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# A Short Survey on the Riemann Hypothesis and Some Attempts to Prove It

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*Article*

# A Short Survey on the Riemann Hypothesis and Some Attempts to Prove It

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**Abstract:** A short survey on the Riemann Hypothesis is presented, and various even recent attempts aimed at proving it are described.

**Keywords:** zeros of the Riemann zeta function; the Riemann Hypothesis; sum over all inverse zeta's nontrivial zeros

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## 1. Introduction

The Riemann zeta function is usually defined as the Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it, \tag{1.1}$$

[50, Ch. I, sec. 1.1, (1.1.1), p. 1], [16, Ch.1, sec. 1.2, (1), p. 6], which converges for all $s \in \mathbf{C}$ with $\sigma = \mathrm{Re}\{s\} > 1$, and as its meromorphic continuation in the complex plane, where it has only a simple pole at $s = 1$. It can also be defined by the Euler product formula,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}s > 1,$$

where $p$ runs through all primes, $p = 2, 3, 5, 7, \ldots$, [50, Ch. I, sec. 1.1, (1.1.2), p. 1], [16, sec. 2.1, (1), p. 6].

While it is known that $\zeta(s)$ vanishes at the infinitely many negative even integers $s_n := -2n$, $n \in \mathbf{N}$, called the *trivial* zeros, and that, in addition, infinitely (countably) many zeros exist on the critical line, $\mathrm{Re}\{s\} = 1/2$, a result due to Hardy [25], [16, Ch. 11, sec. 11.1, p. 226]), it is not known to date whether other *nontrivial* zeros exist in $\mathbf{C}$, necessarily in the critical strip, $\Sigma := \{(\sigma, t) \in \mathbf{C} : 0 < \sigma < 1, -\infty < t < +\infty\}$. That no such zeros exist is what is called the Riemann Hypothesis (RH).

The importance of the RH rests on in its relation with the distribution of primes [16, Sec. 1.11, p. 22], [26], but also, by now, on many equivalent formulations, as well as on a large number of issues whose validity would immediately follow from its validity.

To study the zeros of the zeta function, the so-called (Riemann's) xi function,

$$\xi(s) := \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \tag{1.2}$$

[50, Ch. II, (2.1.12), p. 16], has also been widely used. In fact, for instance Hardy proved that $\xi(s)$ has infinitely (countably) many zeros on the critical line $\mathrm{Re}\{s\} = 1/2$, [25], [16, Ch. 11, sec. 11.1, p. 226]).

Many properties of and generalities on the zeta function can be found in a number of books; see [7,16,50], e.g. Several surveys, recording advances and possible paths to prove the RH have appeared regularly in the literature over the time; see, for instance, [3,4,7,14,48], without any claim to be exhaustive. Indeed, the literature concerning the Riemann Hypothesis (RH) is extremely broad, with at least a dozen contributions appearing in qualified literature in the first half of 2024 alone. Fortunately, unlike in the case of Fermat's Last Theorem, where any amateur, regardless of their incompetence or lack of mathematical skills, could naively attempt to tackle the problem, addressing

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the RH requires a solid understanding of basic mathematics, specifically, knowledge that goes beyond that of a typical sophomore.

The following list is far from being exhaustive. Typically, they are (recent or less recent) failures.

## 2. About Some Attempts to Prove the RH

Many attempts aiming either to prove or to disprove the correctness of the RH have been made since the 1859 work of Bernhard Riemann [46], resorting to a variety of methods, some of which somewhat sophisticated.

### *2.1. Zero-Free Regions*

The interesting idea of identifying "zero-free" regions inside the critical strip should be mentioned. The first of such regions was constructed in 1899 by de la Vallée-Poussin [16, sec. 5.2, p. 79], [18], but one of the best results has been considered for long time that achieved by Vinogradov and Korobov (see the review paper [18] for references and other cases). In 2015, Mossinghoff and Trudgian have obtained the largest known zero-free region [36].

