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Short Note

Axiomatic Characterization of Utilitarian Social Utility for Wealth Distributions

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Abstract: We show that the von-Neumann Morgenstern theory of expected utility can provide an axiomatic characterization of utilitarian social utility functions of wealth distributions based on a utility of wealth function.

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1. Introduction

The purpose of this note is to show that the von-Neumann Morgenstern theory of expected utility can provide an axiomatic characterization of utilitarian social utility functions of wealth distributions based on a utility of wealth function that is often used in welfare economics. Significant contributions to welfare economics using utility of wealth functions, is the voluminous literature emanating from the seminal paper of Atkinson (1970) on economic inequality. Our axiomatic characterization is analogous to Proposition 3 in Lahiri (2023). A connection- if there be any- between utility of wealth and utility of monetary gains and losses under risk incorporating loss aversion, is briefly mentioned in Lahiri (2024). In Lahiri (2024), we suggest that the "marginal utility of wealth" as used in welfare economics, could be some convex combination of the "constant average utility of gains" and the "constant average utility of losses" for an individual whose monetary wealth is the one for which the marginal utility is being sought. There may be a difference between the two constant average utilities due to "loss aversion" and this is discussed in the seminal work of Kahneman and Tversky (1979).

2. Framework of Analysis

Consider a society consisting of individuals each initially endowed with non-negative wealth whose monetary value is represented by 'w'. To facilitate, mathematical analysis, we assume, that there is a positive real number M (sufficiently large) which is the satiation level of wealth, i.e., individuals in the society are indifferent between all wealth greater than or equal to M.

A **wealth distribution** is a finite array $\xi = <(w_1, p_1),\ldots, (w_n, p_n)>$ for some positive integer 'n', with $(w_j, p_j) \in (0, M] \times \mathbb{R}_{++}$ for each $j \in \{1,\ldots,n\}$ and $\sum_{j=1}^{n} p_j \in (0, 1]$.

The interpretation of the pair (w_j, p_j) in a wealth distribution $\xi = <(w_1, p_1),\ldots, (w_n, p_n)>$ is that the share (proportion) of individuals in the society with wealth $w_j > 0$ is $p_j >$ Further, a share $1 - \sum_{j=1}^{n} p_j$ which may be positive or zero, have "no" wealth at all.

Let $\mathcal{W}$ denote the set of all wealth distributions.

The following are wealth distributions analogous to what Rubinstein (1988) refers to as a lottery.

A **binary wealth distribution** is a wealth distribution of the form $<(w, p)>$ where $w \in (0, M]$ and $p \in (0, 1]$.

The interpretation of a binary wealth distribution $<(w, p)>$ is that a share p of the population have a monetary wealth of $w > 0$ and the rest have no wealth at all.

For the sake of notational simplicity, we will write a binary wealth distribution $<(w, p)>$ as (w, p).

Let $\mathcal{W}^1$ denote the set of all binary wealth distributions.

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We assume that there is a preference relation (i.e., transitive and complete (and hence reflexive) binary relation) $\succcurlyeq$ on $\mathcal{W}\cup\{(M, 0)\}$ whose symmetric part of is denoted by $\sim$ and whose asymmetric part is denoted by $\succ$.

A **utilitarian social utility function** for $\succcurlyeq$ is a function W: $\mathcal{W}\cup\{(M,0)\} \rightarrow \mathbb{R}$ such there exists a function $u: [0, M] \rightarrow \mathbb{R}$ for which the following are satisfied:

(i) $u(0) = 0$.

(ii) For all $\xi = <(w_1, p_1), \ldots, (w_n, p_n) \in \mathcal{W}$, $W(\xi) = \sum_{j=1}^{n} p_j u(w_j)$.

(iii) For all $\xi_1, \xi_2 \in \mathcal{W}$: $\xi_1 \succcurlyeq \xi_2$ if and only if $W(\xi_1) \geq W(\xi_2)$.

u is called a **utility function for wealth** representing $\succcurlyeq$.

Assumption 1 (Strong Monotonicity): For all $(w,p), (\tilde{w},q) \in \mathcal{W}^1$: $w \geq \tilde{w}, p \geq q$ with at least one strict inequality implies $(w,p) \succ (\tilde{w},q)$.

Assumption 2 (Indifference): For all $w \in (0, M]$, there exists $r \in (0,1]$ such that $(w, 1) \sim (M, r)$.

Assumption 3 (Common Ratio Property): For all $(w,p), (\tilde{w},q) \in \mathcal{W}^1$ and $t \in (0,1]$: $(w,p) \succcurlyeq (\tilde{w},q)$ implies $(w,tp) \succcurlyeq (\tilde{w},tq)$.

Assumption 3 is an interesting property that can be found in Allais (1952).

We now invoke the following additional assumption for our desired axiomatic characterization.

Assumption 4 (Substitution): If $< w_1, p_1), \ldots, (w_n, p_n) >$, $< (\tilde{w}_1, q_1), \ldots, (\tilde{w}_n, q_n) > \in \mathcal{W}$, $(\bar{w}, r) \in \mathcal{W}^1$, $(\bar{w}, r) \sim (w_j, p_j)$ for some $j \in \{1, \ldots, n\}$ and $r \leq p_j$, then $< (w_1, p_1), \ldots, (w_n, p_n) > \sim < (\tilde{w}_1, q_1), \ldots, (\tilde{w}_m, q_m) >$ where $(w_k, p_k) = (\tilde{w}_k, q_k)$ for all $k \in \{1, \ldots, n\} \backslash \{j\}$ and $(\tilde{w}_j, q_j) = (\bar{w}, r)$.

3. The Main Result

The following proposition which is our main result, is mathematically identical to proposition 3 in Lahiri (2023).

Proposition 1: If $\succcurlyeq$ on $\mathcal{W}$ satisfies Assumptions 1 to 4, then there exists a utilitarian social utility function for $\succcurlyeq$.

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