = B_2L_{B_0}, C_{22} = C_2L_{B_0}. \tag{10}$$

Then, dual quaternion matrix Equation (1) is consistent if and only if

$$R_{A_0}C_0 = 0, C_0L_{B_0} = 0, \tag{11}$$
$$R_{A_3}C_{21} = 0, R_{A_0}C_2L_{B_0} = 0, C_{22}L_{B_3} = 0. \tag{12}$$

In this case, the general solution X of the dual quaternion matrix equation matrix Equation (1) can be expressed as $X = X_0 + X_1\varepsilon$, where

$$X_0 = A_0^\dagger C_0B_0^\dagger + L_{A_0}U_1 + U_2R_{B_0}, \tag{13}$$
$$X_1 = A_0^\dagger(C_2 - A_2U_1B_0 - A_0U_2B_2)B_0^\dagger + L_{A_0}W_1 + W_2R_{B_0}, \tag{14}$$
$$U_1 = A_3^\dagger C_{21}B_0^\dagger + L_{A_3}Q_1 + Q_2R_{B_0}, \tag{15}$$
$$U_2 = A_0^\dagger C_{22}B_3^\dagger + L_{A_0}Q_3 + Q_4R_{B_3}, \tag{16}$$

where $Q_i(i = \overline{1,4}), W_i(i = 1,2)$ are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.

Lemma 3 ([25]). *Let A, B and C be arbitrary matrices over $\mathbb{H}$ with appropriate dimensions. Then the following equalities hold:*
(1) $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$.
(2) $L_A = (L_A)^2 = (L_A)^*, R_A = (R_A)^2 = (R_A)^*$.
(3) $L_A(BL_A)^\dagger = (BL_A)^\dagger, (R_AC)^\dagger R_A = (R_AC)^\dagger$.

Lemma 4 ([25]). *Assume A, B be arbitrary matrices over $\mathbb{H}$ with appropriate dimensions. Then they satisfy the following equalities:*
(1) $||A + B||^2 = ||A||^2 + ||B||^2 + 2\mathrm{Re}[\mathrm{tr}B^*A]$.
(2) $\mathrm{Re}[\mathrm{tr}AB] = \mathrm{Re}[\mathrm{tr}BA]$.

Proposition 2. *Given $A = (a_{ij}) \neq 0 \in \mathbb{H}^{m\times n}$, for an arbitrary matrix $X \in \mathbb{H}^{m\times n}$, $\mathrm{Re}[\mathrm{tr}X^*A]$ can take any real value.*

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Proof. Since $A \neq 0$, there exists $a_{ij} = q_0 + q_1 i + q_2 j + q_3 k \neq 0, q_t \in \mathbb{R}, (t = 0, 1, 2, 3)$. Then we discuss in two cases.
(1) If $q_0 \neq 0$, let $X = mE_{ij}$, where $m \in \mathbb{R}$ and E_{ij} represents the matrix whose (i, j) element is 1 and other elements are all 0, we have

$$\text{Re}[\text{tr}X^* A] = \text{Re}[ma_{ij}] = mq_0.$$

Thus we can change $\text{Re}[\text{tr}X^* A]$ to any real value through alter the value of m.
(2) If $q_1 \neq 0$ or $q_2 \neq 0$ or $q_3 \neq 0$, with out loss generality, suppose $q_1 \neq 0$. Let $X = (mi)E_{ij}$, where $m \in \mathbb{R}$, we have

$$\text{Re}[\text{tr}X^* A] = \text{Re}[(-mi))a_{ij}] = mq_1.$$

Thus we can change $\text{Re}[\text{tr}X^* A]$ to any real value through alter the value of m. □

3. The Least-Norm Solution to the Matrix Equation (1)

In this section, starting from the general solution to the dual quaternion matrix Equation (1), we explore the least-norm solution to this matrix equation over the dual quaternion algebra.

Theorem 1. *Suppose that dual quaternion matrix Equation (1) is consistent, let*

$$A_2 = A_1 L_{A_0}, B_2 = R_{B_0} B_1, C_2 = C_1 - A_0 A_0^\dagger C_0 B_0^\dagger B_1 - A_1 A_0^\dagger C_0 B_0^\dagger B_0,$$
$$A_3 = R_{A_0} A_2, C_{21} = R_{A_0} C_2, B_3 = B_2 L_{B_0}, C_{22} = C_2 L_{B_0}.$$

Then the least-norm solution to the matrix Equation (1) can be discussed in two cases:
(1) If $A_0^\dagger C_0 B_0^\dagger \neq 0$ and $A_3^\dagger C_2 B_0^\dagger = A_0^\dagger C_2 B_3^\dagger = 0$, then the least-norm solution $X = X_0 + X_1 \varepsilon$ to the matrix Equation (1) can be expressed as

$$X_0 = A_0^\dagger C_0 B_0^\dagger, \tag{17}$$
$$X_1 = A_0^\dagger C_2 B_0^\dagger + L_{A_0} W_1 + W_2 R_{B_0}, \tag{18}$$

where $W_i (i = 1, 2)$ are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.
In this case, the least-norm of the general solution can be expressed as

$$||X||_{min} = ||A_0^\dagger C_0 B_0^\dagger|| + \frac{\text{Re}[\text{tr}(A_0^\dagger C_2 B_0^\dagger)^* (A_0^\dagger C_0 B_0^\dagger)]}{||A_0^\dagger C_0 B_0^\dagger||} \varepsilon.$$

