Article

Not peer-reviewed version

The Generalized Characteristic Polynomial of the $K_{m, n}$-Complement of a Bipartite Graph

Weiliang Zhao and Helin Gong *

Posted Date: 19 December 2024

doi: 10.20944/preprints202412.1620.v1

Keywords: bipartite graph; $K_{m,n}$-complement; generalized matrix; balanced bipartite subgraph

Article

The Generalized Characteristic Polynomial of the $K_{m,n}$-Complement of a Bipartite Graph

Weiliang Zhao [1,*] and Helin Gong [2]

1 Department of Fundamental Courses, Zhejiang Industry Polytechnic College, Shaoxing 312000, Zhejiang, P.R.China
2 Department of Mathematics, Shaoxing University, Shaoxing 312000, Zhejiang, P.R.China
* Correspondence: helingong@126.com

Abstract: The generalized matrix of a graph G is defined as $M(G) = A(G) - tD(G)$ ($t \in \mathbb{R}$, $A(G)$ and $D(G)$ respectively denote the adjacency matrix and the degree matrix of G), and the generalized characteristic polynomial of G is merely the characteristic polynomial of $M(G)$. Let $K_{m,n}$ be the complete bipartite graph. Then the $K_{m,n}$-complement of a subgraph G in $K_{m,n}$ is defined as the graph obtained by removing all edges of an isomorphic copy of G from $K_{m,n}$. In this paper, by using a determinant expansion on the sum of two matrices (one of which is a diagonal matrix), a general method for computing the generalized characteristic polynomial of the $K_{m,n}$-complement of a bipartite subgraph G was provided. Furthermore, when G is the k edge-disjoint union or a graph with rank no more than 4, the explicit formula for the generalized characteristic polynomial of the $K_{m,n}$-complements of G is given.

Keywords: bipartite graph; $K_{m,n}$-complement; generalized matrix; balanced bipartite subgraph

---

1. Introduction

In the present paper, we only consider undirected, simple, and connected graphs unless otherwise stated. Let $G = (V(G), E(G))$ (or shortly (V, E)) be a graph with vertex set $V(G)$ and edge set $E(G)$. The adjacency matrix of G is defined as $A(G) = (a_{ij})$, where a_{ij} equals the number of edges connecting vertices v_i and v_j when $i \neq j$ and 0 when $i = j$. The rank of a graph G, denoted by $r(G)$, is defined to be the rank of its adjacency matrix $A(G)$. The degree matrix $D(G)$ is defined as the diagonal matrix $\mathrm{diag}(d_1, d_2, \ldots, d_n)$, where $n = |V|$, and d_i equals the number of edges incident to vertex v_i. In the literature [8], Cvetković et al. introduced a bivariate polynomial, $\phi(G; \lambda, t) = \det(\lambda I - (A(G) - tD(G)))$ (abbreviated as $\phi(G)$). Wang et al. [18] referred to it as the generalized characteristic polynomial of G. It is natural to define $A(G) - tD(G)$ in the variable t as the generalized matrix of a graph G, denoted by $M(G) = (m_{ij}(G))_{|V|\times|V|}$. To be specific,it is easy to see

$$m_{ij}(G) = \begin{cases} -td_G(v_i) & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the generalized characteristic polynomial of graph G is exactly the characteristic polynomial of the generalized matrix $M(G)$. That is, the polynomial $\phi(G) = \phi(G; \lambda, t) = \det(\lambda I - M(G)) = \det(\lambda I - (A(G) - tD(G)))$ is referred to as the generalized characteristic polynomial of G. Note that $A(G) - tD(G)$ with $t \in \mathbb{R}$ encodes several well-known graph matrices, such as the adjacency matrix, the Laplacian matrix, and the normalized Laplacian matrix. It is evident that the generalized characteristic polynomial of a graph generalizes several well-known polynomial invariants of graphs, for example:

- The characteristic polynomial of the adjacency matrix of a graph G is given by $\phi(G; \lambda, 0) = \det(\lambda I - A(G))$;

 doi:10.20944/preprints202412.1620.v1

- The characteristic polynomial of the Laplacian matrix $D(G) - A(G)$ of G is $(-1)^{|V|}\phi(G; -\lambda, 1) = \det(-\lambda I - A(G) + D(G))$;
- The characteristic polynomial of the unsigned Laplacian matrix $D(G) + A(G)$ of graph G is $\phi(G; \lambda, -1) = \det(\lambda I - A(G) - D(G))$;
- The characteristic polynomial of the normalized Laplacian matrix $I - D_G^{-\frac{1}{2}} A_G D_G^{-\frac{1}{2}}$ is $(-1)^{|V|}\phi(G; 0, -\lambda + 1) = \det(-A(G) - (\lambda - 1)D(G))$.

