Hypothesis

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An Original Presentation of an Old Theory: Lorentz Transformation Generalized to Allow for an Accelerating Coordinate System

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Essay

An Original Presentation of an Old Theory: Lorentz Transformation Generalized to Allow for an Accelerating Coordinate System

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Abstract: The traditional version of the Lorentz transformation is used to relate the coordinates of an event (a point in spacetime) between different inertial coordinate systems in the absence of any gravitational effects. It is limited to those cases in which both reference frames are inertial, so they have a constant velocity relative to each other. This report utilizes a familiar postulate, regarding the equivalence between coordinates measured in an accelerating reference frame and coordinates measured in a reference frame instantaneously at rest with the accelerating frame, together with the traditional version of the Lorentz transformation to derive a generalized version that allows one of the reference frames to be accelerating. This is not a new topic but the presentation and all derivations in this report are the author's own inventions. By defining suitable quantities and introducing suitable notation, the generalized version can be written in a way that is almost as simple as the traditional (constant velocity) version when calculating the coordinates of an event in an inertial system when given the coordinates in an accelerating system. Unfortunately, calculations of the inverse transformation, i.e., calculating the coordinates in the accelerating system when given coordinates in an inertial system, are more cumbersome. Worse yet, while a suitably selected history and future ensure the existence of an inverse transformation, there can exist spacetime points for which it is not unique. However, the metric tensor can be derived in the accelerating system for the general case and is included in this report. This is used to calculate time dilations and Doppler effects that are outside the scope of inertial coordinate systems.

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1. Introduction

The traditional version of the Lorentz transformation is used to relate the coordinates of an event (a point in spacetime) between different inertial coordinate systems in the absence of any gravitational effects. It is limited to those cases in which both reference frames are inertial, so they have a constant velocity relative to each other. This report utilizes a familiar postulate (stated below in (1.1)) together with the traditional version of the Lorentz transformation to derive a generalized version that allows one of the reference frames to be accelerating. This is not a new topic but the presentation and all derivations in this report are the author's own inventions. After starting with familiar first principles, the report is self-contained in that derivations are provided for all conclusions. This accounts for the scarcity of references. By defining suitable quantities and introducing suitable notation, the generalized version can be written in a way that is almost as simple as the traditional (constant velocity) version when calculating the coordinates of an event in an inertial system when given the coordinates in an accelerating system. Unfortunately, calculations of the inverse transformation, i.e., calculating the coordinates in the accelerating system when given coordinates in an inertial system, are more cumbersome. Worse yet, while a suitably selected history and future ensure the existence of an inverse transformation, there can exist spacetime points for which it is not unique. However, the metric tensor can be derived in the accelerating system for the general case and is included in this report. This is used to calculate time dilations and Doppler effects that are outside the scope of inertial coordinate systems.

This report frequently uses the phrase "as seen by an observer". In the context of this report, the word "seen" does not refer to a literal visual image. A visual image of an event (an "event" is a point in spacetime) is received by an observer after a time delay, the time required for a light signal sent by the event to reach the observer. In the context of this report, the phrase "as seen by an observer" does

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not refer to a visual image but, instead, means that the observer assigned spacetime coordinates to the event.[1] For example, the phrase "the observer sees a clock to be running slow" means that if the observer assigns a time coordinate to one tick of that clock, and another time coordinate to the next tick of that clock, the time between consecutive ticks is longer than the time between consecutive ticks of the observer's own clock.

This report considers two observers. One, called the home observer, uses a clock at rest at the origin of his coordinate system, which is an inertial coordinate system (recognized to be inertial by freely moving particles having constant velocities).[2] The other observer, called the traveler (with travel defined to be relative to home), uses a clock at the origin of his coordinate system, called the traveler's system. The traveler's system is not rotating but may have a translational acceleration relative to the home system. A spacetime point denoting the start of the traveler's journey has an arbitrary initial spatial translation relative to the home system and an arbitrary initial velocity in the home system, but the clocks are synchronized so that the traveler's time coordinate and home time coordinate of the start of the journey are both zero. The goal is to relate, for an arbitrary event, the traveler's coordinates of that event to the home coordinates of the same event.

The traveler's clock traces out a worldline as seen by the home observer. Information that is assumed to be known includes the three-dimensional (spatial) vector function, denoted $\boldsymbol{X}(t)$, which is a parameterization of the traveler's worldline expressing the spatial coordinates $\boldsymbol{x}$, in the home system, of a point on the traveler's worldline, in terms of the coordinate time t in the home system.[3] From this parametrization, the velocity of the traveler's clock as seen by the home observer is the function $\boldsymbol{V}(t)$ calculated from $\boldsymbol{V}(t) = d\boldsymbol{X}(t)/dt$.

All conclusions in this report are derived from two postulates. One, taken from [1] with minor paraphrasing, is the statement:

$$\left[\begin{array}{l}\text{Carefully constructed clocks (e.g., atomic clocks), when accelerated, will tick}\\ \text{at the same rate as unaccelerated clocks moving momentarily along with them,}\\ \text{and sufficiently rigid accelerated measuring rods will measure the same lengths}\\ \text{as unaccelrated rods moving momentarily along with them.}\end{array}\right] \tag{1.1}$$

The second postulate that all results in this report are derived from is the Lorentz transformation applicable to the case in which the traveler has a constant velocity $\boldsymbol{v}$ as seen by the home observer. This is presented in various formats in [2] through [8] but the format preferred for this work is taken from [9] and given as follows. Consider an event with home coordinates denoted $(\boldsymbol{x}_1, t_1)$ and traveler coordinates denoted $(\overline{\boldsymbol{x}}_1, \overline{t}_1)$, and a second event with home coordinates denoted $(\boldsymbol{x}_2, t_2)$ and traveler coordinates denoted $(\overline{\boldsymbol{x}}_2, \overline{t}_2)$. The standard (constant velocity) Lorentz transformation in [9] relates the displacement between these events expressed in each of the two systems according to [4]

$$t_2 - t_1 = \gamma\left(\overline{t}_2 - \overline{t}_1 + \frac{\boldsymbol{v} \circ [\overline{\boldsymbol{x}}_2 - \overline{\boldsymbol{x}}_1]}{c^2}\right) \tag{1.2a}$$

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[1] If the only information available to the observer with which to assign spacetime coordinates to an event is a visual image, it is assumed that the observer accounted for the travel time of the light signal when assigning these coordinates so that the assigned time coordinate does not include this travel time.

[2] Gravity is not considered here so we say freely moving instead of free falling.

[3] The notation used here uses bold block font for three-dimensional spatial vectors. Bold cursive font will be used later for 4-vectors discussed later.

[4] The open symbol $\circ$ denotes the Euclidean dot product between three-dimensional (spatial) vectors. The unit vector $\boldsymbol{n}$ is a unit vector in the context of this dot product. This dot product is distinguished from a four-dimensional dot product, denoted by a solid dot, defined later.

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$$\boldsymbol{x}_2 - \boldsymbol{x}_1 = \overline{\boldsymbol{x}}_2 - \overline{\boldsymbol{x}}_1 + (\gamma - 1)\left[(\overline{\boldsymbol{x}}_2 - \overline{\boldsymbol{x}}_1)\circ \boldsymbol{n}\right]\boldsymbol{n} + \gamma\, \boldsymbol{v}\,(\overline{t}_2 - \overline{t}_1) \tag{1.2b}$$

where the parameter γ and the unit vector $\boldsymbol{n}$ are defined by

$$\gamma \equiv \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}, \qquad \boldsymbol{n} \equiv \frac{\boldsymbol{v}}{v} \tag{1.3}$$

2. Analysis

Let P denote an arbitrary spacetime point on the traveler's worldline. The coordinates of P in the home system are denoted $(\boldsymbol{x}_P, t_P)$, and the coordinates in the traveler's system are denoted $(\overline{\boldsymbol{x}}_P, \overline{t}_P)$. The traveler's clock is at the traveler's origin and the worldline considered here is the worldline of the traveler's clock so

$$\overline{\boldsymbol{x}}_P = \boldsymbol{0} \tag{2.1}$$

Also, even if the traveler's clock is accelerating, the hypothesis (1.1) states that it is still true that increments of proper time for displacements along the worldline of the traveler's clock equal increments of time measured by the traveler's clock. With proper time and the traveler's clock both set to zero when the traveler starts his journey, the proper time along the worldline up to the point P equals $\overline{t}_P$. The proper time is an invariant that can be calculated in the home system, giving

$$\overline{t}_P = \tau(t_P) \tag{2.2}$$

where τ is the proper time calculated in the home system according to

$$\tau(t) = \int_0^t \frac{1}{\gamma(\xi)}\, d\xi \tag{2.3}$$

with $\gamma(t)$ defined by

$$\gamma(t) \equiv \frac{1}{\sqrt{1 - \dfrac{V^2(t)}{c^2}}} \tag{2.4}$$

Now consider an inertial system that is "local" with the traveler's system at the arbitrary point P on the traveler's worldline. This statement is defined to mean that there exists a time at which the origins of the two systems are at the same location and with each axis of the two systems aligned, the two systems are at rest relative to each other at this time, the spacetime coordinates of the origins at this time are the spacetime coordinates of P, and the clock in the inertial system is set to $\overline{t}_P$ at this time. We will call this new inertial system the P-system. We use the postulate, stated in (1.1), that the traveler's clock will momentarily (while at P) tick at the same rate as the P-system's clock, and the traveler's measuring rods will momentarily (while at P) measure the same lengths as the P-system's measuring rods. To utilize this local inertial system in the analysis we use the following approach. Instead of selecting an arbitrary event and attempting to calculate traveler coordinates of it, we work in the opposite direction by selecting traveler coordinates and then identify what the event is that has those coordinates (this identification can be done by finding the home coordinates). Because the spacetime point P is arbitrary, it is sufficiently general to confine our attention to those events that the P-system declares to be simultaneous with the event P. The event to be constructed will be called E, with traveler coordinates denoted $(\overline{\boldsymbol{x}}_E, \overline{t}_E)$ and home coordinates denoted $(\boldsymbol{x}_E, t_E)$. With the P-

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system declaring E to be simultaneous with P, we conclude from the hypothesis (1.1) that the traveler also declares E to be simultaneous with P, [5] implying that

$$\bar{t}_E = \bar{t}_P \tag{2.5}$$

A constraint that the event E must satisfy to be treated in this analysis is that there must exist a point P on the traveler's worldline such that the inertial system local with the traveler's system at P sees E to be simultaneous with P. Given that E is such an "allowed" event, so that E is simultaneous with P as seen by the P-system, the coordinates of E in the P-system are the same as the coordinates in the traveler's system. Therefore, the coordinates in the P-system are also $(\bar{\boldsymbol{x}}_E, \bar{t}_E)$. Also, the coordinates of P in the P-system are $(\bar{\boldsymbol{x}}_P, \bar{t}_P) = (\boldsymbol{0}, \bar{t}_P)$. The P-system has velocity $\boldsymbol{V}(t_P)$ as seen in the home system, so the transformation from barred coordinates to unbarred coordinates is obtained from (1.2) by replacing subscripts (with 2 replaced by E and 1 replaced by P), replacing $\boldsymbol{v}$ with $\boldsymbol{V}(t_P)$, and replacing γ with $\gamma(t_P)$. The results, after simplification by using (2.1) and (2.5) are

$$t_E - t_P = \gamma(t_P)\frac{\boldsymbol{V}(t_P)\circ\bar{\boldsymbol{x}}_E}{c^2} \tag{2.6a}$$

$$\boldsymbol{x}_E - \boldsymbol{x}_P = \bar{\boldsymbol{x}}_E + \left[\gamma(t_P) - 1\right](\bar{\boldsymbol{x}}_E \circ \boldsymbol{n})\boldsymbol{n} \tag{2.6b}$$

3. Summary and Discussion of the Transformation to Home Coordinates

Information assumed to be available includes the functions $\boldsymbol{X}(t)$ and $\boldsymbol{V}(t)$ explained in Section 1. From these we construct the functions $\gamma(t)$ and $\tau(t)$ via (2.4) and (2.3), so all these functions, which refer to the traveler's worldline as seen by the home observer, are regarded as given. The goal here is to calculate the home coordinates of an arbitrary event E when the given information, in addition to the above functions, consists of the traveler's coordinates $(\bar{\boldsymbol{x}}_E, \bar{t}_E)$. The following steps are used:

The first step calculates t_P. This is done by combining (2.5) with (2.2) so t_P is calculated from the given $\bar{t}_E$ via

$$\bar{t}_E = \tau(t_P) \tag{3.1}$$

Note that $\gamma(t)$ is a strictly increasing function so there exists a unique t_P satisfying (3.1) providing only that $\bar{t}_E$ is between zero and the largest display of the traveler's clock (finite if its worldline has a stopping point). With t_P now solved, we calculate $\boldsymbol{x}_P$ via

$$\boldsymbol{x}_P = \boldsymbol{X}(t_P) \tag{3.2}$$

With this done, all terms appearing in (2.6) have been evaluated except for the terms, $\boldsymbol{x}_E$ and t_E, to be solved. They can now be solved by writing (2.6) as

$$t_E = t_P + \gamma(t_P)\frac{\boldsymbol{V}(t_P)\circ\bar{\boldsymbol{x}}_E}{c^2} \tag{3.3a}$$

$$\boldsymbol{x}_E = \boldsymbol{x}_P + \bar{\boldsymbol{x}}_E + \left[\gamma(t_P) - 1\right]\left[\bar{\boldsymbol{x}}_E \circ \boldsymbol{V}(t_P)\right]\frac{\boldsymbol{V}(t_P)}{V^2(t_P)} \tag{3.3b}$$

It is interesting that (3.3a) implies that $t_E = t_P$ if either $\boldsymbol{V}(t_P) = \boldsymbol{0}$ or $\bar{\boldsymbol{x}}_E = \boldsymbol{0}$. This could have been anticipated even before deriving (3.3). Recall that the events P and E are always simultaneous as seen by the P-system, because this is a condition that was imposed on E. If $\boldsymbol{V}(t_P) = \boldsymbol{0}$, the P-system is

[5] While the P-system declares the events E and P to be simultaneous, it is not necessarily true that the home observer declares them to be simultaneous.

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stationary relative to home, so P and E are also simultaneous as seen by the home system. Therefore, we could have anticipated that $t_E = t_P$ when $\mathbf{V}(t_P) = \mathbf{0}$ even before deriving (3.3). Now suppose $\overline{\mathbf{x}}_E = \mathbf{0}$. Recall again that the event E is simultaneous with P as seen by the P-system, and therefore also as seen by the traveler, so they have the same time coordinates in the traveler's system. Also, P is on the traveler's worldline, so its spatial coordinates are zero in the traveler's system. Therefore, $\overline{\mathbf{x}}_E = \mathbf{0}$ if and only if P and E have the same spacetime coordinates in the traveler's system, in which case they are the same spacetime points and therefore have the same spacetime coordinates in all systems. Therefore, we could have anticipated that $t_E = t_P$ and $\mathbf{x}_E = \mathbf{x}_P$ when $\overline{\mathbf{x}}_E = \mathbf{0}$ even before deriving (3.3).

A topic that can be discussed here is time dilation. However, there are two versions of time dilation because the traveler and home observer can disagree on simultaneity.[6] Specifically, they can disagree on which readings on their two clocks are simultaneous. One way to clearly state which version of time dilation is being considered starts with the realization that a clock display is not only a time coordinate for some observer, but it can also be regarded as an event (think of a 5:00 o'clock whistle which is clearly an event). The version of time dilation that is being considered is clearly stated when stating which clock is the one whose displays are the events for which coordinates are to be determined. Here we discuss the version of time dilation in which displays on the traveler's clock are not only traveler time coordinates of the displays but also events. The goal is to calculate home time coordinates of these events, so we will calculate how fast the traveler's clock is ticking as seen by the home observer. This calculation is very simple. The traveler's worldline is taken to be the worldline of the traveler's clock (the clock defines the traveler's origin) so traveler clock displays are events for which $\overline{\mathbf{x}}_E = \mathbf{0}$. From the previous paragraph we conclude that $t_E = t_P$ so (3.1) becomes

$$\bar{t}_E = \tau(t_E) \quad \text{(when the events are traveling clock displays denoted } \bar{t}_E\text{)} \tag{3.4a}$$

which can be inverted via (2.3) and (2.4) to solve for t_E. Also, differentiating (3.4a) while using (2.3) gives

$$\frac{d\,t_E}{d\,\bar{t}_E} = \gamma(t_E) \quad \text{(when the events are traveling clock displays denoted } \bar{t}_E\text{)} \tag{3.4b}$$

The instantaneous result (3.4b) is the same simple result that is obtained without acceleration. This is a time dilation in that dt_E, which is the time seen by the home observer that is needed to change the display of the traveler's clock, is greater than $d\,\bar{t}_E$, which is the change of the display of the traveler's clock and therefore also equal to the time seen by the traveler needed to change the display of the traveler's clock. A larger time between displays corresponds to a slower clock, so the home observer sees the traveler's clock to be running slow. Unfortunately, the version of time dilation in which displays on the home clock are events for which traveler time coordinates are to be calculated, i.e., that calculates how fast the home clock is ticking as seen by the traveler, is not so simple but an example is given in Section 12.

