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Article

Geometric Properties of a General Kohn-Nirenberg Domain in \mathbb{C}^n

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Abstract: The Kohn-Nirenberg domains are unbounded domains in \mathbb{C}^n . In this article, we modify the Kohn-Nirenberg domain $\Omega_{K,L} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re} z_n + g |z_n|^2 + \sum_{j=1}^{n-1} (|z_j|^p + K_j |z_j|^{p-q} \operatorname{Re} z_j^q + L_j |z_j|^{p-2q} \operatorname{Im} z_j^{2q}) < 0\}$ and discuss the existence of supporting surface and peak function at the origin.

Keywords: Kohn-Nirenberg domain; peak function; supporting surface

MSC: 32Q02

1. Introduction

Consider a domain $\Omega \in \mathbb{C}^n$ with smooth boundary $\partial\Omega$. As the boundary point is a strongly pseudoconvex point $p \in \partial\Omega$, we can find a local system of holomorphic coordinates. Hakim [1] and Pflug [2] showed that every strongly pseudoconvex point of $\partial\Omega$ is a peak point. But this property fails for weakly pseudoconvex boundary point in general. Kohn and Nirenberg have an example that is defined with the boundary point $(0, 0) \in \partial\Omega$. The representing example [3] is $\Omega = \{z = (z_1, z_2)^T \in \mathbb{C}^2 : r(z) := \operatorname{Re} z_2 + |z_1|^8 + \frac{15}{7} |z_1|^2 \operatorname{Re} z_1^6 < 0\}$, which is a pseudoconvex domain with point in the boundary that does not admit any peak function, supporting surface and the boundary can not be convexifiable by any local holomorphic coordinates [4–6]. The existence of supporting functions and smooth peak functions and the properties of convexifiability have been done by Pflug [2], by Kolař [7], by J. Han [8], D. Zhao [9] and by J. Byun and H. R. Cho [10]. In [11,12], Taeyong Ahn etc. provide a tool to construct global holomorphic peaks from local holomorphic supporting functions for a class of unbounded domains in \mathbb{C}^n . But it is still an open question whether any Kohn-Nirenberg domain is biholomorphic to a bounded domain. In order to better understand the properties of the Kohn-Nirenberg domain, in [14], Simone Calamai provides some new examples of Kohn-Nirenberg domain that develop some properties and theories about convexifiability in \mathbb{C}^2 .

Let $\mathcal{O}^\alpha(\Omega)$ ($0 \leq \alpha \leq \infty$) denote the space of functions holomorphic on Ω and of class C^α on $\bar{\Omega}$. Recall a point $\xi \in \partial\Omega$ is a peak point to Ω if there is a function $f \in \mathcal{O}^\alpha(\Omega)$ satisfying $f(\xi) = 1$ and $|f(z)| < 1$ for all $z \in \bar{\Omega} \setminus \{\xi\}$. We call f a peak function. A holomorphic supporting surface for Ω at ξ is a complex manifold M of co-dimension 1 with the property: there exists a neighborhood $N(\xi)$ of ξ such that $\bar{\Omega} \cap N(\xi) \cap M = \{\xi\}$. In [9], D. Zhao etc. considered a general modification of the Kohn-Nirenberg domain near the origin in \mathbb{C}^n , namely, the domain $\Omega_K = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re} z_n + \sum_{j=1}^{n-1} (|z_j|^p + K_j |z_j|^{p-q} \operatorname{Re} z_j^q) < 0\}$, where $K = (K_1, \dots, K_{n-1}) \in \mathbb{R}^{n-1}$, $p, q \in \mathbb{Z}^+$ and $p - q - 2 \geq 0$. They proved the following sufficient condition.

Theorem 1 ([9]). *Given the above domain Ω_K with $|K_j| \leq \frac{p^2}{p^2 - q^2}$ ($1 \leq j \leq n - 1$), then*

1. Ω_K is a pseudoconvex domain.