Given an undirected graph $G = (V, E)$, if E is considered as a set of symmetric directed edges, meaning that if $e \in E$, then $\bar{e} \in E$, where $\bar{e}$ is the reverse edge of e, then G can also be viewed as a directed graph. For $e \in E$, let $h(e)$ denote the head of the directed edge e and $t(e)$ the tail of e. A closed walk in G is defined as a sequence of edges $C = (e_1, \ldots, e_k)$ such that $h(e_i) = t(e_{i+1})$ for $i \in \mathbb{Z}/k\mathbb{Z}$. Here $k = |C|$ is the length of C and $cbc(C) = \#\{i \in \{1, \ldots, k\} \mid e_{i+1} = e_i\}$ is called the cyclic bump count of C. The notation $[C]$ is referred to as the equivalence class of the closed walk C under edge permutation, meaning that $(e_1, \ldots, e_k) \sim (e_2, \ldots, e_k, e_1)$. If none of the representatives of $[C]$ can be expressed as C^k (for $k \geq 2$), then the cycle C is said to be irreducible. The set of all irreducible cycles is denoted by $\mathcal{C}$. The Bartholdi zeta function of a graph G is defined as (see [4] for details)

$$Z_G(\lambda, t) = \prod_{[C] \in \mathcal{C}} \frac{1}{1 - \lambda^{cbc(C)} t^{|C|}}.$$

The function $Z_G(t) = Z_G(0, t)$ is referred to as the (Ihara–Selberg) zeta function [10], which was introduced by Ihara to study the zeta function of a regular graph and its reciprocal, and the reciprocal of the zeta function of a regular graph was generalized to the reciprocal of the Bartholdi zeta function for a general graph G as below:

$$Z_G(\lambda, t)^{-1} = \left(1 - (1-\lambda)^2 t^2\right)^{|E|-|V|} \det\left(I - tA_G + (1-\lambda)(D_G - (1-\lambda)I)t^2\right).$$

In particular, the reciprocal of the zeta function for a general graph G is given by:

$$Z_G(t)^{-1} = (1 - t^2)^{|E|-|V|} \det\left(I - tA_G + t^2(D_G - I)\right).$$

The zeta function encodes significant structural information about the graph, such as the number of vertices, edges, and loops. Moreover, the number of spanning trees (the complexity of the graph) $\tau(G)$ satisfies the equation [14]:

$$\left.\frac{\partial f_G(t)}{\partial t}\right|_{t=1} = 2(|E| - |V|)\tau(G), \quad \text{where} \quad f_G(t) = \det\left(I - tA_G + t^2(D_G - I)\right).$$

For a comprehensive treatment of many aspects of Zeta function, refer to [17].

Zeta functions of certain classes of graphs have received considerable attention, such as the line graph of semi-regular bipartite graphs [15], the middle graph of semi-regular bipartite graphs [16], the cone graph of regular graphs [2], and various special join graphs of regular graphs [5,7]. It is not difficult to prove that $\phi(G; \lambda, t)$ determines the reciprocal of the zeta function, and vice versa [11]. Let $G = (V, E)$ be a subgraph of the complete bipartite graph $K_{m,n}$. The $K_{m,n}$-complement of G is defined as the graph obtained from $K_{m,n}$ by deleting all edges of G in $K_{m,n}$, i.e., $K_{m,n} - E(G)$. In this paper, we shall show a computational method for deriving the formula for the generalized characteristic polynomial of the $K_{m,n}$-complement of any bipartite graph G, and further give an explicit formula for the generalized characteristic polynomials of the $K_{m,n}$-complement of a bipartite graph with rank less than or equal to 4.

.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 December 2024 doi:10.20944/preprints202412.1620.v1

2. Notations and Terminology

Let $G = (V, E)$ be a graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E = \{e_1, \ldots, e_m\}$. For two vertices $v_i, v_j \in V$, if v_i and v_j are adjacent, we denote this as $v_i \sim v_j$. The neighborhood of a vertex v_i in G is defined as $N_G(v_i) = \{v_j \in V \mid v_i \sim v_j\}$, and the degree of vertex v_i in G is denoted by $d_i = d_G(v_i) = |N_G(v_i)|$. The complement of the graph $G = (V, E)$ is denoted as $G^c = (V, E^c)$, where $E^c = \{v_iv_j \mid v_i, v_j \in V, v_iv_j \notin E\}$. If $G = (V, E)$ and $G' = (V', E')$ with $V' \subseteq V$ and $E' = \{(u, v) \mid u, v \in V', (u, v) \in E\}$, then G' is referred to as an induced subgraph of G.