In 2006, Freitas established a Li-type criterion to obtain a necessary and sufficient condition for the existence of zero-free strips, contained inside the critical strip, $0 < \mathrm{Re}\{z\} < 1$. To this kind of approach, a numerical rigorous result by Platt and Trudgian belongs, but it will be reported later, in more detail.

### *2.2. The Hilbert-Pólya Conjecture*

Among the perhaps most intriguing ideas proposed to prove the RH, one should recall the Hilbert-Pólya conjecture, according to which the RH is true because non-trivial zeros of the zeta function correspond (in a certain canonical way) to the eigenvalues of some positive operator. This conjecture has been often regarded as a most promising approach. However, very little is known about its origins, some people attributed its formulation to Hilbert and Pólya, independently, some time in the 1910s.

Andrew Odlyzko had a correspondence with George Polya in 1981-1982 [42], where he asked a reference, time, and reasoning about this issue. George Pólya replied that he had spent two years in Göttingen ending around the begin of 1914, when he tried to learn analytic number theory from Landau. The latter asked him one day: "You know some physics. Do you know a physical reason that the Riemann hypothesis should be true." This would be the case, he answered, if the nontrivial zeros of the Xi-function were so connected with the physical problem that the Riemann hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real." Pólya added that he never published this remark.

That's it! It may be too little, to vague, to base some investigations on this, but several researchers, mostly physicists, took it seriously, even though still unsuccessfully.

There are also some theorems by Pólya's [44,45], [16, Ch. 12, sec. 12.5, p. 270], that perhaps could be related to the aforementioned plan.

### *2.3. The Keiper-Li's Criterion*

The Li's, or better Keiper-Li's criterion concerns the positivity of a certain numerical sequence [2,10,12,23,30]. In 1997, Xian-Jin Li established a necessary and sufficient condition for the validity of the RH, since then known as the "Li's criterion" [30]; see also [29]. This has the form of a set of inequalities, namely,

$$\lambda_n := \sum_{\rho} \left[ 1 - \left( 1 - \frac{1}{\rho} \right)^n \right] \geq 0 \quad \text{for } n = 1, 2, \ldots, \tag{2.1}$$

the sum being extended to *all* nontrivial zeros, $\rho$, of the Riemann zeta function, and understood as in (3.4). In [2], the conditions in (2.1) are formulated with the $\lambda_n$'s strictly positive.

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In some sense, there are two ways to evaluate the $\lambda_n$'s. One is to look at their definition,

$$\lambda_n := \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left[ s^{n-1} \ln \xi(s) \right]_{s=1}, \tag{2.2}$$

which comes from the representation

$$\ln \xi\left(\frac{1}{1-x}\right) = \ln \xi\left(\frac{x}{x-1}\right) = -\ln 2 + \sum_{n=1}^{\infty} \lambda_n x^n$$

of the xi function (1.2), [27, sec. 4.2], [29, sec. 4,eq. (20),p. 767]. Other equivalent formulations for the $\lambda_n$'s exist; see, e.g., [11], and also [12,13,33].

The first 3,300 Keiper-Li coefficients were evaluated by K. Maślanka in [32]. A computation for up to $n = 100,000$, made in [27], shows an agreement on the first two decimals with a certain "Keiper's conjecture" (hence with the Riemann hypothesis), providing $\lambda_{100,000} = 4.62580782406902231409416038...$ (plus about 2,900 more accurate digits). Keiper's asymptotic approximation yields in fact $\lambda_{100,000} \approx 4.626132$, [28, pp. 63-65].

Another way to evaluate the $\lambda_n$'s is using the successively obtained formula (2.1); see [30]. Using the latter, however, the validity or not of the RH *cannot* be established since it involves all nontrivial zeros, without knowing whether any of them exist or not off the critical line. In [51], Voros established an alternative asymptotic behavior, basing on (2.1).