(2) If $A_0^\dagger C_0 B_0^\dagger = A_3^\dagger C_2 B_0^\dagger = A_0^\dagger C_2 B_3^\dagger = 0$, then the least-norm solution $X = X_0 + X_1 \varepsilon$ to the matrix Equation (1) can be expressed as

$$X_0 = 0, \tag{19}$$
$$X_1 = A_0^\dagger C_1 B_0^\dagger. \tag{20}$$

In this case, the least-norm of the general solution can be expressed as

$$||X||_{min} = ||A_0^\dagger C_1 B_0^\dagger|| \varepsilon.$$

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Proof. By Lemma 2, the general solution $X = X_0 + X_1\varepsilon$ to matrix Equation (1) can be expressed as follows:

$$\begin{aligned}
X_0 &= A_0^\dagger C_0 B_0^\dagger + L_{A_0}U_1 + U_2R_{B_0},\\
X_1 &= A_0^\dagger(C_2 - A_2U_1B_0 - A_0U_2B_2)B_0^\dagger + L_{A_0}W_1 + W_2R_{B_0},\\
U_1 &= A_3^\dagger C_{21}B_0^\dagger + L_{A_3}Q_1 + Q_2R_{B_0},\\
U_2 &= A_0^\dagger C_{22}B_3^\dagger + L_{A_0}Q_3 + Q_4R_{B_3},
\end{aligned}$$

where $Q_i(i = \overline{1,4}), W_i(i = 1,2)$ are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.

According to Proposition 1, we can find that the solution for $\|X\|$ can be translate into analyzing $\|X_0\|$, $\|X_1\|$ and $\mathrm{Re}[\mathrm{tr}X_1^*X_0]$. Then, by considering the total order relation for dual numbers, we understand that to minimize $\|X\|$, we first need to focus on minimizing the real part of $\|X\|$ i.e. $\|X_0\|$.

In view of Lemma 3 and Lemma 4, we have

$$\begin{aligned}
||X_0||^2 &= ||A_0^\dagger C_0B_0^\dagger + L_{A_0}U_1 + U_2R_{B_0}||^2\\
&= ||A_0^\dagger C_0B_0^\dagger||^2 + ||L_{A_0}U_1 + U_2R_{B_0}||^2 + 2\mathrm{Re}[\mathrm{tr}(L_{A_0}U_1 + U_2R_{B_0})^*A_0^\dagger C_0B_0^\dagger]\\
&= ||A_0^\dagger C_0B_0^\dagger||^2 + ||L_{A_0}U_1 + U_2R_{B_0}||^2 + 2\mathrm{Re}[\mathrm{tr}(U_1^*L_{A_0}A_0^\dagger C_0B_0^\dagger + \mathrm{tr}R_{B_0}U_2^*A_0^\dagger C_0B_0^\dagger)]\\
&= ||A_0^\dagger C_0B_0^\dagger||^2 + ||L_{A_0}U_1 + U_2R_{B_0}||^2 + 2\mathrm{Re}[\mathrm{tr}U_2^*A_0^\dagger C_0B_0^\dagger R_{B_0}]\\
&= ||A_0^\dagger C_0B_0^\dagger||^2 + ||L_{A_0}U_1 + U_2R_{B_0}||^2,
\end{aligned} \tag{21}$$

then,

$$\begin{aligned}
||L_{A_0}U_1 + U_2R_{B_0}||^2 &= ||L_{A_0}(A_3^\dagger C_{21}B_0^\dagger + L_{A_3}Q_1 + Q_2R_{B_0})\\
&\quad + (A_0^\dagger C_{22}B_3^\dagger + L_{A_0}Q_3 + Q_4R_{B_3})R_{B_0}||^2\\
&= ||L_{A_0}L_{A_3}Q_1 + L_{A_0}Q_2R_{B_0} + L_{A_0}Q_3R_{B_0} + Q_4R_{B_3}R_{B_0}||^2\\
&\quad + ||L_{A_0}A_3^\dagger C_{21}B_0^\dagger + A_0^\dagger C_{22}B_3^\dagger R_{B_0}||^2 + K_1\\
&= ||L_{A_0}L_{A_3}Q_1 + L_{A_0}Q_2R_{B_0} + L_{A_0}Q_3R_{B_0} + Q_4R_{B_3}R_{B_0}||^2\\
&\quad + ||L_{A_0}A_3^\dagger C_{21}B_0^\dagger||^2 + ||A_0^\dagger C_{22}B_3^\dagger R_{B_0}||^2 + K_1\\
&\quad + 2\mathrm{Re}[\mathrm{tr}(A_0^\dagger C_{22}B_3^\dagger R_{B_0})^*(L_{A_0}A_3^\dagger C_{21}B_0^\dagger)]\\
&= ||L_{A_0}L_{A_3}Q_1 + L_{A_0}Q_2R_{B_0} + L_{A_0}Q_3R_{B_0} + Q_4R_{B_3}R_{B_0}||^2\\
&\quad + ||L_{A_0}A_3^\dagger R_{A_0}C_2B_0^\dagger||^2 + ||A_0^\dagger C_2L_{B_0}B_3^\dagger R_{B_0}||^2 + K_1\\
&\quad + 2\mathrm{Re}[\mathrm{tr}(L_{A_0}A_0^\dagger C_{22}B_3^\dagger R_{B_0})^*(A_3^\dagger C_{21}B_0^\dagger)]\\
&= ||L_{A_0}L_{A_3}Q_1 + L_{A_0}Q_2R_{B_0} + L_{A_0}Q_3R_{B_0} + Q_4R_{B_3}R_{B_0}||^2\\
&\quad + ||A_3^\dagger C_2B_0^\dagger||^2 + ||A_0^\dagger C_2B_3^\dagger||^2 + K_1,
\end{aligned} \tag{22}$$

