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On Summable Formal Power Series Solutions to Some Initial Value Problem with Infinite Order Irregular Singularity and Mahler Transforms

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Abstract: In this paper, we examine a nonlinear partial differential equation in complex time t and complex space z combined with so-called Mahler transforms acting on time. This equation is endowed with a leading term represented by some infinite order formal differential operator of irregular type which enables the construction of a formal power series solution in t obtained by means of a Borel-Laplace procedure known as $k-$summability. The so-called $k-$sums are shown to solve some related differential functional equations involving integral transforms which stem from the analytic deceleration operators appearing in the multisummability theory for formal power series.

Keywords: asymptotic expansion; Borel-Laplace transform; Fourier transform; initial value problem; Mahler transform; formal power series; nonlinear integro-differential equation; nonlinear partial differential equation; singular perturbation

MSC: 35R10; 35C10; 35C15; 35C20

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1. Introduction

This work is dedicated to the study of a nonlinear initial value problem that combines partial derivatives and so-called Mahler transforms with a leading term expressed by means of a formal differential operator of infinite order, with the shape

$$Q(\partial_z)u(t,z) = \cosh(\alpha_D(t^{k+1}\partial_t)^2)R_D(\partial_z)u(t,z) + P(t,z,t^{k+1}\partial_t,\partial_z,\{\mathfrak{m}_{l_2,t}\}_{l_2\in I})u(t,z) + Q_1(\partial_z)u(t,z)\times Q_2(\partial_z)u(t,z) + f(t,z) \quad (1)$$

for prescribed vanishing initial condition $u(0,z)\equiv 0$. The constituents comprising (1) are described as follows.

- The constant $\alpha_D > 0$ is a positive real number and $k \geq 1$ is a given natural number.
- The elements $Q(X), R_D(X), Q_1(X)$ and $Q_2(X)$ stand for polynomials with complex coefficients.
- The expression $P(t,z,V_1,V_2,\{W_{l_2}\}_{l_2\in I})$ represents a polynomial in t,V_1,V_2, a linear map in the arguments W_{l_2}, for $l_2 \in I$, where I denotes some finite subset of the positive natural numbers $\mathbb{N}\setminus\{0\}$ and a bounded holomorphic function with respect to z on a horizontal strip in $\mathbb{C}$ of the form $H_\beta = \{z\in\mathbb{C}/|\mathrm{Im}(z)| < \beta\}$, for a given real number $\beta > 0$.
- The forcing term $f(t,z)$ embodies a polynomial function in t with bounded holomorphic coefficients on H_β.
- The symbol $\mathfrak{m}_{l_2,t}$ is tagged as the Mahler transform and acts on time t through

$$\mathfrak{m}_{l_2,t}u(t,z) = u(t^{l_2},z) \quad (2)$$

  for all $l_2 \in I$.

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The operators (2) arise from the so-called Mahler equations which are linear functional equations of the form

$$\sum_{k=0}^{n} a_k(z)y(z^{l^k}) = 0 \quad (3)$$

for some given integers $l \geq 2$, $n \geq 1$ and rational coefficients $a_k(z) \in \mathbb{C}(z)$. The study of these equations is nowadays a very active field of research. Many authors have recently contributed to the understanding of the structure of their solutions and have established bridges with other branches of mathematics such as automata theory or transcendence results in number theory. For the links with automatic sequences and the famous Cobham's theorem, we refer to the seminal paper [1] by B. Adamczewski and J.P. Bell. For Galoisian aspects and hypertranscendence results related to (3), we refer to the recent work [2] by B. Adamczewski, T. Dreyfus and C. Hardouin. The algebraic structure of the solutions involving so-called Hahn series has been investigated in the series of papers [11,12] by J. Roques and [6] by C. Faverjon and J. Roques.

Mixed type equations comprising Mahler and differential operators have been much less examined, however recent substantial contributions show that they represent a propitious direction for upcoming research. Indeed, the case of coupled systems of linear differential equations and Mahler equations of the form

$$\begin{cases} xY'(x) = A(x)Y(x) \\ Y(x^q) = B(x)Y(x) \end{cases}$$

where $A(x), B(x)$ are $n \times n$ matrices with rational coefficients in $\mathbb{C}(x)$ and integer $q \geq 2$ are considered in the work [13] by R. Schäfke and M. Singer and the general form of their meromorphic solutions on the universal covering $\widetilde{\mathbb{C} \setminus \{0\}}$ are unveiled. In the paper [10], S. Ōuchi addresses functional equations containing both difference and Mahler operators of the form

$$u(z) + \sum_{j=2}^{m} a_j u(z + z^p \varphi_j(z)) = f(z)$$

for some integer $p \geq 2$, complex coefficients $a_j \in \mathbb{C}^*$ and given holomorphic maps $\varphi_j(z)$ and $f(z)$ near the origin in $\mathbb{C}$. He establishes the existence of a formal power series solution $\hat{u}(z) \in \mathbb{C}[[z]]$ that is proved to be $p-$summable in suitable directions (see Definition 3 of this work or the textbooks [3] and [4] for the definition of $p-$summability). More recently, in a work in progress [14], H. Yamazawa extends the above statement to more general functional equations with shape

$$u(z) + L(u(z + z^p)) = f(z)$$

where L is a general linear differential operator of finite order with holomorphic coefficients on a disc D_r, with radius $r > 0$ centered at 0, for which formal power series solutions $\hat{u}(z)$ are shown to be multisummable in appropriate multidirections in the sense defined in [3], Chapter 6.

The results reached in this paper are holding the line of our previous joint study [9] with A. Lastra where we addressed the next nonlinear problem

$$Q(\partial_z)y(t,z,\epsilon) = \exp(\alpha\epsilon^q t^{q+1}\partial_t)R(\partial_z)y(t,z,\epsilon) + H(t,\epsilon,\{m_{\kappa,t,\epsilon}\}_{\kappa\in J},\epsilon^q t^{q+1}\partial_t,\partial_z)y(t,z,\epsilon) \\ + Q_1(\partial_z)y(t,z,\epsilon) \times Q_2(\partial_z)y(t,z,\epsilon) + f(t,z,\epsilon) \quad (4)$$

for given vanishing initial data $y(0,z,\epsilon) \equiv 0$. The constant $\alpha > 0$ represents some well chosen positive real number and q is taken in the open interval $(1/2, 1)$. Here Q, R, H, Q_1 and Q_2 stand for polynomials and the forcing term f is built up in a similar way as above. The symbol $m_{\kappa,t,\epsilon}$, for

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$\kappa \in J$ (where J stands for a finite subset of the positive real numbers $\mathbb{R}_{+}^{*}$), is labeled as the Moebius transform operating on time t by means of

$$m_{\kappa,t,\epsilon}y(t,z,\epsilon) = y(\frac{t}{1+\kappa\epsilon t},z,\epsilon).$$

An additional dependence with respect to a complex parameter $\epsilon \in \mathbb{C}^{*}$ is assumed compared to (1) which gives (4) the quality of a singularly perturbed equation.

For some suitable bounded sector $\mathcal{T}$ edged at 0 in $\mathbb{C}^{*}$ and a set $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ of bounded sectors edged at 0 whose union contains a full neighborhood of 0 in $\mathbb{C}^{*}$, we construct genuine bounded holomorphic solutions $y_p(t,z,\epsilon)$ to (4) on the product $\mathcal{T} \times H_{\beta} \times \mathcal{E}_p$, expressed through a Laplace transform of order q and Fourier inverse transform

$$y_p(t,z,\epsilon) = \frac{q}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\mathfrak{d}_p}} \omega^{\mathfrak{d}_p}(u,m,\epsilon) \exp\left(-(\frac{u}{\epsilon t})^{k}\right) e^{\sqrt{-1}zm} \frac{du}{u} dm$$

along well chosen halflines $L_{\mathfrak{d}_p} = [0,+\infty)e^{\sqrt{-1}\mathfrak{d}_p}$ with $\mathfrak{d}_p \in \mathbb{R}$, where $\omega^{\mathfrak{d}_p}(u,m,\epsilon)$ represents a function called *Borel-Fourier map* featuring exponential growth of order q on some sector containing $L_{\mathfrak{d}_p}$ with respect to u, showing exponential decay relatively to m on $\mathbb{R}$ and relying analytically on ϵ near 0. Furthermore, the partial maps $\epsilon \mapsto y_p(t,z,\epsilon)$ are shown to share on the sectors $\mathcal{E}_p$ a common asymptotic expansion

$$\hat{y}(t,z,\epsilon) = \sum_{n \geq 0} y_n(t,z)\epsilon^n$$

which represents a formal power series in ϵ with bounded holomorphic coefficients y_n, $n \geq 0$, on the product $\mathcal{T} \times H_{\beta}$. This asymptotic expansion turns out to be (at most) of Gevrey order $1/q$ meaning that constants $C, M > 0$ can be singled out with

$$\sup_{t \in \mathcal{T}, z \in H_{\beta}} |y_p(t,z,\epsilon) - \sum_{n=0}^{N-1} y_n(t,z)\epsilon^n| \leq CM^N \Gamma(1+\frac{N}{q})|\epsilon|^N$$

for all integers $N \geq 1$, whenever $\epsilon \in \mathcal{E}_p$.

The leading term of (4) consists in a formal differential operator of infinite order with respect to t,

$$\exp(\alpha\epsilon^q t^{q+1}\partial_t)R(\partial_z) = \sum_{p \geq 0} \frac{(\alpha\epsilon^q)^p}{p!}(t^{q+1}\partial_t)^{(p)}R(\partial_z) \quad (5)$$

where $(t^{q+1}\partial_t)^{(p)}$ stands for the $p-$th iterate of the irregular differential operator $t^{q+1}\partial_t$. The reason for the appearance of such a principal term with infinite order is triggered by the presence of the Moebius transforms $m_{\kappa,t,\epsilon}$, $\kappa \in J$, which forbids leading finite order differential operators.

In the present contribution, our aim is to carry out a similar procedure by means of Fourier-Laplace transforms in order to construct solutions to (1) and to related problems to (1). However, the occurence of the Mahler transforms $\{\mathfrak{m}_{l_2,t}\}_{l_2 \in I}$ in the main term P of (1) modifies utterly the whole picture in comparison with [9].

As a first major change, the choice of a principal term with shape (5) is now insufficient to guarantee the construction of solutions to (1) in our framework. We supplant it by an exponential formal differential operator of higher order

$$\cosh(\alpha_D(t^{k+1}\partial_t)^2)R_D(\partial_z) = \frac{1}{2}\left(\exp(\alpha_D(t^{k+1}\partial_t)^2) + \exp(-\alpha_D(t^{k+1}\partial_t)^2)\right)R_D(\partial_z).$$

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The reasons for such an option will be motivated later on in the introduction.