In order to prove (or to disprove) the RH via Li's criterion, one should instead use (2.2), e.g, which does not involve explicitly the nontrivial zeros themselves.

Trying to make explicit the Li's representation of the parameters $\lambda_n$ (in [11, eq. (9),p. 527], an integral representation for $\xi^{(j)}(1)$ is given), one may consider resorting to the celebrated Faa' di Bruno formula, which yields the $n$-th derivative of a given composite function. This uses sums over set partitions and is extremely involved, indeed it seems to be a hopeless task.

### *2.4. Horizontal Monotonicity*

In [34], it is recalled that the so-called "horizontal behavior" of $|\zeta(s)|$ was considered by Saidak and Zvengrowski [47], and earlier by Spira [49]. It was observed that, at the beginning of the article on the Riemann zeta function in the Wolfram MathWorld [53], a plot shows horizontal "ridges" of $|\zeta(\sigma + it)|$, for $0 < \sigma < 1$ and $1 < t < 100$. If indeed such ridges decrease strictly monotonically for $0 < \sigma < 1/2$ the RH would be proved to be true (cf. [6,47]). This behavior is easy to prove outside the critical strip, but nobody has been able so far to prove it *inside* the critical strip.

### *2.5. Hyperbolicity of Jensen Polynomials*

In 1927, G. Pólya proved that the RH is equivalent to the so-called hyperbolicity of Jensen polynomials for the Riemann zeta function $\zeta$ at its point of symmetry. This is called the Pólya-Jensen criterion. Advances in this direction are based in the newly discovered fact that these polynomials can be well approximated in terms of Hermite polynomials [22]. This result was initially praised by E. Bombieri [4], but the approach was then considered not useful, hence a not too promising route by D.W. Farmer [17]).

### *2.6. Basing on New Bounds for Large Values of Dirichlet Polynomials*

Very recently, in [24], L. Guth and J. Maynard have posted a significant new result on the arxiv. Using largely Fourier analysis, the authors' breakthrough towards the Riemann hypothesis rests on improving a result deemed insurmountable for more than 50 years, that is, making the first substantial improvement to a classical 1940 bound of Ingham regarding the zeroes of the Riemann zeta function. Field medalist Terence Tao considered this work a remarkable breakthrough. Yet, he still judged it very far from fully proving the RH.

## 3. Numerical Approach

A continuous progress has been made over the years in computing numerically more and more zeros *on* the critical line, and exploring the possible existence of zeros *off* the critical line; see, e.g., [31,39]. Of course, no numerical approach can provide acceptable proofs of the RH, unless they are accompanied by rigorous and suitably sharp error bounds, but the numerical path may be useful to formulate conjectures and develop insight. For instance, simulations and visualizations suggest the possible "horizontal monotonicity" of the modulus of $\zeta(s)$, but unfortunately nobody has been able to prove to occur inside the critical strip [34].

Relevant computations of the sum of the series $\sum_k t_k^{-2}$ and $\sum_k (1/4 + t_k^2)^{-1}$, $1/2 + it_k$ being the nontrivial zeros on the critical line, and estimates (with bounds) for some more general sums were carried out in [9,37]. Some attention has also been devoted to computing the sums $\sigma_k := \sum_\rho \rho^{-k}$, extended over all nontrivial zeros, for fixed $k$'s, $k \in \mathbf{N}$. The $\sigma_k$'s are also related to the Li's coefficients, $\lambda_n$, defined below; see [29, eq. (27),p. 768], e.g.

Here are a few examples. In 1982, R.P. Brent et al. have tested the RH computationally and confirmed its validity for the first $200,000,001$ zeros [8]. Such zeros are all located in the critical strip, up to the ordinate $t = 81,702,130.19$. In 2002, S. Wedeniwski has shown that the first $10^{12}$ nontrivial zeros lie on the critical line [55]. In [52], it is recalled that in 2004 X. Gourdon [21] used a fast method, developed by Odlyzko and Schönhage, to verify that the first $10^{13}$ nontrivial zeros of the zeta function lie on the critical line, which fact validates the RH up to about the ordinate $2.4 \times 10^{12}$, in the critical strip; see [20]. Actually, A.M. Odlyzko in [40,41] shows the heights of the zeros of the Riemann zeta function numbered $10^{22} + 1$ up to $10^{22} + 10^4$.