4. A Check for Consistency

Note that (3.1) through (3.3) are as simple to write as (1.2) while having enough generality to apply to an accelerating traveler. However, (3.1) through (3.3) have a different format than (1.2). As a check for consistency, we show that the two formats are equivalent for the case in which the traveler has a constant velocity so that (1.2) also applies. Specifically, we will show that (3.1) through (3.3) produce (1.2) when the traveler's velocity is constant.

Given a constant traveler velocity there is no loss of generality by orienting the coordinate axis so that the velocity is in the x-direction with x-component denoted v. In this case, γ is a constant given by (1.3), so the definition (2.3) of τ together with (3.1) gives

$$t_P = \gamma\,\bar{t}_E \quad \text{(constant velocity)} \tag{4.1}$$

[6] There is a third version of time dilation in Section 14.

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Also, the vector equation (3.2) has components

$$x_P = v t_P, \quad y_P = 0, \quad z_P = 0 \quad \text{(constant velocity)}$$

where we treat the case in which the traveler starts the journey from the home origin. The above can be written via (4.1) as

$$x_P = \gamma v \bar{t}_E, \quad y_P = 0, \quad z_P = 0 \quad \text{(constant velocity)} \tag{4.2}$$

Next, the scaler equation (3.3a) becomes

$$t_E = t_P + \gamma \frac{v \bar{x}_E}{c^2} \quad \text{(constant velocity)} \tag{4.3}$$

while the vector equation (3.3b) has the components

$$x_E = x_P + \gamma \bar{x}_E, \quad y_E = \bar{y}_E, \quad z_E = \bar{z}_E, \quad \text{(constant velocity)} \tag{4.4}$$

The last step substitutes (4.1) into (4.3) and uses (4.2) together with (4.4) to get

$$t_E = \gamma \left(\bar{t}_E + \frac{v \bar{x}_E}{c^2} \right) \quad \text{(constant velocity)} \tag{4.5a}$$

$$x_E = \gamma \left(\bar{x}_E + v \bar{t}_E \right), \quad y_E = \bar{y}_E, \quad z_E = \bar{z}_E, \quad \text{(constant velocity)} \tag{4.5b}$$

which is the familiar format for expressing the Lorentz transformation applicable to inertial reference frames.

5. A Simple Application (Twin Paradox)

A simple application of the transformation given by (3.1) through (3.3) is the case in which the traveler departs from home with a constant speed v but changes direction as needed to return home. The goal is to compare clock readings (home versus traveler) when the traveler returns home. This comparison was already given by the time-dilation equation (3.4a) but the goal here is to provide another derivation of the same result. We first continue with the derivation that starts with (3.4a). When v is constant this equation reduces to

$$t_E = \gamma \bar{t}_E \quad (\text{constant } v, E \text{ is display of } \bar{t}_E \text{ on traveler's clock}) \tag{5.1}$$

. (5.1)

In (5.1), the display $\bar{t}_E$ on the traveler's clock is the event (also the traveler's time coordinate of the event) while t_E is the home time coordinate of the event.

Next, we use a different derivation of the same result (5.1) just to demonstrate consistency. Instead of the event E consisting of the traveler's clock displaying the reading $\bar{t}_E$, which is the interpretation of E in (3.4) and (5.1), here we take the event E to be the traveler's return home. In this interpretation, $\bar{t}_E$ is the traveler's time coordinate of his return home while t_E is the home time coordinate of the traveler's return home. The first step in the analysis notes that a constant γ in (2.3) together with (3.1) gives

$$t_P = \gamma \bar{t}_E \quad (\text{constant } v) \tag{5.2}$$

But the event E is a point on the traveler's worldline, so $\bar{\mathbf{x}}_E = \mathbf{0}$ and (3.3a) gives

$$t_E = t_P \quad (E \text{ on traveler's world line}) \tag{5.3}$$

Combining (5.3) with (5.2) produces agreement with (5.1).

6. The Inverse Transformation

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When traveler coordinates of an event are given, the home coordinates are obtained from a simple procedure in Section 3. We now consider the case in which the home coordinates are given, and the goal is to calculate the traveler's coordinates. Using the transformation in either direction, one of the unknowns to be solved is t_P. When the traveler coordinates are given, t_P is solved via (3.1). When the home coordinates x_E and t_E are the givens, we need to construct an equation that contains the unknown t_P with all other terms known. This is done by taking the dot product of (3.3b) with $V(t_p)$ to get

$$\boldsymbol{V}(t_P)\circ\boldsymbol{x}_E = \boldsymbol{V}(t_P)\circ\boldsymbol{x}_P + \boldsymbol{V}(t_P)\circ\overline{\boldsymbol{x}}_E + \left[\gamma(t_P)-1\right]\left[\overline{\boldsymbol{x}}_E\circ\boldsymbol{V}(t_P)\right] =$$

$$\boldsymbol{V}(t_P)\circ\boldsymbol{x}_P + \gamma(t_P)\left[\overline{\boldsymbol{x}}_E\circ\boldsymbol{V}(t_P)\right]$$

which gives

$$\gamma(t_P)\boldsymbol{V}(t_P)\circ\overline{\boldsymbol{x}}_E = \boldsymbol{V}(t_P)\circ\left(\boldsymbol{x}_E-\boldsymbol{x}_P\right)$$

Substituting this into (3.3a) while using (3.2) gives

$$t_E = t_P + \frac{1}{c^2}\boldsymbol{V}(t_P)\circ\left(\boldsymbol{x}_E-\boldsymbol{X}(t_P)\right) \tag{6.1}$$

With the parameters t_E and x_E given, and the functions $\boldsymbol{V}$ and $\boldsymbol{X}$ given, (6.1) is the equation governing t_P. An unfortunate property of (6.1) is that there are example traveler worldlines in which (6.1) does not have a unique solution for t_P for some inputs t_E and x_E. In some cases, no solution exists. For some other cases, solutions exist but are not unique. This is discussed in more detail in Section 8. For the remainder of this discussion we assume that event E is one in which (6.1) has a unique solution for t_P and this solution has been found (by some numerical root finding routine if necessary).

Given that t_P satisfying (6.1) can be found and has been found, we calculate $\overline{t}_E$ via (3.1). The last step of the inverse transformation solves for $\overline{\boldsymbol{x}}_E$ as follows. To shorten the notation, we define, for an arbitrary vector $\boldsymbol{W}$, a parallel part $\boldsymbol{W}_{||}$ and a perpendicular part $\boldsymbol{W}_{\perp}$, with respect to the direction of $\boldsymbol{V}(t_P)$, according to

$$\boldsymbol{W}_{||} \equiv \left(\boldsymbol{W}\circ\boldsymbol{V}(t_P)\right)\frac{\boldsymbol{V}(t_P)}{V^2(t_P)}, \quad \boldsymbol{W}_{\perp} \equiv \boldsymbol{W}-\boldsymbol{W}_{||} \tag{6.2}$$

Using this notation we can write (3.3b) as

$$\boldsymbol{x}_E-\boldsymbol{x}_P = \overline{\boldsymbol{x}}_E + \left[\gamma(t_P)-1\right]\overline{\boldsymbol{x}}_{E,||} = \overline{\boldsymbol{x}}_{E,\perp} + \gamma(t_P)\overline{\boldsymbol{x}}_{E,||} \tag{6.3}$$

The parallel part of the left side equals the parallel part of the far right side, and the perpendicular part of the left side equals the perpendicular part of the far right side so

$$\left(\boldsymbol{x}_E-\boldsymbol{x}_P\right)_{||} = \gamma(t_P)\overline{\boldsymbol{x}}_{E,||}, \quad \text{or} \quad \overline{\boldsymbol{x}}_{E,||} = \frac{1}{\gamma(t_P)}\left(\boldsymbol{x}_E-\boldsymbol{x}_P\right)_{||} \tag{6.4a}$$

and

$$\overline{\boldsymbol{x}}_{E,\perp} = \left(\boldsymbol{x}_E-\boldsymbol{x}_P\right)_{\perp} \tag{6.4b}$$

Now express $\overline{\boldsymbol{x}}_E$ as the sum of the parallel part plus perpendicular part while using (6.4) to get

$$\overline{\boldsymbol{x}}_E = \overline{\boldsymbol{x}}_{E,\perp} + \overline{\boldsymbol{x}}_{E,||} = \left(\boldsymbol{x}_E-\boldsymbol{x}_P\right)_{\perp} + \frac{1}{\gamma(t_P)}\left(\boldsymbol{x}_E-\boldsymbol{x}_P\right)_{||}$$

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Now use the second equation in (6.2) to write this as

$$\bar{\boldsymbol{x}}_E = \boldsymbol{x}_E - \boldsymbol{x}_P - (\boldsymbol{x}_E - \boldsymbol{x}_P)_{||} + \frac{1}{\gamma(t_P)}(\boldsymbol{x}_E - \boldsymbol{x}_P)_{||} = \boldsymbol{x}_E - \boldsymbol{x}_P + \left[\frac{1}{\gamma(t_P)} - 1\right](\boldsymbol{x}_E - \boldsymbol{x}_P)_{||}$$

Finally, we use the first equation in (6.2) to write this as

$$\bar{\boldsymbol{x}}_E = \boldsymbol{x}_E - \boldsymbol{x}_P + \left[\frac{1}{\gamma(t_P)} - 1\right]\left[(\boldsymbol{x}_E - \boldsymbol{x}_P)\circ \boldsymbol{V}(t_P)\right]\frac{\boldsymbol{V}(t_P)}{V^2(t_P)}$$

Before summarizing the above results, we digress by noting two interesting properties of (6.1). One property is that $t_E = t_P$ if $\boldsymbol{V}(t_P) = 0$. This could have been anticipated even before deriving (6.1) as already explained in Section 3. The second interesting property deserves more discussion. That property is the implication that if $\boldsymbol{x}_E = \boldsymbol{X}(t_P)$ then $t_E = t_P$. This could have been anticipated even before deriving (6.1) as follows. Suppose $\boldsymbol{x}_E = \boldsymbol{X}(t_P)$, i.e., $\boldsymbol{x}_E = \boldsymbol{x}_P$. Then the events E and P have the same spatial coordinates in the home system. They also have the same time coordinates in the P-system (by choice of point P) so it is not surprising that E and P are the same spacetime point, implying $t_E = t_P$. This conclusion is made rigorous by (6.1), showing that it is true that the condition $\boldsymbol{x}_E = \boldsymbol{X}(t_P)$ implies $t_E = t_P$. Therefore, the condition $\boldsymbol{x}_E = \boldsymbol{X}(t_P)$ implies the simultaneous conditions $\boldsymbol{x}_E = \boldsymbol{x}_P$ and $t_E = t_P$, which implies that E and P are the same spacetime points, which implies that they have the same coordinates in all systems. A variety of implications follow by combining this implication with others already stated (e.g., that the event E is on the traveler's worldline if and only if $\bar{\boldsymbol{x}}_E = \boldsymbol{0}$, and the conclusion from Section 3 that $t_E = t_P$ and $\boldsymbol{x}_E = \boldsymbol{x}_P$ when $\bar{\boldsymbol{x}}_E = \boldsymbol{0}$). Putting the implications all together, we obtain the following summary of implications:

$$\left.\begin{array}{l} \left[\boldsymbol{x}_E = \boldsymbol{X}(t_P)\right] \text{ if and only if } [E \text{ is the same spacetime point as } P] \\ \text{if and only if } [E \text{ is a point on the traveler's worldline}] \text{ if and only if} \\ \left[\bar{\boldsymbol{x}}_E = \boldsymbol{0}\right]. \\ \\ \text{If any of the above are satisfied then } t_E = t_P. \\ \text{If } \boldsymbol{V}(t_P) = \boldsymbol{0} \text{ then } t_E = t_P. \end{array}\right\} \quad (6.5)$$

We now summarize the results derived in this section. It is assumed that we are given $\boldsymbol{x}_E$ and t_E, and the goal is to calculate $\bar{\boldsymbol{x}}_E$ and $\bar{t}_E$. The first step calculates t_P from

$$t_E = t_P + \frac{1}{c^2}\boldsymbol{V}(t_P)\circ(\boldsymbol{x}_E - \boldsymbol{X}(t_P)) \quad (6.6)$$

where it is understood that we are treating the case in which there exists such a t_P (see Section 8 for more discussion). Having done this, the quantities $\boldsymbol{X}(t_P)$, $\boldsymbol{V}(t_P)$, $\gamma(t_P)$, and $\tau(t_P)$ all become knowns quantities, via the given worldline of the traveler in the home system together with (2.3) and (2.4), in addition to $\boldsymbol{x}_E$ and t_E being known quantities. The next set of steps calculate $\boldsymbol{x}_P$, $\bar{t}_E$, and $\bar{\boldsymbol{x}}_E$ from

$$\boldsymbol{x}_P = \boldsymbol{X}(t_P) \quad (6.7)$$

$$\bar{t}_E = \tau(t_P) \quad (6.8a)$$

$$\bar{\boldsymbol{x}}_E = \boldsymbol{x}_E - \boldsymbol{x}_P + \left[\frac{1}{\gamma(t_P)} - 1\right]\left[(\boldsymbol{x}_E - \boldsymbol{x}_P)\circ \boldsymbol{V}(t_P)\right]\frac{\boldsymbol{V}(t_P)}{V^2(t_P)} \quad (6.8b)$$

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7. Another Check for Consistency

The goal of this section is to verify, merely as a check for mistakes, that (6.6) through (6.8) reduce to the familiar Lorentz transformation when the traveler has a constant velocity.

Given a constant traveler velocity there is no loss of generality by orienting the coordinate axis so that the velocity is in the x-direction with x-component denoted v. For this case, only x-components of terms in (6.6) have contributions and using $X(t_P) = vt_P$ with (6.6) gives

$$t_E = t_P + \frac{1}{c^2} v\left(x_E - v t_P\right)$$

with solution

$$t_P = \frac{1}{1 - \dfrac{v^2}{c^2}}\left(t_E - \frac{v x_E}{c^2}\right)$$

or

$$t_P = \gamma^2\left(t_E - \frac{v x_E}{c^2}\right) \quad \text{(constant velocity)} \tag{7.1}$$

Also, the vector equation (6.7) has the components

$$x_P = v t_P, \quad y_P = 0, \quad z_P = 0 \quad \text{(constant velocity)}$$

which can be written via (7.1) as

$$x_P = v\gamma^2\left(t_E - \frac{v x_E}{c^2}\right), \quad y_P = 0, \quad z_P = 0 \quad \text{(constant velocity)} \tag{7.2}$$

Next, the scaler equation (6.8a), together with a constant γ in (2.3) gives

$$\bar{t}_E = \frac{1}{\gamma} t_P$$

and using (7.1) gives

$$\bar{t}_E = \gamma\left(t_E - \frac{v x_E}{c^2}\right) \quad \text{(constant velocity)} \tag{7.3a}$$

Finaly, the vector equation (6.8b) has the components

$$\bar{x}_E = \frac{1}{\gamma}\left(x_E - x_P\right), \quad \bar{y}_E = y_E - y_P, \quad \bar{z}_E = z_E - x_P, \quad \text{(constant velocity)}$$

Substituting (7.2) into this and simplifying terms via the identity

$$\frac{1}{\gamma^2} + \frac{v^2}{c^2} = 1$$

gives

$$\bar{x}_E = \gamma\left(x_E - v t_E\right), \quad \bar{y}_E = y_E, \quad \bar{z}_E = z_E, \quad \text{(constant velocity)} \tag{7.3b}$$

The transformation given by (7.3) is the familiar form for the case of a constant velocity.