A graph $G = (V, E)$ is a bipartite graph if and only if there exists a bipartition of V into (V_1, V_2) (namely $V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset$) such that no two vertices within V_1 or V_2 are adjacent. If the sizes of the bipartition sets are equal, i.e., $|V_1| = |V_2|$, then G is said to be a balanced bipartite. If $G = (V, E)$ is a bipartite graph with a bipartition (V_1, V_2), the bipartite complement of G, denoted as G^{bc}, has vertex set $V(G^{bc}) = V(G)$ and edge set $E(G^{bc}) = \{xy \mid x \in V_1, y \in V_2, xy \notin E(G)\}$. For a bipartite graph G, its adjacency matrix is given by

$$A(G) = \begin{bmatrix} 0 & B(G) \\ B^T(G) & 0 \end{bmatrix},$$

where $B(G) = (b_{ij})_{m\times n}$ is the bipartite adjacency matrix of G that defines the vertex adjacency relationship between the bipartite sets V_1 and V_2. Specifically,

$$b_{ij} = \begin{cases} 1 & v_i \sim v_j,\ v_i \in V_1,\ v_j \in V_2 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1 ([12]). *Let G be a balanced bipartite graph. If G has a unique perfect matching, then the bipartite adjacency matrix $B(G)$ has determinant 1 or -1.*

Lemma 2 ([6]). *Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ matrix, and let D be a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$, i.e. $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$. Then*

$$\det(A + D) = \sum_{\theta\subseteq[n]} \det(A_\theta)\det(D_{\overline{\theta}}), \tag{1}$$

where θ is a subset of $[n] = \{1, 2, \ldots, n\}$ and $\overline{\theta}$ is the complement of θ in $[n]$, namely $\overline{\theta} = \{k \mid k \in [n], k \notin \theta\}$; A_θ is the submatrix formed by the rows and columns of A indexed by θ. By convention, $\det(A_\emptyset) = 1$.

The following lemma immediately follows from Lemma 2.

Lemma 3 ([6]). *If D is an invertible matrix, then the determinant of the matrix $A + D$ can be expressed as*

$$\det(A + D) = \det(D) \sum_{\theta\subseteq[n]} \frac{\det(A_\theta)}{\det(D_\theta)}. \tag{2}$$

Lemma 4 ([3]). *Let A be an $n \times n$ matrix. If there exists a $p \times q$ zero submatrix in A such that $p + q \geq n + 1$, then $\det(A) = 0$.*

3. The Generalized Characteristic Polynomial

For the sake of simplicity, the complete graph, cycle, and path on n vertices are denoted by K_n, C_n, and P_n, respectively. Notationally, for $m, n \in \mathbb{Z}$, $[m] = \{1, 2, \ldots, m\}$ and $[m + 1, m + n] = \{m + 1, m + 2, \ldots, m + n\}$; I_n and $J_{m\times n}$ (or J_n) respectively denote the $n \times n$ identity matrix and the $m \times n$ (or $n \times n$) matrix of all ones. For the rest of this paper, we will use $K_{m,n}$ to symbolize the complete bipartite graph with bipartite partition (X, Y), where $X = \{v_1, v_2, \ldots, v_m\}$ and $Y = \{v_{m+1}, v_{m+2}, \ldots, v_{m+n}\}$,

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 December 2024 doi:10.20944/preprints202412.1620.v1

and $\mathscr{B}_{m,n}$ to symbolize the set of all bipartite graphs with bipartite partition (X,Y) such that $|X| = m$ and $|Y| = n$. Note that a graph $G \in \mathscr{B}_{m,n}$ if and only if its bipartite complement $G^{bc} \in \mathscr{B}_{m,n}$. Let G be a subgraph of the complete bipartite graph $K_{m,n}$. The $K_{m,n}$-complement of a subgraph G in $K_{m,n}$ is defined as the graph obtained from $K_{m,n}$ by deleting all edges of G in $K_{m,n}$, denoted by $K_{m,n} - G$.

Theorem 1. *Let $G = (V,E)$ be a subgraph of $K_{m,n}$ with bipartite partition (X,Y) mentioned previously, and G has a bipartite partition (V_1, V_2) such that $V_1 \subseteq X$ ($|V_1| = s$), $V_2 \subseteq Y$ ($|V_2| = t$). Then, we have*

$$\begin{aligned}
&\phi(K_{m,n} - G) \\
&= (nt+\lambda)^{m-s}(mt+\lambda)^{n-t} \prod_{v\in V_1}((n-d_G(v))t+\lambda) \prod_{v\in V_2}((m-d_G(v))t+\lambda) \\
&\cdot\left[1 + \sum_{Q^{bc}\in\mathcal{G}(H^{bc})} \frac{(-1)^{\frac{|V(Q^{bc})|}{2}}\left(\det(J_k - B(Q^{bc}))\right)^2}{\prod_{v\in V(Q^{bc})\cap X}((n-d_G(v))t+\lambda)\prod_{v\in V(Q^{bc})\cap Y}((m-d_G(v))t+\lambda)}\right] \\
&= (nt+\lambda)^{m-s}(mt+\lambda)^{n-t} \prod_{v\in V_1}((n-d_G(v))t+\lambda) \prod_{v\in V_2}((m-d_G(v))t+\lambda) \\
&\cdot\left[1 + \sum_{Q\in\mathcal{G}(K_{m,n}-G)} \frac{(-1)^{\frac{|V(Q)|}{2}}\left(\det(B(Q))\right)^2}{\prod_{v\in V(Q)\cap X}((n-d_G(v))t+\lambda)\prod_{v\in V(Q)\cap Y}((m-d_G(v))t+\lambda)}\right],
\end{aligned} \tag{3}$$

where $\mathcal{G}(H^{bc})$ is the set of all induced balanced bipartite subgraphs Q^{bc} in H such that the bipartite complement Q of Q^{bc} has a nonsingular biadjacency matrix $B(Q)$; $\mathcal{G}(H)$ is the set of all nonempty induced balanced bipartite subgraphs Q in H (i.e., $K_{m,n} - G$) such that $B(Q)$ is nonsingular.