2. If $q \mid p$ or $q \nmid p$ but $|K_j| < 1$ ($1 \leq j \leq n-1$), there exists a C^∞ -peak function and a supporting surface at the origin $0 \in \partial\Omega_K$.
3. If $q \nmid p$ and $\sum_{j=1}^{n-1} |K_j| > n-1$, there does not exist any C^∞ -peak function and supporting surface at $0 \in \partial\Omega_K$.

In fact, the the above general Kohn-Nirenberg domain Ω_K is a special case of decoupled domain in \mathbb{C}^n [13]. Based on the modificationn of the domain Ω_K , we define a general Kohn-Nirenberg type domain as follows.

$$\Omega_{K,L} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re}z_n + g|z_n|^2 + \sum_{j=1}^{n-1} (|z_j|^p + K_j|z_j|^{p-q} \operatorname{Re}(z_j^q) + L_j|z_j|^{p-2q} \operatorname{Im}(z_j^{2q})) < 0\}$$

where $K = (K_1, K_2, \dots, K_{n-1}) \in \mathbb{R}^{n-1}$, $L = (L_1, L_2, \dots, L_{n-1}) \in \mathbb{R}^{n-1}$, $g \geq 0$, $p \in \mathbb{Z}^+$, $q \in \mathbb{Z}^+$ and $p-2q-1 \geq 0$.

If we do not consider the term $g|z_n|^2$, the general modified domain $\Omega_{K,L}$ is a weighted-bumped domain [12], denoted by Ω_{WB} (use the notation of Definition 2). If we consider this term, the Theorem 4.6 [11] has a argument that there exists global holomorphic supporting function when $g > 0$ and a bound point of Ω_{WB} admits a local holomorphic supporting function. Thus $\Omega_{K,L}$ keeps the main features. It will be interesting to study whether the existence of supporting surface and peak function at the origin in [9] can be generalized to the domain $\Omega_{K,L}$.

For the domain $\Omega_{K,L}$, we study the existence of the holomorphic peak function, supporting surface at the boundary points. The main result of this article is the following theorem.

Theorem 2. Let $\Omega_{K,L}$ be the above domain with $|K_j| + 2|L_j| \leq \frac{p^2}{p^2-q^2}$ ($1 \leq j \leq n-1$), we have

1. $\Omega_{K,L}$ is a pseudoconvex domain.
2. If $q \mid p$ or $q \nmid p$ and $|K_j| + |L_j| < 1$ ($1 \leq j \leq n-1$), there exists holomorphic peak function and supporting surface at the origin $0 \in \partial\Omega_{K,L}$.
3. If $q \nmid p$ and $\sum_{j=1}^{n-1} |K_j| > n-1$, there does not exist any supporting surface at $0 \in \partial\Omega_{K,L}$.

Here, $q \mid p$ means that q divides p , $q \nmid p$ means that q does not divide p . We shall use these notations in this article. The structure of this article is as follows. In Sect. 2, we give some basic definitions for Kohn-Nirenberg domain. In Sect. 3, we provide the proof of theorem 2 about (1)-(3).

2. Basic Definitions and Lemmas

Let $\Omega = \{(z_1, \dots, z_n) \in \mathbb{C}^n : r(z) < 0\}$ be a domain in \mathbb{C}^n with smooth boundary, its defining function is $r(z)$. Let $\mathcal{O}^\alpha(\Omega)$ ($0 \leq \alpha \leq \infty$) be the space of holomorphic functions on Ω and C^α -continuous on $\bar{\Omega}$.

Definition 1. For a point $\xi \in \partial\Omega$ and a vector $t = (t_1, \dots, t_n) \in \mathbb{C}^n$, we write $\partial r_\xi(t) = \sum_{j=1}^n \frac{\partial r}{\partial z_j}(\xi) t_j$. The Levi form of $r(z)$ at ξ applied to $t \in \mathbb{C}^n$ is $L(\xi, t) = \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(\xi) t_j \bar{t}_k$. ξ is called a pseudoconvex point if

$$L(\xi, t) = \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(\xi) t_j \bar{t}_k \geq 0$$

for all $t \in T_\xi \partial\Omega = \{t \in \mathbb{C}^n : \partial r_\xi(t) = \sum_{j=1}^n \frac{\partial r}{\partial z_j}(\xi) t_j = 0\}$, where $T_\xi \partial\Omega = \{t \in \mathbb{C}^n : \partial r_\xi(t) = 0\}$ is the corresponding complex tangent space.