8. Existence and Uniqueness of the Inverse Transformation

Whether or not the inverse transformation in Section 6 exists and is unique is completely determined by the very first step of the calculation, which is to solve (6.6) for t_P. As pointed out in

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Section 2, a constraint that the event E must satisfy to be treated in this analysis is that there must exist a point P on the traveler's worldline such that the inertial system local with the traveler's system at P sees E to be simultaneous with P. There exists a solution for t_P to (6.6) if and only if E is such an "allowed" event. However, it is also possible that E is such an allowed event, so there exists a point P satisfying the above condition, but such a point P is not unique. In this case there are multiple solutions for t_P to (6.6). Lack of existence is possible if the traveler's acceleration continues forever, and an example is given later in Section 12. However, it is shown in this section that if the acceleration always has a finite magnitude and has a finite time duration, solutions must exist but are not always unique.

The goal of this section is to show that if the acceleration always has a finite magnitude and has a finite time duration, there necessarily exists a solution for t_P to (6.6) (albeit not necessarily unique). The first step towards this goal shortens the notation in (6.6) by defining the function $F(t_P,\boldsymbol{x}_E)$, a function of t_P and the vector $\boldsymbol{x}_E$, by

$$F\left(t_P,\boldsymbol{x}_E\right) \equiv t_P + \frac{1}{c^2}\boldsymbol{V}\left(t_P\right)\circ\left[\boldsymbol{x}_E - \boldsymbol{X}\left(t_P\right)\right] \tag{8.1}$$

so that (6.6) becomes

$$t_E = F\left(t_P,\boldsymbol{x}_E\right) \tag{8.2}$$

The traveler's journey begins at $t = 0$ but nothing was said about his prior history. It is convenient to stipulate that, before starting the journey, the traveler was at rest in the home system, so $\boldsymbol{V}(t_P) = \boldsymbol{0}$ if $t_P < 0$. Also, the acceleration is taken to have a finite time duration so there exists some cutoff time in the home system, denoted t_C, such that the acceleration is zero when $t > t_C$. We therefore have

$$\boldsymbol{V}\left(t_P\right) = \begin{cases} \boldsymbol{V}\left(t_C\right) & \text{if} \quad t_P > t_C \\ \boldsymbol{0} & \text{if} \quad t_P < 0 \end{cases} \tag{8.3a}$$

The cutoff for the velocity implies that $\boldsymbol{X}(t_P)$ satisfies

$$\boldsymbol{X}\left(t_P\right) = \boldsymbol{X}\left(t_C\right) + \boldsymbol{V}\left(t_C\right)\left[t_P - t_C\right] \qquad \text{if} \quad t_P > t_C \tag{8.3b}$$

Substituting (8.3) into (8.1) and regrouping terms for the case in which $t_P > t_C$ gives

$$F\left(t_P,\boldsymbol{x}_E\right) = \begin{cases} \dfrac{1}{\gamma^2\left(t_C\right)}\left(t_P - t_C\right) + F\left(t_C,\boldsymbol{x}_E\right) & \text{if} \quad t_P > t_C \\ t_P + \dfrac{1}{c^2}\boldsymbol{V}\left(t_P\right)\circ\left[\boldsymbol{x}_E - \boldsymbol{X}\left(t_P\right)\right] & \text{if} \quad 0 \le t_P \le t_C \\ t_P \quad \text{if} \quad t_P < 0 \end{cases} \tag{8.4}$$

where γ is defined by (2.4). It is evident from inspection of (8.4) that

$$F\left(t_P,\boldsymbol{x}_E\right) \to -\infty \quad \text{as} \quad t_P \to -\infty, \quad \text{and} \quad F\left(t_P,\boldsymbol{x}_E\right) \to +\infty \quad \text{as} \quad t_P \to +\infty \tag{8.5}$$

Given that the magnitude of acceleration is always finite, so the velocity is a continuous function of time, $F(t_P,\boldsymbol{x}_E)$ is a continuous function of each argument. Continuity together with the mapping property (8.5) implies that (8.2) has a solution for t_P for any value of t_E.

Having established that there necessarily exists a solution for t_P to (6.6) if the acceleration always has a finite magnitude and has a finite time duration, the next question asks for the conditions in which the solution is unique under the assumed conditions regarding the acceleration. Note from (8.4) that $F(t_P,\boldsymbol{x}_E)$ is strictly increasing in t_P when $t_P < 0$ and when $t_P > t_C$. If (a big "if") $F(t_P,\boldsymbol{x}_E)$ is strictly increasing in t_P when $0 < t_P < t_C$, then a solution for t_P to (6.6) is unique. However, if (another big "if") a solution for t_P to (6.6) is a value of t_P at which $F(t_P,\boldsymbol{x}_E)$ is strictly decreasing in t_P, then this solution

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for t_P is not unique. Therefore, the question of uniqueness is answered by considerations of whether $F(t_P,\boldsymbol{x}_E)$ is increasing or decreasing in t_P. It will be shown below that (given a nonzero acceleration at some point in time) there necessarily exists home coordinates (t_E, $\boldsymbol{x}_E$) for which a solution for t_P to (6.6) is not unique. A sufficient condition, imposed on $\boldsymbol{x}_E$, for uniqueness will also be given. The arguments below will refer to the derivative of $F(t_P,\boldsymbol{x}_E)$ obtained from (8.1) and given by

$$\frac{\partial}{\partial t_P}F(t_P,\boldsymbol{x}_E)=1-\frac{1}{c^2}V^2(t_P)+\frac{1}{c^2}\boldsymbol{A}(t_P)\circ(\boldsymbol{x}_E-\boldsymbol{X}(t_P)) \tag{8.6}$$

We will show that there exist home coordinates (t_E, $\boldsymbol{x}_E$) for which a solution for t_P to (6.6) is not unique by constructing them. Select a t_P for which $\boldsymbol{A}(t_P) \neq 0$ but is otherwise arbitrary. Now select any $\boldsymbol{x}_E$ satisfying

$$1-\frac{1}{c^2}V^2(t_P)+\frac{1}{c^2}\boldsymbol{A}(t_P)\circ(\boldsymbol{x}_E-\boldsymbol{X}(t_P))<0 \quad \text{(for discussion)} \tag{8.7}$$

Note that such an $\boldsymbol{x}_E$ satisfying (8.7) necessarily exists when $\boldsymbol{A}(t_P) \neq 0$. Also note, from (8.6), that this choice for $\boldsymbol{x}_E$ makes $F(t_P,\boldsymbol{x}_E)$ strictly decreasing in t_P at the selected value of t_P. Now use (8.2) (equivalent to (6.6)) to define t_E. These steps have constructed a set of home coordinates (t_E, $\boldsymbol{x}_E$) such that $F(t_P,\boldsymbol{x}_E)$ strictly decreasing in t_P at the solution t_P, implying that the solution is not unique. The conclusion is that, given that there is some point in time at which the acceleration is not zero. There exist home coordinates (t_E, $\boldsymbol{x}_E$) for which a solution for t_P to (6.6) is not unique.

A fairly simple constraint imposed on the home coordinates (t_E, $\boldsymbol{x}_E$) that is a sufficient condition for the solution for t_P to (6.6) to be unique is for $\boldsymbol{x}_E$ to be selected to make the right side of (8.6) positive for all t_P. This constraint makes $F(t_P,\boldsymbol{x}_E)$ strictly increasing in t_P for all t_P, ensuring uniqueness.

9. Some Identities for Later Use

The next section calculates the metric tensor in the traveling system. That analysis is less cumbersome if some identities are available, and one goal of this section is to make them available. A few other miscellaneous identities useful in later sections are also derived.

The first goal is to derive expressions for the derivatives of parallel parts and perpendicular parts of arbitrary spatial vectors as they are defined in (6.2). This is accomplished in several steps. The first step calculates the derivative of the magnitude of the velocity vector. The second step uses this result to calculate the derivative of the unit vector in the direction of the velocity vector, and the last step calculates derivative of parallel and perpendicular parts of arbitrary vectors.

Note from (6.2) that parallel and perpendicular parts of an arbitrary spatial vector $\boldsymbol{W}$ can be written as

$$\boldsymbol{W}_{||}\equiv(\boldsymbol{W}\circ\boldsymbol{n}(t_P))\boldsymbol{n}(t_P), \quad \boldsymbol{W}_{\perp}\equiv\boldsymbol{W}-\boldsymbol{W}_{||} \tag{9.1}$$

where the unit vector $\boldsymbol{n}$ is defined by

$$\boldsymbol{n}(t_P)\equiv\frac{\boldsymbol{V}(t_P)}{V(t_P)} \tag{9.2}$$

The first step that is used to calculate derivatives of parallel and perpendicular parts of $\boldsymbol{W}$ notes that

$$2V(t_P)\frac{dV(t_P)}{dt_P}=\frac{dV^2(t_P)}{dt_P}=\frac{d\boldsymbol{V}(t_P)\circ\boldsymbol{V}(t_P)}{dt_P}=2\boldsymbol{V}(t_P)\circ\frac{d\boldsymbol{V}(t_P)}{dt_P}=2\boldsymbol{V}(t_P)\circ$$

which gives

$$\frac{dV(t_P)}{dt_P}=\boldsymbol{A}(t_P)\circ\boldsymbol{n}(t_P) \tag{9.3}$$

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Now use

$$\frac{d\,\boldsymbol{n}(t_P)}{d\,t_P}=\frac{d}{d\,t_P}\left(\frac{\boldsymbol{V}(t_P)}{V(t_P)}\right)=\frac{1}{V(t_P)}\frac{d\boldsymbol{V}(t_P)}{d\,t_P}-\frac{\boldsymbol{V}(t_P)}{V^2(t_P)}\frac{dV(t_P)}{d\,t_P}=$$

$$\frac{1}{V(t_P)}\boldsymbol{A}(t_P)-\frac{\boldsymbol{n}(t_P)}{V(t_P)}\frac{dV(t_P)}{d\,t_P}\quad .$$

Substituting (9.3) into the above gives

$$\frac{d\,\boldsymbol{n}(t_P)}{d\,t_P}=\frac{1}{V(t_P)}\left[\boldsymbol{A}(t_P)-\left(\boldsymbol{A}(t_P)\circ\boldsymbol{n}(t_P)\right)\boldsymbol{n}(t_P)\right] \tag{9.4}$$

From the definition (9.1) of parallel and perpendicular parts, we recognize the square bracket in (9.4) as the perpendicular part of the acceleration so

$$\frac{d\,\boldsymbol{n}(t_P)}{d\,t_P}=\frac{1}{V(t_P)}\boldsymbol{A}_\perp(t_P) \tag{9.5}$$

The next step takes the derivative of the parallel part of an arbitrary vector $\boldsymbol{W}$ using

$$\frac{d\,\boldsymbol{W}_{||}(t_P)}{d\,t_P}=\frac{d}{d\,t_P}\left(\boldsymbol{W}(t_P)\circ\boldsymbol{n}(t_P)\right)\boldsymbol{n}(t_P)=\left(\frac{d\,\boldsymbol{W}(t_P)}{d\,t_P}\circ\boldsymbol{n}(t_P)\right)\boldsymbol{n}(t_P)+$$

$$\left(\boldsymbol{W}(t_P)\circ\frac{d\,\boldsymbol{n}(t_P)}{d\,t_P}\right)\boldsymbol{n}(t_P)+\left(\boldsymbol{W}(t_P)\circ\boldsymbol{n}(t_P)\right)\frac{d\,\boldsymbol{n}(t_P)}{d\,t_P}$$

The first term on the right is seen to be the parallel part of the derivative of $\boldsymbol{W}$. Substituting (9.5) into the second and third terms on the right produces

$$\frac{d\,\boldsymbol{W}_{||}(t_P)}{d\,t_P}=\left(\frac{d\,\boldsymbol{W}(t_P)}{d\,t_P}\right)_{||}+$$

$$\frac{1}{V(t_P)}\left[\left(\boldsymbol{W}(t_P)\circ\boldsymbol{A}_\perp(t_P)\right)\boldsymbol{n}(t_P)+\left(\boldsymbol{W}(t_P)\circ\boldsymbol{n}(t_P)\right)\boldsymbol{A}_\perp(t_P)\right] \tag{9.6}$$

The derivative of the perpendicular part is obtained from

$$\frac{d\,\boldsymbol{W}_\perp(t_P)}{d\,t_P}=\frac{d\,\boldsymbol{W}(t_P)-\boldsymbol{W}_{||}(t_P)}{d\,t_P}=\frac{d\,\boldsymbol{W}(t_P)}{d\,t_P}-\frac{d\,\boldsymbol{W}_{||}(t_P)}{d\,t_P}$$

Substituting (9.6) into the above far right term while using

$$\left(\frac{d\,\boldsymbol{W}(t_P)}{d\,t_P}\right)_\perp=\frac{d\,\boldsymbol{W}(t_P)}{d\,t_P}-\left(\frac{d\,\boldsymbol{W}(t_P)}{d\,t_P}\right)_{||}$$

gives

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$$\frac{d\boldsymbol{W}_{\perp}(t_P)}{dt_P} = \left(\frac{d\boldsymbol{W}(t_P)}{dt_P}\right)_{\perp} - \frac{1}{V(t_P)}\left[\left(\boldsymbol{W}(t_P)\circ\boldsymbol{A}_{\perp}(t_P)\right)\boldsymbol{n}(t_P) + \left(\boldsymbol{W}(t_P)\circ\boldsymbol{n}(t_P)\right)\boldsymbol{A}_{\perp}(t_P)\right]. \tag{9.7}$$

Another identity useful in later sections is an expression for the derivative of $\gamma(t_P)$. Using (2.4) gives

$$\frac{d\gamma(t_P)}{dt_P} = \frac{d}{dt_P}\left(1-\frac{V^2(t_P)}{c^2}\right)^{-1/2} = \frac{1}{2c^2}\left(1-\frac{V^2(t_P)}{c^2}\right)^{-3/2}\frac{dV^2(t_P)}{dt_P} = \frac{1}{2c^2}\left(1-\frac{V^2(t_P)}{c^2}\right)^{-3/2}\frac{d\boldsymbol{V}(t_P)\circ\boldsymbol{V}(t_P)}{dt_P}$$

which finally gives

$$\frac{d\gamma(t_P)}{dt_P} = \frac{1}{c^2}\gamma^3(t_P)\boldsymbol{V}(t_P)\circ\boldsymbol{A}(t_P) \tag{9.8}$$

Still another identity that will be useful later and is trivial to prove is

$$\gamma^2(t_P) - 1 = \frac{1}{c^2}\gamma^2(t_P)V^2(t_P) \tag{9.9}$$

Now consider an arbitrary trajectory, or worldline, seen in an arbitrary inertial system. This could be the worldline of the traveler's clock in the home system but not necessarily. Along this worldline we can define increments of proper time, denoted $d\tau$, and increments of coordinate time, denoted dt. From these increments we can define the derivative $dt/d\tau$, which is a derivative defined on a curve. Two different expressions for this derivative are derived. Both start with the equation

$$(d\tau)^2 = (dt)^2 - \frac{1}{c^2}(d\boldsymbol{x}\circ d\boldsymbol{x}) \tag{9.10a}$$

and then divide by $(d\tau)^2$ to get

$$1 = \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\frac{d\boldsymbol{x}}{d\tau}\circ\frac{d\boldsymbol{x}}{d\tau} \tag{9.10b}$$

One expression for $dt/d\tau$ is obtained by rearranging terms in (9.10) to get

$$\frac{dt}{d\tau} = \sqrt{1+\frac{1}{c^2}\frac{d\boldsymbol{x}}{d\tau}\circ\frac{d\boldsymbol{x}}{d\tau}} \tag{9.11a}$$

A second expression for the same derivative is obtained by using the chain rule to write (9.10) as

$$1 = \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\frac{dt}{d\tau}\frac{d\boldsymbol{x}}{dt}\circ\frac{dt}{d\tau}\frac{d\boldsymbol{x}}{dt} = \left[1-\frac{1}{c^2}\frac{d\boldsymbol{x}}{dt}\circ\frac{d\boldsymbol{x}}{dt}\right]\left(\frac{dt}{d\tau}\right)^2$$

and rearranging terms to get

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$$\frac{dt}{d\tau}=\frac{1}{\sqrt{1-\frac{1}{c^2}\frac{d\boldsymbol{x}}{dt}\circ\frac{d\boldsymbol{x}}{dt}}}=\gamma \tag{9.11b}$$

with γ defined by the second equality in (9.11b), consistent with the definition in earlier sections. Combining (9.11a) with (9.11b) gives

$$\frac{1}{\sqrt{1-\frac{1}{c^2}\frac{d\boldsymbol{x}}{dt}\circ\frac{d\boldsymbol{x}}{dt}}}=\sqrt{1+\frac{1}{c^2}\frac{d\boldsymbol{x}}{d\tau}\circ\frac{d\boldsymbol{x}}{d\tau}} \tag{9.11c}$$

when $d\tau$ is defined by (9.10).