Proof. Let $H = K_{m,n} - G \in \mathcal{B}_{m,n}$. Note that H has the same bipartite partition as $K_{m,n}$, that is, $(X,Y) = \{v_1, v_2, \ldots, v_m\} \cup \{v_{m+1}, \ldots, v_{m+n}\}$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$. Obviously, $H^{bc} \cong G \cup (m+n-|V|)K_1 \in \mathscr{B}_{m,n}$. The generalized matrix $M(H) = A(H) - tD(H) = (m_{ij}(H))_{(m+n)\times(m+n)}$ is given by

$$m_{ij}(H) = \begin{cases} t(d_{H^{bc}}(v_i) - n) & \text{if } i = j \in [m], \\ t(d_{H^{bc}}(v_i) - m) & \text{if } i = j \in [m+1, m+n], \\ 1 & \text{if } i \neq j \text{ and } v_iv_j \in E(H), \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$M(H) = \begin{bmatrix} -tD_1(H) & J_{m\times n} - B(H^{bc}) \\ J_{n\times m} - B^T(H^{bc}) & -tD_2(H) \end{bmatrix}$$

where $D_1(H) = \text{diag}(n - d_{H^{bc}}(v_1), n - d_{H^{bc}}(v_2), \ldots, n - d_{H^{bc}}(v_m))$ and $D_2(H) = \text{diag}(m - d_{H^{bc}}(v_{m+1}), m - d_{H^{bc}}(v_{m+2}), \ldots, m - d_{H^{bc}}(v_{m+n}))$.

Let $\mathbf{1}_{V_1}$ be the $(m+n)\times 1$ column vector such that the i-th element is 1 for $1 \le i \le m$ and 0 for $m+1 \le i \le m+n$. Similarly, let $\mathbf{1}_{V_2}$ be the $(m+n)\times 1$ vector where the i-th element is 0 for $1 \le i \le m$ and 1 for $m+1 \le i \le m+n$.

$$\begin{aligned}
\lambda I_{m+n} - M(H) &= \begin{bmatrix} tD_1(H) + \lambda I_m & B(H^{bc}) - J_{m\times n} \\ B^T(H^{bc}) - J_{n\times m} & tD_2(H) + \lambda I_n \end{bmatrix} \\
&= tD(H) + \lambda I_{m+n} + A(H^{bc}) - \mathbf{1}_{V_1}\mathbf{1}_{V_2}^T - \mathbf{1}_{V_2}\mathbf{1}_{V_1}^T \\
&= D' + A'
\end{aligned}$$

Let $D' = tD(H) + \lambda I_{m+n}$ and $A' = A(H^{bc}) - \mathbf{1}_{V_1}\mathbf{1}_{V_2}^T - \mathbf{1}_{V_2}\mathbf{1}_{V_1}^T$. Then, we have

$$\det(D') = \prod_{v\in X}((n - d_{H^{bc}}(v))t + \lambda) \prod_{v\in Y}((m - d_{H^{bc}}(v))t + \lambda) \neq 0.$$

By Lemma 2, the following equality holds:

 doi:10.20944/preprints202412.1620.v1

$$\det(\lambda I_{m+n} - M(H)) = \det(D' + A') = \det(D') \sum_{\theta \subseteq [m+n]} \frac{\det(A'_\theta)}{\det(D'_\theta)}. \tag{4}$$

Now, we claim two facts:

Fact 1: The first-order principal submatrix of A' is zero. The third-order principal submatrices of A' are in the form:

$$\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix},$$

and both of their determinants are zero. The fifth-order principal submatrices of A' are in one of the following forms:

$$\begin{bmatrix} 0 & * & * & * & * \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ * & * & * & * & 0 \end{bmatrix},$$

and each of their determinants equals zero. Analogously, the odd-order principal submatrices of A' are in the form:

$$A'_\theta = \begin{bmatrix} 0_{p\times p} & A_1 \\ A_1^T & 0_{q\times q} \end{bmatrix} \quad (p + q = |\theta|).$$

Note that $2p \geq n+1$ or $2q \geq n+1$. According to Lemma 4, we conclude that $\det(A'_\theta) = 0$, indicating that all odd-order principal submatrices of A' are singular.