If the Levi form is positive at boundary point ξ , i.e. $L(\xi, t) > 0$, we call ξ a strong pseudoconvex point.

If all the boundary points are (strong) pseudoconvex points, the domain Ω is called a (strong) pseudoconvex domain.

Definition 2 ([12]). For domain $\Omega_{WB} \in \mathbb{C}^n$, if $\operatorname{Re}z_n + \mathcal{P}(z_1, z_2, \dots, z_{n-1}) < 0$ ($(z_1, z_2, \dots, z_n) \in \mathbb{C}^n$), where

(a) \mathcal{P} is a real-valued, weighted polynomial on Ω_{WB} .

(b) All the boundary points of Ω_{WB} are pseudoconvex points and all but the origin are strong pseudoconvex points.

Lemma 1 ([9]). For any real number K with $|K| \leq \frac{m^2}{m^2-1}$ ($1 < m \in \mathbb{Z}^+$), there exist constants $b \in \mathbb{R}$ and $C > 0$ such that

$$1 + K\cos t + b\cos(mt) > C, \quad \text{for all } t \in [0, 2\pi)$$

where $b = b(m)$ and $C = C(m)$ depend only on m .

Lemma 2. If $|K| + 2|L| \leq \frac{m^2}{m^2-1}$ ($1 < m \in \mathbb{Z}^+$), there exist constants $b \in \mathbb{R}$ and $C > 0$ such that

$$1 + K\cos(t) + L\sin(2t) + b\cos(mt) > C, \quad \text{for } \forall t \in [0, 2\pi)$$

Proof. If $|K| + 2|L| \leq \frac{m^2}{m^2-1}$, then $|K + 2L\sin(t)| \leq |K| + 2|L| \leq \frac{m^2}{m^2-1}$. From Lemma 1 and $K\cos(t) + L\sin(2t) = (K + 2L\sin(t))\cos(t)$, we have constants $b \in \mathbb{R}$ and $C > 0$ such that

$$1 + K\cos(t) + L\sin(2t) + b\cos(mt) > C, \quad \text{for } t \in [0, 2\pi)$$

where $b = b(m)$ and $C = C(m)$ depend only on m .

Set the domain $\Omega_{K,L} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : r(z) < 0\}$, its defining function $r(z)$ on \mathbb{C}^n is as follows

$$r(z) = \operatorname{Re}z_n + g|z_n|^2 + \sum_{j=1}^{n-1} (|z_j|^p + K_j|z_j|^{p-q}\operatorname{Re}z_j^q + L_j|z_j|^{p-2q}\operatorname{Im}z_j^{2q})$$

where $K = (K_1, \dots, K_{n-1}) \in \mathbb{R}^{n-1}$, $L = (L_1, \dots, L_{n-1}) \in \mathbb{R}^{n-1}$, $g \geq 0$, $p \in \mathbb{Z}^+$, $q \in \mathbb{Z}^+$ and $p - 2q - 1 \geq 0$. \square

Lemma 3. If $(p^2 - q^2)|K_j| + (p^2 - 4q^2)|L_j| \leq p^2(1 \leq j \leq n - 1)$ ($g \geq 0$), then $\Omega_{K,L}$ is a pseudoconvex domain.