10. The Metric Tensor in the Traveling System

This section calculates the metric tensor in the traveling system. This metric, denoted $\overline{g}$, is obtained by starting with the Lorentz metric, denoted g, applicable to the home system and then transforming it via the transformation of a double covariant tensor. This transformation is defined by

$$\overline{g}_{i,j}=\sum_{k,l=0}^{3}\frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}g_{k,l} \quad \text{for } i,j=0,1,2,3 \tag{10.1}$$

where we use the notation $x^0 = ct_E$, $x^1 = x_E$, $x^2 = y_E$, $x^3 = z_E$, with corresponding notation for the traveler coordinates. The Lorentz metric is diagonal with $g_{0,0} = 1$ and all other diagonal elements equal to –1, so (10.1) becomes

$$\overline{g}_{i,j}=\frac{\partial x^0}{\partial \overline{x}^i}\frac{\partial x^0}{\partial \overline{x}^j}-\sum_{k=1}^{3}\frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^k}{\partial \overline{x}^j} \quad \text{for } i,j=0,1,2,3$$

Recognizing the sum on the right as a three dimensional (spatial) vector dot product gives

$$\overline{g}_{i,j}=c^2\frac{\partial t_E}{\partial \overline{x}^i}\frac{\partial t_E}{\partial \overline{x}^j}-\frac{\partial \boldsymbol{x}_E}{\partial \overline{x}^i}\circ\frac{\partial \boldsymbol{x}_E}{\partial \overline{x}^j} \quad \text{for } i,j=0,1,2,3 \tag{10.2}$$

It is evident from (3.1) that partial derivatives that treat $\overline{t}_E, \overline{x}_E, \overline{y}_E$, and $\overline{z}_E$ as independent variables (meaning that a partial derivative that varies one variable in the list does so with all others in the list held fixed) also treat $t_P, \overline{x}_E, \overline{y}_E$, and $\overline{z}_E$ as independent variables. Therefore, a derivative with respect $\overline{t}_E$ can be expressed in terms of a derivative with respect to t_P by using

$$\frac{\partial}{\partial \overline{t}_E}=\frac{d t_P}{d\overline{t}_E}\frac{\partial}{\partial t_P}=\left(\frac{d\overline{t}_E}{dt_P}\right)^{-1}\frac{\partial}{\partial t_P}$$

Using (3.1) and (2.3) to evaluate the derivative in the parenthesis on the right allows the above to be written as

$$\frac{\partial}{\partial \overline{t}_E}=\gamma(t_P)\frac{\partial}{\partial t_P} \tag{10.3}$$

Using (10.3) with (10.2), components of the metric tensor can be written as

$$\overline{g}_{0,0}=\gamma^2(t_P)\left[\left(\frac{\partial t}{\partial t_P}\right)^2-\frac{1}{c^2}\frac{\partial \boldsymbol{x}}{\partial t_P}\circ\frac{\partial \boldsymbol{x}}{\partial t_P}\right], \quad \overline{g}_{0,1}=\gamma(t_P)\left[c\frac{\partial t}{\partial t_P}\frac{\partial t}{\partial \overline{x}}-\frac{1}{c}\frac{\partial \boldsymbol{x}}{\partial t_P}\circ\frac{\partial \boldsymbol{x}}{\partial \overline{x}}\right] \tag{10.4a}$$

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$$\bar{g}_{0,2} = \gamma(t_P)\left[c\frac{\partial t}{\partial t_P}\frac{\partial t}{\partial \bar{y}} - \frac{1}{c}\frac{\partial \boldsymbol{x}}{\partial t_P}\circ\frac{\partial \boldsymbol{x}}{\partial \bar{y}}\right], \quad \bar{g}_{0,3} = \gamma(t_P)\left[c\frac{\partial t}{\partial t_P}\frac{\partial t}{\partial \bar{z}} - \frac{1}{c}\frac{\partial \boldsymbol{x}}{\partial t_P}\circ\frac{\partial \boldsymbol{x}}{\partial \bar{z}}\right] \tag{10.4b}$$

$$\bar{g}_{1,1} = c^2\left(\frac{\partial t}{\partial \bar{x}}\right)^2 - \frac{\partial \boldsymbol{x}}{\partial \bar{x}}\circ\frac{\partial \boldsymbol{x}}{\partial \bar{x}}, \qquad \bar{g}_{1,2} = c^2\frac{\partial t}{\partial \bar{x}}\frac{\partial t}{\partial \bar{y}} - \frac{\partial \boldsymbol{x}}{\partial \bar{x}}\circ\frac{\partial \boldsymbol{x}}{\partial \bar{y}} \tag{10.4c}$$

$$\bar{g}_{1,3} = c^2\frac{\partial t}{\partial \bar{x}}\frac{\partial t}{\partial \bar{z}} - \frac{\partial \boldsymbol{x}}{\partial \bar{x}}\circ\frac{\partial \boldsymbol{x}}{\partial \bar{z}}, \qquad \bar{g}_{2,2} = c^2\left(\frac{\partial t}{\partial \bar{y}}\right)^2 - \frac{\partial \boldsymbol{x}}{\partial \bar{y}}\circ\frac{\partial \boldsymbol{x}}{\partial \bar{y}} \tag{10.4d}$$

$$\bar{g}_{2,3} = c^2\frac{\partial t}{\partial \bar{y}}\frac{\partial t}{\partial \bar{z}} - \frac{\partial \boldsymbol{x}}{\partial \bar{y}}\circ\frac{\partial \boldsymbol{x}}{\partial \bar{z}}, \qquad \bar{g}_{3,3} = c^2\left(\frac{\partial t}{\partial \bar{z}}\right)^2 - \frac{\partial \boldsymbol{x}}{\partial \bar{z}}\circ\frac{\partial \boldsymbol{x}}{\partial \bar{z}} \tag{10.4e}$$

where we shortened the notation by omitting the subscripts E to the event coordinates $\bar{x}_E, \bar{y}_E, \bar{z}_E$ and t_E, x_E, y_E, and z_E. The remaining off-diagonal elements of the metric tensor are implied by the symmetry condition $\bar{g}_{i,j} = \bar{g}_{j,i}$.

The next step is to calculate the derivatives in (10.4). One equation needed for this is (3.3a), which is written below with the E subscripts omitted as

$$t = t_P + \gamma(t_P)\frac{\boldsymbol{V}(t_P)\circ\bar{\boldsymbol{x}}}{c^2} \tag{10.5}$$

The other needed equation is (3.3b). It is convenient to write it in terms of parallel and perpendicular parts. Doing so while omitting the E subscripts to shorten the notation gives

$$\boldsymbol{x} = \boldsymbol{X}(t_P) + \bar{\boldsymbol{x}}_\perp + \gamma(t_P)\bar{\boldsymbol{x}}_\| \tag{10.6}$$

The first derivative to be calculated is obtained from (10.5), (9.8), and a direct application of the product rule which gives

$$\frac{\partial t}{\partial t_P} = 1 + \frac{1}{c^4}\gamma^3(t_P)\left[\boldsymbol{V}(t_P)\circ\boldsymbol{A}(t_P)\right]\left[\boldsymbol{V}(t_P)\circ\bar{\boldsymbol{x}}\right] + \frac{1}{c^2}\gamma(t_P)\boldsymbol{A}(t_P)\circ\bar{\boldsymbol{x}}$$

which can also be written as

$$\frac{\partial t}{\partial t_P} = 1 + \frac{1}{c^4}\gamma^3(t_P)V^2(t_P)\left[\boldsymbol{n}(t_P)\circ\boldsymbol{A}(t_P)\right]\left[\boldsymbol{n}(t_P)\circ\bar{\boldsymbol{x}}\right] + \frac{1}{c^2}\gamma(t_P)\boldsymbol{A}(t_P)\circ\bar{\boldsymbol{x}}$$

Now use

$$\boldsymbol{A}_\|(t_P)\circ\bar{\boldsymbol{x}} = \left[\boldsymbol{A}(t_P)\circ\boldsymbol{n}(t_P)\right]\boldsymbol{n}(t_P)\circ\bar{\boldsymbol{x}} = \left[\boldsymbol{n}(t_P)\circ\boldsymbol{A}(t_P)\right]\left[\boldsymbol{n}(t_P)\circ\bar{\boldsymbol{x}}\right]$$

to write the above as

$$\frac{\partial t}{\partial t_P} = 1 + \frac{1}{c^4}\gamma^3(t_P)V^2(t_P)\boldsymbol{A}_\|(t_P)\circ\bar{\boldsymbol{x}} + \frac{1}{c^2}\gamma(t_P)\boldsymbol{A}(t_P)\circ\bar{\boldsymbol{x}}$$

Next, use

$$\boldsymbol{A}(t_P)\circ\bar{\boldsymbol{x}} = \boldsymbol{A}_\|(t_P)\circ\bar{\boldsymbol{x}} + \boldsymbol{A}_\perp(t_P)\circ\bar{\boldsymbol{x}}$$

to write the above as

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$$\frac{\partial t}{\partial t_P}=1+\frac{1}{c^2}\gamma(t_P)\left[\gamma^2(t_P)\frac{V^2(t_P)}{c^2}+1\right]\boldsymbol{A}_{\|}(t_P)\circ\overline{\boldsymbol{x}}+\frac{1}{c^2}\gamma(t_P)\boldsymbol{A}_{\perp}(t_P)\circ\overline{\boldsymbol{x}}$$

Finally, use the identity (9.9) to write the above as

$$\frac{\partial t}{\partial t_P}=1+\frac{1}{c^2}\gamma^3(t_P)\boldsymbol{A}_{\|}(t_P)\circ\overline{\boldsymbol{x}}+\frac{1}{c^2}\gamma(t_P)\boldsymbol{A}_{\perp}(t_P)\circ\overline{\boldsymbol{x}} \tag{10.7a}$$

For the next derivative it is convenient to write (10.6) as

$$\boldsymbol{x}=\boldsymbol{X}(t_P)+\overline{\boldsymbol{x}}+\left[\gamma(t_P)-1\right]\overline{\boldsymbol{x}}_{\|}$$

So

$$\frac{\partial \boldsymbol{x}}{\partial t_P}=\boldsymbol{V}(t_P)+\frac{\partial\gamma(t_P)}{\partial t_P}\overline{\boldsymbol{x}}_{\|}+\left[\gamma(t_P)-1\right]\frac{\partial\overline{\boldsymbol{x}}_{\|}}{\partial t_P}$$

Using (9.6) and (9.8) gives

$$\frac{\partial \boldsymbol{x}}{\partial t_P}=\boldsymbol{V}(t_P)+\frac{1}{c^2}\gamma^3(t_P)\left[\boldsymbol{V}(t_P)\circ\boldsymbol{A}(t_P)\right]\overline{\boldsymbol{x}}_{\|}+$$

$$\frac{1}{V(t_P)}\left[\gamma(t_P)-1\right]\left[\left(\overline{\boldsymbol{x}}\circ\boldsymbol{A}_{\perp}(t_P)\right)\boldsymbol{n}(t_P)+\left(\overline{\boldsymbol{x}}\circ\boldsymbol{n}(t_P)\right)\boldsymbol{A}_{\perp}(t_P)\right].$$

Regrouping terms and using $\overline{\boldsymbol{x}}_{\|}=\left[\overline{\boldsymbol{x}}\circ\boldsymbol{n}(t_P)\right]\boldsymbol{n}(t_P)$ and $\boldsymbol{V}(t_P)=V(t_P)\boldsymbol{n}(t_P)$ allows the above to be rewritten as

Once more we use

$$\frac{\partial \boldsymbol{x}}{\partial t_P}=$$

$$\left\{V(t_P)+\frac{1}{c^2}\gamma^3(t_P)V(t_P)\left[\boldsymbol{n}(t_P)\circ\boldsymbol{A}(t_P)\right]\left[\overline{\boldsymbol{x}}\circ\boldsymbol{n}(t_P)\right]+\frac{1}{V(t_P)}\left[\gamma(t_P)-1\right]\left[\overline{\boldsymbol{x}}\circ\boldsymbol{A}_{-}\right.\right.$$

$$+\frac{1}{V(t_P)}\left[\gamma(t_P)-1\right]\left(\overline{\boldsymbol{x}}\circ\boldsymbol{n}(t_P)\right)\boldsymbol{A}_{\perp}(t_P).$$

to write the above as

$$\frac{\partial \boldsymbol{x}}{\partial t_P} =$$

$$\left\{ V(t_P) + \frac{1}{c^2}\gamma^3(t_P)V(t_P)\boldsymbol{A}_{\|}(t_P)\circ\overline{\boldsymbol{x}} + \frac{1}{V(t_P)}\left[\gamma(t_P)-1\right]\left[\overline{\boldsymbol{x}}\circ\boldsymbol{A}_{\perp}(t_P)\right]\right\}\boldsymbol{n}(t_P)$$

$$+\frac{1}{V(t_P)}\left[\gamma(t_P)-1\right]\left(\overline{\boldsymbol{x}}\circ\boldsymbol{n}(t_P)\right)\boldsymbol{A}_{\perp}(t_P).$$

The next derivatives are simpler to calculate but some notation is needed. Let $\boldsymbol{e}_{(x)}$, $\boldsymbol{e}_{(y)}$, and $\boldsymbol{e}_{(z)}$ be the unit vectors (in the context of three dimensional Euclidean geometry) in the directions of the x-axis, y-axis, and z-axis, respectively, in the traveling system. The subscripts appear in parentheses to emphasize that they are vector names or labels as opposed to vector components. Using this notation we easily obtain from (10.5) that

$$\frac{\partial t}{\partial \overline{x}} = \frac{1}{c^2}\gamma(t_P)\boldsymbol{e}_{(x)}\circ\boldsymbol{V}(t_P) \tag{10.7c}$$

$$\frac{\partial t}{\partial \overline{y}} = \frac{1}{c^2}\gamma(t_P)\boldsymbol{e}_{(y)}\circ\boldsymbol{V}(t_P) \tag{10.7d}$$

$$\frac{\partial t}{\partial \overline{z}} = \frac{1}{c^2}\gamma(t_P)\boldsymbol{e}_{(z)}\circ\boldsymbol{V}(t_P) \tag{10.7e}$$

To calculate the remaining derivatives, we note that partial derivatives that hold t_P fixed, so they hold $\boldsymbol{n}(t_P)$ fixed, commute with the taking of parallel parts or perpendicular parts. Specifically,

$$\frac{\partial \overline{\boldsymbol{x}}_{\|}}{\partial \overline{x}} = \left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \overline{x}}\right)_{\|} = \boldsymbol{e}_{(x)\|}, \qquad \frac{\partial \overline{\boldsymbol{x}}_{\perp}}{\partial \overline{x}} = \left(\frac{\partial \overline{\boldsymbol{x}}}{\partial \overline{x}}\right)_{\perp} = \boldsymbol{e}_{(x)\perp}$$

with analogous results for $\overline{y}$ and $\overline{z}$ derivatives. Using this fact with (10.6) easily gives

$$\frac{\partial \boldsymbol{x}}{\partial \overline{x}} = \boldsymbol{e}_{(x)\perp} + \gamma(t_P)\boldsymbol{e}_{(x)\|} \tag{10.7f}$$

$$\frac{\partial \boldsymbol{x}}{\partial \overline{y}} = \boldsymbol{e}_{(y)\perp} + \gamma(t_P)\boldsymbol{e}_{(y)\|} \tag{10.7g}$$

$$\frac{\partial \boldsymbol{x}}{\partial \overline{z}} = \boldsymbol{e}_{(z)\perp} + \gamma(t_P)\boldsymbol{e}_{(z)\|} \tag{10.7h}$$

Various products of derivatives are prepared in advance to assist with the evaluation of the metric tensor. We first shorten the notation by defining T-functions that are characterized by the property of being zero if either $\overline{\boldsymbol{x}}$ or $A(t_P)$ are null vectors. They are defined by

$$T_1(t_P,\overline{\boldsymbol{x}}) \equiv \frac{1}{c^2}\gamma^3(t_P)\boldsymbol{A}_{\|}(t_P)\circ\overline{\boldsymbol{x}} \tag{10.8a}$$

$$T_2(t_P,\overline{\boldsymbol{x}}) \equiv \frac{1}{V^2(t_P)}\boldsymbol{A}_{\perp}(t_P)\circ\overline{\boldsymbol{x}} \tag{10.8b}$$