Under fitting conditions on the shape of our main equation (1) itemized in the statement of Theorem 1, Section 5, we can construct a formal power series $\hat{u}(t,z) = \sum_{n \geq 1} u_n(z) t^n$ whose coefficients u_n, $n \geq 0$, are bounded holomorphic on H_β, which solves (1) with vanishing initial data $\hat{u}(0,z) \equiv 0$. This formal series is built up through a Borel-Laplace method similar to the classical $k-$summability approach discussed in [3], that we call m_k-summability, whose basic results are recalled in Subsection 3.2.1. It means that we can exhibit analytic maps $u^d(t,z)$ on products $S_{d,\vartheta,R} \times H_\beta$ where $S_{d,\vartheta,R}$ stands for a bounded sector edged at 0 with some small radius $R > 0$, well chosen bisecting direction $d \in \mathbb{R}$ (among a set Θ_{Q,R_D} explicitely depicted in Lemma 2) and opening ϑ slightly larger than π/k, such that

- the map $u^d(t,z)$ is expressed as Fourier-Laplace transform

$$u^d(t,z) = \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_d} \omega^d(\tau,m) \exp\left(-\left(\frac{\tau}{t}\right)^k\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm$$

  where $\omega^d(\tau,m)$ is called the m_k-Borel transform of $\hat{u}(t,z)$ with respect to t (see Definition 3) which

  - defines an analytic map with respect to τ with (at most) exponential growth of order k on a union $D_\rho \cup S_d$, where D_ρ is a disc with small radius $\rho > 0$ and S_d is an unbounded sector bisected by d, edged at 0 with small aperture,
  - represents a continuous function relatively to m on $\mathbb{R}$ with exponential decay at infinity.
- the partial map $t \mapsto u^d(t,z)$ is the unique holomorphic map on $S_{d,\vartheta,R}$ which has the formal series $\hat{u}(t,z)$ as asymptotic expansion of Gevrey order $1/k$, meaning that one can find two constants $C, M > 0$ for which

$$\sup_{z \in H_\beta} |u^d(t,z) - \sum_{n=1}^{N-1} u_n(z) t^n| \leq C M^N \Gamma(1 + \frac{N}{k}) |t|^N$$

  holds for all integers $N \geq 2$, provided that $t \in S_{d,\vartheta,R}$.

Another substantial contrast between the problems (1) and (4) lies in the observation that the holomorphic maps $u^d(t,z)$ do not (in general) obey the main equation solved by $\hat{u}(t,z)$ (see the concluding remark of the work). Instead, $u^d(t,z)$ is shown to solve two different related functional differential equations (225) or (226) depending on the location of the unbounded sector S_d in $\mathbb{C}$, see Theorem 2 in Section 5.

In Subsection 3.2.2, we show that the action of the Mahler operator $\mathfrak{m}_{l_2,t}$, for $l_2 \geq 2$, on the formal series $\hat{u}(t,z)$ is described by some integral operator acting on its m_k-Borel transform $\tau \mapsto \omega^d(\tau,m)$,

$$\hat{\mathcal{D}}_{k,k/l_2}(\omega^d)(\tau^{l_2}) := -\frac{k^2/l_2}{2\sqrt{-1}\pi} \int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}} \omega^d(\xi,m) \mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2}) d\xi \qquad (6)$$

along a closed Hankel path $\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}$ confined nearby the origin in $\mathbb{C}$. These operators are derived from a version of the *analytic deceleration operators* introduced by J. Écalle, which turn out to be the inverse for the composition of the so-called *analytic acceleration operators* which play a central role in the theory of multisummability, see [3], Chapters 5 and 6. As shown in Proposition 7, it comes out that the kernel $\tau \mapsto \mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})$ appearing in (6) has (at most) an exponential growth rate of

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order $\frac{kl_2}{l_2-1}$ on the sector S_d, which implies that the analytic map $\tau \mapsto \hat{\mathcal{D}}_{k,k/l_2}(\omega^d)(\tau^{l_2})$ itself owns upper bounds of the form

$$C(m)\exp(K|\tau|^{\frac{kl_2}{l_2-1}}) \tag{7}$$

provided that $\tau \in S_d$, for some constant $K > 0$ and map $m \mapsto C(m)$ with exponential decay on $\mathbb{R}$, see Sublemma 1 and 2. As a result, the presence of an infinite order operator with the shape $\cosh(\alpha_D(t^{k+1}\partial_t)^2)$ in the leading term of (1) seems mandatory since it acts in the Borel plane on $\tau \mapsto \omega^d(\tau,m)$ as the multiplication by the map $\tau \mapsto \cosh(\alpha_D(k\tau^k)^2)$ whose exponential growth rate on the sector S_d is of order $2k$ which exceeds $\frac{kl_2}{l_2-1}$, for $l_2 \geq 2$, and compensates the bounds (7). Furthermore, we have favored the cosh function instead of the exponential function exp since it allows a larger choice of sectors S_d in both the left and right halfplanes $\mathbb{C}_- = \{z \in \mathbb{C}/\mathrm{Re}(z) < 0\}$ and $\mathbb{C}_+ = \{z \in \mathbb{C}/\mathrm{Re}(z) > 0\}$.

The exceeding growth rate (7) coming from the action of the Mahler operator $\mathfrak{m}_{l_2,t}$ in the Borel plane also compromises the m_k-sum $u^d(t,z)$ of $\hat{u}(t,z)$ to become a genuine solution to (1) since only functions with (at most) exponential growth of order k are Laplace transformable. However, we can exhibit some modified functional equations displayed in (225) or (226) involving the analytic transforms (6) that u^d is shown to obey.

2. Layout of the Main Initial Value Problem

The main problem under study in this work is described as follows

$$\begin{aligned} Q(\partial_z)u(t,z) &= \cosh(\alpha_D(t^{k+1}\partial_t)^2)R_D(\partial_z)u(t,z) \\ &+ \sum_{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}} a_{\underline{l}}(z)\big(t^{l_0}(t^{k+1}\partial_t)^{l_1}R_{\underline{l}}(\partial_z)u\big)(t^{l_2},z) \\ &+ c_{Q_1Q_2}Q_1(\partial_z)u(t,z)\times Q_2(\partial_z)u(t,z) + f(t,z) \end{aligned} \tag{8}$$

for given vanishing initial data $u(0,z) \equiv 0$. Below, we display a list of conditions we set on the building blocks of (8). Namely,

- The constant $k \geq 1$ is a natural number. We set α_D as a positive real number and $c_{Q_1Q_2} \in \mathbb{C}^*$ stands for a non vanishing complex number. Constraints will be set on these constants that will be disclosed later on in the work, see Proposition 9, Section 4.
- The set $\mathcal{A}$ represents a finite subset of $\mathbb{N}^3$ which is asked to fulfill the next record of restrictions.
  1. For all $\underline{l} = (l_0,l_1,l_2) \in \mathcal{A}$, the next inequality
  $$l_2 \leq k \tag{9}$$
  holds.
  2. Provided that $\underline{l} = (l_0,l_1,l_2) \in \mathcal{A}$, the next constraint
  $$l_0 + kl_1 \geq \frac{k}{l_2} \tag{10}$$
  is required.
- The maps $Q(X), R_D(X), R_{\underline{l}}(X)$ for $\underline{l} \in \mathcal{A}$ along with $Q_1(X)$ and $Q_2(X)$ are polynomials with complex coefficients. Their degrees are required to obey the next inequalities
$$\deg(Q) = \deg(R_D) \geq \deg(R_{\underline{l}}) \tag{11}$$

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for all $\underline{l} \in \mathcal{A}$. In addition, we impose that

$$\deg(R_D) \geq \max\left(\deg(Q_1), \deg(Q_2)\right). \tag{12}$$

Besides, the next technical assumption of geometric nature is set on the polynomials $Q(X)$ and $R_D(X)$. We ask for the existence of a bounded sectorial annulus

$$S_{Q,R_D} = \{z \in \mathbb{C}^* / r_{Q,R_D,1} \leq |z| \leq r_{Q,R_D,2} \ , \ |\arg(z) - d_{Q,R_D}| \leq \eta_{Q,R_D}\} \tag{13}$$

with bisecting direction $d_{Q,R_D} \in \mathbb{R}$, aperture $\eta_{Q,R_D} > 0$ and with inner and outer radii $0 < r_{Q,R_D,1} < r_{Q,R_D,2}$ fulfilling the inclusion

$$\{\frac{Q(\sqrt{-1}m)}{R_D(\sqrt{-1}m)} / m \in \mathbb{R}\} \subset S_{Q,R_D}. \tag{14}$$

Precise requirements on the shape of the sectorial annulus S_{Q,R_D} will be exposed later on in Section 4, see Lemma 2.

– The differential operator of infinite order $\cosh(\alpha_D(t^{k+1}\partial_t)^2)$ is defined as the sum

$$\cosh(\alpha_D(t^{k+1}\partial_t)^2) = \frac{1}{2}\left(\exp(\alpha_D(t^{k+1}\partial_t)^2) + \exp(-\alpha_D(t^{k+1}\partial_t)^2)\right), \tag{15}$$

where each term is defined as a formal expansion

$$\exp(\pm\alpha_D(t^{k+1}\partial_t)^2) = \sum_{p\geq 0} \frac{(\pm\alpha_D)^p}{p!}(t^{k+1}\partial_t)^{(2p)}$$

where $(t^{k+1}\partial_t)^{(2p)}$ stands for the $2p$-th iterate of the differential operator $t^{k+1}\partial_t$.

In order to describe the properties of the coefficients $a_{\underline{l}}(z)$ for $\underline{l} \in \mathcal{A}$ and forcing term $f(t,z)$, we need to recall the definition of some Banach space of continous functions introduced in the work [5] and the action of the Fourier inverse transform on these spaces. The next two definitions already appear in the work [7].

Definition 1. *Let β, μ be positive real numbers. We set $E_{(\beta,\mu)}$ as the vector space of continuous functions $h : \mathbb{R} \to \mathbb{C}$ such that the norm*

$$||h(m)||_{(\beta,\mu)} = \sup_{m\in\mathbb{R}} (1+|m|)^{\mu} \exp(\beta|m|)|h(m)|$$

is finite. We observe that the space $E_{(\beta,\mu)}$ equipped with the norm $||.||_{(\beta,\mu)}$ represents a Banach space. Furthermore, for given elements f, g of $E_{(\beta,\mu)}$, let us denote

$$(f * g)(m) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(m-m_1)g(m_1)dm_1 \tag{16}$$

the convolution product of f, g. Assume that $\mu > 1$. Then, $f * g$ belongs to $E_{(\beta,\mu)}$ and the next inequality

$$||f * g||_{(\beta,\mu)} \leq C_\mu ||f||_{(\beta,\mu)} ||g||_{(\beta,\mu)}$$

holds for some constant $C_\mu > 0$ relying on μ. In particular, the Banach space $(E_{(\beta,\mu)}, ||.||_{(\beta,\mu)})$ equipped with the product $*$ turns out to become a Banach algebra.