where

$$K_1 = 2\mathrm{Re}[\mathrm{tr}(L_{A_0}L_{A_3}Q_1 + L_{A_0}Q_2R_{B_0} + L_{A_0}Q_3R_{B_0} + Q_4R_{B_3}R_{B_0})^*(A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger)].$$

According to Lemma 3 and Lemma 4, we have

$$\begin{aligned}
\mathrm{Re}[\mathrm{tr}(L_{A_0}L_{A_3}Q_1)^*(A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger) &= \mathrm{Re}[\mathrm{tr}(Q_1^*L_{A_3}L_{A_0}A_3^\dagger C_2B_0^\dagger + Q_1^*L_{A_3}L_{A_0}A_0^\dagger C_2B_3^\dagger)]\\
&= \mathrm{Re}[\mathrm{tr}Q_1^*L_{A_3}A_3^\dagger C_2B_0^\dagger]\\
&= 0,
\end{aligned}$$

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$$\begin{aligned}\mathrm{Re}[\mathrm{tr}(L_{A_0}Q_2R_{B_0})^*(A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger) &= \mathrm{Re}[\mathrm{tr}(R_{B_0}Q_2^*L_{A_0}A_3^\dagger C_2B_0^\dagger + R_{B_0}Q_2^*L_{A_0}A_0^\dagger C_2B_3^\dagger)]\\ &= \mathrm{Re}[\mathrm{tr}Q_2^*L_{A_0}A_3^\dagger C_2B_0^\dagger R_{B_0}]\\ &= 0,\end{aligned}$$

$$\begin{aligned}\mathrm{Re}[\mathrm{tr}(L_{A_0}Q_3R_{B_0})^*(A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger) &= \mathrm{Re}[\mathrm{tr}(R_{B_0}Q_3^*L_{A_0}A_3^\dagger C_2B_0^\dagger + R_{B_0}Q_3^*L_{A_0}A_0^\dagger C_2B_3^\dagger)]\\ &= \mathrm{Re}[\mathrm{tr}Q_3^*L_{A_0}A_3^\dagger C_2B_0^\dagger R_{B_0}]\\ &= 0,\end{aligned}$$

$$\begin{aligned}\mathrm{Re}[\mathrm{tr}(Q_4R_{B_3}R_{B_0})^*(A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger) &= \mathrm{Re}[\mathrm{tr}(R_{B_0}R_{B_3}Q_4^*A_3^\dagger C_2B_0^\dagger + R_{B_0}R_{B_3}Q_4^*A_0^\dagger C_2B_3^\dagger)]\\ &= \mathrm{Re}[\mathrm{tr}(R_{B_3}Q_4^*A_3^\dagger C_2B_0^\dagger R_{B_0} + Q_4^*A_0^\dagger C_2B_3^\dagger R_{B_0}R_{B_3})]\\ &= \mathrm{Re}[\mathrm{tr}Q_4^*A_0^\dagger C_2B_3^\dagger R_{B_3}]\\ &= 0.\end{aligned}$$

Thus, $K_1 = 0$. By (21) and (22), we obtain

$$\begin{aligned}||X_0||^2 &= ||A_0^\dagger C_0B_0^\dagger||^2 + ||A_3^\dagger C_2B_0^\dagger||^2 + ||A_0^\dagger C_2B_3^\dagger||^2\\ &\quad + ||L_{A_0}L_{A_3}Q_1 + L_{A_0}Q_2R_{B_0} + L_{A_0}Q_3R_{B_0} + Q_4R_{B_3}R_{B_0}||^2\\ &\geq ||A_0^\dagger C_0B_0^\dagger||^2 + ||A_3^\dagger C_2B_0^\dagger||^2 + ||A_0^\dagger C_2B_3^\dagger||^2,\end{aligned} \tag{23}$$

with equality iff

$$L_{A_0}L_{A_3}Q_1 + L_{A_0}Q_2R_{B_0} + L_{A_0}Q_3R_{B_0} + Q_4R_{B_3}R_{B_0} = 0, \tag{24}$$

i.e.

$$X_0 = A_0^\dagger C_0B_0^\dagger + A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger. \tag{25}$$

Therefore,

$$\begin{aligned}||X_0||_{min} &= ||A_0^\dagger C_0B_0^\dagger + A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger||\\ &= \sqrt{||A_0^\dagger C_0B_0^\dagger||^2 + ||A_3^\dagger C_2B_0^\dagger||^2 + ||A_0^\dagger C_2B_3^\dagger||^2}.\end{aligned} \tag{26}$$