Proof. For boundary point $\xi = (z_1, \dots, z_n) \in \partial\Omega$ and tangent vector $t = (t_1, \dots, t_n)$, we compute the Levi form and get

$$r_{z_j} = \frac{1}{2}p z_j^{\frac{1}{2}p-1}(\bar{z}_j)^{\frac{1}{2}p} + \frac{K_j(p+q)}{4} z_j^{\frac{1}{2}p+\frac{1}{2}q-1}(\bar{z}_j)^{\frac{1}{2}p-\frac{1}{2}q} + \frac{K_j(p-q)}{4} z_j^{\frac{1}{2}p-\frac{1}{2}q-1}(\bar{z}_j)^{\frac{1}{2}p+\frac{1}{2}q} + \frac{L_j(p+2q)}{4i} z_j^{\frac{1}{2}p+q-1}(\bar{z}_j)^{\frac{1}{2}p-q} - \frac{L_j(p-2q)}{4i} z_j^{\frac{1}{2}p-q-1}(\bar{z}_j)^{\frac{1}{2}p+q}$$

where $i = \sqrt{-1}$. $r_{z_j\bar{z}_k} = 0$ ($j \neq k$) and $r_{z_n\bar{z}_n} = g$.

For $1 \leq j \leq n-1$, there is

$$\begin{aligned} r_{z_j \bar{z}_j} &= \frac{1}{4} p^2 (z_j)^{\frac{1}{2}p-1} + \frac{K_j(p+q)(p-q)}{8} (z_j)^{\frac{1}{2}p+\frac{1}{2}q-1} (\bar{z}_j)^{\frac{1}{2}p-\frac{1}{2}q-1} + \frac{K_j(p-q)}{8} (p+q) \\ &\quad (z_j)^{\frac{1}{2}p-\frac{1}{2}q-1} (\bar{z}_j)^{\frac{1}{2}p+\frac{1}{2}q-1} + \frac{L_j(p+2q)(p-2q)}{8i} (z_j)^{\frac{1}{2}p+q-1} (\bar{z}_j)^{\frac{1}{2}p-q-1} - \\ &\quad \frac{L_j(p-2q)(p+2q)}{8i} (z_j)^{\frac{1}{2}p-q-1} (\bar{z}_j)^{\frac{1}{2}p+q-1} \\ &= \frac{1}{4} \left\{ p^2 |z_j|^{p-2} + K_j(p^2 - q^2) (z_j)^{\frac{1}{2}p-\frac{1}{2}q-1} (\bar{z}_j)^{\frac{1}{2}p-\frac{1}{2}q-1} \left[\frac{(z_j)^q + (\bar{z}_j)^q}{2} \right] + L_j \right. \\ &\quad \left. (p^2 - 4q^2) (z_j)^{\frac{1}{2}p-q-1} (\bar{z}_j)^{\frac{1}{2}p-q-1} \left[\frac{(z_j)^{2q} - (\bar{z}_j)^{2q}}{2i} \right] \right\} \\ &= \frac{1}{4} \left\{ p^2 |z_j|^{p-2} + (p^2 - q^2) K_j |z_j|^{p-q-2} \operatorname{Re}(z_j)^q + (p^2 - 4q^2) L_j |z_j|^{p-2q-2} \operatorname{Im}(z_j)^{2q} \right\} \end{aligned}$$

If $(p^2 - q^2)|K_j| + (p^2 - 4q^2)|L_j| \leq p^2$ and $g \geq 0$, we have the Levi form

$$\begin{aligned} L(\xi, t) &= \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} (\xi) t_j \bar{t}_k \\ &= \sum_{j=1}^{n-1} r_{z_j \bar{z}_j} |t_j|^2 + g |t_n|^2 \\ &\geq \frac{1}{4} \sum_{j=1}^{n-1} (p^2 - (p^2 - q^2)) |K_j| - (p^2 - 4q^2) |L_j| |z_j|^{p-2} |t_j|^2 + g |t_n|^2 \\ &\geq 0. \end{aligned}$$

Thus the Levi form $L(\xi, t)$ is semi-positive definite, which proves that $\Omega_{K,L}$ is pseudoconvex. \square

3. Holomorphic Peak Function and Supporting Surface of the General Modified Domain $\Omega_{K,L}$

Here we prove the main Theorem 2 about (1)-(3).