As it is well known, in order to prove the RH it suffices to show that no zero of the zeta function exists in the *semi-infinite open critical half-strip*, $H := \{(\sigma, t) \in \mathbf{C} : 1/2 < \sigma < 1, t > 0\}$. In fact, if a nontrivial zero, say $\rho$, exists in $\Sigma$, then also its conjugate, $\bar{\rho}$, would be a zero (since $\zeta(s)$ is meromorphic and real on the reals), as well as its symmetric one with respect to the critical line $\mathrm{Re}\{s\} = 1/2$, that is $1 - \rho$, and thus $1 - \bar{\rho}$ as well. This follows from the functional equation obeyed by the zeta function,

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi}{2} s\right) \Gamma(1-s)\, \zeta(1-s) \tag{3.1}$$

[50, Ch. II, (2.1.1), p. 13], [16, eq. (4), p. 13], also written, changing $s$ into $1 - s$, $\zeta(1-s) = 2^{1-s}\pi^{-s} \cos \left(\frac{\pi}{2} s\right) \Gamma(s) \zeta(s)$, [50, Ch. II, (2.1.8), p. 16].

Therefore, nontrivial zeros come in groups of four, if they exist in the critical strip, off the critical line, and in pair if they are on the critical line. Therefore, if $s_0 := \sigma_0 + it_0$ with $1/2 < \sigma_0 < 1$ and $t_0 > 0$, is a zero, the Weierstrass factorization of the entire function $\xi$, would contain terms like

$$\{(s - s_0)(s - \bar{s_0})[s - (1 - s_0)][s - (1 - \bar{s_0})]\}^{m_0}, \tag{3.2}$$

where $m_0 \in \mathbf{N}$ would be the multiplicity of $s_0$.

On the other hand, it is common knowledge too that no zeros exist on the intersection of the critical strip and the real line, as well as on the lines $\mathrm{Re}\{s\} = 0$, $\mathrm{Re}\{s\} = 1$. One can thus confine the discussion to the set $H$, but this region can be restricted further. In fact, it is also known that no nontrivial zero exists in $H$ with an ordinate less than some value. For instance, in [35, Ex. 10.2.1, (c), p. 353] it is recorded that, if $\rho_0 = \sigma_0 + it_0$ is a nontrivial zero with $\sigma_0 \neq 1/2$, then $|t_0| > 9.2518^+$. Actually, in a recent paper, Platt and Trudgian have shown numerically, but *rigorously*, using interval arithmetic, that the RH is true up to about the height $3 \times 10^{12}$ (cf. [20,21]), that is, all zeros in the critical strip with imaginary part $0 < t \leq 3 \times 10^{12}$ have real part $\mathrm{Re}\{s\} = 1/2$ (and are all simple), [43]. Consequently, one can confine any investigation for ruling out the existence of nontrivial zeros just to $H_{t^*} := H \cap \{t > t^*\}$, for some $t^*$ (equal to $9.2518^+$, but, if useful, even equal to $3 \times 10^{12}$).