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Definition 2. *Let $f \in E_{(\beta,\mu)}$ with $\beta > 0$, $\mu > 1$. The inverse Fourier transform of f is given by the next integral transform*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(m) \exp(\sqrt{-1}xm) dm \tag{17}$$

for all $x \in \mathbb{R}$. We observe that the function $\mathcal{F}^{-1}(f)$ extends to an analytic bounded function on all strips

$$H_{\beta'} = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta'\}, \tag{18}$$

for given $0 < \beta' < \beta$.
a) We set the function $m \mapsto \phi(m) = \sqrt{-1}mf(m)$ which belongs to the space $E_{(\beta,\mu-1)}$. Then, the next differential identity

$$\partial_z \mathcal{F}^{-1}(f)(z) = \mathcal{F}^{-1}(\phi)(z) \tag{19}$$

occurs for all $z \in H_{\beta'}$ with $0 < \beta' < \beta$.
b) Let g be an element of $E_{(\beta,\mu)}$ and set ψ as the convolution product of f and g given by the expression (16). Then, the next product formula

$$\mathcal{F}^{-1}(f)(z)\mathcal{F}^{-1}(g)(z) = \mathcal{F}^{-1}(\psi)(z) \tag{20}$$

holds for all $z \in H_{\beta'}$ provided that $0 < \beta' < \beta$.

- The coefficients $a_{\underline{l}}(z)$ are crafted as follows. For all $\underline{l} \in \mathcal{A}$, we consider maps $m \mapsto A_{\underline{l}}(m)$ that belong to the space $E_{(\beta,\mu)}$, for given $\beta > 0$ and where $\mu > 0$ is subjected to the conditions

  $$\mu > \deg(R_{\underline{l}}) + 1 \ , \ \ \mu > \max(\deg(Q_1) + 1, \deg(Q_2) + 1) \tag{21}$$

  for all $\underline{l} \in \mathcal{A}$. For later use, we denote

  $$\mathbf{A}_{\underline{l}} = ||A_{\underline{l}}||_{(\beta,\mu)}. \tag{22}$$

  The upper size of $\mathbf{A}_{\underline{l}}$ will be fixed later in due course of the paper, see Proposition 9, Section 4. We set

  $$a_{\underline{l}}(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A_{\underline{l}}(m) e^{\sqrt{-1}zm} dm \tag{23}$$

  as the inverse Fourier transform of $m \mapsto A_{\underline{l}}(m)$. From Definition 2, it defines a bounded holomorphic function on all the strips $H_{\beta'}$ for any prescribed $0 < \beta' < \beta$.
- The forcing term $f(t,z)$ is built up in the next way. Let $J \in \mathbb{N}^*$ be a subset of the positive natural numbers. For $j \in J$, we mind a map $m \mapsto \mathcal{F}_j(m)$ which belongs to the space $E_{(\beta,\mu)}$ for the real numbers $\beta > 0$ and $\mu > 1$ satisfying (21) given in the previous item. We introduce the notation

  $$\mathbf{F}_j = ||\mathcal{F}_j||_{(\beta,\mu)} \tag{24}$$

  for $j \in J$. The forcing term $f(t,z)$ is defined as the next polynomial in t

  $$f(t,z) = \sum_{j \in J} F_j(z) \Gamma(j/k) t^j \tag{25}$$

  where

  - The symbol $\Gamma(x)$ stands for the classical Gamma function $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$, for any $x > 0$.

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- The coefficients $F_j(z)$, $j \in J$, are the inverse Fourier transforms

$$F_j(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \mathcal{F}_j(m) e^{\sqrt{-1}zm} dm \tag{26}$$

of $\mathcal{F}_j$. From Definition 2, $z \mapsto F_j(z)$ defines a bounded holomorphic map on any strip $H_{\beta'}$ with $0 < \beta' < \beta$.

We introduce the next polynomial in the variable τ with coefficients in $E_{(\beta,\mu)}$,

$$\mathcal{F}(\tau, m) = \sum_{j \in J} \mathcal{F}_j(m) \tau^j. \tag{27}$$

According to the definition of the Gamma function, we observe that the forcing term $f(t,z)$ has an integral representation as a Laplace transform of order k and inverse Fourier integral

$$f(t,z) = \frac{k}{(2\pi)^{1/2}} \int_{L_{d_1}} \int_{-\infty}^{+\infty} \mathcal{F}(\tau, m) \exp\left(-\left(\frac{\tau}{t}\right)^k\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \tag{28}$$

where $L_{d_1} = [0, +\infty)e^{\sqrt{-1}d_1}$ stands for any halfline in direction $d_1 \in \mathbb{R}$ that depends on the variable t through the restriction $\cos(k(d_1 - \arg(t))) > 0$. Such a representative will be useful in the next section 3.

3. Reduction of the Main Problem to an Integral Equation

In this section, we perform two important reductions of our initial value problem. In the first subsection, we reduce our problem to the study of a differential/convolution equation involving Mahler transforms by means of a Fourier transform. In the second subsection, we further reduce the problem to an integral equation through the application of formal Borel/Laplace transforms of order k. This second reduction is essential in the achievement of our first main result, Theorem 1 in Section 5.

3.1. First Reduction to a Differential/Convolution Equation with Mahler Transforms

We search for solutions to (8) in the form of an inverse Fourier transform

$$u(t,z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} U(t,m) e^{\sqrt{-1}zm} dm \tag{29}$$

for some expression $U(t,m)$ such that the partial maps $m \mapsto U(t,m)$ belong to the space $E_{(\beta,\mu)}$ for $\beta, \mu > 0$ prescribed in Section 2. The precise shape of $U(t,m)$ will be unveiled in the next subsection. With the help of Definition 2, we reach the next

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Proposition 1. *The integral expression $u(t,z)$ given by (29) formally solves (8)* **if** *the map $U(t,m)$ obeys the next differential/convolution equation comprising Mahler transforms*

$$Q(\sqrt{-1}m)U(t,m) = \cosh(\alpha_D(t^{k+1}\partial_t)^2)R_D(\sqrt{-1}m)U(t,m)$$
$$+ \sum_{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}} \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)(t^{l_0}(t^{k+1}\partial_t)^{l_1}U)(t^{l_2},m_1)R_{\underline{l}}(\sqrt{-1}m_1)dm_1$$
$$+ c_{Q_1Q_2}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} U(t,m-m_1)Q_1(\sqrt{-1}(m-m_1))U(t,m_1)Q_2(\sqrt{-1}m_1)dm_1$$
$$+ \sum_{j\in J}\mathcal{F}_j(m)\Gamma(j/k)t^j \quad (30)$$

for given initial data $U(0,m) \equiv 0$.

3.2. Reduction to an Integral Equation

We seek for a solution to the reduced equation (30) expressed as a formal power series

$$\hat{U}(t,m) = \sum_{n\geq 1} U_n(m)t^n \quad (31)$$

in the time variable t with coefficients $m \mapsto U_n(m)$ that belong to $E_{(\beta,\mu)}$ for $\beta,\mu > 0$ assigned in Section 2. In the next two subsections, we provide the required prefatory material for our second step of reduction.

3.2.1. Essentials on Banach Valued m_k-Summable Formal Power Series

The objective of this subsection is to remind the reader the notion of m_k-summability and its basic properties as decribed in our previous work [7] which is a slight adjustment of the concept of $k-$summability discussed in the textbook [4].

Definition 3. *Let $(\mathbb{E},||.||_{\mathbb{E}})$ be a complex Banach space. We select an integer $k \geq 1$ and define the sequence $m_k(n) = \Gamma(n/k)$, for all $n \geq 1$. A formal power series*

$$\hat{U}(t) = \sum_{n\geq 1} a_n t^n \in t\mathbb{E}[[t]]$$

is called m_k-summable with respect to t in the direction $d \in \mathbb{R}$ if

- *The so-called formal m_k-Borel transform of $\hat{U}(t)$ defined by the power series*

$$\mathcal{B}_{m_k}(\hat{U})(\tau) = \sum_{n\geq 1}\frac{a_n}{\Gamma(n/k)}\tau^n \in \tau\mathbb{E}[[\tau]] \quad (32)$$

  is convergent on a disc D_ρ for some $\rho > 0$.
- *The convergent series $\mathcal{B}_{m_k}(\hat{U})(\tau)$ can be analytically continued with respect to τ (as a function still denoted $\mathcal{B}_{m_k}(\hat{U})(\tau)$) on some unbounded sector*

$$S_{d,\delta} = \{\tau \in \mathbb{C}^*/|d - \arg(\tau)| < \delta\}$$

  with aperture 2δ and bisecting direction $d \in \mathbb{R}$. Moreover, two constants $C > 0$ and $K > 0$ can be found such that

$$||\mathcal{B}_{m_k}(\hat{U})(\tau)||_{\mathbb{E}} \leq Ce^{K|\tau|^k} \quad (33)$$

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for all $\tau \in S_{d,\delta}$.

Let $\hat{U}(t)$ *be a* m_k-*summable formal power series with respect to* t *in a direction* d. *We define the Laplace transform of order* k *in direction* d *of the* m_k-*Borel transform* $\mathcal{B}_{m_k}(\hat{U})(\tau)$ *by the integral transform*

$$\mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{U}))(t) = k\int_{L_\gamma} \mathcal{B}_{m_k}(\hat{U})(\tau)\exp\big(-(\frac{\tau}{t})^k\big)\frac{d\tau}{\tau} \tag{34}$$

along a halfline $L_\gamma = [0,+\infty)e^{\sqrt{-1}\gamma} \subset S_{d,\delta}\cup\{0\}$, *where* γ *relies on* t *and matches the inequality* $\cos(k(\gamma-\arg(t))) > \Delta_1$ *for some constant* $\Delta_1 > 0$.

The function $t \mapsto \mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{U}))(t)$ *is bounded holomorphic on any bounded sector*

$$S_{d,\theta,R^{1/k}} = \{t \in \mathbb{C}^*/|t| < R^{1/k} \ , \ |d-\arg(t)| < \theta/2\} \tag{35}$$

where $\frac{\pi}{k} < \theta < \frac{\pi}{k} + 2\delta$ *and* $0 < R < \frac{\Delta_1}{K}$, *for* K *appearing in* (33). *This function is called the* m_k-*sum of* $\hat{U}(t)$ *in the direction* d.

The above definition of m_k-sum of a formal power series is justified by the next proposition (see also Proposition 11 p. 75 from [4] known as Watson's Lemma)

Proposition 2. *Let* $\hat{U}(t)$ *be a* m_k-*summable formal power series with respect to* t *in some direction* d. *Then, the Laplace transform* $t \mapsto \mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{U}))(t)$ *is the unique holomorphic map on* $S_{d,\theta,R^{1/k}}$ *which has the formal power series* $\hat{U}(t)$ *as asymptotic expansion of Gevrey order* $1/k$ *with respect to* t *on* $S_{d,\theta,R^{1/k}}$ *for any given opening* θ *under the condition* $\frac{\pi}{k} < \theta < \frac{\pi}{k} + 2\delta$. *It means that one can find two constants* $C, M > 0$ *for which the next inequality*

$$||\mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{U}))(t) - \sum_{p=1}^{N-1} a_p t^p||_{\mathbb{E}} \le CM^N\Gamma(1+\frac{N}{k})|t|^N \tag{36}$$

holds for all integers $N \ge 2$, *all* $t \in S_{d,\theta,R^{1/k}}$.