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$$\boldsymbol{T}_{\perp}\left(t_P,\overline{\boldsymbol{x}}\right) \equiv \frac{1}{V\left(t_P\right)}\left[\gamma\left(t_P\right)-1\right]\left(\overline{\boldsymbol{x}}\circ\boldsymbol{n}\left(t_P\right)\right)\boldsymbol{A}_{\perp}\left(t_P\right) \tag{10.8c}$$

where the perpendicular symbol is included as a reminder that the vector $\boldsymbol{T}_{\perp}\left(t_P,\overline{\boldsymbol{x}}\right)$ is orthogonal (in the context of three-dimensional Euclidean geometry) to $\boldsymbol{n}(t_P)$ (recall that $\boldsymbol{n}(t_P)$ is the unit vector, in the context of three-dimensional Euclidean geometry, in the direction of $\boldsymbol{V}(t_P)$). With these definitions we can write (10.7a) and (10.7b) as

$$\frac{\partial t}{\partial t_P} = 1 + T_1\left(t_P,\overline{\boldsymbol{x}}\right) + \frac{V^2\left(t_P\right)}{c^2}\gamma\left(t_P\right)T_2\left(t_P,\overline{\boldsymbol{x}}\right) \tag{10.9a}$$

$$\frac{\partial \boldsymbol{x}}{\partial t_P} = V\left(t_P\right)\left\{1 + T_1\left(t_P,\overline{\boldsymbol{x}}\right) + \left[\gamma\left(t_P\right)-1\right]T_2\left(t_P,\overline{\boldsymbol{x}}\right)\right\}\boldsymbol{n}\left(t_P\right) + \boldsymbol{T}_{\perp}\left(t_P,\overline{\boldsymbol{x}}\right) \tag{10.9b}$$

We now list several products of derivatives. We temporarily shorten the notation by not displaying the arguments of the various functions. Full notation will be restored in final results derived for the metric tensor. It is evident from (10.9a) that

$$\left(\frac{\partial t}{\partial t_P}\right)^2 = \left(1 + T_1 + \gamma\frac{V^2}{c^2}T_2\right)^2 \tag{10.10a}$$

Also, orthogonality between $\boldsymbol{T}_{\perp}\left(t_P,\overline{\boldsymbol{x}}\right)$ and $\boldsymbol{n}(t_P)$ together with (10.9b) gives

$$\frac{\partial \boldsymbol{x}}{\partial t_P}\circ\frac{\partial \boldsymbol{x}}{\partial t_P} = V^2\left(1 + T_1 + \left[\gamma-1\right]T_2\right)^2 + \boldsymbol{T}_{\perp}\circ\boldsymbol{T}_{\perp} \tag{10.10b}$$

We next use (10.9a) with (10.7c) to get

$$\frac{\partial t}{\partial t_P}\frac{\partial t}{\partial \overline{x}} = \frac{1}{c^2}\gamma V\left(1 + T_1 + \frac{V^2}{c^2}\gamma T_2\right)\boldsymbol{e}_{(x)}\circ\boldsymbol{n} \tag{10.11a}$$

Next use (10.9b) with (10.7f) while paying attention to which vectors are parallel to each other, which are orthogonal to each other, and use the fact that $\boldsymbol{n}\left(t_P\right)\circ\boldsymbol{e}_{(x)\|} = \boldsymbol{n}\left(t_P\right)\circ\boldsymbol{e}_{(x)}$ to get

$$\frac{\partial \boldsymbol{x}}{\partial t_P}\circ\frac{\partial \boldsymbol{x}}{\partial \overline{x}} = \gamma V\left(1 + T_1 + \left[\gamma-1\right]T_2\right)\boldsymbol{n}\circ\boldsymbol{e}_{(x)} + \boldsymbol{T}_{\perp}\circ\boldsymbol{e}_{(x)\perp} \tag{10.11b}$$

The next products considered are obtained from (10.7c) and (10.7f) which give

$$\left(\frac{\partial t}{\partial \overline{x}}\right)^2 = \frac{1}{c^4}\gamma^2V^2\left[\boldsymbol{e}_{(x)}\circ\boldsymbol{n}\right]^2 \tag{10.12a}$$

$$\frac{\partial \boldsymbol{x}}{\partial \overline{x}}\circ\frac{\partial \boldsymbol{x}}{\partial \overline{x}} = \left[\boldsymbol{e}_{(x)\perp} + \gamma\boldsymbol{e}_{(x)\|}\right]\circ\left[\boldsymbol{e}_{(x)\perp} + \gamma\boldsymbol{e}_{(x)\|}\right]$$

Expanding the product in the second equation gives

$$\frac{\partial \boldsymbol{x}}{\partial \overline{x}}\circ\frac{\partial \boldsymbol{x}}{\partial \overline{x}} = \boldsymbol{e}_{(x)\perp}\circ\boldsymbol{e}_{(x)\perp} + \gamma^2\boldsymbol{e}_{(x)\|}\circ\boldsymbol{e}_{(x)\|}$$

But we also have

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$$1 = \boldsymbol{e}_{(x)} \circ \boldsymbol{e}_{(x)} = \left[\boldsymbol{e}_{(x)||} + \boldsymbol{e}_{(x)\perp}\right] \circ \left[\boldsymbol{e}_{(x)||} + \boldsymbol{e}_{(x)\perp}\right] = \boldsymbol{e}_{(x)||} \circ \boldsymbol{e}_{(x)||} + \boldsymbol{e}_{(x)\perp} \circ \boldsymbol{e}_{(x)\perp}$$

Or

$$\boldsymbol{e}_{(x)\perp} \circ \boldsymbol{e}_{(x)\perp} = 1 - \boldsymbol{e}_{(x)||} \circ \boldsymbol{e}_{(x)||}$$

so the above equation becomes

$$\frac{\partial \boldsymbol{x}}{\partial \bar{x}} \circ \frac{\partial \boldsymbol{x}}{\partial \bar{x}} = 1 + \left[\gamma^2 - 1\right] \boldsymbol{e}_{(x)||} \circ \boldsymbol{e}_{(x)||}$$

We now use the identity (9.9) to write the above as

$$\frac{\partial \boldsymbol{x}}{\partial \bar{x}} \circ \frac{\partial \boldsymbol{x}}{\partial \bar{x}} = 1 + \frac{1}{c^2} \gamma^2 V^2 \boldsymbol{e}_{(x)||} \circ \boldsymbol{e}_{(x)||} =$$

$$1 + \frac{1}{c^2} \gamma^2 V^2 \left[\left(\boldsymbol{e}_{(x)} \circ \boldsymbol{n}\right) \boldsymbol{n}\right] \circ \left[\left(\boldsymbol{e}_{(x)} \circ \boldsymbol{n}\right) \boldsymbol{n}\right] = 1 + \frac{1}{c^2} \gamma^2 V^2 \left[\boldsymbol{e}_{(x)} \circ \boldsymbol{n}\right]^2$$

or

$$\frac{\partial \boldsymbol{x}}{\partial \bar{x}} \circ \frac{\partial \boldsymbol{x}}{\partial \bar{x}} = 1 + \frac{1}{c^2} \gamma^2 V^2 \left[\boldsymbol{e}_{(x)} \circ \boldsymbol{n}\right]^2 \qquad (10.12b)$$

The last pair of products considered are obtained from (10.7c), (10.7d), (10.7f), and (10.7g). These give

$$\frac{\partial t}{\partial \bar{x}} \frac{\partial t}{\partial \bar{y}} = \frac{1}{c^4} \gamma^2 V^2 \left[\boldsymbol{e}_{(x)} \circ \boldsymbol{n}\right] \left[\boldsymbol{e}_{(y)} \circ \boldsymbol{n}\right] \qquad (10.13a)$$

$$\frac{\partial \boldsymbol{x}}{\partial \bar{x}} \circ \frac{\partial \boldsymbol{x}}{\partial \bar{y}} = \left[\boldsymbol{e}_{(x)\perp} + \gamma \boldsymbol{e}_{(x)||}\right] \circ \left[\boldsymbol{e}_{(y)\perp} + \gamma \boldsymbol{e}_{(y)||}\right]$$

Expanding the product in the second equation gives

$$\frac{\partial \boldsymbol{x}}{\partial \bar{x}} \circ \frac{\partial \boldsymbol{x}}{\partial \bar{y}} = \boldsymbol{e}_{(x)\perp} \circ \boldsymbol{e}_{(y)\perp} + \gamma^2 \boldsymbol{e}_{(x)||} \circ \boldsymbol{e}_{(y)||}$$

But we also have

$$0 = \boldsymbol{e}_{(x)} \circ \boldsymbol{e}_{(y)} = \left[\boldsymbol{e}_{(x)||} + \boldsymbol{e}_{(x)\perp}\right] \circ \left[\boldsymbol{e}_{(y)||} + \boldsymbol{e}_{(y)\perp}\right] = \boldsymbol{e}_{(x)||} \circ \boldsymbol{e}_{(y)||} + \boldsymbol{e}_{(x)\perp} \circ \boldsymbol{e}_{(y)\perp}$$

or

$$\boldsymbol{e}_{(x)\perp} \circ \boldsymbol{e}_{(y)\perp} = -\boldsymbol{e}_{(x)||} \circ \boldsymbol{e}_{(y)||}$$

so the above equation becomes

$$\frac{\partial \boldsymbol{x}}{\partial \bar{x}} \circ \frac{\partial \boldsymbol{x}}{\partial \bar{y}} = \left[\gamma^2 - 1\right] \boldsymbol{e}_{(x)||} \circ \boldsymbol{e}_{(y)||}$$

We now use the identity (9.9) to write the above as

$$\frac{\partial \boldsymbol{x}}{\partial \bar{x}} \circ \frac{\partial \boldsymbol{x}}{\partial \bar{y}} = \frac{1}{c^2} \gamma^2 V^2 \boldsymbol{e}_{(x)\parallel} \circ \boldsymbol{e}_{(y)\parallel} =$$

$$\frac{1}{c^2} \gamma^2 V^2 \left[\left(\boldsymbol{e}_{(x)} \circ \boldsymbol{n} \right) \boldsymbol{n} \right] \circ \left[\left(\boldsymbol{e}_{(y)} \circ \boldsymbol{n} \right) \boldsymbol{n} \right] = \frac{1}{c^2} \gamma^2 V^2 \left(\boldsymbol{e}_{(x)} \circ \boldsymbol{n} \right) \left(\boldsymbol{e}_{(y)} \circ \boldsymbol{n} \right)$$

or

$$\frac{\partial \boldsymbol{x}}{\partial \bar{x}} \circ \frac{\partial \boldsymbol{x}}{\partial \bar{y}} = \frac{1}{c^2} \gamma^2 V^2 \left[\boldsymbol{e}_{(x)} \circ \boldsymbol{n} \right] \left[\boldsymbol{e}_{(y)} \circ \boldsymbol{n} \right] \tag{10.13b}$$

We can now assemble various products of derivatives to produce the metric tensor. The two equations in (10.10) together with (10.4) give

$$\bar{g}_{0,0} = \gamma^2 (t_P) \left\{ 1 + T_1 (t_P, \bar{\boldsymbol{x}}) + \gamma \frac{V^2 (t_P)}{c^2} T_2 (t_P, \bar{\boldsymbol{x}}) \right\}^2$$

$$- \gamma^2 (t_P) \frac{V^2 (t_P)}{c^2} \left\{ 1 + T_1 (t_P, \bar{\boldsymbol{x}}) + \left[\gamma (t_P) - 1 \right] T_2 (t_P, \bar{\boldsymbol{x}}) \right\}^2 - \gamma^2 (t_P) \frac{1}{c^2} \boldsymbol{T}_\perp (t_P, \bar{\boldsymbol{x}}) \circ \boldsymbol{T}_\perp \tag{10.14a}$$

.

Another component is obtained from the two equations in (10.11) together with (10.4) which give

$$\bar{g}_{0,1} = \frac{1}{c} \gamma \left\{ V \left[\gamma - \left(1 - \frac{V^2}{c^2} \right) \gamma^2 \right] T_2 \boldsymbol{n} \circ \boldsymbol{e}_{(x)} - \boldsymbol{T}_\perp \circ \boldsymbol{e}_{(x)} \right\}$$

where we used $\boldsymbol{T}_\perp \circ \boldsymbol{e}_{(x)} = \boldsymbol{T}_\perp \circ \left[\boldsymbol{e}_{(x)\parallel} + \boldsymbol{e}_{(x)\perp} \right] = \boldsymbol{T}_\perp \circ \boldsymbol{e}_{(x)\perp}$. Expressing the parenthesis in terms of γ gives.

$$\bar{g}_{0,1} = \frac{1}{c} \gamma (t_P) \left\{ V (t_P) \left[\gamma (t_P) - 1 \right] T_2 (t_P, \bar{\boldsymbol{x}}) \boldsymbol{n} (t_P) \circ \boldsymbol{e}_{(x)} - \boldsymbol{T}_\perp (t_P, \bar{\boldsymbol{x}}) \circ \boldsymbol{e}_{(x)} \right\} \tag{10.14b}$$

$$\bar{g}_{0,2} = \frac{1}{c} \gamma (t_P) \left\{ V (t_P) \left[\gamma (t_P) - 1 \right] T_2 (t_P, \bar{\boldsymbol{x}}) \boldsymbol{n} (t_P) \circ \boldsymbol{e}_{(y)} - \boldsymbol{T}_\perp (t_P, \bar{\boldsymbol{x}}) \circ \boldsymbol{e}_{(y)} \right\} \tag{10.14c}$$

$$\bar{g}_{0,3} = \frac{1}{c} \gamma (t_P) \left\{ V (t_P) \left[\gamma (t_P) - 1 \right] T_2 (t_P, \bar{\boldsymbol{x}}) \boldsymbol{n} (t_P) \circ \boldsymbol{e}_{(z)} - \boldsymbol{T}_\perp (t_P, \bar{\boldsymbol{x}}) \circ \boldsymbol{e}_{(z)} \right\} \tag{10.14d}$$

Similarly

The next diagonal element, $\bar{g}_{1,1}$, is obtained by combining the two equations in (10.12) with (10.4). Including analogous results for $\bar{g}_{2,2}$ and $\bar{g}_{3,3}$ we obtain

$$\bar{g}_{1,1} = \bar{g}_{2,2} = \bar{g}_{3,3} = -1 \tag{10.14e}$$

The off diagonal element $\bar{g}_{1,2}$ is obtained from the two equations in (10.13) together with (10.4). The result is obvious, and including analogous results for other terms gives

$$\bar{g}_{1,2} = \bar{g}_{1,3} = \bar{g}_{2,3} = 0 \tag{10.14f}$$

All other components of the metric tensor are implied by the symmetric condition $\bar{g}_{i,j} = \bar{g}_{j,i}$.

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The equations in (10.14) give the components of the metric tensor as functions of t_P and $\overline{\boldsymbol{x}}$. However, t_P is itself a function of $\overline{t}$, obtained by inverting (3.1) which is written below with the subscript E omitted as

$$\overline{t} = \tau(t_P) \tag{10.15}$$

Recognizing this implicit dependence on $\overline{t}$, (10.14) gives the components of the metric tensor as functions of $\overline{t}$ and $\overline{\boldsymbol{x}}$.

Recall that the T-functions are each zero if either $\overline{\boldsymbol{x}} = 0$ or t_P is a point at which $\boldsymbol{A}(t_P) = 0$. If either of these conditions are satisfied, so all T-functions are zero, the metric reduces to the Lorentz metric. This is seen by a casual inspection for all elements except $\overline{g}_{0,0}$. That $\overline{g}_{0,0} = 1$ when all T-functions are zero is seen by noting that (10.14a) reduces to

$$\overline{g}_{0,0} = \gamma^2(t_P) - \gamma^2(t_P)\frac{V^2(t_P)}{c^2}$$

when all T-functions are zero. This gives $\overline{g}_{0,0} = 1$ via (9.9), confirming that the metric reduces to the Lorentz metric if either $\overline{\boldsymbol{x}} = 0$ or t_P is a point at which $\boldsymbol{A}(t_P) = 0$.

11. 4-Vectors

Previous sections explain how various quantities can be calculated when the given information is the trajectory of the traveler's clock relative to the home system and expressed in terms of the home system coordinates. However, the next section gives an example in which this trajectory is not the given information. Instead, it is necessary to deduce this trajectory from other information that is given. Quantities called 4-vectors provide computational conveniences that make this deduction easier, so a review of 4-vectors is given in this section. This review emphasizes a distinction between vectors and vector components. Also, the notation used here uses bold font for 4-vectors, as was previously done for three-dimensional spatial vectors, but distinguishes between them by using cursive font for 4-vectors and block letters for three-dimensional spatial vectors.

