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Article

Some remarks on the inverse problem in the variational calculus within the functional and antiexact differential forms approach

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Abstract: We present a slightly more broader framework of variational calculus to accommodate differential equations that are not variational as they stand. We discuss two approaches: The first one utilizes antiexact differential forms as obstruction to variationality, make them vanish that gives constraints for all possible variations. The approach we discuss describes of differential equations introducing new functions that make equations variational and then reduce them using a functional constraints. The latter approach incorporates via a not completely standard scheme the classical Dirac reduction approach.

Keywords: calculus of variations; variationality; homotopy operator; Dirac reduction; Lagrangian representation; Dirac constraints; Poisson operator; symplectic structure

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1. Introduction

Calculus of variations is currently a well-established vast discipline with methods ranging from functional analysis [13,17,44] through geometric formulation in jet spaces in terms of variational bicomplex [6–8,8,20–24,30,39,40,42,43,45,46].

One of the main problems in the calculus of variations is the Inverse Problem (IP). In basic formulation: given a system of differential equations, check if they are variational, i.e., if they are Euler-Lagrange equations. In this formulation, the solution of the problem is affirmative usually by modification of original problem, e.g., by adding new equations that correct non-variationality of original equations. One way is to use Hamiltonian structures, see e.g., [36] for further references. In a more restricted problem it reads as follows: given the differential equations 'as they stand' (without any alteration), check if they are the Euler-Lagrange equations for some Lagrangian. The solution to this classical problem dates back to the works of Helmholtz [18], where the well-known Helmholtz conditions were formulated. A recent summary is presented in [20,29,40,47], the formulation in terms of exterior differential systems in [9,32,33], the summary from the viewpoint of classical mechanics is presented in [38], and the perspective from the functional-analytic viewpoint is given in the classical book by Vainberg [44].

In general we will focus on differential expressions $\mathcal{E}[u] \in J_u(\Omega;\mathbb{R}^m)$, $u \in M$, over a domain $\Omega \subset \mathbb{R}^n$, understood as a system of differential equations on a mapping u. It can be composed into functional one-form as

$$\alpha[u] = \langle R[u]\mathcal{E}[u]|du\rangle, \tag{1}$$

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where $R[u] \in \mathrm{End}(\mathbb{R}^m)$ is an *a priori* chosen operator, mixing the orders of separate equations. The usual way one defines vertical exterior derivative $\hat{d} : \Lambda^k_*(J(\Omega;\mathbb{R}^m)) \to \Lambda^{k+1}_*(J(\Omega;\mathbb{R}^m)), k \in \mathbb{Z}_+$, and a related functional form

$$A[u] := \int_\Omega dx \alpha[u], \tag{2}$$

where, by definition, $\Lambda^k_*(J(\Omega;\mathbb{R}^m)) := \Lambda^k(J(\Omega;\mathbb{R}^m))/D\Lambda^{k-1}(J(\Omega;\mathbb{R}^m))$, where $D := \sum_{j=1}^n dx_j \wedge \frac{d}{dx_j}$ is the total differential operator. Naturally, two functional forms (2) are assumed to be equivalent modulo a divergence term $\sum_{j=1}^n \frac{d}{dx_j}\Lambda^k_u(J(\Omega;\mathbb{R}^m))$.

One can define complementary vertical homotopy operator $H : \Lambda^{k+1}_*(J(\Omega;\mathbb{R}^m)) \to \Lambda^k_*(J(\Omega;\mathbb{R}^m))$, defined [25,26,30,44] as

$$H\omega = \int_0^1 i_{\mathcal{K}}\omega|_{\tau(s,u)}\, ds, \tag{3}$$

where $\omega \in \Lambda^{k+1}_*(J(\Omega;\mathbb{R}^m))$, $\mathcal{K} = (u-u_0)^I \partial_I$, and $\tau(s,u) = u_0 + s(u-u_0), s \in [0,1]$, is the linear homotopy between $u \in J_u(\Omega;\mathbb{R}^m)$ and the center $u_0 \in J_{u_0}(\Omega;\mathbb{R}^m)$ of this homotopy. We assume that of the jet-manifold $J(\Omega;\mathbb{R}^m)$ is connected and star-shaped. This homotopy operator proves to be nilpotent $H^2 = 0$, moreover, there is a homotopy invariance formula

$$\hat{d}H + H\hat{d} = I - s^*_{u_0}, \tag{4}$$

where s_{u_0} is an injection to a specific solution $u_0 \in M$.

Similar to [11,12,25,26] one can make use of the homotopy operator (3) to define the set of so called "*antiexact*" forms

$$\mathcal{A} := Ker(H) \subset \Lambda_*(J(\Omega;\mathbb{R}^m)), \tag{5}$$

likewise exact forms $\mathcal{E} := Ker(\hat{d})$. Suitably, the space of vertical forms can be decomposed into the direct sum as $\Lambda_*(J(\Omega;\mathbb{R}^m)) = \mathcal{E} \oplus \mathcal{A}$. In addition, one can define projector operators [11,12,25,26],

$$\hat{d}H \; : \; \Lambda_*(J(\Omega;\mathbb{R}^m)) \to \mathcal{E} \subset \Lambda_*(J(\Omega;\mathbb{R}^m)), \tag{6}$$
$$H\hat{d} \; : \; \Lambda_*(J(\Omega;\mathbb{R}^m)) \to \mathcal{A} \subset \Lambda_*(J(\Omega;\mathbb{R}^m)), \tag{7}$$

for which $\left(\hat{d}H\right)^2 = \hat{d}H, \left(H\hat{d}\right)^2 = H\hat{d}$ on $\Lambda_*(J(\Omega;\mathbb{R}^m))$.