Proof of Theorem 2 about (1). Let $\Omega_{K,L} = \{z \in \mathbb{C}^n : \operatorname{Re} z_n + g|z_n|^2 + \sum_{j=1}^{n-1} (|z_j|^p + K_j|z_j|^{p-q} \operatorname{Re} z_j^q + L_j|z_j|^{p-2q} \operatorname{Im} z_j^{2q}) < 0\}$. Suppose $|K_j| + 2|L_j| \leq \frac{p^2}{p^2 - q^2} (1 \leq j \leq n-1, g \geq 0)$, then $(p^2 - q^2)|K_j| + (p^2 - 4q^2)|L_j| \leq (p^2 - q^2)|K_j| + 2(p^2 - q^2)|L_j| \leq p^2 (1 \leq j \leq n-1, g \geq 0)$. So Lemma 3 implies that $\Omega_{K,L}$ is pseudoconvex. \square

Remark 1. Let $\Omega_{K,L}$ be the above domain. If $|K_j| + 2|L_j| \leq \frac{p^2}{p^2 - q^2} (1 \leq j \leq n-1, g \geq 0)$, it is easy to see the origin is a weakly pseudoconvex (not strong pseudoconvex) boundary point of $\Omega_{K,L}$.

Proof of Theorem 2 about (2). Let $\Omega_{K,L}$ be the above domain with $|K_j| + 2|L_j| \leq \frac{p^2}{p^2 - q^2} (1 \leq j \leq n-1, g \geq 0)$.

For $r(z) = \operatorname{Re} z_n + g|z_n|^2 + \sum_{j=1}^{n-1} (|z_j|^p + K_j|z_j|^{p-q} \operatorname{Re} z_j^q + L_j|z_j|^{p-2q} \operatorname{Im} z_j^{2q})$, we consider two cases.

1. The case $q \mid p (p = mq)$. Let $z_j = r_j e^{i\theta_j} (1 \leq j \leq n-1)$ and $z_n = u + iv$. In polar coordinate system, we have $r(z_1, z_2, \dots, z_n) = u + gu^2 + gv^2 + \sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j)] r_j^p$. Then we consider the coordinate transformation $z_1^* = z_1; z_2^* = z_2; \dots; z_{n-1}^* = z_{n-1}; z_n^* = z_n - \sum_{j=1}^{n-1} b_j z_j^p$, and $b_j \in \mathbb{R}$. In the new coordinate system, after dropping the stars we have

$$\begin{aligned} r(z_1, z_2, \dots, z_n) &= u + g(u + \sum_{j=1}^{n-1} b_j \cos(p\theta_j) r_j^p)^2 + g(v + \sum_{j=1}^{n-1} b_j \sin(p\theta_j) r_j^p)^2 + \\ &\quad \sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j) + b_j \cos(p\theta_j)] r_j^p \end{aligned}$$

Note that $|K_j| + 2|L_j| \leq \frac{p^2}{p^2 - q^2} = \frac{m^2}{m^2 - 1}$ ($p = mq$). Lemma 2 implies that there exists $C > 0$ such that

$$\sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j) + b_j \cos(p\theta_j)] r_j^p > C \sum_{j=1}^{n-1} r_j^p$$

The point 0 belongs to the set $\{(z_1, z_2, \dots, 0) : z_j \in \mathbb{C}, 1 \leq j \leq n-1\} \cap N(0) \cap \overline{\Omega}_{K,L}$, where $N(0)$ is a neighborhood of 0. For all $0 \neq z \in \{(z_1, z_2, \dots, 0) : z_j \in \mathbb{C}, 1 \leq j \leq n-1\} \cap N(0) \cap \overline{\Omega}_{K,L}$, there exists j such that $z_j \neq 0$. We have

$$r(z_1, z_2, \dots, 0) = g \left\{ \left(\sum_{j=1}^{n-1} b_j \cos(p\theta_j) r_j^p \right)^2 + \left(\sum_{j=1}^{n-1} b_j \sin(p\theta_j) r_j^p \right)^2 \right\} + 0 + \sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j) + b_j \cos(p\theta_j)] r_j^p > C \sum_{j=1}^{n-1} r_j^p > 0$$