We now discuss $||X||$ under two different cases based on conditions (24) and (25).
1) If $A_0^\dagger C_0B_0^\dagger \neq 0$ or $A_3^\dagger C_2B_0^\dagger \neq 0$ or $A_0^\dagger C_2B_3^\dagger \neq 0$, by (23), we have

$$||X_0||^2 = ||A_0^\dagger C_0B_0^\dagger||^2 + ||A_3^\dagger C_2B_0^\dagger||^2 + ||A_0^\dagger C_2B_3^\dagger||^2 > 0.$$

Thus, $X_0 \neq 0$ i.e. X is appreciable. Consequently,

$$||X|| = ||X_0|| + \frac{\mathrm{Re}[\mathrm{tr}X_1^*X_0]}{||X_0||}\varepsilon.$$

Next, we turn our attention to the dual part of $||X||$. Having already examined $||X_0||$ in the above, we will now delve into the analysis of $\mathrm{Re}[\mathrm{tr}X_1^*X_0]$ in the following part.

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By (14), (15), (16) and (24), we have

$$\begin{aligned}
X_1 &= A_0^\dagger(C_2 - A_2U_1B_0 - A_0U_2B_2)B_0^\dagger + L_{A_0}W_1 + W_2R_{B_0} \\
&= A_0^\dagger(C_2 - A_2(A_3^\dagger C_{21}B_0^\dagger + L_{A_3}Q_1 + Q_2R_{B_0})B_0 \\
&\quad - A_0(A_0^\dagger C_{22}B_3^\dagger + L_{A_0}Q_3 + Q_4R_{B_3})B_2)B_0^\dagger + L_{A_0}W_1 + W_2R_{B_0} \\
&= A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_2(A_3^\dagger C_{21}B_0^\dagger + L_{A_3}Q_1)B_0B_0^\dagger - A_0^\dagger A_0(A_0^\dagger C_{22}B_3^\dagger + Q_4R_{B_3})B_2B_0^\dagger \\
&\quad + L_{A_0}W_1 + W_2R_{B_0} \\
&= A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_2A_3^\dagger C_{21}B_0^\dagger - A_0^\dagger A_2L_{A_3}Q_1B_0B_0^\dagger - A_0^\dagger C_{22}B_3^\dagger B_2B_0^\dagger \\
&\quad - A_0^\dagger A_0Q_4R_{B_3}B_2B_0^\dagger + L_{A_0}W_1 + W_2R_{B_0} \\
&= A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_2A_3^\dagger C_{21}B_0^\dagger - A_0^\dagger C_{22}B_3^\dagger B_2B_0^\dagger + L_{A_0}W_1 + W_2R_{B_0} \\
&\quad - A_0^\dagger A_1(L_{A_0}L_{A_3}Q_1)B_0B_0^\dagger - A_0^\dagger A_0(Q_4R_{B_3}R_{B_0})B_1B_0^\dagger
\end{aligned}$$

$$\begin{aligned}
&= A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger + L_{A_0}W_1 + W_2R_{B_0} \\
&\quad + A_0^\dagger A_1(L_{A_0}Q_2R_{B_0} + L_{A_0}Q_3R_{B_0} + Q_4R_{B_3}R_{B_0})B_0B_0^\dagger \\
&\quad + A_0^\dagger A_0(L_{A_0}L_{A_3}Q_1 + L_{A_0}Q_2R_{B_0} + L_{A_0}Q_3R_{B_0})B_1B_0^\dagger \\
&= A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger + L_{A_0}W_1 + W_2R_{B_0}. \qquad (27)
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Re}[\mathrm{tr}X_1^*X_0] &= \mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger + L_{A_0}W_1 + W_2R_{B_0})^* \\
&\quad (A_0^\dagger C_0B_0^\dagger + A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger)] \\
&= a + K, \qquad (28)
\end{aligned}$$

where,

$$\begin{aligned}
a &= \mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger)^*(A_0^\dagger C_0B_0^\dagger + A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger)] \\
&= \mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger)^*(A_0^\dagger C_0B_0^\dagger)] \\
&\quad + \mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger)^*(L_{A_0}A_3^\dagger C_2B_0^\dagger)] \\
&\quad + \mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger)^*(A_0^\dagger C_2B_3^\dagger R_{B_0})] \\
&= \mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger)^*(A_0^\dagger C_0B_0^\dagger)] \\
&\quad + \mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger)^*L_{A_0}(A_3^\dagger C_2B_0^\dagger)] \\
&\quad + \mathrm{Re}[\mathrm{tr}R_{B_0}(A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger)^*(A_0^\dagger C_2B_3^\dagger)] \\
&= \mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger)^*(A_0^\dagger C_0B_0^\dagger)] \\
&\quad + \mathrm{Re}[\mathrm{tr}(C_2B_0^\dagger - A_1A_3^\dagger C_2B_0^\dagger - C_2B_3^\dagger B_1B_0^\dagger)^*(L_{A_0}A_0^\dagger)^*(A_3^\dagger C_2B_0^\dagger)] \\
&\quad + \mathrm{Re}[\mathrm{tr}(B_0^\dagger R_{B_0})^*(A_0^\dagger C_2 - A_0^\dagger A_1A_3^\dagger C_2 - A_0^\dagger C_2B_3^\dagger B_1)^*(A_0^\dagger C_2B_3^\dagger)] \\
&= \mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2B_0^\dagger - A_0^\dagger A_1A_3^\dagger C_2B_0^\dagger - A_0^\dagger C_2B_3^\dagger B_1B_0^\dagger)^*(A_0^\dagger C_0B_0^\dagger)] \qquad (29)
\end{aligned}$$