Proceeding to a variational representation of a priori given set of equations $\mathcal{E}[u] \subset J_u(\Omega;\mathbb{R}^m)$, $u \in M$, over a domain $\Omega \subset \mathbb{R}^n$, one can redefine its lack of variationality in terms of the related non-zero antiexact part: namely, the associated functional one-form (1) can not be represented as a vertical differential $\hat{d}\mathcal{L}$ for some Lagrangian $\mathcal{L} : J(\Omega;\mathbb{R}^m) \to \mathbb{R}$, if the related antiexact part is nontrivial, that is $H\hat{d}\alpha[u] \neq 0, u \in M$.

The paper is organized as follows: in the next section we present a hybrid varaitional problem that is based on antiexact forms, then next section provides some another ways to treating the inverse variational problem by means of introducing some auxiliary a priori Lagrangian one-forms reduced on suitably constructed submanifolds via the corresponding Dirac type constraints.

2. General Approach to a Hybrid Variation Problem

In general, an arbitrary smooth differential system $\mathcal{E} \subset J(\Omega;\mathbb{R}^m)$ can be rewritten as a functional 1-form

$$\alpha[u] = \langle \mathcal{E}[u] | du \rangle, \tag{8}$$

which is naturally decomposable into the direct sum components as

$$\alpha[u] = \hat{d}H\alpha[u] \oplus H\hat{d}\alpha[u], \tag{9}$$

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within which the density

$$\mathcal{L}[u] := H\alpha[u], \tag{10}$$

is called a *quasi-Lagrangian*. The splitting (9) can be rewritten as

$$\alpha[u] = \hat{d}\mathcal{L}[u] \oplus \beta[u], \tag{11}$$

where, by definition, $\beta[u] := H\hat{d}\alpha[u], u \in M$. We can then interpret the identity (11) in the following way: the differential system under regard can be considered as a solution for the following '*optimization*' problem:

$$0 = i_X\alpha[u] = i_X\hat{d}\mathcal{L} \oplus i_X\beta[u], \tag{12}$$

for vector fields $X \in \Gamma(J(\Omega;\mathbb{R}^m))$. As the condition (12), in general, is not solvable for all vector fields $X \in \Gamma(J(\Omega;\mathbb{R}^m)$, it is natural to reduce the whole system $\alpha[u] = 0$ on the functional submanifold

$$M_\beta := \{u \in M : \beta[u] = 0\}. \tag{13}$$

The obtained this way differential system

$$0 \sim \alpha[u] = \hat{d}\mathcal{L}|_{M_\beta} = 0 \tag{14}$$

becomes *a priori* Lagrangian on the functional submanifold $M_\beta \subset M$, as

$$\alpha[u]|_{M_\beta} = \hat{d}\mathcal{L}_\beta, \tag{15}$$

where the Lagrangian $\mathcal{L}_\beta := \mathcal{L}|_{M_\beta} \in \Lambda^0(M_\beta)$ is well defined on M_β. The latter partially solves the inverse problem of variational representation for the one-form $\alpha \in \tilde{\Lambda}^1(J(\Omega;\mathbb{R}^m))$, suitably reduced on the functional submanifold $M_\beta \subset \mathcal{S}(\Omega;\mathbb{R})$. A slight modification of this scheme, based on the hybrid variational analysis, is worked out below.

2.1. The Reduction Scheme and a Related Hybrid Variational Problem

Let us consider a smooth differential system $\mathcal{E}[u] \subset J_u(\Omega;\mathbb{R}^m)$, $u \in M$, on the jet -manifold $J(\Omega;\mathbb{R}^m)$ and analyze its virtually assumed Lagrangian structure, that is the existence of such a smooth mapping $\mathcal{L} : J(\Omega;\mathbb{R}^m) \to \mathbb{R}$, that this differential system $\mathcal{E}[u]$ is equivalent to the gradient $\mathrm{grad}\mathcal{L}[u] \subset T^*_u(M)$, which can be written down in the simplest case as

$$\mathrm{grad}\mathcal{L}[u] = A[u]\mathcal{E}[u], \tag{16}$$

where one assumes that $\mathcal{E}[u] \in T^*_u(M)$ and $A[u] \in End(T^*_u(M))$, $u \in M$, is some nondegenerate operator endomorphism of the cotangent space $T^*(M)$. In the general case it is also well known that the problem under regard is not practically resolvable, if the differential system $\mathcal{E}[u] = F\,(\mathrm{grad}\mathcal{L}[u])$ for some nonlinear and nondegenerate mapping $F : T^*_u(M) \to T^*_u(M)$. Yet, if we are interested in representing a given differential system $\mathcal{E} \subset J(\Omega;\mathbb{R}^m)$ within some kind of a *hybrid* variational formalism, one can try to express the relationship (16) in the differential geometric language on the jet-manifold $J(\Omega;\mathbb{R}^m)$ as a differential form

$$a[u] := \langle A[u]\mathcal{E}[u]|du\rangle = \hat{d}\mathcal{L}[u] \oplus H\hat{d}\,\langle A[u]\mathcal{E}[u]|du\rangle, \tag{17}$$

where $\mathcal{L}[u] := H\langle A[u]\mathcal{E}[u]|du\rangle \in \Lambda^0_*(J_u(\Omega;\mathbb{R}^m))$, a mapping $H : \Lambda^1_*(J_u(\Omega;\mathbb{R}^m)) \to \Lambda^0_*(J_u(\Omega;\mathbb{R}^m))$ at $u \in M$ denotes the usual [1,11,12,25,26,44] Poincare homotopy operator and $\langle\cdot|\cdot\rangle$ is the natural bi-linear

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form on $T^*(M) \times T(M)$. The representation (17) makes it possible to write down the following hybrid variational problem

$$u = \arg \inf_{u\, \in\, M_\beta} \left(\int_\Omega d^n x \mathcal{L}[u] \right) \tag{18}$$

on a functional submanifold $M_\beta \subset M$, defined via the functional relationship

$$M_\beta := \{ u \in M : \beta[u] := \ H\hat{d}\ \langle A[u]\mathcal{E}[u]|du\rangle = 0 \}. \tag{19}$$

The latter easily gives rise to the gradient relationship

$$\left(\mathrm{grad}\mathcal{L}_\beta[u] |\ | X \right) = 0 \tag{20}$$

for the Lagrangian $\mathcal{L}_\beta[u] := \mathcal{L}[u]|_{M_\beta}$ and all $X \in T(M_\beta)$, completely equivalent to that of (17), reduced on the submanifold $M_\beta \subset M$. Thus, one can formulate the following proposition.