Fact 2: By definition, $A' = A(H^{bc}) - \mathbf{1}_{V_1}\mathbf{1}_{V_2}^T - \mathbf{1}_{V_2}\mathbf{1}_{V_1}^T$ or equivalently

$$A' = \begin{bmatrix} 0 & B(H^{bc}) - J_{m\times n} \\ B^T(H^{bc}) - J_{n\times m} & 0 \end{bmatrix}.$$

Suppose θ is a subset of $[m+n]$ such that $\frac{|\theta|}{2} = |\theta \cap [m]| = |\theta \cap [m+1, m+n]|$. Let A'_θ be an even-order principal submatrix of A'. We denote by $A'_{\theta\cap[m],\theta\cap[m+1,m+n]}$ the submatrix of A' formed by the rows indexed by $\theta \cap [m]$ and the columns indexed by $\theta \cap [m+1, m+n]$. Furthermore,

$$\det(A'_\theta) = (-1)^{\frac{|\theta|}{2}\times\frac{|\theta|}{2}} \left(\det(A'_{\theta\cap[m],\theta\cap[m+1,m+n]})\right)^2,$$

This reduces to

$$\begin{aligned}\det(A'_\theta) &= (-1)^{\frac{|\theta|}{2}\times\frac{|\theta|}{2}} \left(\det(B_\theta - J_{\frac{|\theta|}{2}})\right)^2 \\ &= (-1)^{\left(\frac{|\theta|}{2}\right)^2} \left(\det(J_{\frac{|\theta|}{2}} - B_\theta)\right)^2 \\ &= (-1)^{\left(\frac{|\theta|}{2}\right)^2} (\det(\overline{B}_\theta))^2 \\ &= (-1)^{|\theta|} (\det(\overline{B}_\theta))^2,\end{aligned}$$

where B_θ is the $\frac{|\theta|}{2} \times \frac{|\theta|}{2}$ submatrix obtained from $B(H^{bc})$ by deleting the rows that are not in $\theta \cap [m]$ and the columns that are not in $\theta \cap [m+1, m+n]$. Similarly, $\overline{B}_\theta$ is the matrix resulting from $B(H)$ by the same deletion of rows and columns, and it is easy to see $B_\theta + \overline{B}_\theta = J_{\frac{|\theta|}{2}}$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 December 2024 doi:10.20944/preprints202412.1620.v1

Set $k = \frac{|\theta|}{2}$, and let Q be a balanced bipartite induced subgraph of H with $2k$ vertices, and the bipartite complement Q^{bc} corresponding to Q is also a balanced bipartite induced subgraph of H^{bc}. Obviously, $B(Q) + B(Q^{bc}) = J_k$. Let $\mathcal{G}_k(H^{bc})$ denote the set of induced balanced bipartite subgraphs Q^{bc} in H with $2k$ vertices such that the bipartite complement Q of Q^{bc} has a nonsingular biadjacency matrix $B(Q)$, that is, the rank of $B(Q)$ is k ($k \le r(A')/2$). Moreover, we denote by $\mathcal{G}(H^{bc})$ the set of all induced balanced bipartite subgraphs Q^{bc} in H such that $B(Q)$ is nonsigular. We denote by $\mathcal{G}(H)$ the set of all nonempty induced balanced bipartite subgraphs Q in H (i.e., $K_{m,n} - G$) such that $B(Q)$ is nonsingular; that is to say, if Q is such a nonempty induced balanced bipartite subgraph in H on $2k$ vertices, then the $k \times k$ matrix $B(Q)$ satisfies the condition $r(J_k - B(Q^{bc})) = k$, and it's worth noting that the induced subgraph Q may not necessarily be nonsingular.

Observe that (i) $r(A') = 2r(J_{m\times n} - B(H^{bc})) = 2r(B(H^{bc}) - J_{m\times n}) \le 2r(B(H^{bc})) + 2r(J_{m\times n}) = 2r(B(G)) + 2 = r(A(G)) + 2$, where $r(M)$ symbolizes the rank of the matrix M; (ii) $B(Q) = J_k - B(Q^{bc})$. Then, by Lemma 2.3, we conclude that