is a constant, and

$$\begin{aligned}
K &= \mathrm{Re}[\mathrm{tr}(L_{A_0}W_1 + W_2R_{B_0})^*(A_0^\dagger C_0B_0^\dagger + A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger)] \\
&= \mathrm{Re}[\mathrm{tr}(W_1^*L_{A_0}A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger R_{B_0}W_2^*)] \\
&= \mathrm{Re}[\mathrm{tr}(W_1^*A_3^\dagger C_2B_0^\dagger + A_0^\dagger C_2B_3^\dagger W_2^*)]. \qquad (30)
\end{aligned}$$

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Next, we explore the situation of K in two cases.
1.1) If $A_3^\dagger C_2 B_0^\dagger = A_0^\dagger C_2 B_3^\dagger = 0$, then substituting these into (29) and (30), we obtain

$$K = 0, a = \mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2 B_0^\dagger)^*(A_0^\dagger C_0 B_0^\dagger)].$$

According to (25) and (27), the least-norm solution to the dual quaternion matrix Equation (1) can be expressed as

$$\begin{aligned} X_0 &= A_0^\dagger C_0 B_0^\dagger + A_3^\dagger C_2 B_0^\dagger + A_0^\dagger C_2 B_3^\dagger = A_0^\dagger C_0 B_0^\dagger, \\ X_1 &= A_0^\dagger C_2 B_0^\dagger - A_0^\dagger A_1 A_3^\dagger C_2 B_0^\dagger - A_0^\dagger C_2 B_3^\dagger B_1 B_0^\dagger + L_{A_0} W_1 + W_2 R_{B_0} \\ &= A_0^\dagger C_2 B_0^\dagger + L_{A_0} W_1 + W_2 R_{B_0}, \end{aligned}$$

where $W_i (i = 1, 2)$ are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.

In this case, the least-norm of the general solution can be expressed as

$$\begin{aligned} ||X||_{min} &= ||X_0|| + \frac{\mathrm{Re}[\mathrm{tr} X_1^* X_0]}{||X_0||}\varepsilon \\ &= ||A_0^\dagger C_0 B_0^\dagger|| + \frac{\mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2 B_0^\dagger)^*(A_0^\dagger C_0 B_0^\dagger)]}{||A_0^\dagger C_0 B_0^\dagger||}\varepsilon. \end{aligned}$$

1.2) If $A_3^\dagger C_2 B_0^\dagger \neq 0$ or $A_0^\dagger C_2 B_3^\dagger \neq 0$, by (30) and Proposition 2, K can take any real value. Thus, the dual part of $||X||$,

$$\frac{\mathrm{Re}[\mathrm{tr} X_1^* X_0]}{||X_0||} = \frac{a + K}{||A_0^\dagger C_0 B_0^\dagger + A_3^\dagger C_2 B_0^\dagger + A_0^\dagger C_2 B_3^\dagger||}$$

can take any real value. Therefore, the dual quaternion matrix Equation (1) does not have a least-norm solution in this case.
2) If $A_0^\dagger C_0 B_0^\dagger = A_3^\dagger C_2 B_0^\dagger = A_0^\dagger C_2 B_3^\dagger = 0$, by (23), we obtain

$$||X_0||^2 = ||A_0^\dagger C_0 B_0^\dagger||^2 + ||A_3^\dagger C_2 B_0^\dagger||^2 + ||A_0^\dagger C_2 B_3^\dagger||^2 = 0.$$

Thus, $X_0 = 0$ i.e. X is infinitesimal. Therefore, $||X|| = ||X_1||\varepsilon$.

According to (27), we have

$$\begin{aligned} X_1 &= A_0^\dagger C_2 B_0^\dagger - A_0^\dagger A_1 A_3^\dagger C_2 B_0^\dagger - A_0^\dagger C_2 B_3^\dagger B_1 B_0^\dagger + L_{A_0} W_1 + W_2 R_{B_0} \\ &= A_0^\dagger (C_1 - A_0 A_0^\dagger C_0 B_0^\dagger B_1 - A_1 A_0^\dagger C_0 B_0^\dagger B_0) B_0^\dagger + L_{A_0} W_1 + W_2 R_{B_0} \\ &= A_0^\dagger C_1 B_0^\dagger + L_{A_0} W_1 + W_2 R_{B_0}. \end{aligned}$$