Proposition 1. *Any smooth differential system $\mathcal{E}[u] \subset T^*(M), u \in M,$ on the jet -manifold $J(\Omega; \mathbb{R}^m)$ admits the hybrid variational representation*

$$u = \arg \inf_{u \in M_\beta} \left(\int_\Omega d^n x \ \mathcal{L}[u] \right) \tag{21}$$

on the functional submanifold $M_\beta \subset M$, defined by the relationship (19).

As a simple example one can consider the Burgers type dissipative evolution equation

$$\mathcal{E}[u] \sim u_t - u_{xx} + vuu_x = 0, \tag{22}$$

on the jet-manifold $J(\mathbb{R}^2; \mathbb{R})$ for a function $u \in M_u$, where a parametric function $v \in M_v$ satisfies the adjoint evolution equation

$$v_t + v_{xx} + uvv_x = 0 \tag{23}$$

on the related jet-manifold $J(\mathbb{R}^2; \mathbb{R})$. Having taken the nondegenerate operator endomorphism $A[u, v] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in End(T^*(M))$, where $M := M_u \times M_v$, we can easily check that the submanifold $M_\beta = M$ is defiend by the differential form $\beta[u] := H\hat{d}\langle A[u]\mathcal{E}[u]|du\rangle = 0$, vanishing identically. The latter makes it possible to state that the corresponding hybrid variational interpretation (21) becomes a true variational problem for the combined Burgers type system (22) and (23).

2.2. An optimal control problem aspect and the related Dirac type reduction scheme

As a typical example, let us consider a dynamical system

$$v_t = K[v] \tag{24}$$

on a toric functional manifold $M_v \subset C(\mathbb{T}^p; \mathbb{R}^m)$, which *a priori* is not of variational type, make its smooth functional parametrical extension

$$v_t = K[v, u], \quad K[v, u]|_{\beta[u,v]=0} = K[v] \tag{25}$$

with respect to a toric functional variable $u \in M_u \subset C(\mathbb{T}^p; \mathbb{R}^n)$ for some smooth differential functional relationship $\beta[u, v] = 0$ on the product $M_v \times M_u$, and pose the following Bellman-Pontriagin type

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optimal control problem [3,31] subject to some smooth Lagrangian density $\mathcal{L} : J_{(v,u)}(\mathbb{T}^p;\mathbb{R}^2)$ on a temporal interval $[0,T] \subset \mathbb{R}_+$:

$$v = \arg\inf_{v\in M_v} \int_0^T dt \int_{\mathbb{T}^p} \mathcal{L}[v,u] d^p x, \tag{26}$$

for a fixed $u \in M_u$ under the condition that the evolution flow (25) possesses a smooth conserved quantity $\gamma = \int_{\mathbb{T}^p} \gamma[v,u]d^p x \in \mathcal{D}(M_v \times M_u)$, that is $d\gamma/dt = 0$ on the combined manifold $M_v \times M_u$ for all $t \in [0,T]$. The latter, in particular, means that we need to determine such an additional evolution flow

$$u_t = F[v,u] \tag{27}$$

on the extended control manifold M_u, which will ensure the existence of the mentioned above smooth conserved quantity $\gamma \in \mathcal{D}(M_v \times M_u)$. The problem above is solved [31] by means of construction of the extended Lagrangian functional

$$\begin{aligned} \mathrm{L}[v,\psi] \quad : \quad = \int_0^T dt \int_0^{2\pi} (\mathcal{L}[v,u] + \langle \psi | (v_t - K[v,u], v_t - F[v,u])^{\intercal} \rangle \; + \\ + \langle \mu(x,t) | \partial \mathrm{grad} \gamma[u,v]/\partial t \rangle + \; \langle \mathrm{grad} \gamma[u,v] | (K[v,u], F[v,u])^{\intercal} \rangle) d^p x, \end{aligned} \tag{28}$$

supplemented with Lagrangian multipliers $\mu \in C_0^1(\mathbb{T}^p \times [0,t];\mathbb{R}^2)$ and $\psi \in C_0^1([0,T]; T^*(M_v \times M_u))$ almost everywhere with respect to the temporal parameter $t \in [0,T]$, and next determining its critical points:

$$\begin{aligned} \text{ffi}\mathrm{L}[v;\mu,\psi] = 0 \sim \mathrm{grad}\mathcal{L}[v,u] \; - \; \psi_t - \\ -(K[v,u], F[v,u])^{\intercal',*} \psi \; + \\ + \; \partial \mathrm{grad}\gamma/\partial t + \; (K[v,u], F[v,u])^{\intercal',*} \mathrm{grad}\gamma = 0 \end{aligned} \tag{29}$$

for all $(v,u) \in M_v \times M_u$ jointly with the condition $d\gamma/dt = 0$ for $t \in [0,T]$. The obtained functional relationship (29) under the condition $\mu(0) = 0 = \mu(T)$ reduces to the following generalized Noether-Lax condition

$$(K[v,u], F[v,u])^{\intercal',*} \psi = \mathrm{grad}\mathcal{L}[v,u] \tag{30}$$

on the Lagrangian multiplier $\psi \in C_0^1([0,T]; T^*(M_v \times M_u))$, as the following Noether-Lax equality

$$\partial \mathrm{grad}\gamma/\partial t + \; (K[v,u], F[v,u])^{\intercal',*} \mathrm{grad}\gamma = 0 \tag{31}$$

holds *a priori* for any smooth conservation law $\gamma \in \mathcal{D}(M_v \times M_u)$ of the joint dynamical system

$$v_t = K[v,u], \; u_t = F[v,u] \tag{32}$$

on the combined manifold $M_v \times M_u$.