This is a contradiction with the definition $r(z_1, z_2, \dots, z_n) \leq 0$ and implies that

$$\{(z_1, z_2, \dots, 0) : z_j \in \mathbb{C}, 1 \leq j \leq n-1\} \cap N(0) \cap \overline{\Omega}_{K,L} = \{0\}.$$

Thus in the new coordinates, the complex manifold $\{(z_1, z_2, \dots, 0) : z_j \in \mathbb{C}, 1 \leq j \leq n-1\}$ is a holomorphic supporting surface at the origin $0 \in \partial\Omega_{K,L}$. The holomorphic supporting function is $f(z_1, z_2, \dots, z_n) = z_n$ at 0. And the corresponding holomorphic peak function is $h(z_1, z_2, \dots, z_n) = e^{z_n + z_n^2}$ for the origin 0.

In fact, it is obvious that $h(0) = 1$. Put $z_n = u + iv$. For $\forall z \in \overline{\Omega}_{K,L} - \{0\}$, we have $r(z_1, z_2, \dots, z_n) \leq 0$, i.e.

$$u + g(u + \sum_{j=1}^{n-1} b_j \cos(p\theta_j) r_j^p)^2 + g(v + \sum_{j=1}^{n-1} b_j \sin(p\theta_j) r_j^p)^2 + \sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j) + b_j \cos(p\theta_j)] r_j^p \leq 0$$

Thus

$$u \leq - \sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j) + b_j \cos(p\theta_j)] r_j^p \leq -C \sum_{j=1}^{n-1} r_j^p \leq 0.$$

If $u = 0$, then $v < 0$ and $|h(z_1, z_2, \dots, z_n)| = e^{u+u^2-v^2} = e^{-v^2} < 1$.

If $u < 0$, then $|h(z_1, z_2, \dots, z_n)| = e^{u+u^2-v^2} = e^{-v^2} < 1$.

So, the function $h(z_1, z_2, \dots, z_n)$ is a local peak function at $0 \in \partial\Omega_{K,L}$. Further, Hakim and Sibony [1,2] show that there is a global peak function with the same regularity as $h(z_1, z_2, \dots, z_n)$.

- The case $q \nmid p$ and $|K_j| + |L_j| < 1$ ($1 \leq j \leq n-1$). There exists holomorphic peak function and supporting surface at the origin. Note that $\sum_{j=1}^{n-1} (|z_j|^p + K_j |z_j|^{p-q} \operatorname{Re}(z_j^q) + L_j |z_j|^{p-2q} \operatorname{Im}(z_j^{2q})) \geq \sum_{j=1}^{n-1} [1 - (|K_j| + |L_j|)] |z_j|^p$, similar to case (I), we have the complex manifold $\{(z_1, z_2, \dots, 0) : z_j \in \mathbb{C}, 1 \leq j \leq n-1\}$, which is also a holomorphic supporting surface at the origin. At the same time, $h(z_1, z_2, \dots, z_n) = e^{z_n + z_n^2}$ is a local peak function at $0 \in \partial\Omega_{K,L}$. In [1,2], Hakim and Sibony show that there is a global peak function with the same regularity as h .

□

Proof of Theorem 2 about (3). Assume that there exists supporting surface at the origin $0 \in \partial\Omega_{K,L}$. The support surface M as a complex manifold of co-dimension 1 implies that there are an open neighbourhood $N(0) \subset \mathbb{C}^n$ and holomorphic function f on $N(0)$ such that

- $M \cap N(0) = \{z \in N(0) : f(z) = 0\}$;
- $\operatorname{rank} \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right) = 1$.

We shall study two different cases.