$$\begin{aligned}
&\phi(K_{m,n} - G) = \det(D' + A') \\
=&\prod_{v\in X}((n - d_{H^{bc}}(v))t + \lambda)\prod_{v\in Y}((m - d_{H^{bc}}(v))t + \lambda) \\
&\cdot\Big[1 + \sum_{k=1}^{r(A(G))/2+1}\sum_{Q^{bc}\in\mathcal{G}_k(H^{bc})}\frac{(-1)^{k^2}\big(\det(J_k - B(Q^{bc}))\big)^2}{\prod_{v\in V(Q^{bc})\cap X}((n - d_{H^{bc}}(v))t + \lambda)\prod_{v\in V(Q^{bc})\cap Y}((m - d_{H^{bc}}(v))t + \lambda)}\Big] \\
=&(nt + \lambda)^{m-s}(mt + \lambda)^{n-t}\prod_{v\in V_1}((n - d_G(v))t + \lambda)\prod_{v\in V_2}((m - d_G(v))t + \lambda) \\
&\times\Big[1 + \sum_{Q^{bc}\in\mathcal{G}(H^{bc})}\frac{(-1)^{\frac{|V(Q^{bc})|}{2}}\big(\det(J_k - B(Q^{bc}))\big)^2}{\prod_{v\in V(Q^{bc})\cap X}((n - d_G(v))t + \lambda)\prod_{v\in V(Q^{bc})\cap Y}((m - d_G(v))t + \lambda)}\Big] \\
=&(nt + \lambda)^{m-s}(mt + \lambda)^{n-t}\prod_{v\in V_1}((n - d_G(v))t + \lambda)\prod_{v\in V_2}((m - d_G(v))t + \lambda) \\
&\cdot\Big[1 + \sum_{Q\in\mathcal{G}(K_{m,n}-G)}\frac{(-1)^{\frac{|V(Q)|}{2}}\big(\det(B(Q))\big)^2}{\prod_{v\in V(Q)\cap X}((n - d_G(v))t + \lambda)\prod_{v\in V(Q)\cap Y}((m - d_G(v))t + \lambda)}\Big].
\end{aligned}$$

This completes the proof. □

4. An Application

Our main result in this section gives an application of Theorem 1. A permutation matrix is a square matrix obtained from the same size identity matrix by a permutation of rows. Two n by n matrices A and B are said to be permutationally equivalent if there exist n by n permutation matrices P, Q such that $PAQ = B$.

Theorem 2. *With notations mentioned in Theorem 1, if G has its rank $r(A(G)) \le 4$, then*

$$\begin{aligned}
\phi(K_{m,n} - G) = \;& (nt + \lambda)^{m-s}(mt + \lambda)^{n-t}\prod_{v\in V_1}((n - d_G(v))t + \lambda)\prod_{v\in V_2}((m - d_G(v))t + \lambda) \\
&\times\Big[1 + \sum_{Q\in\Omega_{\mathcal{F}_1}(K_{m,n}-G)}\frac{1}{\prod_{v\in V(Q)\cap X}((n-d_G(v))t+\lambda)\prod_{v\in V(Q)\cap Y}((m-d_G(v))t+\lambda)} \\
&+ \sum_{Q\in\Omega_{\mathcal{F}_2}(K_{m,n}-G)}\frac{-1}{\prod_{v\in V(Q)\cap X}((n-d_G(v_i))t+\lambda)\prod_{v\in V(Q)\cap Y}((m-d_G(v_i))t+\lambda)} \\
&+ \sum_{Q\in\Omega_{\{C_6\}}(K_{m,n}-G)}\frac{-2}{\prod_{v\in V(Q)\cap X}((n-d_G(v))t+\lambda)\prod_{v\in V(Q)\cap Y}((m-d_G(v))t+\lambda)}\Big],
\end{aligned} \tag{5}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 December 2024 doi:10.20944/preprints202412.1620.v1

where $\Omega_Q(G)$ is the set of all induced bipartite subgraphs of G that are isomorphic to the graph Q, and $\Omega_{\mathcal{F}}(G)$ is the set of all induced bipartite subgraphs of G that are isomorphic to certain graph in the family $\mathcal{F}$. Here, $\mathcal{F}_1 = \{2K_2, Q_7\}$, $\mathcal{F}_2 = \{K_2, 3K_2, K_2 \cup P_4, P_6, Q_3, Q_6\}$, where Q_3, Q_6 and Q_7 are illustrated in Figure 1, Figure 2 and Figure 3.

Proof. From the proof of Theorem 1, we know that

$$\det(\lambda I_{m+n} - M(H)) = \det(D' + A') = \det(D') \sum_{\theta \subseteq [m+n]} \frac{\det(A'_\theta)}{\det(D'_\theta)} \tag{6}$$

where $H = K_{m,n} - G$, $A' = A(H^{bc}) - \mathbf{1}_{V_1}\mathbf{1}_{V_2}^T - \mathbf{1}_{V_2}\mathbf{1}_{V_1}^T$, or

$$A' = \begin{bmatrix} 0 & B(H^{bc}) - J_{m\times n} \\ B^T(H^{bc}) - J_{n\times m} & 0 \end{bmatrix}.$$

Hence we only need to study the summation

$$\sum_{\theta \subseteq [m+n]} \frac{\det(A'_\theta)}{\det(D'_\theta)} \tag{7}$$

or equivalently

$$1 + \sum_{Q \in \mathcal{G}(K_{m,n}-G)} \frac{(-1)^{\frac{|V(Q)|}{2}} \left(\det(B(Q))\right)^2}{\prod_{v \in V(Q)\cap X} ((n - d_G(v))t + \lambda) \prod_{v \in V(Q)\cap Y} ((m - d_G(v))t + \lambda)} \Bigg]. \tag{8}$$