A solution $\psi \in T^*(M_v \times M_u)$ to the condition (30) allows the unique representation as the direct sum $\psi = \bar{\psi} \oplus \varphi$ of its skew symmetric $\bar{\psi} \in T^*(M_v \times M_u)$ and strictly symmetric $\varphi \in T^*(M_v \times M_u)$ components, satisfying, respectively, the following differential-functional equations:

$$\bar{\psi}_t + (K[v,u], F[v,u])^{\intercal',*} \bar{\psi} = \mathrm{grad}\; \mathcal{L}[v,u], \tag{33}$$

where, by definition, $\bar{\psi}' \neq \bar{\psi}'^{,*}$ on $M_v \times M_u$, and

$$\varphi_t + (K[v,u], F[v,u])^{\intercal',*} \varphi = 0, \tag{34}$$

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where, by definition, $\varphi' = \varphi'^{,*}$ on $M_v \times M_u$ for all $u \in M_u$. Under the *a priori* assumed condition that the evolution flow (32) is a Hamiltonian system on the functional manifold M_v with respect to the related symplectic structure mapping $\Omega : T(M_v \times M_u) \to T^*(M_v \times M_u)$, the differential-functional equation (33) is always [1,2,4] solvable, giving rise to the known differential-geometric relationship

$$\Omega = \bar{\psi}' - \bar{\psi}'^{,*}, \tag{35}$$

subject to which the following compatible vector field representation

$$(K, F)^\intercal = -\Omega^{-1}\text{grad}\,[(\bar{\psi}|(K,F)^\intercal) - \mathcal{L}] \tag{36}$$

holds on $M_v \times M_u$. Simultaneously, the differential-functional equation (34) is also always [1,2,4] solvable under the condition that $\varphi = \text{grad}\,\gamma \in T^*(M_v \times M_u)$ for some conserved quantity $\gamma \in D(M_v \times M_u)$ of the evolution flow (25) regardless of whether the evolution flow (25) on M_v is Hamiltonian or not. The latter means, evidently, that it is also of variational type, which can be suitably reduced via the Dirac scheme on the functional submanifold

$$M_F := \{(v,u) \in M_v \times M_u : \beta[u,v] = 0, \beta'_u F[v,u] + \beta'_v K[v,u] = 0\}, \tag{37}$$

concerving its variational type on $M_F \subset M_v \times M_u$, following from the stated above Hamiltonian representation (36).

2.3. Example: Burgers Equation

Concerning the diffusion type Burgers equation example, considered before,

$$v_t = v_{xx} + 2vv_x := K[v] \tag{38}$$

on the functional manifold $M_v \subset C(\mathbb{R};\mathbb{R})$, treated within the optimal control problem scheme above, there is suggested the following way of embedding the flow (38) into the Dirac type constrained variariational picture:

1. we parametrically extend the flow (38) by means of the simple relationship $\beta[u,v] = uv - v = 0$ as
$$v_t = v_{xx} + 2(uv)v_x := K[v,u], \quad K[v,u]|_{\beta[v,u]=0} = K[v], \tag{39}$$
and pose the optimal control problem for the flow (39): to detect such an evolution flow
$$u_t \stackrel{?}{=} F[v,u], \tag{40}$$
on the functional parameter $u \in M_u \subset C(\mathbb{R};\mathbb{R})$, under which the joint dynamical system
$$\begin{aligned} v_t &= v_{xx} + 2uvv_x, \\ u_t &\stackrel{?}{=} F[v,u], \end{aligned} \tag{41}$$
becomes Hamiltonian on the combined functional manifold $M_v \times M_u$;
2. having solved the optimal problem above, we apply to the obtained Hamiltonian system (41) the Dirac type reduction on the functional submanifold $M_F := \{(v,u) \in M_v \times M_u : uK + vF - K = 0\}$, turning back the joint dynamical system (41) to its previous form (38), yet already upon the functional submanifold $M_F \subset M_v \times M_u$;
3. as a result, owing to the fact that the reduced on the submanifold $M_F \subset M_v \times M_u$, dynamical system (38) persists to be Hamiltonian too, it will represent a true Burgers dynamical system as the one *a priori* representable in the variational Lagrangian form on this submanifold.

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Subject to this Burgers dynamical system (38) the analytic scheme above gives rise to the following evolution flow

$$v_t = -v_{xx} + 2vu_xu = F[v,u] \tag{42}$$

on the parametric functional manifold M_u, which jointly with the flow (38) represents the following Hamiltonian system:

$$\left.\begin{array}{l} v_t = v_{xx} + 2uvv_x, \\ u_t = -u_{xx} + 2uvu_x \end{array}\right\} = -\Omega^{-1}\text{grad } H[v,u], \tag{43}$$

where $\Omega : T(M_v \times M_u) \to T^*(M_v \times M_u)$ is the corresponding canonical symplectic structure mapping on $M_v \times M_u$:

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{44}$$

and $H \in \mathcal{D}(M_v \times M_u)$ is the related to it conserved Hamiltonian function:

$$H = \int_0^{2\pi} dx[u_xv_x + (v^2uu_x - u^2vv_x)/2]. \tag{45}$$

Having now applied the classical Dirac type reduction scheme to the Hamiltonian system (43) upon the submanifold $M_F := \{(v,u) \in M_v \times M_u : uK + vF = K\}$, one obtains a true Burgers dynamical system (38) as a Hamiltonian system on this submanifold $M_F \subset M_v$, *a priori* possessing, respectively, the related variational Lagrangian representation. To demonstrate this property, we will make use of the fact that the obtained Hamiltonian system possesses [4,5,34,35] a countable hierarchy of functionally independent conserved quantities $\gamma_j \in D(M_v \times M_u), j \in \mathbb{Z}_+$, amongst them the Hamiltonian (45), whose functional gradients are calculated analytically via the recursion scheme:

$$\text{grad}\gamma_j[u,v] = \Lambda^j \text{grad}H[u,v], \tag{46}$$

where the gradient recursion operator $\Lambda : T^*(M_v \times M_u) \to T^*(M_v \times M_u)$ is given by the following integro-differential operator expression:

$$\Lambda = \begin{pmatrix} -\partial - u\partial^{-1}v_x + \partial u\partial^{-1}v & u^2 - u\partial^{-1}u_x - \partial u\partial^{-1}u \\ v^2 - v\partial^{-1}v_x - \partial v\partial^{-1}v & \partial - v\partial^{-1}u_x + \partial v\partial^{-1}u \end{pmatrix}. \tag{47}$$

Having calculated the conserved gradient expression $\text{grad}\gamma_1[u,v] = \Lambda\text{grad}H[u,v] \in T^*(M_v \times M_u)$:

$$grad\gamma_1[v,u] = \begin{pmatrix} -u_{3x} + 2(uu_xv)_x + (\partial u\partial^{-1}v - u\partial^{-1}v_x)(u_{xx} - 2uu_xv) + \\ +u^2v_{xx} + 2u^3vv_x - u\partial^{-1}u_x(v_{xx} + 2vv_xu) - \\ -\partial u\partial^{-1}vuv_{xx} - 2\partial u\partial^{-1}(u^2vv_x) \\ \\ v^2u_{xx} - 2uu_xv^3 - v\partial^{-1}v_xu_{xx} + 2v\partial^{-1}v_xvuu_x - \\ -\partial v\partial^{-1}vu_{xx} + 2\partial v\partial^{-1}v^2uu_x + v_{3x} + \\ +2(uvv_x)_x - v\partial^{-1}u_xv_{xx} - 2v\partial^{-1}u_xuvv_x + \\ +\partial v\partial^{-1}uv_{xx} + 2\partial v\partial^{-1}u^2vv_x \end{pmatrix} \tag{48}$$

and taken into account that it is invariant with respect to the vector field (43) and ensuing from the linear Noether-Lax relationship

$$\frac{\partial}{\partial t}grad\gamma_1[v,u] + K'^{,*}[v,u]grad\gamma_1[v,u] = 0, \tag{49}$$

one can proceed to the invariant reduction of the Burgers evolution flow (38) upon the functional submanifold $M_F \subset M_v \times M_u$. Preliminarily, we need to take the invariant Lagrangian function

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density $\mathcal{L}_1 := \gamma_1 + cH + c_0 uv$ and reduce our Hamiltonian flow (43) on the 6-dimensional invariant submanifold

$$M_1 := \{(u,v) \in M_v \times M_u : \text{grad}\mathcal{L}_1[u,v] = 0\} \sim J^2(\mathbb{R}/\{2\pi\mathbb{Z}\};\mathbb{R}^2),$$

as a flow on $J^2(\mathbb{R}/\{2\pi\mathbb{Z}\};\mathbb{R}^2) \subset M_v \times M_u$, taking into account [4,14–16] the classical Gelfand-Dickey relationship:

$$d\mathcal{L}_1[u,v] = \langle \text{grad}\mathcal{L}_1[v,u] | (dv, du)^\intercal \rangle + d\alpha^{(1)}[v,u]/dx, \tag{50}$$

determining on the submanifold M_1 the nondegenerate symplectic structure $\omega^{(2)} = d\alpha^{(1)}[v,u] \in \Lambda^2(M_1)$. From (50) one easily ensues that the Hamiltonian flow (43) on the submanifold M_1, being Hamiltonian with respect to the constructed above symplectic structure $\omega^{(2)} \in \Lambda^2(M_1)$, can be reduced via the Dirac scheme upon the submanifold

$$M_{1,F} := \{(v,u) \in M_1 : uK + vF = K\}, \tag{51}$$

thus reducing it to the initial Burgers flow (42), equivalently representing it as a Lagrangian variational problem on the jet-submanifold $M_{1,F} \subset M_1 \sim J^2(\mathbb{R}/\{2\pi\mathbb{Z}\};\mathbb{R}^2)$.

3. Schwinger's Variational Principle as a Constrained Problem

In this section we study Schiwnger's 'third way' of formulating variational problem [10,28,41]. In this approach we make independent variations of field, its tangent and cotangent components. At first it seems that one requires space $T(M) \times T^*(M)$ for this variation, however, one use only $T(M)$ and a constraint.

Consider a functional manifold M and a Lagrangian

$$\mathcal{L} : \mathbb{R} \times (M \times T(M)) \to \mathbb{R}, \tag{52}$$

representable as smooth mapping $\mathcal{L}[t,\phi,\dot\phi]$ in local coordinates $(t,\phi,\dot\phi) \in \mathbb{R}\times(M \times T(M))$. Its Schwinger extension is defined as a mapping $\mathcal{L} \to \mathcal{L}_S$, where $\mathcal{L}_S : \mathbb{R}\times(M \times T(M)) \times T^*(M) \to \mathbb{R}$ is some analytical expression, whose simplest form looks as

$$\mathcal{L}_S(t,\phi,\dot\phi;p)) = \langle p|(\dot\phi - v)\rangle + \mathcal{L}[\phi,v]. \tag{53}$$

where variables $\phi \in M$, $v \in T(M))$ and $p \in T^*(M)$ are assumed to be independent. Then the following proposition holds.

Proposition 2. *The least action variation of the Schwinger's Lagrangian functional extension (53) is equivalent to that of the Lagrangian functional (52).*

Proof. In $\mathcal{L}$ replace $\dot\phi$ by an arbitrary element $v \in \Gamma(TM)$, and introduce a vector of Lagrange multipliers $\pi = \{\pi^i\}$ for the constraint $\dot\phi = v \in T(M)$. This gives (53).