In the next proposition, we recall some crucial identities for the formal m_k-Borel transform under the action of differential operators of irregular type, multiplication by a monomial and products (see Proposition 6 from [7]).

Proposition 3. *Let* $(\mathbb{E}, ||.||_{\mathbb{E}})$ *be a complex Banach algebra whose product is denoted* $*$. *Let* $k, l \ge 1$ *be natural numbers. Let* $\hat{U}_j(t)$, $j = 1,2$, *be elements of* $t\mathbb{E}[[t]]$. *The next formal identities hold*

$$\mathcal{B}_{m_k}(t^{k+1}\partial_t\hat{U}_1(t))(\tau) = k\tau^k\mathcal{B}_{m_k}(\hat{U}_1)(\tau),$$
$$\mathcal{B}_{m_k}(t^l\hat{U}_1(t))(\tau) = \frac{\tau^k}{\Gamma(l/k)}\int_0^{\tau^k}(\tau^k - s)^{\frac{l}{k}-1}\mathcal{B}_{m_k}(\hat{U}_1)(s^{1/k})\frac{ds}{s},$$
$$\mathcal{B}_{m_k}(\hat{U}_1(t)\hat{U}_2(t))(\tau) = \tau^k\int_0^{\tau^k}\mathcal{B}_{m_k}(\hat{U}_1)((\tau^k-s)^{1/k})\mathcal{B}_{m_k}(\hat{U}_2)(s^{1/k})\frac{1}{(\tau^k-s)s}ds \tag{37}$$

where in the last formula, the product of formal power series is built up by means of the product $*$ *in the Banach algebra* $\mathbb{E}$.

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In the next proposition, we provide the counterpart of the above proposition for the action of differential operators of irregular type, multiplication by a monomial and products for m_k-sums of formal power series. Its proof is similar to the one given for Lemma 2 of [9].

Proposition 4. *Let $(\mathbb{E}, ||.||_{\mathbb{E}}, *)$ be a complex Banach algebra. Let $k, l \geq 1$ be natural numbers. We consider $\hat{U}_j(t)$, $j = 1, 2$, two elements of $t\mathbb{E}[[t]]$ that are assumed to be m_k-summable in some direction $d \in \mathbb{R}$. For $j = 1, 2$, we set*

$$U_j^d(t) = k \int_{L_d} \omega_j(\tau) \exp\big(-(\frac{\tau}{t})^k\big) \frac{d\tau}{\tau}$$

the m_k-sum of $\hat{U}_j(t)$ in direction d, where $\omega_j(\tau)$ denotes the m_k-Borel transform of $\hat{U}_j(t)$. The next identities hold whenever $t \in S_{d,\theta,R^{1/k}}$ provided that $\theta > \pi/k$ and is taken close enough to π/k and the radius $R > 0$ is chosen in the vicinity of the origin,

$$t^{k+1}\partial_t U_j^d(t) = k \int_{L_d} \{k\tau^k \omega_j(\tau)\} \exp\big(-(\frac{\tau}{t})^k\big) \frac{d\tau}{\tau},$$

$$t^l U_j^d(t) = k \int_{L_d} \Big\{\frac{\tau^k}{\Gamma(l/k)} \int_0^{\tau^k} (\tau^k - s)^{\frac{l}{k}-1} \omega_j(s^{1/k}) \frac{ds}{s}\Big\} \exp\big(-(\frac{\tau}{t})^k\big) \frac{d\tau}{\tau},$$

$$U_1^d(t) U_2^d(t) = k \int_{L_d} \Big\{\tau^k \int_0^{\tau^k} \omega_1((\tau^k - s)^{1/k}) \omega_2(s^{1/k}) \frac{1}{(\tau^k - s)s} ds\Big\} \exp\big(-(\frac{\tau}{t})^k\big) \frac{d\tau}{\tau}, \quad (38)$$

where in the closing formula, the product of $\mathbb{E}-$valued functions is built up by means of the product $*$ of the algebra $\mathbb{E}$.

3.2.2. Action of the Mahler operators on formal m_k-Borel transforms

The aim of this subsection is twofold. We first derive a formula describing the action of the Mahler operators on formal m_k-Borel transforms for formal power series through so-called formal deceleration operators. Then, for later use in Section 3.2.3, under some additional assumptions, we provide an integral representation of these deceleration operators constructed with the help with some kernel function. At last, we provide some important analytic features of this kernel function.

The next definition is a slightly modified version of the *deceleration operators* defined in the textbook [3], p. 46.

Definition 4. *Let $(\mathbb{E}, ||.||_{\mathbb{E}})$ be a complex Banach space. Let $1 \leq k < k'$ be two rational numbers. We define the $(k', k)-$deceleration operator $\hat{\mathcal{D}}_{k',k}$ from $\tau\mathbb{E}[[\tau]]$ into $h\mathbb{E}[[h]]$ by the formula*

$$\hat{\mathcal{D}}_{k',k}(\hat{f})(h) := \sum_{n \geq 1} f_n \frac{\Gamma(n/k')}{\Gamma(n/k)} h^n \quad (39)$$

for all elements $\hat{f}(\tau) = \sum_{n \geq 1} f_n \tau^n$ of $\tau\mathbb{E}[[\tau]]$.

Remark: This formal deceleration operator $\hat{\mathcal{D}}_{k',k}$ turns out to be the inverse for the composition of the so-called *acceleration operator* $\hat{\mathcal{A}}_{k',k}$ acting on formal series $\hat{f}(\tau) = \sum_{n \geq 1} f_n \tau^n$ through

$$\hat{\mathcal{A}}_{k',k}(\hat{f})(h) := \sum_{n \geq 1} f_n \frac{\Gamma(n/k)}{\Gamma(n/k')} h^n$$

introduced in the paper [8], Section 4.3, by A. Lastra and the author and which stands for an adjusted version of the classical acceleration operators as defined in [3], Chapter 5.

The next proposition discloses a formula for the formal m_k-Borel transform under the action of a Mahler operator.

Proposition 5. *Let $(\mathbb{E}, ||.||_{\mathbb{E}})$ be a complex Banach space. Let $p \geq 2$ and $k \geq 1$ be natural numbers with $k \geq p$. Let $\hat{U}(t)$ be an element of $t\mathbb{E}[[t]]$. We set $\hat{V}(t) = \hat{U}(t^p)$ the element of $t\mathbb{E}[[t]]$ obtained by applying the Mahler operator $t \mapsto t^p$ to $\hat{U}(t)$. The next identity holds*

$$\mathcal{B}_{m_k}(\hat{V})(\tau) = \hat{\mathcal{D}}_{k,k/p}(\mathcal{B}_{m_k}(\hat{U}))(\tau^p) \tag{40}$$

Proof. Let us expand $\hat{U}$ as $\hat{U}(t) = \sum_{n\geq 1} u_n t^n$. Hence, $\hat{V}(t) = \hat{U}(t^p) = \sum_{n\geq 1} u_n t^{pn}$. According to the first item of Definition 3, we observe that

$$\mathcal{B}_{m_k}(\hat{V})(\tau) = \sum_{n\geq 1} u_n \frac{\tau^{pn}}{\Gamma(\frac{pn}{k})}. \tag{41}$$

On the other hand, the m_k-Borel transform of $\hat{U}$ writes $\mathcal{B}_{m_k}(\hat{U})(\tau) = \sum_{n\geq 1} u_n \tau^n/\Gamma(n/k)$ and by Definition 4, we deduce that

$$\hat{\mathcal{D}}_{k,k/p}(\mathcal{B}_{m_k}(\hat{U}))(h) = \sum_{n\geq 1} \frac{u_n}{\Gamma(n/k)} \frac{\Gamma(n/k)}{\Gamma(\frac{n}{k/p})} h^n = \sum_{n\geq 1} \frac{u_n}{\Gamma(\frac{pn}{k})} h^n \tag{42}$$

At last, the combination of (41) and (42) yields (40). □

In the next proposition, we display an integral representation for the $(k',k)-$deceleration operator $\hat{\mathcal{D}}_{k',k}$ under further assumptions on its source space.

Proposition 6. *Let $(\mathbb{E}, ||.||_{\mathbb{E}})$ be a complex Banach space. Let $1 \leq k < k'$ be two rational numbers. We consider an element $\hat{f}(\tau)$ of $\tau\mathbb{E}[[\tau]]$ which is assumed to be convergent on some disc D_ρ with $\rho > 0$. For a given $h \in \mathbb{C}^*$, we attach the next two items.*

- *We choose a direction $\gamma_h \in \mathbb{R}$ and a positive real number $\Delta_1 > 0$ such that*

$$\cos(k'(\gamma_h - \arg(h))) > \Delta_1. \tag{43}$$

- *We consider the so-called closed Hankel path denoted $\tilde{\gamma}_{k,h}$ depicted as the union of*
  - *the oriented segment $\tilde{\gamma}_{k,h,1} = [0, (\rho/2)e^{\sqrt{-1}(\gamma_h + \frac{\pi}{2k} + \frac{\delta'}{2})}]$*
  - *the oriented arc of circle*

$$\tilde{\gamma}_{k,h,3} = \{(\rho/2)e^{\sqrt{-1}\theta}/\theta \in [\gamma_h + \frac{\pi}{2k} + \frac{\delta'}{2}, \gamma_h - \frac{\pi}{2k} - \frac{\delta'}{2}]\}$$

  - *the oriented segment $\tilde{\gamma}_{k,h,2} = [(\rho/2)e^{\sqrt{-1}(\gamma_h - \frac{\pi}{2k} - \frac{\delta'}{2})}, 0]$*

  where $\delta' > 0$ is a positive real number taken close to 0.

Then, the $(k',k)-$deceleration operator $\hat{\mathcal{D}}_{k',k}$ has the following integral representation

$$\hat{\mathcal{D}}_{k',k}(\hat{f})(h) = -\frac{k'k}{2\sqrt{-1}\pi} \int_{\tilde{\gamma}_{k,h}} \hat{f}(\xi)\mathbb{D}_{k',k}(\xi, h) d\xi \tag{44}$$

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for all $h \in \mathbb{C}^*$, where the kernel $\mathbb{D}_{k',k}(\xi, h)$ is expressed by means of the integral

$$\mathbb{D}_{k',k}(\xi,h) = \frac{1}{\xi^{k+1}} \int_{L_{\gamma_h}} u^{k-1} \exp\left((\frac{u}{\xi})^k - (\frac{u}{h})^{k'} \right) du \tag{45}$$

along a the halfline $L_{\gamma_h} = [0, +\infty)e^{\sqrt{-1}\gamma_h}$.