Observe that $r(A') = 2r(B(H^{bc}) - J_{m\times n}) \le 2r(B(H^{bc})) + 2r(J_{m\times n}) = 2r(B(G)) + 2 = r(A(G)) + 2 \le 6$. The following cases need to be discussed:

Case 1: If $\theta = \emptyset$, then

$$\frac{\det(A'_\theta)}{\det(D'_\theta)} = 1.$$

Case 2: If $|\theta| = 2$, then

$$A'_\theta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consequently, $Q \cong K_2$ and $Q^{bc} \cong K_2^{bc}$ (the trivial graph on two vertices). The contribution of this case to the summation part of Equation (7) is given by

$$\sum_{Q \in \Omega_{K_2}(K_{m,n}-G)} \frac{-1}{\prod_{v \in V(Q)\cap X} ((n - d_Q(v))t + \lambda) \prod_{v \in V(Q)\cap Y} ((m - d_Q(v))t + \lambda)}.$$

Case 3 If $|\theta| = 4$, then $|\theta \cap [m]| = |\theta \cap [m+1, m+n]| = 2$. Let $\theta = \{i, j, k, l\}$ with $i < j < k < l$, where $i, j \in [m]$ and $k, l \in [m+1, m+n]$. In this case, the matrix A' is in the form

$$A'_\theta = \begin{bmatrix} 0 & 0 & a_{ik} & a_{il} \\ 0 & 0 & a_{jk} & a_{jl} \\ a_{ik} & a_{jk} & 0 & 0 \\ a_{il} & a_{jl} & 0 & 0 \end{bmatrix},$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 December 2024 doi:10.20944/preprints202412.1620.v1

where $\begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix}$ is permutationally equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ since $\det(A'_\theta) \neq 0$. This indicates that the induced subgraph Q in $K_{m,n} - G$ satisfies that $Q \cong 2K_2$ or P_4. The contribution of this case to the summation part of Equation (7) is given by

$$\sum_{Q \in \Omega_{\{2K_2, P_4\}}(K_{m,n}-G)} \frac{1}{\prod\limits_{v \in V(Q) \cap V_1} ((n - d_Q(v))t + \lambda) \prod\limits_{v \in V(Q) \cap V_2} ((m - d_Q(v))t + \lambda)}.$$

Case 4 If $|\theta| = 6$, then $|\theta \cap [m]| = |\theta \cap [m+1, m+n]| = 3$. Suppose $B_3(Q^{bc})$ is the 3×3 principal submatrix of $B(Q^{bc})$ in A' such that

$$A'_\theta = \begin{bmatrix} 0 & B_3(Q^{bc}) - J_3 \\ (B_3(Q^{bc}) - J_3)^T & 0 \end{bmatrix}.$$

Moreover,

$$\det(A'_\theta) = (-1)^{3 \times 3} \det\left(B_3(Q^{bc}) - J_3\right) \det\left((B_3(Q^{bc}) - J_3)^T\right) = -(\det(J_3 - B_3(Q^{bc})))^2.$$

By exhaustive search, there exist 174 nonsingular 0–1 matrices of order 3, each of which is permutationally equivalent to one of the following seven matrices or their transposes:

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, B_4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, B_6 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B_7 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

In [9], it is proved by the authors that B_k corresponds to the induced subgraph Q_k in $K_{m,n} - G$, where $Q_1 \cong 3K_2$, $Q_2 \cong K_2 \cup P_4$, Q_3, $Q_4 \cong P_6$, $Q_5 \cong C_6$, Q_6, or Q_7 (see Figure 1, Figure 2 and Figure 3). By Lemma 1 or simple calculations, we know that $(\det(J_3 - B_k))^2 = 1$ for $k = 1, 2, 3, 4, 5, 7$ and $(\det(J_3 - B_6))^2 = 4$. Hence, the contribution of this case to Equation (7) is given by

$$\sum_{Q \in \Omega_{\{Q_1, Q_2, Q_3, Q_4, Q_6, Q_7\}}(K_{m,n}-G)} \frac{-1}{\prod\limits_{v \in V(Q) \cap X} ((n - d_Q(v))t + \lambda) \prod\limits_{v \in V(Q) \cap Y} ((m - d_Q(v))t + \lambda)}$$
$$+ \sum_{Q \in \Omega_{C_6}(K_{m,n}-G)} \frac{-4}{\prod\limits_{v \in V(Q) \cap X} ((n - d_Q(v))t + \lambda) \prod\limits_{v \in V(Q) \cap Y} ((m - d_Q(v))t + \lambda)}.$$

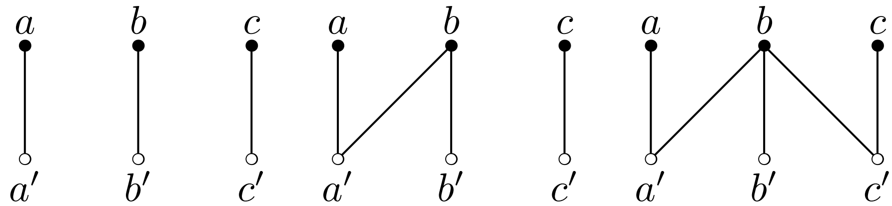

Fig. 1. $Q_1 \cong 3K_2$, $Q_2 \cong K_2 \cup P_4$, Q_3.