Thus,

$$\begin{aligned} ||X_1||^2 &= ||A_0^\dagger C_1 B_0^\dagger + L_{A_0} W_1 + W_2 R_{B_0}||^2 \\ &= ||A_0^\dagger C_1 B_0^\dagger||^2 + ||L_{A_0} W_1 + W_2 R_{B_0}||^2 + 2\mathrm{Re}[\mathrm{tr}(L_{A_0} W_1 + W_2 R_{B_0})^*(A_0^\dagger C_1 B_0^\dagger)] \\ &= ||A_0^\dagger C_1 B_0^\dagger||^2 + ||L_{A_0} W_1 + W_2 R_{B_0}||^2 + 2\mathrm{Re}[\mathrm{tr}(W_1^* L_{A_0} A_0^\dagger C_1 B_0^\dagger + \mathrm{tr} A_0^\dagger C_1 B_0^\dagger R_{B_0} W_2^*)] \\ &= ||A_0^\dagger C_1 B_0^\dagger||^2 + ||L_{A_0} W_1 + W_2 R_{B_0}||^2 \\ &\geq ||A_0^\dagger C_1 B_0^\dagger||^2, \end{aligned} \tag{31}$$

with equality iff

$$L_{A_0} W_1 + W_2 R_{B_0} = 0, \tag{32}$$

i.e.

$$X_1 = A_0^\dagger C_1 B_0^\dagger. \tag{33}$$

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Therefore,

$$||X_1||_{min} = ||A_0^\dagger C_1 B_0^\dagger||. \tag{34}$$

Thus, the least-norm solution to the dual quaternion matrix Equation (1) can be expressed as

$$X_0 = 0,$$
$$X_1 = A_0^\dagger C_1 B_0^\dagger.$$

In this case, the least-norm of the general solution can be expressed as

$$||X||_{min} = ||X_1||\varepsilon = ||A_0^\dagger C_1 B_0^\dagger||\varepsilon. \quad \square$$

Through the proof process described, we observe that if $A_3^\dagger C_2 B_0^\dagger \neq 0$ or $A_0^\dagger C_2 B_3^\dagger \neq 0$, the dual quaternion matrix Equation (1) does not have a least-norm solution. We will illustrate this situation in the following.

Remark 1. *If $A_3^\dagger C_2 B_0^\dagger \neq 0$ or $A_0^\dagger C_2 B_3^\dagger \neq 0$, the dual quaternion matrix Equation (1) has a minimum real part of norm solution, and the dual part of norm can take any real value. The solution $X = X_0 + X_1\varepsilon$ can be expressed as*

$$X_0 = A_0^\dagger C_0 B_0^\dagger + A_3^\dagger C_2 B_0^\dagger + A_0^\dagger C_2 B_3^\dagger, \tag{35}$$
$$X_1 = A_0^\dagger C_2 B_0^\dagger - A_0^\dagger A_1 A_3^\dagger C_2 B_0^\dagger - A_0^\dagger C_2 B_3^\dagger B_1 B_0^\dagger + L_{A_0} W_1 + W_2 R_{B_0}, \tag{36}$$

where $W_i (i = 1, 2)$ are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions. In this case,

$$\begin{aligned} ||X|| &= ||X_0||_{min} + \frac{a+K}{||X_0||_{min}}\varepsilon \\ &= ||A_0^\dagger C_0 B_0^\dagger + A_3^\dagger C_2 B_0^\dagger + A_0^\dagger C_2 B_3^\dagger|| + \frac{a+K}{||A_0^\dagger C_0 B_0^\dagger + A_3^\dagger C_2 B_0^\dagger + A_0^\dagger C_2 B_3^\dagger||}\varepsilon, \end{aligned}$$

where

$$a = \mathrm{Re}[\mathrm{tr}(A_0^\dagger C_2 B_0^\dagger - A_0^\dagger A_1 A_3^\dagger C_2 B_0^\dagger - A_0^\dagger C_2 B_3^\dagger B_1 B_0^\dagger)^*(A_0^\dagger C_0 B_0^\dagger)]$$

is a constant and

$$K = \mathrm{Re}[\mathrm{tr}(W_1^* A_3^\dagger C_2 B_0^\dagger + A_0^\dagger C_2 B_3^\dagger W_2^*)]$$

can take any real value.

As several applications of Theorem 1, we consider the least-norm solutions to the matrix equations $AX = B$ and $XC = D$ over the dual quaternion algebra.

Corollary 1. *Suppose that dual quaternion matrix equation $AX = B$ is consistent, let*

$$A_2 = A_1 L_{A_0}, B_2 = B_1 - A_1 A_0^\dagger B_0,$$
$$A_3 = R_{A_0} A_2, B_3 = R_{A_0} B_2.$$

Then the least-norm solution to the matrix equation $AX = B$ can be discussed in two cases:
(1) If $A_0^\dagger B_0 \neq 0$ and $A_3^\dagger B_2 = 0$, then the least-norm solution $X = X_0 + X_1\varepsilon$ to the matrix equation $AX = B$ can be expressed as

$$X_0 = A_0^\dagger B_0, \tag{37}$$
$$X_1 = A_0^\dagger B_2 + L_{A_0} W_1, \tag{38}$$

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where W_1 is an arbitrary matrix over $\mathbb{H}$ with appropriate dimensions.