From the variation with respect to ϕ, v and the multipliers p independently we obtain

$$\tfrac{d}{dt}p - \tfrac{\partial\mathcal{L}}{\partial\phi} = 0, p = \tfrac{\partial\mathcal{L}}{\partial v}, \dot\phi = v. \tag{54}$$

This gives Euler-Lagrange equations for the density $\mathcal{L}$ and the definition of momentum π. □

One can see that $p \in T^*(M)$, by construction, is the canonical momentum playing the role of a Lagrange multiplier subject to the tangent element $v \in T(M)$. Since the Hamiltonian function is defined as $\mathcal{H}[\phi,p] = \langle p|\dot\phi\rangle - \mathcal{L}[\phi,v])|_{p=\delta\mathcal{L}[\phi,v]/\delta v}$ modulo the determining relationship

$$\delta\mathcal{L}_S[t,\phi,v;p]/\delta v = 0 \sim p = \delta\mathcal{L}[t,\phi,v]/\delta v,$$

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one gets right away the classical Hamiltonian equations

$$dx/dt = \delta\mathcal{H}[t,\phi,p]/\delta p, \quad dp/dt = -\delta\mathcal{H}[t,\phi,p]/\delta\phi.$$

Turn back now to our problem of Lagrangian representation of a given evolution equation

$$K[v, v_t] \; = 0 \tag{55}$$

on a jet-manifold $J(\mathbb{R}; M)$, whose Lagrangian form is either not known or not existing on the whole. To suggest a partial solution to this problem, one can consider a close enough to (55) Lagrangian evolution equation

$$\tilde{K}[v, v_t] \; = 0$$

on a jet-manifold $J(\mathbb{R}; M)$, whose invariant reduction on the functional submanifold

$$M_\beta := \{v \in M \; : \beta[v] = 0, \; \langle \beta'[v] | v_t \rangle = 0\}, \tag{56}$$

defined by the *evolution invariant* constraints $\beta[v] = 0 \in \Lambda$, $\langle \beta'[v]|w\rangle = 0$, will coincide with the given evolution equation (55). This means that there exists some smooth extended Schwinger type functional

$$\mathrm{L}_{\lambda,\mu}[v, w; p] := \int_0^T dt \int_\Omega [\langle p|v_t - w\rangle + \mathcal{L}[v, w]d^n x + (\lambda|\beta[v]) + \big(\mu|\langle\beta'[v]|w\rangle\big), \tag{57}$$

on the whole manifold M, for which the least action condition

$$\delta(\mathrm{L}_{\lambda,\mu}[v, w; p] + (\lambda|\beta[v]) + \big(\mu|\langle\beta'[v]|w\rangle\big)) = 0 \tag{58}$$

with respect to variables $(v, p; w) \in M \times T^*(M) \times T(M)$ and the corresponding Lagrangian multipliers $(\lambda, \mu) \in \Lambda^* \times T^*(\Lambda)$ reduces on the submanifold $M_\beta \subset M$ to the given evolution equation (55). This means that on the submanifold $M_\beta \subset M$

$$\begin{gathered} p = \; \mathrm{grad}_w \mathcal{L}[v, w] + \beta'[v]^* \mu, \\ p_t + \mathrm{grad}_v \mathcal{L}[v, w] + \beta^{',*}[v]\lambda[v, p] + \langle \beta'^{*,\prime}[v] | (\mu, w)\rangle = 0. \end{gathered} \tag{59}$$

The latter makes it possible to deduce from (59) the multiplier

$$\mu[v, w; p] = \; \beta^{'}[v]^{*,-1}(p - \mathrm{grad}_w \mathcal{L}[v, w]) \; \in T^*(\Lambda) \tag{60}$$

and, suitably, the next multiplier $\lambda \in \Lambda^*$. Thus, one can formulate the following proposition.

Proposition 3. *A given evolution equation (55), reduced on the invariant functional submanifold (56), allows Lagrangian representation (58), specified by means of the functional parameter (60).*

4. Conclusions

We proposed some ways of formulation of the variational problem for problems that are not variational. One uses antiexact forms to construct a constraints for space of all possible variations. The other approach extends the number of variables and equations to make the new system variational and then reduce the extended jet space to submanifold that vanish these additional variables. Still, the algorithmic way of such extension is to be found.

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Appendix A Dirac Constraints

To unify all notions related to Dirac's theory of constraints, we give a short summary. We will base on a few resources [1,19,27,37].

We consider a finite-dimensional symplectic manifold $(P, \omega^{(2)})$ $\dim(P) = 2n < \infty$, with respect to the symplectic form $\omega^{(2)} \in \Lambda^2(P)$, endowed with a set of smooth functional constraints $\{\phi_i : P \to \mathbb{R} : \phi_i = 0 : i = \overline{1,p}, p < n\}$. These constraints naturally determine the smooth submanifold $N \subset P$ under condition of independence of these constraints, i.e., $d\phi_1 \wedge \ldots \wedge d\phi_p|_N \neq 0$.

Any vector field $X \in T(P)$, defined by a smooth mapping $f : P \to \mathbb{R}$ via the relationship $i_X \omega^{(2)} = -df$ and called *Hamiltonian*, can be projected to the vector $\bar{X} \in T(N)$ on the invariant submanifold $N \subset P$, defined by the projected Hamiltonian $\bar{H}$ by $i_{\bar{X}} \omega^{(2)} = -d\bar{H}$. In particular, this mapping projects any trajectory $c \in T(P)$ onto $\bar{c} \in T(N)$, as presented in Fig. A1.