A good illustrative example of a 4-vector is the 4-velocity, denoted **V** in the present discussion (**V** will be an arbitrary 4-vector in later discussions in this section) of some particle at some point on its worldline. This will be defined after explaining one of the distinctions between it and the three-dimensional velocity $\boldsymbol{V}$ which is the derivative of spatial coordinates with respect to the time coordinate. The three-dimensional velocity of the same particle is a different vector in different reference frames that move relative to each other. In contrast, the 4-velocity is the same vector in different reference frames, it is only its components with respect to different basis vectors that are different. Therefore the 4-velocity is completely defined when we specify its components in any convenient set of basis vectors. It is convenient to use the basis vectors assigned to the home system to specify **V**, but we must first decide on what the home system basis vectors will be. We will use the same basis vectors for the home system that were denoted $\boldsymbol{e}_{(x)}$, $\boldsymbol{e}_{(y)}$, and $\boldsymbol{e}_{(z)}$ that were used in the traveling system,[7] except that we include another vector $\boldsymbol{e}_{(ct)}$ in the direction of the time axis. We also change notation by using $\mathbf{e}_{(0)}$, $\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$, and $\mathbf{e}_{(3)}$ in cursive font and with integers for indices, instead of $\boldsymbol{e}_{(ct)}$, $\boldsymbol{e}_{(x)}$, $\boldsymbol{e}_{(y)}$, and $\boldsymbol{e}_{(z)}$, to emphasize four dimensions. As discussed above, the 4-velocity is completely determined in all reference frames if we specify its components in the home basis vectors. It is therefore completely defined by the equation

$$\mathcal{V} = \frac{dx^0}{d\tau}\boldsymbol{e}_{(0)} + \frac{dx^1}{d\tau}\boldsymbol{e}_{(1)} + \frac{dx^2}{d\tau}\boldsymbol{e}_{(2)} + \frac{dx^3}{d\tau}\boldsymbol{e}_{(3)}$$

where $d\tau$ is an increment of proper time along the particle's worldline, and we use the notation

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[7] It will be seen later that different basis vectors will be needed for the traveling system in order for vector components to obey the transformations of contravariant vector components.

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$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z \tag{11.1}$$

The velocity vector was selected for illustration of a 4-vector but in the remainder of this section, **V** is an arbitrary 4-vector. It is uniquely determined (as shown later) in all systems if we specify the numbers V^0, V^1, V^2, and V^3 satisfying

$$\mathcal{V} = V^0 \boldsymbol{e}_{(0)} + V^1 \boldsymbol{e}_{(1)} + V^2 \boldsymbol{e}_{(2)} + V^3 \boldsymbol{e}_{(3)} \tag{11.2}$$

The coefficients to the basis vectors in such an expansion are called the contravariant components of **V**.[8] We use the notation in which $\mathbf{V}^{\,i}$ denotes the i^{th} contravariant component in whatever inertial system that was selected to be called the unbarred system and that uses the basis vectors in (11.2). Therefore (11.2) can also be written as

$$\mathcal{V} = \mathcal{V}^0 \boldsymbol{e}_{(0)} + \mathcal{V}^1 \boldsymbol{e}_{(1)} + \mathcal{V}^2 \boldsymbol{e}_{(2)} + \mathcal{V}^3 \boldsymbol{e}_{(3)} \tag{11.3}$$

Now consider the contravariant components in another system, called the barred system, that uses basis vectors $\overline{\boldsymbol{e}}_{(0)}$, $\overline{\boldsymbol{e}}_{(1)}$, $\overline{\boldsymbol{e}}_{(2)}$, and $\overline{\boldsymbol{e}}_{(3)}$. Again, the contravariant components in that system are the coefficients in the expansion in those basis vectors, so if we let $\mathcal{V}^{\bar{i}}$ denote the i^{th} contravariant component in the barred system we have

$$\mathcal{V} = \mathcal{V}^{\bar{0}} \overline{\boldsymbol{e}}_{(0)} + \mathcal{V}^{\bar{1}} \overline{\boldsymbol{e}}_{(1)} + \mathcal{V}^{\bar{2}} \overline{\boldsymbol{e}}_{(2)} + \mathcal{V}^{\bar{3}} \overline{\boldsymbol{e}}_{(3)} \tag{11.4}$$

However, in order for contravariant components in different systems to be related by the transformations traditionally used to define contravariant components, the basis vectors in the barred system are not arbitrary. They are tangent vectors to coordinate curves and are uniquely determined after the coordinates in the barred system have been selected. Denoting these coordinates as $(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$, or more briefly as $\bar{\mathcal{X}}$, the barred basis vectors, which can be functions of the barred coordinates, are defined by

$$\overline{\boldsymbol{e}}_{(i)}(\bar{\mathcal{X}}) \equiv \frac{\partial \mathcal{X}}{\partial \bar{x}^i} \quad \text{for } i = 0,1,2,3 \tag{11.5a}$$

where $\mathcal{X}$ is defined by

$$\mathcal{X} = x^0 \boldsymbol{e}_{(0)} + x^1 \boldsymbol{e}_{(1)} + x^2 \boldsymbol{e}_{(2)} + x^3 \boldsymbol{e}_{(3)} \tag{11.5b}$$

so (11.5a) becomes

$$\overline{\boldsymbol{e}}_{(i)}(\bar{\mathcal{X}}) \equiv \frac{\partial x^0}{\partial \bar{x}^i} \boldsymbol{e}_{(0)} + \frac{\partial x^1}{\partial \bar{x}^i} \boldsymbol{e}_{(1)} + \frac{\partial x^2}{\partial \bar{x}^i} \boldsymbol{e}_{(2)} + \frac{\partial x^3}{\partial \bar{x}^i} \boldsymbol{e}_{(3)} = \sum_j \frac{\partial x^j}{\partial \bar{x}^i} \boldsymbol{e}_{(j)} \quad \text{for } i = 0,1,2,3 \tag{11.6}$$

with the understanding that a summation that does not explicitly show the range of the index uses the range of 0 to 3 (the Einstein summation convention is not used because there are exceptions to the rule, and it is not difficult to include a summation symbol to avoid any possible confusion).[9]

---

[8] In contrast, covariant components are defined in terms of dot products. There is no distinction between contravariant and covariant components when the basis vectors are an orthonormal set, but it will be seen later that the basis vectors used here are not an orthonormal set as defined by a solid dot-product denoted • and defined later (the basis vectors are mutually orthogonal but not all of them have unit norm), so there is a distinction between contravariant and covariant components.

[9] It might be noted that [10] stated that the derivative $\partial / \partial \bar{x}^0$ is a derivative with respect to proper time, making $\overline{\boldsymbol{e}}_{(0)}$ in (11.6) the 4-velocity. In reality, $\bar{x}^0$ is proper time only on the worldline of the

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An important and convenient consequence of defining the barred basis vectors by (11.6) is that the transformation between basis vectors can easily be inverted to solve for the unbarred basis vectors in terms of the barred basis vectors. To perform this inversion, note that the Kronecker delta can be expressed as

$$\delta_k{}^j = \frac{\partial x^j}{\partial x^k}$$

When the selected spacetime coordinates are in neighborhoods for which there is an invertible transformation between barred and unbarred coordinates, the chain rule applied to the above gives

$$\delta_k{}^j = \sum_i \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial x^k} \tag{11.7}$$

Now multiply both sides of (11.6) by $\partial \bar{x}^i / \partial x^k$ and sum in *i* to get

$$\sum_i \frac{\partial \bar{x}^i}{\partial x^k}\bar{\boldsymbol{e}}_{(i)}(\bar{x}) = \sum_{j,i}\frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial x^k}\boldsymbol{e}_{(j)}$$

Next, we use (11.7) to write this as

$$\sum_i \frac{\partial \bar{x}^i}{\partial x^k}\bar{\boldsymbol{e}}_{(i)}(\bar{x}) = \sum_j \delta_k{}^j \boldsymbol{e}_{(j)} = \boldsymbol{e}_{(k)}$$

Changing dummy symbols gives

$$\boldsymbol{e}_{(i)} = \sum_j \frac{\partial \bar{x}^j}{\partial x^i}\bar{\boldsymbol{e}}_{(j)}(\bar{x}) \tag{11.8}$$

which is the inverse transform between basis vectors.

We can now confirm that contravariant components satisfy the transformation typically used to define contravariant components. Combine (11.3) with (11.4) to get

$$\sum_i \mathcal{V}^{\bar{i}}\,\bar{\boldsymbol{e}}_{(i)}(\bar{x}) = \sum_i \mathcal{V}^i \boldsymbol{e}_{(i)} \tag{11.9}$$

Now substitute (11.8) into the right side of (11.9) and interchange dummy symbols to get

$$\sum_i \mathcal{V}^{\bar{i}}\,\bar{\boldsymbol{e}}_{(i)}(\bar{x}) = \sum_{i,j}\mathcal{V}^j \frac{\partial \bar{x}^i}{\partial x^j}\bar{\boldsymbol{e}}_{(i)}(\bar{x})$$

The barred basis vectors are linearly independent so the above gives

$$\mathcal{V}^{\bar{i}} = \sum_j \mathcal{V}^j \frac{\partial \bar{x}^i}{\partial x^j} \tag{11.10}$$

which is the transformation typically taken to be the definition of contravariant vector components.

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traveler's origin. Therefore, $\bar{\boldsymbol{e}}_{(0)}$ is the 4-velocity when evaluated at $\bar{\boldsymbol{x}}_E = \boldsymbol{0}$ but is more complicated when evaluated at $\bar{\boldsymbol{x}}_E \neq \boldsymbol{0}$. The general construction of the barred basis vectors combines (11.6) with (10.7).

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Covariant vector components can also be defined but to help with notation we start with a definition of the solid dot product denoted $\bullet$. This dot product between two arbitrary 4-vectors **V** and **U** is defined by

$$\mathcal{V}\bullet\mathcal{U}\equiv\sum_{i,j}g_{i,j}\mathcal{V}^{i}\mathcal{U}^{j} \tag{11.11a}$$

where the superscripts on the right denote contravariant components in the unbarred system, and g is the metric tensor in the unbarred system. The unbarred system is inertial and rectangular, so its metric tensor is the Lorentz metric, which allows us to rewrite (11.11) as

$$\mathcal{V}\bullet\mathcal{U}=\mathcal{V}^{0}\mathcal{U}^{0}-\sum_{i=1}^{3}\mathcal{V}^{i}\mathcal{U}^{i} \tag{11.11b}$$

However, the notation in (11.11a) is more convenient than the notation in (11.11b) for deriving an expression for the dot product in the barred system. Recall that the barred metric tensor is defined by the transformation (10.1), and we already established the transformation (11.10) for contravariant components of 4-vectors. We can show via (11.7) that the composite of these transformations applied to the right side of (11.11a) produces

$$\mathcal{V}\bullet\mathcal{U}=\sum_{i,j}\overline{g}_{i,j}\mathcal{V}^{\overline{i}}\mathcal{U}^{\overline{j}} \tag{11.11c}$$

In summary, three expressions derived (so far) for solid dot products are given by the three equations in (11.11). [10]

Special interesting cases are dot products between basis vectors. Recall that the contravariant component $\boldsymbol{e}_{(i)}{}^{j}$ is the j^{th} coefficient in the expansion of $\boldsymbol{e}_{(i)}$ in the unbarred basis vectors. Similarly, the contravariant component $\overline{\boldsymbol{e}}_{(i)}{}^{\overline{j}}$ is the j^{th} coefficient in the expansion of $\overline{\boldsymbol{e}}_{(i)}$ in the barred basis vectors.[11] Therefore we have

$$\boldsymbol{e}_{(i)}{}^{j}=\overline{\boldsymbol{e}}_{(i)}{}^{\overline{j}}=\delta_{i}^{j} \tag{11.12a}$$

Also, the definition of contravariant components as coefficients in expansions together with (11.6) and (11.8) give

$$\overline{\boldsymbol{e}}_{(i)}{}^{j}(\overline{x})=\frac{\partial x^{j}}{\partial\overline{x}^{i}},\qquad \boldsymbol{e}_{(i)}{}^{\overline{j}}(x)=\frac{\partial\overline{x}^{j}}{\partial x^{i}} \tag{11.12b}$$

where a coordinate dependence of $\boldsymbol{e}_{(i)}{}^{\overline{j}}$ is indicated and is due to a coordinate dependence of the basis vectors. Substituting (11.12a) into (11.11a) for the unbarred case, and into (11.11c) for the barred case, gives

$$\boldsymbol{e}_{(i)}\bullet\boldsymbol{e}_{(j)}=g_{i,j},\quad \overline{\boldsymbol{e}}_{(i)}\bullet\overline{\boldsymbol{e}}_{(j)}=\overline{g}_{i,j} \tag{11.13}$$

Listing the individual products for the unbarred case gives

$$\boldsymbol{e}_{(0)}\bullet\boldsymbol{e}_{(0)}=1,\quad \boldsymbol{e}_{(i)}\bullet\boldsymbol{e}_{(i)}=-1\text{ for }i=1,2,3,\quad \boldsymbol{e}_{(i)}\bullet\boldsymbol{e}_{(j)}=0\text{ for }i\neq j \tag{11.14}$$

Note that the middle equation is one of those exceptions to the Einstein summation convention and is an example of the reason for not using that convention.

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[10] Symmetry of the transformation (10.1) together with symmetry of the Lorentz metric implies symmetry of all metrics so the order of dot-product multiplication is seen to be reversible in any of the expressions used for it.

[11] We sometimes shorten the notation by not displaying the coordinate dependence of the barred basis vectors, but when that is done, it must be remembered that they may be functions of coordinates.

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The covariant components of an arbitrary 4-vector **V** in the unbarred and barred systems, with the i^{th} component denoted $\mathcal{V}_i$ in the unbarred system and denoted $\mathcal{V}_{\bar{i}}$ in the barred system, are defined by

$$\mathcal{V}_i \equiv \mathcal{V} \bullet \boldsymbol{e}_{(i)} \qquad \mathcal{V}_{\bar{i}} \equiv \mathcal{V} \bullet \bar{\boldsymbol{e}}_{(i)} \tag{11.15}$$

The transformation between barred and unbarred covariant components is easily derived by starting with (11.6) to get

$$\mathcal{V} \bullet \bar{\boldsymbol{e}}_{(i)} = \mathcal{V} \bullet \sum_j \frac{\partial x^j}{\partial \bar{x}^i} \boldsymbol{e}_{(j)} = \sum_j \frac{\partial x^j}{\partial \bar{x}^i} \mathcal{V} \bullet \boldsymbol{e}_{(j)}$$

and using (11.15) gives

$$\mathcal{V}_{\bar{i}} = \sum_j \frac{\partial x^j}{\partial \bar{x}^i} \mathcal{V}_j \tag{11.16}$$

which is the transformation traditionally used to define covariant vector components.[12]

Covariant and contravariant components of vectors can be related to each other by using (11.11) to get

$$\mathcal{V} \bullet \boldsymbol{e}_{(i)} \equiv \sum_{j,k} g_{j,k} \mathcal{V}^j \boldsymbol{e}_{(i)}{}^k, \quad \mathcal{V} \bullet \boldsymbol{e}_{(i)} = \mathcal{V}^0 \boldsymbol{e}_{(i)}{}^0 - \sum_{j=1}^{3} \mathcal{V}^j \boldsymbol{e}_{(i)}{}^j, \quad \mathcal{V} \bullet \bar{\boldsymbol{e}}_{(i)} = \sum_{j,k} \bar{g}_{j,k} \mathcal{V}^{\bar{j}} \bar{\boldsymbol{e}}_{(i)}{}^{\bar{k}}$$

Substituting (11.12a) and (11.15) into the above gives

$$\mathcal{V}_i = \sum_j g_{j,i} \mathcal{V}^j = \sum_j g_{i,j} \mathcal{V}^j \tag{11.17a}$$

$$\mathcal{V}_0 = \mathcal{V}^0, \qquad \mathcal{V}_i = -\mathcal{V}^i \quad \text{for } i = 1, 2, 3 \tag{11.17b}$$

$$\mathcal{V}_{\bar{i}} = \sum_j \bar{g}_{j,i} \mathcal{V}^{\bar{j}} = \sum_j \bar{g}_{i,j} \mathcal{V}^{\bar{j}} \tag{11.17c}$$

where we used the fact that the metric tensors are symmetric.