In this case, the least-norm of the general solution can be expressed as

$$||X||_{min} = ||A_0^\dagger B_0|| + \frac{\mathrm{Re}[\mathrm{tr}(A_0^\dagger B_2)^*(A_0^\dagger B_0)]}{||A_0^\dagger B_0||}\varepsilon.$$

(2) *If $A_0^\dagger B_0 = A_3^\dagger B_2 = 0$, then the least-norm solution $X = X_0 + X_1\varepsilon$ to the matrix equation $AX = B$ can be expressed as*

$$X_0 = 0, \tag{39}$$
$$X_1 = A_0^\dagger B_1. \tag{40}$$

In this case, the least-norm of the general solution can be expressed as

$$||X||_{min} = ||A_0^\dagger B_1||\varepsilon.$$

Remark 2. *If $A_3^\dagger B_2 \neq 0$, the dual quaternion matrix equation $AX = B$ has a minimum real part of norm solution, and the dual part of norm can take any real value. The solution $X = X_0 + X_1\varepsilon$ can be expressed as*

$$X_0 = A_0^\dagger B_0 + A_3^\dagger B_2, \tag{41}$$
$$X_1 = A_0^\dagger B_2 - A_0^\dagger A_1 A_3^\dagger B_2 + L_{A_0} W_1, \tag{42}$$

where W_1 is an arbitrary matrix over $\mathbb{H}$ with appropriate dimensions. In this case,

$$\begin{aligned} ||X|| &= ||X_0||_{min} + \frac{a+K}{||X_0||_{min}}\varepsilon \\ &= ||A_0^\dagger B_0 + A_3^\dagger B_2|| + \frac{a+K}{||A_0^\dagger B_0 + A_3^\dagger B_2||}\varepsilon, \end{aligned}$$

where $a = \mathrm{Re}[\mathrm{tr}(A_0^\dagger B_2 - A_0^\dagger A_1 A_3^\dagger B_2)^* A_0^\dagger B_0]$ is a constant and $K = \mathrm{Re}[\mathrm{tr}(W_1^* A_3^\dagger B_2)]$ can take any real value.

Corollary 2. *Suppose that dual quaternion matrix equation $XC = D$ is consistent, let*

$$C_2 = R_{C_0}C_1, D_2 = D_1 - D_0C_0^\dagger C_1,$$
$$C_3 = C_2 L_{C_0}, D_3 = D_2 L_{C_0}.$$

Then the least-norm solution to the matrix equation $XC = D$ can be discussed in two cases:
(1) *If $D_0C_0^\dagger \neq 0$ and $D_2C_3^\dagger = 0$, then the least-norm solution $X = X_0 + X_1\varepsilon$ to the matrix equation $XC = D$ can be expressed as*

$$X_0 = D_0C_0^\dagger, \tag{43}$$
$$X_1 = D_2C_0^\dagger + W_1 R_{C_0}, \tag{44}$$

where W_1 is an arbitrary matrix over $\mathbb{H}$ with appropriate dimensions.

In this case, the least-norm of the general solution can be expressed as

$$|X||_{min} = ||D_0C_0^\dagger|| + \frac{\mathrm{Re}[\mathrm{tr}(D_2C_0^\dagger)^*(D_0C_0^\dagger)]}{||D_0C_0^\dagger||}\varepsilon.$$

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(2) *If $D_0C_0^\dagger = D_2C_3^\dagger = 0$, then the least-norm solution $X = X_0 + X_1\varepsilon$ to the matrix equation $XC = D$ can be expressed as*

$$X_0 = 0, \tag{45}$$
$$X_1 = D_1C_0^\dagger. \tag{46}$$

In this case, the least-norm of the general solution can be expressed as

$$||X||_{min} = ||D_1C_0^\dagger||\varepsilon.$$

Remark 3. *If $D_2C_3^\dagger \neq 0$, the dual quaternion matrix equation $XC = D$ has a minimum real part of norm solution, and the dual part of norm can take any real value. The solution $X = X_0 + X_1\varepsilon$ can be expressed as*

$$X_0 = D_0C_0^\dagger + D_2C_3^\dagger, \tag{47}$$
$$X_1 = D_2C_0^\dagger - D_2C_3^\dagger C_1C_0^\dagger + W_1R_{C_0}, \tag{48}$$

where W_1 is an arbitrary matrix over $\mathbb{H}$ with appropriate dimensions. In this case,

$$\begin{aligned} ||X|| &= ||X_0||_{min} + \frac{a+K}{||X_0||_{min}}\varepsilon \\ &= ||D_0C_0^\dagger + D_2C_3^\dagger|| + \frac{a+K}{||D_0C_0^\dagger + D_2C_3^\dagger||}\varepsilon, \end{aligned}$$

where $a = \mathrm{Re}[\mathrm{tr}(D_2C_0^\dagger - D_2C_3^\dagger C_1C_0^\dagger)^*(D_0C_0^\dagger)]$ is a constant and $K = \mathrm{Re}[\mathrm{tr}(D_2C_3^\dagger W_1^*)]$ can take any real value.

4. Numerical Examples

In this part, we will present two numerical examples to illustrate the conclusion in the previous section.