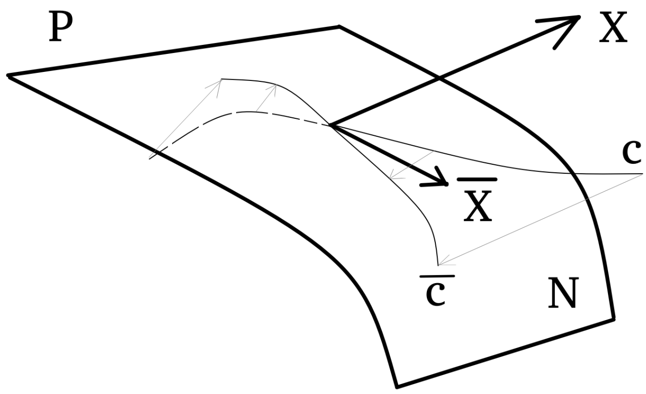

Figure A1. The projection $\bar{X}$ of X from the symplectic space P onto a submanifold N. Then the trajectory c is also projected onto the trajectory $\bar{c}$ in N.

The first step is to find the projection $X|_N := \bar{X} \in T(N)$ on the submanifold $N \subset M$, defined by the conditions $\langle d\phi_i | \bar{X} \rangle |_N = 0, i = \overline{1,p}$. Thus, one can write down the decomposition

$$X = \sum_{j=\overline{1,p}} \lambda_j \nabla_s \phi_j + \bar{X}, \tag{A1}$$

where $\lambda_j \in \mathbb{R}, j = \overline{1,p},$ are so called Lagrange multipliers and $\nabla_s \phi_j \in T(P), j = \overline{1,p}$, are the related symplectic skew-gradient vector fields, connected with the symplectic structure $\omega^{(2)} \in \Lambda^2(P)$ via the relationship $i_{\nabla_s \phi_j} \omega^{(2)} = -d\phi_j$ for $j = \overline{1,p}$. Choosing now $X := \nabla_s f \in T(P)$ as generated by a smooth function $f : P \to \mathbb{R}$, and taking into account the definition of its Poisson bracket $\{f, h\} := dh(\nabla_s f)$ with an arbitrary smooth function $h : P \to \mathbb{R}$, we easily obtain that

$$\{f, \phi_k\} = -\sum_{j=\overline{1,p}} \{\phi_k, \phi_j\} \lambda_j \tag{A2}$$

for every $k = \overline{1,p}$. If the matrix $\Phi := \{ \{\phi_i, \phi_j\} : i, j = \overline{1,n}\}$ is nondegenerate on the submanifold $N \subset P$, it allows to determine the vector $\lambda \in \mathbb{R}^p$ of the Lagrange multipliers as

$$\lambda = -\Phi^{-1}\{f, \phi\}, \tag{A3}$$

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where $\phi := (\phi_1, \phi_2, ..., \phi_p) : P \in \mathbb{R}^p$ is the related vector of the so called second order constraints functions. Having substituted the result (A3) into the decomposition (A1) and taken its convolution with any differential $dh \in \Lambda^1(N)$, one derives finally the classical Dirac bracket expression

$$\{f,h\}_D = \{f,h\} - \{f,\phi\}\Phi^{-1}\{\phi,h\}|_N \tag{A4}$$

on the submanifold $N \subset P$ for arbitrary smooth functions $f, h : P \to \mathbb{R}$, reduced on the submanifold $N \subset P$.

If we are interested in studying the evolution of a specially chosen vector field $\nabla_s H \in T(P)$, reduced on the invariant submanifold $N \subset P$, the condition $\{H,\phi\}|_N = 0$ should be *a priori* satsified. If it is not a case, that is $\{H,\phi_{j_k}\} = \psi_k|_N \neq 0$, $k = \overline{1,r}$, yet already $\{H,\psi_k\}|_N = 0$ and $\det \Phi_r|_{N_r} \neq 0$, where $\Phi_r := \{(\phi,\psi),(\phi,\psi)\}$, there should be considered the Hamiltonian flow $\nabla_s H \in T(P)$ subject to the Poisson bracket (A4)

$$\partial f/\partial t = \{H,f\} - \sum_{i,j=\overline{1,n}} \{H,\phi_i\}\Phi^{-1}\{\phi_j,f\}. \tag{A5}$$

reduced already on the submanifold $N_r := \{(\phi,\psi) = 0\} \subset N$. If additional constraints $\psi : P \to \mathbb{R}^k$ satisfy the conditions $\psi = (\phi,\psi)C$ for some constant matrix $C \in \mathrm{Hom}(\mathbb{E}^{p+r};\mathbb{E}^r)$, they are called the *first class constraints* and should not be taken into account for constructing the reduced Dirac bracket (A4). The *first class constraints* are responsible for so called gauge transformations. Adding the secondary constraints and reiterating the algorithm, the latter stops at some point. These and other related questions are discussed in detail in physics-oriented books [19,37].

References

1. R. Abraham, J. Marsden, *Foundations of Mechanics*, Second Edition, Benjamin Cummings, NY, 1984
2. V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, NY, 1989.
3. R.E. Bellman, *Dynamic Programming*, Dover, 2003
4. D. Blackmore, A.K. Prykarpatsky and V.H. Samoylenko,*Nonlinear Dynamical Systems of Mathematical Physics*, World Scientific, NJ, 2011.
5. Prykarpatsky, Y.A.; Urbaniak, I.; Kycia R.A.; Prykarpatski A.K. *Dark Type Dynamical Systems: The Integrability Algorithm and Applications*, Algorithms 2022, 15, 266. https:// doi.org/10.3390/a15080266.
6. R. Aldrovandi, R.A. Kraenkel, *On exterior variational calculus,* J. Phys. A: Math. Gen. 21 1329 (1988); DOI: 10.1088/0305-4470/21/6/010
7. R. Aldrovandi, J.G. Pereira, *An Introduction to Geometrical Physics*, 2nd edition, World Scientific 2016; Chapter 20
8. I.M Anderson, *Variational Bicomplex*, unpublished script
9. I. Anderson, G. Thompson, *The inverse problem of the calculus of variations for ordinary differential equations*, 98, 473, Memoirs of the American Mathematical Society, 1992
10. W. Dittrich, *The Development of the Action Principle: A Didactic History from Euler-Lagrange to Schwinger*, Springer, 2021
11. D.G.B. Edelen, *Applied Exterior Calculus*, Dover Publications, Revised edition, 2011
12. D.G.B. Edelen, *Isovector Methods for Equations of Balance*, Springer, 1980
13. I.M. Gelfand, S.V. Fomin, *Calculus of Variations*, Dover Publications, 2000
14. I.M. Gelfand, L.A. Dickey, *Integrable nonlinear equations and Liouville's theorem*, Funct. Anal. Appl.13 (1979), 8–20
15. L.A. Dickey, *Integrable nonlinear equations and Liouville's theorem*, I. Communications in Mathematical Physics, Commun. Math. Phys. 83 (1981), 345–360
16. L.A. Dickey, *Integrable nonlinear equations and Liouville's theorem, II*, Communications in Mathematical Physics, 82 (1981), 361–375
17. M. Giaquinta, S. Hildebrandt, *Calculus of Variations*, 2 vols. Springer 2010
18. H. Helmholtz, *Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung*, J. Reine Angew. Math. 137 (1887)
19. M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, 1994