Several expressions for dot products were listed in (11.11). Two more expressions are obtained by combining (11.17a) with (11.11a) and combining (11.17c) with (11.11c). The results are

$$\mathcal{V} \bullet \mathcal{U} = \sum_i \mathcal{V}_i \mathcal{U}^i \tag{11.18a}$$

$$\mathcal{V} \bullet \mathcal{U} = \sum_i \mathcal{V}_{\bar{i}} \mathcal{U}^{\bar{i}} \tag{11.18b}$$

Now consider an arbitrary worldline, the trajectory in spacetime of some moving point, as seen by the inertial unbarred system. Using equations in Section 2, an increment of proper time along this worldline can be obtained by taking the differential of (2.3) and then using (2.4) to get

$$d\tau = \sqrt{1 - \frac{V^2(t)}{c^2}}\, dt$$

[12] Covariant components defined by other dot products (e.g., a Euclidean dot product in which the unbarred metric tensor is the identity matrix, and the barred metric tensor is defined by (10.1)), will also satisfy (11.16) providing that the definition of the dot product is used consistently as needed to satisfy (11.11a) and (11.11c). The case in which the metric in the unbarred system is the Lorentz metric, and produces (11.11b), is assumed throughout this analysis but is not required by (11.16).

which can also be written as

$$(d\tau)^2 = \left(1 - \frac{V^2(t)}{c^2}\right)(dt)^2 = \frac{1}{c^2}\left(\left(dx^0\right)^2 - d\boldsymbol{x} \circ d\boldsymbol{x}\right) \tag{11.19}$$

If we now define the 4-vector

$$d\chi = dx^0 \boldsymbol{e}_{(0)} + dx^1 \boldsymbol{e}_{(1)} + dx^2 \boldsymbol{e}_{(2)} + dx^3 \boldsymbol{e}_{(3)}$$

and use (11.11b) to express $d\chi \bullet d\chi$ and compare that expression to the right side of (11.19) we find that (11.19) can be written as

$$(d\tau)^2 = \frac{1}{c^2} d\chi \bullet d\chi \tag{11.20}$$

One implication of (11.20) is seen by using (11.11c) to write (11.20) as

$$(d\tau)^2 = \frac{1}{c^2} \sum_{i,j} \bar{g}_{i,j} d\chi^{\bar{i}} d\chi^{\bar{j}}$$

If we now take the barred system to be attached to the moving point, the differentials of spatial coordinates are zero so the above reduces to

$$(d\tau)^2 = \frac{1}{c^2} \bar{g}_{0,0} d\chi^{\bar{0}} d\chi^{\bar{0}}$$

It was concluded at the end of Section 11 that if we take the origin (the location of the clock) of the barred system to be attached to the moving point (the spatial coordinates of the point are zero in the barred system) then $\bar{g}_{0,0} = 1$ so the above becomes

$$d\tau = \frac{1}{c} d\chi^{\bar{0}} \quad \text{(comoving system).} \tag{11.21}$$

This is consistent with the conclusion already used in Section 2, but obtained from the hypothesis (1.1), that the time coordinate of the traveling system at any given point on the worldline of the systems clock is equal to the proper time assigned to that point on the worldline of the systems clock.

A second implication of (11.20) is that $d\tau$ is a scaler invariant. This implies that if $\mathbf{V}$ is an arbitrary 4-vector defined on each point on the given worldline, the derivative $d\mathbf{V}/d\tau$, which is a derivative on a curve, is another 4-vector. From this fact we can derive the product rule for derivatives of dot products. Using the expression (11.11a) for dot products, together with the ordinary product rule and the fact that the Lorentz metric is constant, we obtain

$$\frac{d\mathcal{V} \bullet \mathcal{U}}{d\tau} = \sum_{i,j} g_{i,j} \frac{d\mathcal{V}^i}{d\tau} \mathcal{U}^j + \sum_{i,j} g_{i,j} \mathcal{V}^i \frac{d\mathcal{U}^j}{d\tau} \tag{11.22}$$

We next use the fact that $d\mathcal{V}^i / d\tau$ and $d\mathcal{U}^i / d\tau$ are contravariant components of 4-vectors in the unbarred system. This fact together with (11.11a) gives

$$\frac{d\mathcal{V}}{d\tau} \bullet \mathcal{U} = \sum_{i,j} g_{i,j} \frac{d\mathcal{V}^i}{d\tau} \mathcal{U}^j, \quad \mathcal{V} \bullet \frac{d\mathcal{U}}{d\tau} = \sum_{i,j} g_{i,j} \mathcal{V}^i \frac{d\mathcal{U}^j}{d\tau} \tag{11.23}$$

Combining (11.22) with (11.23) produces the product rule

$$\frac{d\mathcal{V} \bullet \mathcal{U}}{d\tau} = \frac{d\mathcal{V}}{d\tau} \bullet \mathcal{U} + \mathcal{V} \bullet \frac{d\mathcal{U}}{d\tau} \tag{11.24}$$

Note that expressing components in the unbarred system is the simplest way to derive (11.24). If components were expressed in the barred system, there would be two complications. First, the metric tensor in the barred system need not be constant so there would be a third term on the right

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side of the equation analogous to (11.22). Second, $d\mathcal{V}^{\bar{i}}/d\tau$ and $d\mathcal{U}^{\bar{i}}/d\tau$ are not in general contravariant components of 4-vectors in the barred system, so the equation analogous to (11.23) is incorrect. Hence the unbarred system provides the simplest derivation of (11.24).

The previous paragraph stated that $d\mathcal{V}^{\bar{i}}/d\tau$ and $d\mathcal{U}^{\bar{i}}/d\tau$ are not in general contravariant components of 4-vectors in the barred system. It is interesting to find out what the contravariant components of $d\mathbf{V}/d\tau$ are in the barred system. These components can be derived by using the product rule with (11.4) to get

$$\frac{d\mathcal{V}}{d\tau} = \sum_i \frac{d\mathcal{V}^{\bar{i}}}{d\tau}\bar{\boldsymbol{e}}_{(i)} + \sum_i \mathcal{V}^{\bar{i}}\frac{d\bar{\boldsymbol{e}}_{(i)}}{d\tau} \tag{11.25}$$

Note that (11.6) together with the chain rule gives

$$\frac{d\bar{\boldsymbol{e}}_{(i)}(\bar{x})}{d\tau} = \sum_j \left[\frac{d}{d\tau}\left(\frac{\partial x^j}{\partial \bar{x}^i}\right)\right]\boldsymbol{e}_{(j)} = \sum_{j,k}\left(\frac{\partial^2 x^j}{\partial \bar{x}^i \partial \bar{x}^k}\right)\frac{d\bar{x}^k}{d\tau}\boldsymbol{e}_{(j)}$$

To express this in terms of the barred basis vectors we substitute (11.8) into the right side of the above to get

$$\frac{d\bar{\boldsymbol{e}}_{(i)}(\bar{x})}{d\tau} = \sum_{j,k,l}\left(\frac{\partial^2 x^l}{\partial \bar{x}^i \partial \bar{x}^j}\right)\frac{\partial \bar{x}^k}{\partial x^l}\frac{d\bar{x}^j}{d\tau}\bar{\boldsymbol{e}}_{(k)}(\bar{x}) \tag{11.26}$$

To shorten the notation, define the Christoffel symbol by

$$\begin{Bmatrix} k \\ i,j \end{Bmatrix} \equiv \sum_l \left(\frac{\partial^2 x^l}{\partial \bar{x}^i \partial \bar{x}^j}\right)\frac{\partial \bar{x}^k}{\partial x^l} \tag{11.27}$$

so (11.26) becomes

$$\frac{d\bar{\boldsymbol{e}}_{(i)}(\bar{x})}{d\tau} = \sum_{j,k}\begin{Bmatrix} k \\ i,j \end{Bmatrix}\frac{d\bar{x}^j}{d\tau}\bar{\boldsymbol{e}}_{(k)}(\bar{x}) \tag{11.28}$$

From the definition of contravariant components as coefficients in expansions, we conclude from (11.28) that

$$\left(\frac{d\bar{\boldsymbol{e}}_{(i)}(\bar{x})}{d\tau}\right)^{\bar{k}} = \sum_j \begin{Bmatrix} k \\ i,j \end{Bmatrix}\frac{d\bar{x}^j}{d\tau} \tag{11.29}$$

Also, substituting (11.28) into (11.25) gives

$$\frac{d\mathcal{V}}{d\tau} = \sum_i \frac{\delta\mathcal{V}^{\bar{i}}}{\delta\tau}\bar{\boldsymbol{e}}_{(i)} \tag{11.30}$$

where we define

$$\frac{\delta\mathcal{V}^{\bar{i}}}{\delta\tau} \equiv \frac{d\mathcal{V}^{\bar{i}}}{d\tau} + \sum_{j,k}\begin{Bmatrix} i \\ j,k \end{Bmatrix}\frac{d\bar{x}^j}{d\tau}\mathcal{V}^{\bar{k}} \tag{11.31}$$

From the definition of contravariant components as coefficients in expansions, we conclude from (11.30) that

$$\left(\frac{d\mathcal{V}}{d\tau}\right)^{\bar{i}} = \frac{\delta\mathcal{V}^{\bar{i}}}{\delta\tau} \tag{11.32}$$

12. An Example: Constant Acceleration Felt by Traveler

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The example in this section is one in which the velocity of the traveler relative to the home system and expressed as a function of time in the home system is not the immediately given information. Instead, this must be deducted from other information that will be given. Invariance of solid dot products between 4-vectors facilitates this deduction so the given information will be a statement about the 4-vector version of acceleration. This 4-vector, denoted **A**, is defined in general in terms of the home system basis vectors by

$$\mathcal{A} = \sum_i \frac{d^2 x^i}{d\tau^2} \boldsymbol{e}_{(i)} \tag{12.1}$$

The example in this section is the special case in which the acceleration has a constant spatial direction, say in the x-direction. The remaining given information, which implies (after some deduction to follow) the traveler worldline as expressed in home coordinates, is

$$\mathcal{A} \bullet \mathcal{A} = -a^2 \tag{12.2}$$

where a is a positive constant. Expressing the dot product in any inertial system produces

$$-\mathcal{A} \bullet \mathcal{A} = \left(\frac{d^2 x}{d\tau^2} \right)^2 - \left(c \frac{d^2 t}{d\tau^2} \right)^2 = a^2 \tag{12.3}$$

where we now write x instead of x^1. An alternate way to express (12.3) is obtained by first using (9.11a) to derive an expression for $d^2t/d\tau^2$, which is obtained from

$$\frac{d^2 t}{d\tau^2} = \frac{d}{d\tau} \sqrt{1 + \frac{1}{c^2} \left(\frac{dx}{d\tau} \right)^2} = \frac{1}{c^2} \frac{1}{\sqrt{1 + \frac{1}{c^2} \left(\frac{dx}{d\tau} \right)^2}} \frac{dx}{d\tau} \frac{d^2 x}{d\tau^2}$$

and then substitute this into (12.3) and combine some terms to get

$$-\mathcal{A} \bullet \mathcal{A} = \frac{1}{1 + \frac{1}{c^2} \left(\frac{dx}{d\tau} \right)^2} \left(\frac{d^2 x}{d\tau^2} \right)^2 = a^2 \tag{12.4}$$

From (12.4) we can recognize the physical interpretation of the example considered here. This is seen by letting the acceleration refer to the traveler's system and letting the inertial system be the P-system, the inertial system momentarily at rest in the traveler's system. In the P-system, and at the time it is at rest in the traveler's system, the velocity dx/dt is zero, and there is no distinction between τ derivatives and t derivatives, so (12.4) reduces to

$$\frac{d^2 x}{dt^2} = a \quad \text{in } P\text{-system when at rest in traveler's system}$$

If the traveler is carrying an accelerometer, it will measure an acceleration equal to a when at rest in the P-system, but the point P is an arbitrary point on the traveler's worldline, so the physical interpretation of this example 4-vector **A** is that it produces the case in which the traveler feels (or measures via an accelerometer) a constant acceleration equal to a [11].

The next application of (12.4) is to determine the traveler's worldline as seen in the home system. Taking the sign of a to be the same as the sign of $d^2x/d\tau^2$, (12.4) gives

$$\frac{d^2 x}{d\tau^2} = a \sqrt{1 + \frac{1}{c^2} \left(\frac{dx}{d\tau} \right)^2} \tag{12.5}$$

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Endpoint conditions are not important to the analysis as long as singularities are avoided as $a \to 0$. A particular choice that avoids singularities takes the endpoint conditions to be $x = 0$ and $dx/d\tau = 0$ at $\tau = 0$. The solution to (12.5) subject to these endpoint conditions is easily verified to be

$$x = \frac{c^2}{a}\left(\cosh\left(\frac{a}{c}\tau\right) - 1\right) \quad (12.6a)$$

Substituting this into (9.11a) gives

$$\frac{dt}{d\tau} = \cosh\left(\frac{a}{c}\tau\right)$$

The solution satisfying $t = 0$ when $\tau = 0$ is

$$t = \frac{c}{a}\sinh\left(\frac{a}{c}\tau\right) \quad (12.6b)$$

Note that if we take the limit as the acceleration $a \to 0$ we find, via L'Hôpital's rule, that $t \to \tau$ while $x \to 0$ for all τ, which is the expected result for the selected initial conditions without acceleration.

An identity relating hyperbolic functions allows (12.6a) to be written as

$$x = \frac{c^2}{a}\left(\sqrt{1 + \sinh^2\left(\frac{a}{c}\tau\right)} - 1\right)$$

Combining this with (12.6b) gives

$$x = \frac{c^2}{a}\left(\sqrt{1 + \left(\frac{a}{c}t\right)^2} - 1\right) \quad (12.7)$$

A plot of x versus t produced by (12.7) produces a hyperbola, so this example is often called hyperbolic motion [11]. Another important quantity for this example is the coordinate velocity (not a 4-vector) defined to be dx/dt, which we find by differentiating (12.7) to be given by

$$\frac{dx}{dt} = \frac{at}{\sqrt{1 + \left(\frac{a}{c}t\right)^2}} \quad (12.8)$$

Also, substituting (12.8) into (9.11b), using the second equality in (9.11b) gives

$$\gamma = \sqrt{1 + \left(\frac{a}{c}t\right)^2} \quad (12.9)$$

As a reminder that the calculated x and dx/dt refer to the worldline of the traveler's clock, we go back to the notation in Sections 3 and 6. Instead of (12.7) we write

$$X(t_P) = \frac{c^2}{a}\left(\sqrt{1 + \left(\frac{a}{c}t_P\right)^2} - 1\right) \quad (12.10a)$$

Instead of (12.8) we write

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$$V(t_P) = \frac{a\,t_P}{\sqrt{1+\left(\dfrac{a}{c}t_P\right)^2}} \tag{12.10b}$$

Instead of (12.9) we write

$$\gamma(t_P) = \sqrt{1+\left(\frac{a}{c}t_P\right)^2} \tag{12.10c}$$

and instead of (12.6b) we write

$$\tau(t_P) = \frac{c}{a}\sinh^{-1}\left(\frac{a}{c}t_P\right) \tag{12.10d}$$

When difficulties are encountered regarding the existence of traveler coordinates to be calculated for a given spacetime point, i.e., calculated from a given set of home coordinates, they are encountered in the first step of the calculation which is to solve (6.6) for t_P. The example in this section provides an illustration. Substituting (12.10) into (6.6), the equation to be solved for t_P becomes

$$t_E = \left(1+\frac{a}{c^2}x_E\right)\frac{t_P}{\sqrt{1+\left(\dfrac{a}{c}t_P\right)^2}} \tag{12.11}$$

First consider the case in which the parenthesis on the right side of (12.11) is positive. It is easy to show that this condition makes the right side of (12.11) a strictly increasing function of t_P, implying that the solution of (12.11) for t_P is unique if it exists. Existence requires that the left side, t_E, be in the range of the function of t_P on the right. The function on the right is zero at $t_P = 0$, and it is easy to show that the function on the right asymptotically approaches the value $c/a + x_E/c$ as $t_P \to \infty$. We therefore conclude that:

If

$$1+\frac{a}{c^2}x_E > 0$$

then the event E has coordinates in the traveler's system, for the example in which $-\mathcal{A}\bullet\mathcal{A} = a^2$, if and only if

$$0 \le t_E < \frac{c}{a} + \frac{x_E}{c}. \tag{12.12}$$

The upper bound for t_E in (12.12) can be explained in terms of time dilation. Imagine a clock stationary at the location x_E in the home system. This clock as seen by the traveler runs slower and slower as the traveler speeds up in the home system, in such a way that the clock display, as seen by the traveler, asymptotically approaches a finite limiting value. This limiting value is the upper bound in (12.12). A more rigorous discussion of this time dilation can be given, for the example worldline considered in this section, as follows. Recall that (3.4) is the version of time dilation in which the home observer is observing the traveler's clock. For the example in this section, we are now able to calculate the version of time dilation in which the traveler is observing a home clock located at x_E. We start by combining (6.8a) with (12.10d) and invert the equation to get

$$t_P = \frac{c}{a}\sinh\left(\frac{a}{c}\bar{t}_E\right) \tag{12.13}$$

Substituting (12.13) into (12.11) and using some identities for hyperbolic functions gives

$$t_E = \left(\frac{c}{a} + \frac{x_E}{c}\right)\tanh\left(\frac{a}{c}\bar{t}_E\right) \tag{12.14}$$

If we now let the event E be a given display on a home clock located at x_E, then t_E and x_E become the home coordinates of that event. If $t_E + dt_E$ denotes the time coordinate of the next tick (the next event) of the home clock, with the clock still at the spatial coordinate x_E, we relate dt_E to the time increment $d\bar{t}_E$, that the traveler sees to be the time between ticks of the home clock, by differentiating (12.14) while holding x_E fixed (as opposed to holding $\bar{x}_E$ fixed as was done when deriving (3.4b)). The result is

$$\frac{dt_E}{d\bar{t}_E} = \left(1 + \frac{a}{c^2}x_E\right)\frac{1}{\cosh^2\left(\frac{a}{c}\bar{t}_E\right)}$$

Some identities for hyperbolic functions allow (12.14) to be rearranged into

$$\frac{1}{\cosh^2\left(\frac{a}{c}\bar{t}_E\right)} = 1 - \frac{t_E^{\,2}}{\left(\frac{c}{a} + \frac{x_E}{c}\right)^2}$$

and combining the two above equations gives

$$\frac{d\bar{t}_E}{dt_E} = \frac{c}{a}\left(\frac{c}{a} + \frac{x_E}{c}\right)^{-1}\left(1 - \frac{t_E^{\,2}}{\left(\frac{c}{a} + \frac{x_E}{c}\right)^2}\right)^{-1} \quad \text{(when the events are home clock displays)} \tag{12.15}$$

This becomes a time dilation when t_E is sufficiently close to the upper bound in (12.12) so that the right side of (12.15) is greater than 1, which is seen as follows. Recall that t_E is a display on the home clock. When the right side of (12.15) exceeds 1, the time $d\bar{t}_E$ seen by the traveler that is needed to change the display of the home clock is greater than dt_E, which is the change of the display of the home clock and therefore also equal to the time seen by the home observer needed to change the display of the home clock. A larger time between displays corresponds to a slower clock, so the traveler sees the home clock to be running slow when the home clock display is sufficiently close to the allowed upper limit.

Note a lack of symmetry between (3.4b) and (12.15) for the example acceleration considered in this section. When making this comparison we must be careful to recognize that t_E has a different meaning in the derivation of (3.4) than in the derivation of (12.15) and satisfies different equations for the two cases (e.g., $t_E = t_P$ for the former case but not the latter) because the event E has a different meaning for the two cases. In the derivation of (3.4), the event E is a given display on the traveler's clock. In the derivation of (12.15), the event E is a given display on a home clock located at x_E. Hence, we must use caution when comparing (3.4b) to (12.15) by recognizing that terms represented by the same symbols do not have the same meanings. However, despite this subtle issue, there is still an obvious distinction between (3.4b) and (12.15). The expression in (3.4b) does not depend on the relative distance between clocks when all cases being compared have the same relative velocity between clocks. In contrast, the expression in (12.15), which is a time dilation pertaining to a home clock located at x_E, does depend on the location x_E of the home clock.

The upper bound for t_E in (12.12), which limits the spacetime points that can be assigned coordinates in the traveler's system, can be removed by allowing the acceleration to persist for only a finite time, with the traveler having a constant velocity after that time. This was explained in Section 8. It was also explained there that there exist home coordinates at which the transformation to traveler's coordinates is not unique.

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13. Metric Tensor for the Constant Acceleration Felt by Traveler

The next section has a need for the zero-zero component of the metric tensor in the traveler's system and the goal of this section is to calculate the metric tensor for the acceleration example considered. This quantity depends on the space-time point of evaluation and is most conveniently expressed in the traveler's coordinates of a given point. While Sections 2 through 7 and 12 applied *E* subscripts to coordinates to emphasize that they are coordinates of arbitrary events, we shorten the notation here by omitting those subscripts, so it is understood here that $(\bar{t},\bar{x})$ are the traveler's coordinates of an arbitrary event. When expressed as a function of these coordinates, the *i*, *j* component of the metric tensor is denoted $\bar{g}_{i,j}(\bar{t},\bar{x})$. Section 10 showed explicit dependences that various quantities, used to construct the metric tensor, have on $\bar{x}$, but the dependences on $\bar{t}$ was implicit, through a dependence on t_P which in turn is a function of $\bar{t}$. To explicitly show the dependences on $\bar{t}$, note that when the travelers time coordinate $\bar{t}$ is the given quantity, as opposed to home coordinates, $\bar{t}$ is calculated from (12.13) (with the subscript *E* omitted in the notation used here), i.e., calculated from

$$t_P = \frac{c}{a}\sinh\left(\frac{a}{c}\bar{t}\right) \tag{13.1}$$

Quantities previously expressed in terms of t_P can now be expressed explicitly in terms of $\bar{t}$ via the substitution given by (13.1).

Quantities that must be evaluated to calculate the metric tensor include $\gamma(t_P)$ and $V(t_P)$, which, for the acceleration example considered here, were shown in Section 12 to be given by

$$\gamma(t_P) = \sqrt{1+\left(\frac{a}{c}t_P\right)^2} \tag{13.2a}$$

$$V(t_P) = \frac{a\,t_P}{\sqrt{1+\left(\frac{a}{c}t_P\right)^2}} \tag{13.2b}$$

Another quantity that must be evaluated is the three-dimensional acceleration, which is the derivative of *V* with respect to coordinate time in the home system, and can be calculated from

$$A(t_P) = \frac{d\,V(t_P)}{d\,t_P}$$

Using (13.2b) to calculate the derivative in the above gives

$$A(t_P) = \frac{a}{\left(\sqrt{1+\left(\frac{a}{c}t_P\right)^2}\right)^3} \tag{13.2c}$$

We now express the quantities in (13.2) in terms of $\bar{t}$ by using (13.1) together with some identities for hyperbolic functions to get

$$\gamma(t_P) = \cosh\left(\frac{a}{c}\bar{t}\right) \tag{13.3a}$$

$$V(t_P) = c\tanh\left(\frac{a}{c}\bar{t}\right) \tag{13.3b}$$

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$$A(t_P) = \frac{a}{\cosh^3\left(\frac{a}{c}\overline{t}\right)} \tag{13.3c}$$

For the acceleration example considered here, all motion is along the x-axis so the T-functions defined by (10.8) reduce to

$$T_1\left(t_P, \overline{\boldsymbol{x}}\right) = \frac{1}{c^2}\gamma^3\left(t_P\right) A(t_P)\overline{x}$$

together with $T_2\left(t_P, \overline{\boldsymbol{x}}\right) = 0$ and $\boldsymbol{T}_\perp\left(t_P, \overline{\boldsymbol{x}}\right) = \boldsymbol{0}$. Using (13.3) to express $T_1\left(t_P, \overline{\boldsymbol{x}}\right)$ in terms of $\overline{t}$ gives

$$T_1\left(t_P, \overline{\boldsymbol{x}}\right) = \frac{a}{c^2}\overline{x} \tag{13.3d}$$

We see from inspection of (10.14) that when $T_2\left(t_P, \overline{\boldsymbol{x}}\right) = 0$ and $\boldsymbol{T}_\perp\left(t_P, \overline{\boldsymbol{x}}\right) = \boldsymbol{0}$, each component of the metric tensor in the traveler's system equals the corresponding component of the Lorentz metric tensor except the zero-zero component, i.e.,

$$\overline{g}_{i,j} = g_{i,j} \quad \text{if } (i,j) \neq (0,0) \tag{13.4a}$$

where $g_{i,j}$ is the Lorentz metric. Also, when $T_2\left(t_P, \overline{\boldsymbol{x}}\right) = 0$ and $\boldsymbol{T}_\perp\left(t_P, \overline{\boldsymbol{x}}\right) = \boldsymbol{0}$, the zero-zero component given by (10.14a) reduces to

$$\overline{g}_{0,0} = \gamma^2\left(t_P\right)\left(1 - \frac{V^2\left(t_P\right)}{c^2}\right)\left\{1 + T_1\left(t_P, \overline{\boldsymbol{x}}\right)\right\}^2 = \left\{1 + T_1\left(t_P, \overline{\boldsymbol{x}}\right)\right\}^2$$

Substituting (13.3) into the above gives

$$\overline{g}_{0,0} = \left(1 + \frac{a}{c^2}\overline{x}\right)^2 \tag{13.4b}$$

14. Another Kind of Time Dilation and Doppler Effect

Two kinds of time dilation were previously discussed. Section 3 discussed the case in which a given display on the traveler's clock was the event, and the calculated quantity was the home observer's time coordinate of this event (this is equivalent to saying that the calculated quantity is the display on the home clock that is simultaneous with a given display on the traveler's clock when the home observer defines simultaneity). Section 12 discussed the case in which a given display on the home clock was the event, and the calculated quantity was the traveler's time coordinate of this event (this is equivalent to saying that the calculated quantity is the display on the traveler's clock that is simultaneous with a given display on the home clock when the traveler defines simultaneity). We now consider a third case in which two clocks are not moving relative to each other, they are both stationary in the traveler's system, but one clock is spatially displaced relative to the other. One clock, still called the traveler's clock, is at the traveler's origin so the traveler's worldline is the worldline of that clock. The traveler is at the origin of the reference frame and uses this clock to define time coordinates of events. Displays on a second clock, called the displaced clock, are treated as events, and the quantity to be calculated is the traveler's time coordinate of such an event. This is equivalent to calculating the display on the traveler's clock that is simultaneous with a given display on the displaced clock when the traveler located at the traveler's clock defines simultaneity.

A coordinate transformation between the traveler and a displaced traveler has not been derived but the above calculation can be performed by utilizing the metric tensor. Details are as follows. We start with the fact that an increment of time display, denoted $d\overline{\overline{t}}$, on the displaced clock is equal to the increment of proper time, denoted $d\tau$, between nearby points on the worldline of the displaced clock, i.e.,

$$d\bar{\bar{t}} = d\tau \tag{14.1}$$

A second relevant fact is that the increment of proper time is a scaler invariant that can be calculated using the metric tensor of any convenient reference frame. Because the displaced clock is stationary in the traveler's reference frame, it is convenient to use the traveler's metric tensor for this calculation. Using the acceleration example of Sections 12 and 13, let $\bar{x}_C$ denote the location of the displaced clock, along the x-axis, in the traveler's coordinate system. A given point on the worldline of the displaced clock has traveler's coordinates $(\bar{t},\bar{x}_C)$ and a nearby point on the same worldline has coordinates $(\bar{t}+d\bar{t},\bar{x}_C)$. There is no displacement of the spatial coordinate so the relationship between $d\tau$ and coordinate displacements reduces to

$$(d\tau)^2 = \bar{g}_{0,0}(\bar{t},\bar{x}_C)(d\bar{t})^2 \tag{14.2}$$

Combining this with (14.1) while using (13.4b) gives

$$d\bar{\bar{t}} = \left(1+\frac{a}{c^2}\bar{x}_C\right)d\bar{t} \tag{14.3}$$

If the displaced clock is displaced in the direction of the traveler's acceleration, so $\bar{x}_C$ is positive, we have $d\bar{\bar{t}} > d\bar{t}$, so the displaced clock appears to the traveler to be running fast. Displaced in the opposite direction, but not far enough to make the parenthesis in (14.3) negative, makes the displaced clock appear to the traveler to be running slow.

Now consider another situation in which the displaced clock is accompanied by a light source that is stationary in the traveler's system and is at the same location as the displaced clock. There is no motion between the light source and traveler as seen by the accelerating traveler, but the light source and traveler are at different locations.[13] The goal is to derive a Doppler effect as seen by the traveler for this case. This effect is caused by acceleration, which is equivalent to gravity, so we will call this a gravitational Doppler effect. It is interesting to compare this to the effect, which we will call the Lorentz Doppler effect, seen by an inertial system when a light source moves relative to the system. The Lorentz Doppler effect is easily found in the literature, so details need not be included here, and we will merely focus on some points of comparison. The Lorentz effect is a combination of two effects. One is from consecutive wave peaks produced by a moving source being emitted at different locations relative to the observer, which results in different times of travel to the observer. These different travel times contribute to the different arrival times of the wave peaks at the observer location (the other contribution to the different arrival times is the wave period of the source which is the difference in wave peak emission times). This effect, taken by itself, would produce the Doppler effect derived from a nonrelativistic treatment of sound waves in which the observer is at rest in the medium and the light source moves relative to the medium. The second effect that contributes to the Lorentz Doppler effect is a time dilation, which modifies the result derived for sound waves. In contrast, the gravitational Doppler effect is much easier to derive because, as seen below, only time dilation is relevant.

Using the acceleration example of Sections 12 and 13, let $\bar{x}_C$ denote the location of the light source, along the x-axis, in the traveler's coordinate system. Consider two consecutive wave peaks emitted by the light source. They are both emitted at the same location in the traveler's system so they

[13] Although incidental, because the home observer is not relevant in this discussion, it is interesting that, due to length contractions having a velocity dependence, a constant separation between traveler and light source as seen by the traveler (a given condition) does not imply a constant separation as seen by the home observer when velocities change.

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both have the same time of travel, as seen by the traveler, between emission and reaching the traveler. Therefore, according to the traveler, the difference in arrival times between the consecutive peaks, which is the wave period denoted $\bar{T}$ seen by the traveler, is the time difference, seen by the traveler, between emissions of consecutive wave peaks. Recall that $d\bar{\bar{t}}$ in (14.3) is an increment of time display of the displaced clock, while $d\bar{t}$ is the amount of time seen by the traveler to produce that change in clock display. Let the change in displaced clock display $d\bar{\bar{t}}$ equal one clock tick, which is also the time between consecutive wave peak emissions produced by the accompanying light source, so $d\bar{\bar{t}} = T$ where T is the wave period in the reference frame of the light source. Because travel time of the light wave is irrelevant, the wave period $\bar{T}$ seen by the traveler is the difference $d\bar{t}$ in traveler time coordinates between consecutive wave peak emissions, so $d\bar{t} = \bar{T}$. Substituting this together with $d\bar{\bar{t}} = T$ into (14.3) gives

$$T = \left(1 + \frac{a}{c^2}\bar{x}_C\right)\bar{T} \tag{14.4}$$

The frequency f in the reference frame of the light source and frequency $\bar{f}$ at the location of the traveler are related by the inverse relation

$$\bar{f} = \left(1 + \frac{a}{c^2}\bar{x}_C\right) f \tag{14.5a}$$

or

$$\frac{\bar{f}}{f} = \left(1 + \frac{a}{c^2}\bar{x}_C\right) \tag{14.5a}$$

References

1. [1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, Princeton University Press, pp. 164-165, 2017.
2. [2] J. D. Jackson, *Classical Electrodynamics*, Second Edition, John Wiley & Sons, Inc., p. 516, 1975.
3. [3] H. Goldstein, *Classical Mechanics*, Addison-Wesley Publishing Company, pp. 187-191, 1965.
4. [4] G. Arfken, *Mathematical Methods for Physicists*, Second Edition, Academic Press, p. 151, 1970.
5. [5] D. McMahon, *Quantum Field Theory Demystified*, McGraw-Hill, p. 8, 2008.
6. [6] J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, p. 77, 1941.
7. [7] R. Adler, M. Bazen, and M. Schiffer, *Introduction to General Relativity*, Second Edition, McGraw-Hill, p. 7, 1975.
8. [8] R. Penrose, The Emperor’s New Mind, Oxford University Press, p. 200, 1989.
9. [9] D. F. Lawden, *An Introduction to Tensor Calculus and Relativity*, Halsted Press, John Wiley & Sons, Inc., p. 22 (problem 10), 1975.
10. [10] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, Princeton University Press, p. 169, 2017.
11. [11] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, Princeton University Press, p. 166, 2017.