Example 1. *Solving the least-norm solution to the dual quaternion matrix equation $AXB = C$, and providing the norm of the solution, where*

$$A = A_0 + A_1\varepsilon = \begin{bmatrix} 1+i & j-k \\ 2i+k & j \end{bmatrix} + \begin{bmatrix} 2-i+k & k \\ i+j & 0 \end{bmatrix}\varepsilon,$$
$$B = B_0 + B_1\varepsilon = \begin{bmatrix} 1+i+j & i+k \\ j-k & 0 \end{bmatrix} + \begin{bmatrix} 2i-k & i+j \\ 1 & i \end{bmatrix}\varepsilon,$$
$$C = C_0 + C_1\varepsilon = \begin{bmatrix} -3-i+3j+3k & -2-2i \\ -2-5i-j+4k & -4i-2k \end{bmatrix} + \begin{bmatrix} -6+7j+5k & -10+2i-3j-3k \\ -9-7i-3j+6k & -4-7i-2j-6k \end{bmatrix}\varepsilon.$$

Through calculation, we have

$$R_{A_0}C_0 = 0, C_0L_{B_0} = 0,$$
$$R_{A_3}C_{21} = 0, R_{A_0}C_2L_{B_0} = 0, C_{22}L_{B_3} = 0.$$

By Lemma 2, the matrix equation is consistent. On the other hand, we can identify that

$$A_0^\dagger C_0B_0^\dagger \neq 0 \text{ and } A_3^\dagger C_2B_0^\dagger = A_0^\dagger C_2B_3^\dagger = 0.$$

According to Theorem 1, the matrix equation has a least-norm solution, which can be expressed as

$$X = X_0 + X_1\varepsilon = \begin{bmatrix} i+k & 1 \\ 0 & 1+k \end{bmatrix} + \begin{bmatrix} 2i+j & -1 \\ k & 0 \end{bmatrix}\varepsilon.$$

In this case, the norm of solution is $||X|| = \sqrt{5} + \frac{\sqrt{5}}{5}\varepsilon$.

Example 2. *Solving the least-norm solution to the dual quaternion matrix equation $AXB = C$, and providing the norm of the solution, where*

$$\begin{aligned} A &= A_0 + A_1\varepsilon = \begin{bmatrix} 1+k & 2i-j \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} i & 1+k \\ i+j & 2-i+j \end{bmatrix}\varepsilon, \\ B &= B_0 + B_1\varepsilon = \begin{bmatrix} 1+i+j & i-j \\ j+k & 0 \end{bmatrix} + \begin{bmatrix} i-k & 1+i \\ j & 1 \end{bmatrix}\varepsilon, \\ C &= C_0 + C_1\varepsilon = \begin{bmatrix} 2-2i+3j & -1+4i-k \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -4+3i+6j+2k & -3+7i+j-3k \\ -5+i+3j+3k & 2+2i-4k \end{bmatrix}\varepsilon. \end{aligned}$$

Through calculation, we have

$$\begin{aligned} &R_{A_0}C_0 = 0, C_0L_{B_0} = 0, \\ &R_{A_3}C_{21} = 0, R_{A_0}C_2L_{B_0} = 0, C_{22}L_{B_3} = 0. \end{aligned}$$

By Lemma 2, the matrix equation is consistent. On the other hand, we can identify that

$$A_0^{\dagger}C_0B_0^{\dagger} \neq 0, A_3^{\dagger}C_2B_0^{\dagger} \neq 0 \text{ and } A_0^{\dagger}C_2B_3^{\dagger} = 0.$$

According to Remark 1, the matrix equation does not have a least-norm solution and only has minimum real part of norm solution. The solution $X = X_0 + X_1\varepsilon$ can be expressed as

$$\begin{aligned} X_0 &= \begin{bmatrix} 2+j & 0 \\ 1 & i+k \end{bmatrix}, \\ X_1 &= \begin{bmatrix} 0.2857+0.5714i & 0.1429+0.1429k \\ 0.2857-0.1429i+0.4286j-0.8571k & 0.1429i+0.2857j \end{bmatrix} \\ &+ \begin{bmatrix} 0.7143 & -0.1429i+0.4286j \\ 0.1429i-0.4286j & 0.2857 \end{bmatrix} W_1, \end{aligned}$$

where W_1 is an arbitrary matrix over $\mathbb{H}$ with appropriate dimensions. In this case, the norm of solution is $||X|| = 2\sqrt{2} + \frac{1+K}{2\sqrt{2}}\varepsilon$, *where*

$$K = \mathrm{Re}[\mathrm{tr}W_1^* \begin{bmatrix} 1.4286-0.1429i+1.1429j & 0.1429+0.4286i+0.1429j-0.4286k \\ 0.7143+0.2857i-0.8571j+0.1429k & 0.2857i+0.2857k \end{bmatrix}].$$

5. Conclusions

In this paper, we first equivalently transform the expression of the norm of dual quaternion matrices as defined in reference [24]. Using this expression, we discuss the norm of the general solution to matrix equation $AXB = C$ over the dual quaternion algebra when the equation is consistent and establish the conditions for the existence of the least-norm solution. When the equation $AXB = C$ has a least-norm solution, we derive expressions for both the least-norm solution and the least-norm of the general solution in two different cases. If no least-norm solution exists, we consider the minimum real part of norm solution and establish expressions for the minimum real part of norm solution and the

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norm of the solution. In this case, the dual part of norm of solution can take any real value. After that, we investigate the least-norm solution to the dual quaternion matrix equations $AX = B$ and $XC = D$ as an application. Finally, we present two numerical examples to illustrate the main results of this paper. In the future, we will analyze the least-norm solution to more complex and general matrix equations over the dual quaternion algebra.

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