Proof. Let $h \in \mathbb{C}^*$. At first, from the very definition of the Gamma function, we observe that for all natural numbers $n \geq 1$, we have the next integral form for the monomial

$$\Gamma(n/k')h^n = k' \int_{L_{\gamma_h}} u^n \exp\left(-(\frac{u}{h})^{k'} \right) \frac{du}{u} \tag{46}$$

along the halfline L_{γ_h} given in the statement of the proposition 6.

Owing to Proposition 12 from [8] and based on the so-called Hankel formula (see [4], Appendix B.3), we can rewrite the next monomial in an integral form

$$\frac{u^n}{\Gamma(n/k)} = -\frac{k}{2\sqrt{-1}\pi} \int_{\tilde{\gamma}_{k,h}} \xi^n \exp\left((\frac{u}{\xi})^k \right) \frac{u^k}{\xi^{k+1}} d\xi \tag{47}$$

along the Hankel path $\tilde{\gamma}_{k,h}$ detailed in the second item of Proposition 6, for all $u \in L_{\gamma_h}$.

As a result of (46) together with (47), an application of the Fubini theorem yields

$$\begin{aligned} \frac{\Gamma(n/k')}{\Gamma(n/k)} h^n &= k' \int_{L_{\gamma_h}} \frac{u^n}{\Gamma(n/k)} \exp\left(-(\frac{u}{h})^{k'} \right) \frac{du}{u} \\ &= k' \int_{L_{\gamma_h}} \left(-\frac{k}{2\sqrt{-1}\pi} \int_{\tilde{\gamma}_{k,h}} \xi^n \exp\left((\frac{u}{\xi})^k \right) \frac{u^k}{\xi^{k+1}} d\xi \right) \exp\left(-(\frac{u}{h})^{k'} \right) \frac{du}{u} \\ &= -\frac{k'k}{2\sqrt{-1}\pi} \int_{\tilde{\gamma}_{k,h}} \xi^n \frac{1}{\xi^{k+1}} \left(\int_{L_{\gamma_h}} u^k \exp\left((\frac{u}{\xi})^k - (\frac{u}{h})^{k'} \right) \frac{du}{u} \right) d\xi. \end{aligned} \tag{48}$$

At last, the definition of $\hat{\mathcal{D}}_{k',k}(\hat{f})(h)$ given in (39) combined with the integral representation (48) and the uniform convergence of $\hat{f}(\tau)$ on the disc $D_{\rho/2}$ gives rise to the formula (44) and (45). □

In the forthcoming proposition, we provide crucial technical upper bounds for the kernel $\mathbb{D}_{k',k}$ that will be used in the next Section 4, Proposition 9, Lemma 5 and Lemma 6.

Proposition 7. *1) There exists a constant $M_{k',k} > 0$ (depending on k, k') such that the next upper bounds*

$$|\mathbb{D}_{k',k}(\xi,h)| \leq \frac{M_{k',k}}{k|\xi|^{k+1}} |h|^k \left| \frac{h}{\xi} \right|^{\frac{k^2}{k'-k}} \exp\left(|\frac{h}{\xi}|^{\kappa} \right) \tag{49}$$

hold for all $h \in \mathbb{C}^*$, all $\xi \in \tilde{\gamma}_{k,h}$ provided that $|\xi| \leq |h|$, where

$$\kappa = \frac{kk'}{k'-k}. \tag{50}$$

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Observe that $\kappa > k$ since $k' > k$.
2) There exists a constant $M_{k',k,1,2} > 0$ (depending on k, k') such that the upper estimates

$$|\mathbb{D}_{k',k}(\xi, h)| \leq M_{k',k,1,2} \frac{|h|^k}{k|\xi|^{k+1}} \quad (51)$$

hold for all $h \in \mathbb{C}^*$, all $\xi \in \tilde{\gamma}_{k,h,1}$ or $\xi \in \tilde{\gamma}_{k,h,2}$.

Proof. We first express $\mathbb{D}_{k',k}$ as a Laplace transform in the combined variable $(h/\xi)^k$. Indeed, we make the change of variable $u = ht^{1/k}$ in the integral (45) for the variable t belonging to the halfline $L_{\gamma'_h} = [0, +\infty)e^{\sqrt{-1}\gamma'_h}$ with

$$\gamma'_h = k(\gamma_h - \arg(h)) \quad (52)$$

which yields

$$\mathbb{D}_{k',k}(\xi, h) = \frac{1}{\xi^{k+1}} \Big(\int_{L_{\gamma'_h}} h^{k-1} t^{\frac{k-1}{k}} \exp\left((\frac{h}{\xi})^k t - t^{k'/k} \right) \frac{h}{k} t^{\frac{1}{k}-1} dt = \frac{1}{k\xi^{k+1}} h^k \mathbf{D}_{k'/k}((\frac{h}{\xi})^k) \quad (53)$$

where

$$\mathbf{D}_{k'/k}(z) = \int_{L_{\gamma'_h}} \exp(-t^{k'/k}) \exp(zt) dt. \quad (54)$$

In the next lemma, we derive some analytic features and bounds estimates for the Laplace integral $\mathbf{D}_{k'/k}$.

Lemma 1. *a) The map $z \mapsto \mathbf{D}_{k'/k}(z)$ is an entire function in $\mathbb{C}$. Moreover, one can single out some constant $M_{k',k} > 0$ such that*

$$|\mathbf{D}_{k'/k}(z)| \leq M_{k',k} |z|^{\frac{k}{k'-k}} \exp\left(|z|^{\frac{k'}{k'-k}} \right) \quad (55)$$

for all $|z| \geq 1$.
b) For some fixed constant $\Delta_2 > 0$ and the direction γ'_h given in (52), we consider the sector

$$S_{\gamma'_h, \Delta_2} = \{ z \in \mathbb{C}^* / \cos(\arg(z) + \gamma'_h) < -\Delta_2 \}. \quad (56)$$

Then, there exists a constant $M_{k',k,1,2} > 0$ (relying on $k, k', \Delta_1, \Delta_2$, where Δ_1 stems from (43)) such that

$$|\mathbf{D}_{k'/k}(z)| \leq M_{k',k,1,2} \quad (57)$$

for all $z \in S_{\gamma'_h, \Delta_2}$.

Proof. We discuss the first point a). From the Taylor expansion $e^{zt} = \sum_{n \geq 0} (zt)^n / n!$ which converges uniformly on any compact subset of $\mathbb{C}$, we deduce that

$$\mathbf{D}_{k'/k}(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n \quad (58)$$

for any $z \in \mathbb{C}$, with

$$a_n = \int_{L_{\gamma'_h}} t^n \exp(-t^{k'/k}) dt \ , \ n \geq 0.$$

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We make the change of variable $s = t^{k'/k}$ in the above integrals defining a_n and get by definition of the Gamma function that

$$a_n = \frac{k}{k'} \int_{L_{\gamma_h''}} s^{\frac{k}{k'}(n+1)-1} e^{-s} ds = \frac{k}{k'} \Gamma\big(\frac{k}{k'}(n+1)\big) \quad (59)$$

for all $n \geq 0$, where $\gamma_h'' = k'(\gamma_h - \arg(h))$. Combining (58) with (59) yields the expansion

$$\mathbf{D}_{k'/k}(z) = \sum_{n\geq 0} \frac{k}{k'} \frac{\Gamma\big(\frac{k}{k'}(n+1)\big)}{\Gamma(n+1)} z^n \quad (60)$$

for all $z \in \mathbb{C}$. On the other hand, from the Beta integral formula (see [4], Appendix B.3), we remind that

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt \leq 1 \quad (61)$$

for all real numbers $\alpha, \beta \geq 1$. From (61) we deduce that

$$\frac{\Gamma\big(\frac{k}{k'}(n+1)\big)}{\Gamma(n+1)} \leq \frac{1}{\Gamma\big((1-\frac{k}{k'})n + (1-\frac{k}{k'})\big)} \quad (62)$$

for all $n \geq n_{k',k}$, for some integer $n_{k',k} \geq 1$ depending on k, k'. As a result of (60) and (62), we obtain a constant $C_{k',k} > 0$ such that

$$|\mathbf{D}_{k'/k}(z)| \leq C_{k',k} \sum_{n\geq 0} \frac{1}{\Gamma\big((1-\frac{k}{k'})n + (1-\frac{k}{k'})\big)} |z|^n \quad (63)$$

for all $z \in \mathbb{C}$. Now, we call to mind some upper bounds for the so-called Wiman function

$$E_{\alpha,\beta}(x) = \sum_{n\geq 0} \frac{x^n}{\Gamma(\alpha n + \beta)}$$

for prescribed $\alpha \in (0,2)$ and $\beta > 0$ mentioned in our previous work [8], Proposition 1. Namely, some constant $K_{\alpha,\beta} > 0$ can be found such that

$$E_{\alpha,\beta}(x) \leq C_{\alpha,\beta} x^{\frac{1-\beta}{\alpha}} e^{x^{1/\alpha}} \quad (64)$$

for all $x \geq 1$. At last, from (63) and (64), we deduce that awaited bounds (55).

We focus on the second point b). According to the lower bounds (43) and the definition (56) of the sector S_{γ_h',Δ_2}, we reach a constant $M_{k',k,1,2}$ (depending on k', k, Δ_1 and Δ_2) with

$$\begin{aligned} |\mathbf{D}_{k'/k}(z)| &\leq \int_0^{+\infty} \exp\big(-r^{k'/k}\cos(k'(\gamma_h - \arg(h)))\big) \times \exp\big(|z| r \cos(\arg(z) + \gamma_h')\big) dr \\ &\leq \int_0^{+\infty} \exp(-r^{\frac{k'}{k}}\Delta_1) \exp(-|z| r \Delta_2) dr \leq M_{k',k,1,2} \quad (65) \end{aligned}$$

for all $z \in S_{\gamma_h',\Delta_2}$. $\square$

We turn to the first point 1) of Proposition 7. We observe that the inequality (49) is a straight consequence of the factorization (53) and the upper bounds (55).

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We address the second point 2) of Proposition 7. We first observe by construction of $\tilde{\gamma}_{k,h,1}$ and $\tilde{\gamma}_{k,h,2}$ in the second item of Proposition 6, one can find a constant $\Delta_2 > 0$ such that

$$\cos(k(\gamma_h - \arg(\xi))) = \cos(\arg((u/\xi)^k)) < -\Delta_2 \tag{66}$$

for all $u \in L_{\gamma_h}$, provided that $\xi \in \tilde{\gamma}_{k,h,1}$ or $\xi \in \tilde{\gamma}_{k,h,2}$. Besides, for $\Delta_2 > 0$ chosen as in (66), we remark that

$$(h/\xi)^k \in S_{\gamma_h', \Delta_2} \tag{67}$$

for all $\xi \in \tilde{\gamma}_{k,h,1} \cup \tilde{\gamma}_{k,h,2}$. Indeed, (67) is equivalent to

$$\cos(\arg((h/\xi)^k) + k(\gamma_h - \arg(h))) < -\Delta_2$$

which can be rewritten as (66).