- The case $\frac{\partial f}{\partial z_n} = 0$, there is some j such that $\frac{\partial f}{\partial z_j} \neq 0$. The implicit function theorem implies that

$$M = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_j = \phi(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)\}$$

Now let $z_1 = \dots = z_{j-1} = z_{j+1} = \dots = z_n = -\varepsilon$, then

$$\begin{aligned} r(z) &= -\varepsilon + g\varepsilon^2 + (n-2)\varepsilon^p + (-1)^p \sum_{l \neq j} K_l |\phi(-\varepsilon, \dots, -\varepsilon)|^p + K_j |\phi(-\varepsilon, \dots, -\varepsilon)|^{p-q} \\ &\quad \operatorname{Re}[\phi(-\varepsilon, \dots, -\varepsilon)]^q + L_j |\phi(-\varepsilon, \dots, -\varepsilon)|^{p-2q} \operatorname{Re}[\phi(-\varepsilon, \dots, -\varepsilon)]^{2q} \\ &= -\varepsilon + O(\varepsilon^2) < 0. \end{aligned}$$

If ε is small, then $M \cap \overline{\Omega}_{K,L} \neq \{0\}$ in every small neighborhood of 0. Therefore, we have a contradiction with M as a support surface.

2. The case $\frac{\partial f}{\partial z_n} \neq 0$, the implicit function theorem implies that

$$M = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_n = \phi(z_1, \dots, z_{n-1})\}$$

We shall divide this into three different cases.

1. When $\phi(z_1, \dots, z_{n-1}) = \sum_{s=t}^{\infty} F_s(z_1, \dots, z_{n-1})$, $t \geq p+1$, where F_s is the sum of those terms $C_{\alpha_1, \dots, \alpha_{n-1}} z_1^{\alpha_1} \dots z_{n-1}^{\alpha_{n-1}}$ in the power series for which $\alpha_1 + \dots + \alpha_{n-1} = s$.

We let $z_j = \varepsilon e^{i \frac{\chi(K_j)+1}{q} \pi}$, $1 \leq j \leq n-1$, where $\chi(K_j)$ is defined by

$$\chi(K_j) = \begin{cases} 0 & K_j > 0 \\ 1 & K_j < 0 \end{cases} (1 \leq j \leq n-1).$$

And $z_n = \phi(z_1, \dots, z_{n-1})$. If ε is sufficiently small,

$$\begin{aligned} r(z) &= \operatorname{Re}[\phi(\varepsilon e^{i \frac{\chi(K_1)+1}{q} \pi}, \dots, \varepsilon e^{i \frac{\chi(K_{n-1})+1}{q} \pi})] + g|\phi(\cdot)|^2 + (n-1)\varepsilon^p - \varepsilon^p \sum_{j=1}^{n-1} |K_j| \\ &= [(n-1) - \sum_{j=1}^{n-1} |K_j|] \varepsilon^p + O(\varepsilon^{p+1}) < 0. \end{aligned}$$

Hence M is not a supporting surface. It is a contradiction.

2. When $\phi(z_1, \dots, z_{n-1}) = \sum_{s=t}^{\infty} F_s(z_1, \dots, z_{n-1})$, $1 \leq t \leq p-1$, $F_t(z_1, \dots, z_{n-1}) \neq 0$. We can suppose $F_t(t_1, \dots, t_{n-1}) = \lambda_1 \neq 0$ and choose θ such that $\lambda_1 e^{i\theta t} < 0$.

Let $z_1 = t_1 \varepsilon e^{i\theta}$, $z_2 = t_2 \varepsilon e^{i\theta}$, \dots , $z_{n-1} = t_{n-1} \varepsilon e^{i\theta}$, it is easy to see that

$$F_t(z_1, \dots, z_{n-1}) = F_t(t_1, \dots, t_{n-1}) \varepsilon^t e^{i\theta t} = \lambda_1 e^{i\theta t} \varepsilon^t = -|\lambda_1| \varepsilon^t.$$

Then

$$\begin{aligned} r(z) &= \operatorname{Re}[\phi(z_1, \dots, z_{n-1})] + g|\phi(z_1, \dots, z_{n-1})|^2 + \varepsilon^p \sum_{j=1}^{n-1} |t_j|^p + \varepsilon^p \cos q\theta \\ &\quad (\sum_{j=1}^{n-1} K_j |t_j|^p) + \varepsilon^p \sin 2q\theta (\sum_{j=1}^{n-1} L_j |t_j|^p) \\ &= -|\lambda_1| \varepsilon^t + O(\varepsilon^{t+1}) < 0 \end{aligned}$$

if ε is sufficiently small. Hence M is not a supporting surface. It is a contradiction.