20. I. Khavkine, *Presymplectic current and the inverse problem of the calculus of variations*, J. Math. Phys. 54, 111502 (2013); DOI: https://doi.org/10.1063/1.4828666
21. D. Krupka, *Introduction to Global Variational Geometry*, Atlantis Press, 2015
22. D. Krupka, *Global variational theory in fibred spaces*, in Handbook of Global Analysis, Elsevier 2007
23. O. Krupková, G.E. Prince, *Second Order Ordinary Differential Equations in Jet Bundles and the Inverse Problem of the Calculus of Variations*, in Handbook of Global Analysis, Elsevier 2007
24. O. Krupkova, *The Geometry of Ordinary Variational Equations*, Springer 1997
25. R.A. Kycia, *The Poincare Lemma, Antiexact Forms, and Fermionic Quantum Harmonic Oscillator*, Results Math 75, 122 (2020); DOI: 10.1007/s00025-020-01247-8
26. R.A. Kycia, *The Poincare lemma for codifferential, anticoexact forms, and applications to physics*, Results Math 77, 182 (2022); DOI: https://doi.org/10.1007/s00025-022-01646-z
27. J.E. Marsden, T.S. Ratiu, *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*, Springer, 2nd edition 2023
28. K. A. Milton, *Schwinger's Quantum Action Principle: From Dirac's Formulation Through Feynman's Path Integrals, the Schwinger-Keldysh Method, Quantum Field Theory, to Source Theory*, Springer, 2015
29. Z. Muzsnay, G. Thompson, *Inverse problem of the calculus of variations on Lie groups*, Differential Geometry and its Applications, 23, 3, 257–281 (2005); DOI: https://doi.org/10.1016/j.difgeo.2005.05.002
30. P.J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd edition, Springer, 2000
31. L.S. Pontryagin,V.G. Boltyanski, R.S. Gamkrelidze and E.F. Mishchenko, *The Mathematical Theory of Optimal Processes*, Interscience, 1962.
32. T. Doa, G. Prince, *New progress in the inverse problem in the calculus of variations*, Differential Geometry and its Applications, 45, 148–179 (2016); DOI: https://doi.org/10.1016/j.difgeo.2016.01.005
33. G.E. Prince, D.M. King, *The inverse problem in the calculus of variations: nonexistence of Lagrangians*, F. Cantrijn, B. Langerock (Eds.), Differential Geometric Methods on Mechanics and Field Theory: Volume in Honour of Willy Sarlet, Academia Press, Gent, 131–140 (2007)
34. A.K. Prykarpatski, P.Y. Pukach, M.I. Vovk, *Symplectic Geometry Aspects of the Parametrically-Dependent Kardar–Parisi–Zhang Equation of Spin Glasses Theory, Its Integrability and Related Thermodynamic Stability,* Entropy 2023, 25, 308. https://doi.org/10.3390/e25020308
35. A.K. Prykarpatski, V.A. Bovdi, *On the Lie-Algebraic Integrability of the Calogero-Degasperis Dynamical System and Its Generalizations,* Contemporary Mathematics, Contemporary Mathematics, 4(4) 2023, 750-768; http://ojs.wiserpub.com/index.php/CM/,
36. A.K. Prykarpatsky , I.V. Mykytiuk, *Algebraic Integrability of Nonlinear Dynamical Systems on Manifolds*, Springer 1998
37. H.J. Rothe, K.D. Rothe, *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, World Scientific Publishing Company 2010
38. R. M. Santilli, *Foundations of Theoretical Mechanics I*, Springer-Verlag Berlin Heidelberg 1978
39. D. J. Saunders, *The Geometry of Jet Bundles*, Cambridge 1989
40. D.J. Saunders, *Thirty years of the inverse problem in the calculus of variations*, Reports on Mathematical Physics, 66 1 43–53 (2010); DOI: https://doi.org/10.1016/S0034-4877(10)00022-4
41. J. Schwinger at al., *Classical Electrodynamics*, Westview Press, 1998
42. T. Tsujishita, *On variation bicomplexes associated to differential equations*, Osaka J. Math. 19 (1982), 311-363.
43. W.M. Tulczyjew, *The Euler-Lagrange resolution*, in Lecture Notes in Mathematics 836 -48, Springer-Verlag, 1980
44. M.M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Holden-Day 1964
45. A. Vinogradov, *A spectral sequence associated with a non-linear differential equation, and the algebro-geometric foundations of Lagrangian field theory with constraints*, Sov. Math. Dokl. 19 (1978) 144-148
46. R. Vitolo, *Variational sequences*, in Handbook of Global Analysis, Elsevier 2007
47. D. Zenkov, *The Inverse Problem of the Calculus of Variations*, Atlantis Press, 2015