At last, we conclude that the bounds (51) can be derived from the factorization (53) and the upper bounds (57) taking for granted the inclusion (67). □

3.2.3. Statement of the Integral Equation

In this subsection, we denote

$$\hat{W}(\tau, m) = \mathcal{B}_{m_k}(t \mapsto \hat{U}(t, m))(\tau) \tag{68}$$

the formal m_k-Borel transform of the formal power series expansion (31). The object of this subsection is the derivation of some integral equation fulfilled by the formal power series (68) seen as a series with coefficients in the Banach space $\mathbb{E} = E_{(\beta,\mu)}$ endowed with the norm $||.||_{\mathbb{E}} = ||.||_{(\beta,\mu)}$.

In this subsection, we make the assumption that

$$\hat{W} \in E_{(\beta,\mu)}\{\tau\} \tag{69}$$

meaning that $\tau \mapsto \hat{W}(\tau, m)$ is convergent on some disc D_ρ with $\rho > 0$ as a $E_{(\beta,\mu)}-$valued series. We will see in Section 4, where a solution of the integral equation (72) will be constructed in some function space, that this assumption will be satisfied.

In order to improve the legibility of the equation that $\hat{W}(\tau, m)$ is asked to solve

- We introduce the notation

$$\mathbf{C}_{k,l_0,l_1}(\hat{W})(\tau, m) := \frac{\tau^k}{\Gamma(l_0/k)} \int_0^{\tau^k} (\tau^k - s)^{\frac{l_0}{k}-1} (ks)^{l_1} \hat{W}(s^{1/k}, m) \frac{ds}{s} \tag{70}$$

  for all integers $l_0 \geq 1$, $l_1 \geq 0$.
- We define the map

$$P_m(\tau) = Q(\sqrt{-1}m) - \cosh(\alpha_D(k\tau^k)^2) R_D(\sqrt{-1}m) \tag{71}$$

  for all $\tau \in \mathbb{C}$, all $m \in \mathbb{R}$.

Based upon the transformations formula (37) harked back in Proposition 3 and the formal identity for the Mahler transforms reached in (40) of Proposition 5 together with the integral formula (44) derived in Proposition 6 under our assumption (69), we arrive at the next proposition.

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Proposition 8. *The formal power series $\hat{U}(t,m)$ given by (31) solves the equation (30) for vanishing initial data $\hat{U}(0,m) \equiv 0$* **if** *the convergent formal series $\hat{W}(\tau,m)$ given by (68) obeys the next integral equation*

$$
\begin{aligned}
P_m(\tau)\hat{W}(\tau,m) = &\sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0=0,l_2=1}} \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)(k\tau^k)^{l_1}\hat{W}(\tau,m_1)R_{\underline{l}}(\sqrt{-1}m_1)dm_1\\
&+\sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0\geq 1,l_2=1}} \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\mathbf{C}_{k,l_0,l_1}(\hat{W})(\tau,m_1)R_{\underline{l}}(\sqrt{-1}m_1)dm_1\\
&+\sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0=0,l_2>1}} \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\Big(-\frac{k^2/l_2}{2\sqrt{-1}\pi}\int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}}(k\xi^k)^{l_1}\hat{W}(\xi,m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi\Big)\\
&\qquad\times R_{\underline{l}}(\sqrt{-1}m_1)dm_1\\
&+\sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0\geq 1,l_2>1}} \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\Big(-\frac{k^2/l_2}{2\sqrt{-1}\pi}\int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}}\mathbf{C}_{k,l_0,l_1}(\hat{W})(\xi,m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi\Big)\\
&\qquad\times R_{\underline{l}}(\sqrt{-1}m_1)dm_1\\
&+c_{Q_1Q_2}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\Big(\tau^k\int_0^{\tau^k}\hat{W}((\tau^k-s)^{1/k},m-m_1)Q_1(\sqrt{-1}(m-m_1))\\
&\qquad\times\hat{W}(s^{1/k},m_1)Q_2(\sqrt{-1}m_1)\frac{1}{(\tau^k-s)s}ds\Big)dm_1+\sum_{j\in J}\mathcal{F}_j(m)\tau^j
\end{aligned}
\tag{72}
$$

4. Solving the Integral Equation in a Banach Space of Functions with Exponential Growth on Sectors and Decay on the Real Line

In this section, we investigate the existence and unicity of a genuine solution to the above integral equation (72) in the Banach space of functions described in the next definition

Definition 5. *Let S_d be an unbounded sector edged at 0 with bisecting direction $d\in\mathbb{R}$. Let $\nu,\rho>0$ be positive real numbers. We consider the natural number $k\geq 1$ and $\beta,\mu>0$ the real numbers prescribed in Section 2. We denote $F^d_{(\nu,\beta,\mu,k,\rho)}$ the vector space of continuous functions $(\tau,m)\mapsto h(\tau,m)$ on the product $(S_d\cup D_\rho)\times\mathbb{R}$, which are holomorphic with respect to τ on the union $S_d\cup D_\rho$ for which the norm*

$$||h(\tau,m)||_{(\nu,\beta,\mu,k,\rho)} = \sup_{\tau\in S_d\cup D_\rho, m\in\mathbb{R}} (1+|m|)^{\mu}e^{\beta|m|}\frac{1+|\tau|^{2k}}{|\tau|}\exp(-\nu|\tau|^k)|h(\tau,m)| \tag{73}$$

is finite. The space $F^d_{(\nu,\beta,\mu,k,\rho)}$ equipped with the norm $||.||_{(\nu,\beta,\mu,k,\rho)}$ turns out to be a complex Banach space.

These Banach spaces appear for the first time in the previous paper [7] by A. Lastra and the author.

Our strategy consists in rewriting our main integral equation (72) as a fixed point equation (see (195) below) for which a solution can be constructed in the above Banach space given in Definition 5 for well adjusted parameters ν and ρ. In order to recast (72) into (195), we need to divide both sides of (72) by the map $P_m(\tau)$ given in (71) provided that the Borel variable τ is taken in the vicinity of the origin and along a well chosen unbounded sector, given that the fourier mode m is ranged over $\mathbb{R}$.

In the next lemma, we provide some crucial lower bounds for $P_m(\tau)$ on fitting unbounded domains.

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Lemma 2. *Provided that the aperture $\eta_{Q,R_D} > 0$ of the sector S_{Q,R_D} diplayed in (13) and that the difference $|r_{Q,R_D,1} - r_{Q,R_D,2}|$ of the inner and outer radius of S_{Q,R_D} are taken small enough, there exists a non empty subset Θ_{Q,R_D} of $[-\pi,\pi)$ and a small radius $\rho > 0$ with the next features:*

- *For all $d \in \Theta_{Q,R_D}$, one can select an unbounded sector S_d edged at 0 with bisecting direction d.*
- *To the above chosen sector S_d, one can attach two constants $\delta_{S_d,k,\alpha_D} > 0$ (relying on S_d, k and α_D), $\Delta_{S_d,k} > 0$ (depending on S_d and k) with the following lower bounds*

$$|P_m(\tau)| \geq |R_D(\sqrt{-1}m)|\delta_{S_d,k,\alpha_D} \exp(\alpha_D k^2 \Delta_{S_d,k} |\tau|^{2k}) \tag{74}$$

for all $\tau \in S_d \cup D_\rho$, all $m \in \mathbb{R}$.

Proof. For all $m \in \mathbb{R}$, we set $H(m) = Q(\sqrt{-1}m)/R_D(\sqrt{-1}m)$. In a first step, we need to find the complex solutions of the equation

$$\cosh(X) := \frac{e^X + e^{-X}}{2} = H(m). \tag{75}$$

We notice that this equation (75) is equivalent to

$$(e^X)^2 - 2e^X H(m) + 1 = 0. \tag{76}$$

If one sets the quantity

$$\delta(m) = |H^2(m) - 1|^{1/2} \exp\left(\sqrt{-1}\frac{\arg(H^2(m)-1)}{2}\right) \tag{77}$$

for all $m \in \mathbb{R}$, then (76) has two infinite sets of solutions $\{a_l(m)\}_{l\in\mathbb{Z}}$ and $\{b_l(m)\}_{l\in\mathbb{Z}}$ given by explicit expressions

$$a_l(m) = \log|H(m) + \delta(m)| + \sqrt{-1}(\arg(H(m) + \delta(m)) + 2l\pi) \tag{78}$$

for all $l \in \mathbb{Z}$ and $m \in \mathbb{R}$ with

$$b_l(m) = \log|H(m) - \delta(m)| + \sqrt{-1}(\arg(H(m) - \delta(m)) + 2l\pi) \tag{79}$$

for all $l \in \mathbb{Z}$ and $m \in \mathbb{R}$. Namely, owing to the relation

$$(H(m) - \delta(m))(H(m) + \delta(m)) = 1$$

for all $m \in \mathbb{R}$, we observe that both expressions (78) and (79) are well defined since $H(m) - \delta(m)$ and $H(m) + \delta(m)$ are not vanishing quantities and furthermore that the next symmetry occurs

$$b_{-l}(m) = -a_l(m) \tag{80}$$

for all integers $l \in \mathbb{Z}$ and $m \in \mathbb{R}$.

At the next stage, we describe the complex solutions of the equation

$$\cosh(\alpha_D k^2 \tau^{2k}) = H(m). \tag{81}$$

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From the above discussion, we deduce that the complex zeros of (81) are given by the union of the roots of the next algebraic equations

$$\alpha_D k^2 \tau^{2k} = a_l(m) \tag{82}$$

with

$$\alpha_D k^2 \tau^{2k} = b_l(m) \tag{83}$$

for all $l \in \mathbb{Z}$. For each $l \in \mathbb{Z}$, the $2k$ distinct roots of (82) are given by

$$\tau_{h,l}(m) = |\frac{a_l(m)}{\alpha_D k^2}|^{\frac{1}{2k}} \exp\left(\sqrt{-1}(\frac{\arg(a_l(m))}{2k} + \frac{\pi h}{k})\right) \tag{84}$$

and the $2k$ distinct roots of (83) are expressed through

$$v_{h,l}(m) = |\frac{b_l(m)}{\alpha_D k^2}|^{\frac{1}{2k}} \exp\left(\sqrt{-1}(\frac{\arg(b_l(m))}{2k} + \frac{\pi h}{k})\right) \tag{85}$$

for all $0 \leq h \leq 2k-1$, all $m \in \mathbb{R}$. Furthermore, we notice the symmetry relations $-\tau_{h,l}(m) = \tau_{h+k,l}(m)$ with $-v_{h,l}(m) = v_{h+k,l}(m)$ provided that $0 \leq h \leq k$, for any given $l \in \mathbb{Z}$ and $m \in \mathbb{R}$.