3. Then the only remaining case is when

$$\phi(z_1, \dots, z_{n-1}) = \sum_{s=p}^{\infty} F_s(z_1, \dots, z_{n-1}).$$

Let $z_j^l = \varepsilon e^{i \frac{\chi(K_j)+l}{q} \pi}$ ($1 \leq j \leq n-1$), where $\chi(K_j)$ is defined by

$$\chi(K_j) = \begin{cases} 0 & K_j > 0 \\ 1 & K_j < 0 \end{cases}$$

and $z_n^l = \phi(z_1^l, \dots, z_{n-1}^l)$. Then

$$\begin{aligned} r(z^l) &= \operatorname{Re}[\phi(z_1^l, \dots, z_{n-1}^l)] + g|\phi(z_1^l, \dots, z_{n-1}^l)|^2 + (n-1)\varepsilon^p + \varepsilon^p \cos l\pi \left(\sum_{j=1}^{n-1} |K_j|\right) \\ &= \varepsilon^p [\operatorname{Re}(\lambda e^{i\frac{p\pi}{q}l}) + O(\varepsilon) + (n-1) + \cos l\pi (\sum_{j=1}^{n-1} |K_j|)], \end{aligned}$$

where $\lambda = F_p(e^{i\frac{\lambda(K_1)}{q}\pi}, \dots, e^{i\frac{\lambda(K_{n-1})}{q}\pi})$. We take $l = 1, 3, \dots, 2q-1$, then

$$\sum_{l=1,3,\dots,2q-1} r(z^l) = \varepsilon^p \left\{ \operatorname{Re} \lambda \left(\sum_{l=1,3,\dots,2q-1} e^{i\frac{p\pi}{q}l} \right) + O(\varepsilon) + q(n-1) + \sum_{l=1,3,\dots,2q-1} \cos l\pi \left(\sum_{j=1}^{n-1} |K_j| \right) \right\}$$

Since l takes odd integers, we obtain

$$\sum_{l=1,3,\dots,2q-1} \cos l\pi = -q.$$

If $q \nmid p$, we have $\sum_{l=1,3,\dots,2q-1} e^{i\frac{p\pi}{q}l} = 0$. Therefore, when ε is small,

$$\sum_{l=1,3,\dots,2q-1} r(z^l) = \varepsilon^p \left\{ O(\varepsilon) + q[(n-1) - \sum_{j=1}^{n-1} |K_j|] \right\} < 0.$$

Since $M \cap \bar{\Omega}_{K,L} = \{0\}$ and $z^l \in M$, it follows that $r(z^l) > 0$ for all l .

Thus

$$\sum_{l=1,3,\dots,2q-1} r(z^l) > 0.$$

Hence we get a contradiction. This completes the proof.

□

There are still some problems yet to be answered for the domain $\Omega_{K,L}$.

Problem 1. When $q \nmid p$ and $\sum_{j=1}^{n-1} |K_j| > n-1$, the peak functions of the $\Omega_{K,L}$ at the origin is still not clear.

Moreover, the behavior of invariant metrics (e.g., Kobayashi, Carathéodory) on the Kohn-Nirenberg domain could be studied using the techniques in [11]. Such metrics are central to understanding hyperbolic geometry in complex domains, and their properties here might reveal new phenomena in extremal map constructions or boundary asymptotics. Thus, another nature problem is as follows.

Problem 2. The Bergman metric and Carathéodory metrics of $\Omega_{K,L}$ in the statement of the Theorem 2 are positive and complete?

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