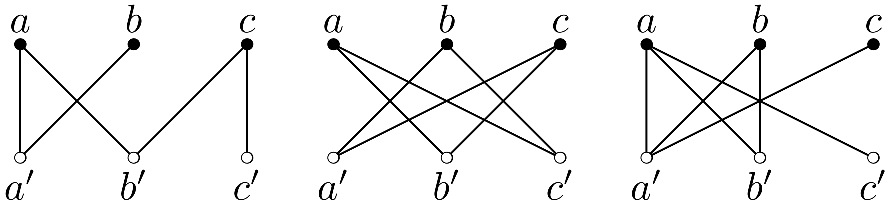

Fig. 2. $Q_4 \cong P_6$, $Q_5 \cong C_6$ and Q_6.

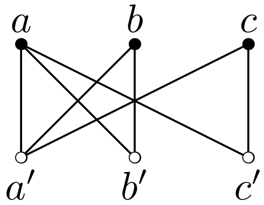

Fig. 3. Q_7.

The proof is completed. □

5. Conclusions

In this paper we studied the computation of the generalized characteristic polynomial or equivalently the zeta function of graphs, and derived a general formula for the generalized characteristic polynomial of the $K_{m,n}$-complement of a bipartite graph. As a by-product, we obtained an explicit formula for the generalized characteristic polynomial of the $K_{m,n}$- tite complement of a bipartite graph with rank no more than 4. In a sense, the formulas obtained in this paper are straightforward and only rely on the use of fundamental linear algebra about the biadjacency matrix of the bipartite graph.

Acknowledgments: This work is partially supported by National Natural Science Foundation of China (No. 12271489, No. 12371332). H.L. Gong is supported by Natural Science Foundation of Zhejiang Province (No. LY21A010006). W.L. Zhao is supported by the Project of the "14th Five-Year Plan" on the Reform of Higher Vocational Education in Zhejiang Province (No. jg20230215).

Conflicts of Interest: There are no conflicts of interests or competing interests.

References

1. H. Bass, The Ihara-Selberg zeta function of a tree lattice. Internat. J. Math. 3 (1992) 717–797.
2. P. Bayati, M. Somodi, On the Ihara zeta functions of cones over regular graphs. Graphs Combin. 29 (2013) 1633–1646.
3. R.B. Bapat, Graphs and Matrices. Hindustan Book Agency & Springer, New Delhi, India (2010).
4. L. Bartholdi, Counting paths in graphs, Enseign.Math. 45 (1999) 83–131.
5. A. Blanchard, E. Pakala, M. Somodi, Ihara zeta function and cospectrality of joins of regular graphs. Discrete Math. 333 (2014) 84–93.
6. J.C. Bruce, Characteristic polynomials by diagonal expansion. Am. Stat. 37 (1983) 233–235.
7. H.Y. Chen, Y. Chen, Bartholdi zeta functions of generalized join graphs. Graphs Combin. 34 (2018) 207–222.
8. D. Cvetković, S. K. Simić, Towards a spectral theory of graphs based on the signless Laplacian. Publ. Inst. Math. 85 (2009) 19–33.
9. H.L. Gong, Y. Gong, J. Ge, The number of spanning trees in K_n-complement of a bipartite graph. J. Algebr. Combin. 60 (2024) 541–554
10. Y. Ihara, On discrete subgroups of the two by two projective linear group over p-adic fields. J. Math. Soc. Japan 18 (1966) 219–235.
11. H.K. Kim, J. Lee, A generalized characteristic polynomial of graphs having a semifree action. Discrete Math. 308 (2008) 555–564.
12. B. Khodakhast, On the determinant of bipartite graphs. Discrete Math. 313 (21) (2013) 2446–2450.

doi:10.20944/preprints202412.1620.v1

13. D.Q. Li, Y.P. Hou, Ihara zeta function and spectrum of the cone over a semiregular bipartite graph. Graphs Combin. 35 (2019) 1503–1517.
14. S. Northshield, A note on the zeta function of a graph. J. Combin. Theory Ser. B 74 (1998) 408–410.
15. I. Sato, Zeta functions and complexities of a semiregular bipartite graph and its line graph. Discrete Math. 307 (2007) 237–245.
16. I. Sato, Zeta functions and complexities of middle graphs of semiregular bipartite graphs. Discrete Math. 355 (2014) 92–99.
17. A. Terras, Zeta Functions of Graphs: A Stroll Through the Garden. Cambridge University Press, Cambridge (2011).
18. W. Wang, F. Li, H.L. Lu, Z.B. Xu, Graphs determined by their generalized characteristic polynomials. Linear Algebra Appl. 434 (2011) 1378–1387.