Bearing in mind from (14), that $H(m)$ belongs to the sector S_{Q,R_D} for all $m \in \mathbb{R}$, provided that the aperture $\eta_{Q,R_D} > 0$ of S_{Q,R_D} and the difference $|r_{Q,R_D,1} - r_{Q,R_D,2}|$ are chosen small enough, there exist directions $d \in \mathbb{R}$ for which an unbounded sector S_d edged at 0 with bisecting direction d can be singled out in a way that

$$S_d \cap \left(\{\tau_{h,l}(m)/0 \leq h \leq 2k-1, l \in \mathbb{Z}, m \in \mathbb{R}\} \cup \{v_{h,l}(m)/0 \leq h \leq 2k-1, l \in \mathbb{Z}, m \in \mathbb{R}\}\right) = \emptyset. \tag{86}$$

For later use, we choose the sector S_d with the further assumption that for all $\theta \in \mathbb{R}$ such that $e^{\sqrt{-1}\theta} \in S_d$, the next condition

$$\cos(2k\theta) \neq 0 \tag{87}$$

holds. We denote Θ_{Q,R_D} the set of all directions d in $(-\pi, \pi)$ for which sectors S_d can be selected fulfilling the above two features (86) and (87).

Now, we explain which constraints are set on the radius $\rho > 0$. Provided that $\rho > 0$ is chosen close enough to 0, one can choose a small radius $\eta_1 > 0$ such that

$$\cosh(\alpha_D k^2 \tau^{2k}) \in D(1, \eta_1) \tag{88}$$

for all $\tau \in D_\rho$, where $D(1, \eta_1)$ stands for the disc centered at 1 with radius η_1. Then, we select the sector S_{Q,R_D} in a way that

$$S_{Q,R_D} \cap D(1, \eta_1) = \emptyset. \tag{89}$$

For the rest of the proof, we choose a sector S_d for $d \in \Theta_{Q,R_D}$ and a disc D_ρ as above.

We first come up with lower bounds for the map $P_m(\tau)$ on the domains $(S_d \cup D_\rho) \cap D_R$, for any prescribed large radius $R > 0$. Namely, we factorize $P_m(\tau)$ in the form

$$P_m(\tau) = R_D(\sqrt{-1}m) \times [H(m) - \cosh(\alpha_D k^2 \tau^{2k})] \tag{90}$$

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for all $\tau \in S_d \cup D_\rho$, all $m \in \mathbb{R}$. Owing to the constraints (86), (88) along with (89) and according to (14), for each given radius $R > 0$ (as large as we want), we can find a constant $\delta_1 > 0$ such that

$$|H(m) - \cosh(\alpha_D k^2 \tau^{2k})| \geq \delta_1 \tag{91}$$

for all $S_d \cup D_\rho$ with $|\tau| \leq R$. Combining (90) and (91) yields lower bounds of the form

$$|P_m(\tau)| \geq |R_D(\sqrt{-1}m)|\delta_1 \tag{92}$$

for all $\tau \in (S_d \cup D_\rho) \cap D_R$ and $m \in \mathbb{R}$.

In the last part of the proof, lower bounds for large values of $|\tau|$ on S_d are exhibited. We write $\tau \in S_d$ in the form $\tau = re^{\sqrt{-1}\theta}$, for radius $r \geq 0$ and angle $\theta \in \mathbb{R}$. Then,

$$\mathrm{Re}(\alpha_D k^2 \tau^{2k}) = \alpha_D k^2 r^{2k} \cos(2k\theta) \tag{93}$$

According to our choice of S_d subjected to (87), two cases arise.

Case 1. There exists a constant $\Delta_{S_d,k,1} > 0$ (depending on S_d and k) such that

$$\cos(2k\theta) > \Delta_{S_d,k,1} \tag{94}$$

for all $\theta \in \mathbb{R}$ with $e^{\sqrt{-1}\theta} \in S_d$. We perform the next factorization

$$\cosh(\alpha_D k^2 \tau^{2k}) = \exp(\alpha_D k^2 \tau^{2k}) \times [\frac{1}{2} + \frac{1}{2}\exp(-2\alpha_D k^2 \tau^{2k})] \tag{95}$$

which allows us to rephrase the next difference as a product

$$\cosh(\alpha_D k^2 \tau^{2k}) - H(m) = \exp(\alpha_D k^2 \tau^{2k})\mathcal{A}(\tau, m) \tag{96}$$

where

$$\mathcal{A}(\tau, m) = [\frac{1}{2} + \frac{1}{2}\exp(-2\alpha_D k^2 \tau^{2k})] \\ \times \left[1 - H(m)\exp(-\alpha_D k^2 \tau^{2k}) \times [\frac{1}{2} + \frac{1}{2}\exp(-2\alpha_D k^2 \tau^{2k})]^{-1}\right] \tag{97}$$

for all $\tau \in S_d$, $m \in \mathbb{R}$. According to (94), we note that

$$|\exp(\alpha_D k^2 \tau^{2k})| = \exp\left(\alpha_D k^2 r^{2k} \cos(2k\theta)\right) \geq \exp\left(\alpha_D k^2 \Delta_{S_d,k,1} |\tau|^{2k}\right) \tag{98}$$

whenever $\tau = re^{\sqrt{-1}\theta} \in S_d$. Besides, observing that $\lim_{|\tau| \to +\infty} |\exp(-\alpha_D k^2 \tau^{2k})| = 0$ and keeping in mind that $r_{Q,R_D,1} \leq |H(m)| \leq r_{Q,R_D,2}$, for all $m \in \mathbb{R}$, we get some constant $A_{S_d,k,\alpha_D} > 0$ and a radius $R_1 > 0$ (large enough) for which

$$|\mathcal{A}(\tau, m)| \geq A_{S_d,k,\alpha_D} \tag{99}$$

for all $\tau \in S_d$ with $|\tau| \geq R_1$ and all $m \in \mathbb{R}$.

Eventually, in gathering the factorizations (90) and (96) together with the lower bounds (98) and (99), we arrive at the lower bounds

$$|P_m(\tau)| \geq |R_D(\sqrt{-1}m)| A_{S_d,k,\alpha_D} \exp\left(\alpha_D k^2 \Delta_{S_d,k,1} |\tau|^{2k}\right) \tag{100}$$

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provided that $\tau \in S_d$ with $|\tau| \geq R_1$ and $m \in \mathbb{R}$. At last, combining these last bounds (100) and (92) for $R = R_1$, we arrive at the awaited lower bounds (74) for some small constant $\delta_{S_d,k,\alpha_D} > 0$ and $\Delta_{S_d,k} := \Delta_{S_d,k,1}$.

Case 2. A constant $\Delta_{S_d,k,2} > 0$ (depending on S_d and k) can be singled out for which

$$\cos(2k\theta) < -\Delta_{S_d,k,2} \quad (101)$$

holds for all $\theta \in \mathbb{R}$ with $e^{\sqrt{-1}\theta} \in S_d$. In that case, we do factorize

$$\cosh(\alpha_D k^2 \tau^{2k}) = \exp(-\alpha_D k^2 \tau^{2k}) \times [\frac{1}{2} + \frac{1}{2}\exp(2\alpha_D k^2 \tau^{2k})] \quad (102)$$

and recast the next difference at a product in the form

$$\cosh(\alpha_D k^2 \tau^{2k}) - H(m) = \exp(-\alpha_D k^2 \tau^{2k})\mathcal{B}(\tau, m) \quad (103)$$

where

$$\mathcal{B}(\tau, m) = [\frac{1}{2} + \frac{1}{2}\exp(2\alpha_D k^2 \tau^{2k})] \times \left[1 - H(m)\exp(\alpha_D k^2 \tau^{2k}) \times [\frac{1}{2} + \frac{1}{2}\exp(2\alpha_D k^2 \tau^{2k})]^{-1}\right] \quad (104)$$

for all $\tau \in S_d$, $m \in \mathbb{R}$. Based on our hypothesis (101), we remark that

$$|\exp(-\alpha_D k^2 \tau^{2k})| = \exp\left(-\alpha_D k^2 r^{2k}\cos(2k\theta)\right) \geq \exp\left(\alpha_D k^2 \Delta_{S_d,k,2}|\tau|^{2k}\right) \quad (105)$$

as long as $\tau = re^{\sqrt{-1}\theta} \in S_d$. On the other hand, since $\lim_{|\tau| \to +\infty} |\exp(\alpha_D k^2 \tau^{2k})| = 0$ and granting that $|H(m)| \in [r_{Q,R_D,1}, r_{Q,R_D,2}]$, for all $m \in \mathbb{R}$, some constant $B_{S_d,k,\alpha_D} > 0$ and a radius $R_2 > 0$ (large enough) can be deduced for which

$$|\mathcal{B}(\tau, m)| \geq B_{S_d,k,\alpha_D} \quad (106)$$

for all $\tau \in S_d$ with $|\tau| \geq R_2$ and all $m \in \mathbb{R}$.

In conclusion, the factorizations (90) and (103) combined with the lower bounds (105) and (106) yield the next lower bounds

$$|P_m(\tau)| \geq |R_D(\sqrt{-1}m)| B_{S_d,k,\alpha_D} \exp\left(\alpha_D k^2 \Delta_{S_d,k,2}|\tau|^{2k}\right) \quad (107)$$

whenever $\tau \in S_d$ with $|\tau| \geq R_2$ and $m \in \mathbb{R}$. Finally, these latter bounds (107) gathered with (92) for $R = R_2$ give rise to the expected lower bounds (74) for some small constant $\delta_{S_d,k,\alpha_D} > 0$ and $\Delta_{S_d,k} := \Delta_{S_d,k,2}$. □

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We now introduce the map $\mathcal{H}$ acting on the Banach spaces of Definition 5 for which the fixed point theorem will be applied, namely

$$\begin{aligned}
\mathcal{H}(\omega(\tau,m)) &:= \sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0=0,l_2=1}} \frac{1}{(2\pi)^{1/2}P_m(\tau)}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)(k\tau^k)^{l_1}\omega(\tau,m_1)R_{\underline{l}}(\sqrt{-1}m_1)dm_1 \\
&+ \sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0\geq 1,l_2=1}} \frac{1}{(2\pi)^{1/2}P_m(\tau)}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\mathbf{C}_{k,l_0,l_1}(\omega)(\tau,m_1)R_{\underline{l}}(\sqrt{-1}m_1)dm_1 \\
&+ \sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0=0,l_2>1}} \frac{1}{(2\pi)^{1/2}P_m(\tau)}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\Big(-\frac{k^2/l_2}{2\sqrt{-1}\pi}\int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}}(k\xi^k)^{l_1}\omega(\xi,m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi\Big) \\
&\qquad \times R_{\underline{l}}(\sqrt{-1}m_1)dm_1 \\
&+ \sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0\geq 1,l_2>1}} \frac{1}{(2\pi)^{1/2}P_m(\tau)}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\Big(-\frac{k^2/l_2}{2\sqrt{-1}\pi}\int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}}\mathbf{C}_{k,l_0,l_1}(\omega)(\xi,m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi\Big) \\
&\qquad \times R_{\underline{l}}(\sqrt{-1}m_1)dm_1 \\
&+ c_{Q_1Q_2}\frac{1}{(2\pi)^{1/2}P_m(\tau)}\int_{-\infty}^{+\infty}\Big(\tau^k\int_0^{\tau^k}\omega((\tau^k-s)^{1/k},m-m_1)Q_1(\sqrt{-1}(m-m_1)) \\
&\qquad \times \omega(s^{1/k},m_1)Q_2(\sqrt{-1}m_1)\frac{1}{(\tau^k-s)s}ds\Big)dm_1 + \sum_{j\in J}\frac{\mathcal{F}_j(m)}{P_m(\tau)}\tau^j. \qquad (108)
\end{aligned}$$

In the next proposition, we show that $\mathcal{H}$ acts on the Banach space $F^d_{(\nu,\beta,\mu,k,\rho)}$ for well chosen parameters and directions d as a shrinking map. This result is central in our work.