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That no nontrivial zeros exist off the critical line, below some value of $t$, can also be shown applying the well-known equation for the sum of the inverse nontrivial zeta's zeros,

$$\sum_{\rho} \frac{1}{\rho} = 1 + \frac{\gamma}{2} - \frac{1}{2} \ln(4\pi) = 0.0230957^{+}, \tag{3.3}$$

where the sum is extended over *all* nontrivial zeros of the zeta function, and $\gamma = 0.5772156^{+}$ is the Euler-Mascheroni constant. This equation was already known to Riemann [16, Ch. 3, sec. 3.8, p. 67, eq. (4)], and appears also in the Li's criterion [2,23,30]. The sum on the left-hand side of (3.3) would diverge, but each term $1/\rho$ is intended to be grouped with its conjugate [15, Ch. 12, p. 81], [1, p. 214], and is understood as

$$\sum_{\rho} := \lim_{T \to +\infty} \sum_{|\mathrm{Im}\{\rho\}| \le T} \tag{3.4}$$

termed "$*$-convergence" by Bombieri and Lagarias [2, p. 275]. Recall again that each nontrivial zero is necessarily accompanied by its complex conjugate as well as by its symmetric one with respect to the critical line (it it lies off this line), and thus by the conjugate of the latter too. Therefore, if a nontrivial zero, say $\rho_0 = \sigma_0 + it_0$, exists in the critical strip, off the critical line, with $1/2 < \sigma_0 < 1$, $t_0 > 0$, then also $\overline{\rho}_0$, $1 - \rho_0$, and $1 - \overline{\rho}_0$ will contribute to the sum on the left-hand side of (3.3) with the amount

$$\frac{1}{\rho_0} + \frac{1}{\overline{\rho}_0} + \frac{1}{1-\rho_0} + \frac{1}{1-\overline{\rho}_0}$$

$$= \frac{2\mathrm{Re}\{\rho_0\}}{|\rho_0|^2} + \frac{2(1-\mathrm{Re}\{\rho_0\})}{|1-\rho_0|^2} = 2\left(\frac{\sigma_0}{\sigma_0^2 + t_0^2} + \frac{1-\sigma_0}{(1-\sigma_0)^2 + t_0^2}\right) \ge \frac{2}{1+t_0^2}. \tag{3.5}$$

For the last inequality, the result of [35, Ex. 10.2, 8. (a), p. 355] was used. This estimate is useful to rule out that some $\rho_0$ might be a nontrivial zero. In fact, if $t_0$ is such that

$$\frac{2}{1+t_0^2} > 1 + \frac{\gamma}{2} - \frac{1}{2} \ln 4\pi - 0.021... = 0.02309957... - 0.021... = 0.002..., \tag{3.6}$$

that is, if (about) $0 < t_0 < \sqrt{999} \approx 31.6069$, the existence of such a zero would violate the validity of (3.3). Note that, if one limits the knowledge of the sum over the first 200 zeros (lying on the critical line) to the value 0.021, with so few digits, as given in [5, p. 249], the last number (on the right-hand side) should be considered as lying between 0.0211 and 0.0219. In fact, correspondingly to the value 0.0211, one would have $0.02309957 - 0.0211 = 0.00199957$, while correspondingly to 0.0219, one would obtain $0.02309957 - 0.0219 = 0.00119957$. In any case, the existence of such a zero would violate (3.3). If, to make another example, $t_0 = 2\pi + 0.1 = 6.3831...$, it is obtained

$$\frac{2}{1+t_0^2} = 2 \times 0.0235547^{+} = 0.0479098^{+},$$

definitely much larger than the discrepancy in (3.6).

**Remark 1.1.** Note that if *more than one* nontrivial zero would exist in $H_{t^*}$, each of them would contribute to the sum in (3.3) with a positive amount; see (3.5). Therefore, the aforementioned discrepancy can only increase.

## 4. Summary

It has been known since long time that *all* nontrivial zeros of the Riemann zeta function must lie in the open critical strip $\Sigma := \{(\sigma, t) \in \mathbf{C} : 0 < \sigma < 1, -\infty < t < +\infty\}$. While the existence of infinitely (countably) many zeros on the critical line, $\mathrm{Re}\{s\} = 1/2$, has also been known since 1914, no proof of either existence or nonexistence in Σ of other nontrivial zeros has been given to date. In this short survey, a few more or less recent attempts at proving the RH are described.

## Compliance with Ethical Standards

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