Proposition 9. *Let $\nu > 0$ be a fixed real number and let β,μ,k be prescribed as in Section 2. Let us assume that the sector S_{Q,R_D} introduced in (13) is selected as in Lemma 2. We choose an unbounded sector S_d for some direction $d \in \Theta_{Q,R_D}$ and a disc D_ρ for a suitably small $\rho > 0$ fixed as in Lemma 2. We make the following additional restriction on the constant α_D which is asked to obey the next inequality*

$$\alpha_D k^2 \Delta_{S_d,k} \geq (2/\rho)^k \qquad (109)$$

where $\Delta_{S_d,k} > 0$ is the constant appearing in (74) from Lemma 2.

Under the assumption that the constants $\mathbf{A}_{\underline{l}} > 0$ for $\underline{l} \in \mathcal{A}$ set up in (22) and the quantity $|c_{Q_1Q_2}|$ are taken adequately small, one can sort a constant $\varpi > 0$ such that the map $\mathcal{H}$ enjoys the next properties

1. *The next inclusion*
$$\mathcal{H}(B_\varpi) \subset B_\varpi \qquad (110)$$
holds where B_ϖ stands for the closed ball of radius $\varpi > 0$ centered at 0 in the space $F^d_{(\nu,\beta,\mu,k,\rho)}$.
2. *The shrinking condition*
$$||\mathcal{H}(\omega_1) - \mathcal{H}(\omega_2)||_{(\nu,\beta,\mu,k,\rho)} \leq \frac{1}{2}||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\rho)} \qquad (111)$$
occurs whenever $\omega_1, \omega_2 \in B_\varpi$.

Proof. We focus on the first item 1. Let ω be an element of $F^{d}_{(\nu,\beta,\mu,k,\rho)}$. By definition, the next inequality

$$|\omega(\tau,m)| \leq ||\omega||_{(\nu,\beta,\mu,k,\rho)}(1+|m|)^{-\mu}e^{-\beta|m|}\frac{|\tau|}{1+|\tau|^{2k}}\exp(\nu|\tau|^{k}) \tag{112}$$

holds for all $\tau \in S_d \cup D_\rho$, all $m \in \mathbb{R}$.

In the next six lemma, we provide upper norm bounds for each piece composing the map $\mathcal{H}$. The elements of the first sum of operators is minded in the next

Lemma 3. *Let $\underline{l} = (l_0, l_1, l_2) \in \mathcal{A}$ with $l_0 = 0$ and $l_2 = 1$. We can find some constant C_1 (relying on $\mu, R_D, R_{\underline{l}}, S_d, k, \alpha_D, l_1$) with*

$$||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\tau^{kl_1}\omega(\tau,m_1)R_{\underline{l}}(\sqrt{-1}m_1)dm_1||_{(\nu,\beta,\mu,k,\rho)} \leq C_1\mathbf{A}_{\underline{l}}||\omega||_{(\nu,\beta,\mu,k,\rho)} \tag{113}$$

for all $\omega \in F^{d}_{(\nu,\beta,\mu,k,\rho)}$.

Proof. We remind from Section 2 that $R_{\underline{l}}(X)$ is a polynomial of degree $\deg(R_{\underline{l}})$ and $R_D(X)$ is a polynomial of degree $\deg(R_D)$ not vanishing on $X = \sqrt{-1}m$, for all $m \in \mathbb{R}$. As a result, two constants $\mathfrak{R}_{\underline{l}}, \mathfrak{R}_D > 0$ can be found with

$$|R_{\underline{l}}(\sqrt{-1}m)| \leq \mathfrak{R}_{\underline{l}}(1+|m|)^{\deg(R_{\underline{l}})}, \quad |R_D(\sqrt{-1}m)| \geq \mathfrak{R}_D(1+|m|)^{\deg(R_D)} \tag{114}$$

for all $m \in \mathbb{R}$.

Let $\omega \in F^{d}_{(\nu,\beta,\mu,k,\rho)}$. Based on the definition of $\mathbf{A}_{\underline{l}}$ given in (22), the lower bounds (74) reached in Lemma 2 together with the upper bounds (112) and the above inequalities (114), we obtain

$$\begin{aligned}
\mathcal{A}_{\tau,m} &= \left|\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\tau^{kl_1}\omega(\tau,m_1)R_{\underline{l}}(\sqrt{-1}m_1)dm_1\right| \\
&\leq \frac{|\tau|^{kl_1}||\omega||_{(\nu,\beta,\mu,k,\rho)}}{\mathfrak{R}_D(1+|m|)^{\deg(R_D)}\delta_{S_d,k,\alpha_D}\exp(\alpha_D k^2\Delta_{S_d,k}|\tau|^{2k})} \times \frac{|\tau|}{1+|\tau|^{2k}}\exp(\nu|\tau|^{k}) \\
&\times \int_{-\infty}^{+\infty}\mathbf{A}_{\underline{l}}(1+|m-m_1|)^{-\mu}e^{-\beta|m-m_1|}(1+|m_1|)^{-\mu}e^{-\beta|m_1|}\mathfrak{R}_{\underline{l}}(1+|m_1|)^{\deg(R_{\underline{l}})}dm_1
\end{aligned} \tag{115}$$

for all $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$. Besides, the triangular inequality

$$|m| \leq |m-m_1| + |m_1| \tag{116}$$

holds for all $m, m_1 \in \mathbb{R}$ and we can choose a constant $M_{S_d,k,l_1,\alpha_D} > 0$ such that

$$\sup_{\tau \in S_d \cup D_\rho}\frac{|\tau|^{kl_1}}{\exp(\alpha_D k^2\Delta_{S_d,k}|\tau|^{2k})} = M_{S_d,k,l_1,\alpha_D}. \tag{117}$$

Gathering (115), (116) and (117), we come up with

$$\mathcal{A}_{\tau,m} \leq \frac{M_{S_d,k,l_1,\alpha_D}}{\delta_{S_d,k,\alpha_D}}||\omega||_{(\nu,\beta,\mu,k,\rho)}\mathbf{A}_{\underline{l}}\frac{\mathfrak{R}_{\underline{l}}}{\mathfrak{R}_D} \times \frac{|\tau|}{1+|\tau|^{2k}}\exp(\nu|\tau|^{k})(1+|m|)^{-\mu}e^{-\beta|m|}\mathcal{A}_1(m) \tag{118}$$

where

$$\mathcal{A}_1(m) = (1+|m|)^{\mu-\deg(R_D)}\int_{-\infty}^{+\infty}\frac{1}{(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu-\deg(R_{\underline{l}})}}dm_1 \tag{119}$$

provided that $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$. Owing to Lemma 2.2 of [5], under the assumptions made in (11) and (21), a constant $C_{1.1} > 0$ can be deduced with

$$\mathcal{A}_1(m) \leq C_{1.1} \tag{120}$$

for all $m \in \mathbb{R}$. At last, joining (118) and (120) yields a constant $C_1 > 0$ for which

$$\mathcal{A}_{\tau,m} \leq C_1 ||\omega||_{(\nu,\beta,\mu,k,\rho)} \mathbf{A}_{\underline{l}} \frac{|\tau|}{1+|\tau|^{2k}} \exp(\nu|\tau|^k)(1+|m|)^{-\mu} e^{-\beta|m|} \tag{121}$$

as long as $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$, which is tantamount to (113). □

The elements of the second sum of operators are considered in the next

Lemma 4. *We set $\underline{l} = (l_0, l_1, l_2) \in \mathcal{A}$ with $l_0 \geq 1$ and $l_2 = 1$. Then, a constant $C_2 > 0$ (depending upon $\mu, R_D, R_{\underline{l}}, S_d, k, \alpha_D, l_0, l_1$) can be picked out such that*

$$\begin{aligned} ||\frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1) \Big[\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{l_0}{k}-1} s^{l_1} \omega(s^{1/k}, m_1) \frac{ds}{s}\Big] \\ \times R_{\underline{l}}(\sqrt{-1}m_1) dm_1 ||_{(\nu,\beta,\mu,k,\rho)} \leq C_2 \mathbf{A}_{\underline{l}} ||\omega||_{(\nu,\beta,\mu,k,\rho)} \end{aligned} \tag{122}$$

for all $\omega \in F^d_{(\nu,\beta,\mu,k,\rho)}$.

Proof. Take ω an element of $F^d_{(\nu,\beta,\mu,k,\rho)}$. On the ground of the definition $\mathbf{A}_{\underline{l}}$ displayed in (22), the lower bounds (74) stated in Lemma 2 together with the upper bounds (112) and the polynomial inequalities (114), we reach

$$\begin{aligned} \mathcal{B}_{\tau,m} = \Big|\frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1) \Big[\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{l_0}{k}-1} s^{l_1} \omega(s^{1/k}, m_1) \frac{ds}{s}\Big] \\ \times R_{\underline{l}}(\sqrt{-1}m_1) dm_1 \Big| \\ \leq \frac{|\tau|^k \int_0^{|\tau|^k} (|\tau|^k - h)^{\frac{l_0}{k}-1} h^{l_1+\frac{1}{k}} e^{\nu h} \frac{dh}{h}}{\mathfrak{R}_D (1+|m|)^{\deg(R_D)} \delta_{S_d,k,\alpha_D} \exp\left(\alpha_D k^2 \Delta_{S_d,k} |\tau|^{2k}\right)} ||\omega||_{(\nu,\beta,\mu,k,\rho)} \\ \times \int_{-\infty}^{+\infty} \mathbf{A}_{\underline{l}} (1+|m-m_1|)^{-\mu} \exp(-\beta|m-m_1|)(1+|m_1|)^{-\mu} \exp(-\beta|m_1|) \mathfrak{R}_{\underline{l}} (1+|m_1|)^{\deg(R_{\underline{l}})} dm_1 \end{aligned} \tag{123}$$

for all $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$. By applying the change of variable $h = |\tau|^k u$, for $0 \leq u \leq 1$, a constant $M_{S_d,k,l_0,l_1,\alpha_D} > 0$ is deduced with

$$\begin{aligned} \frac{|\tau|^k \int_0^{|\tau|^k} (|\tau|^k - h)^{\frac{l_0}{k}-1} h^{l_1+\frac{1}{k}} \frac{dh}{h}}{\exp\left(\alpha_D k^2 \Delta_{S_d,k} |\tau|^{2k}\right)} \times \frac{1+|\tau|^{2k}}{|\tau|} \\ = \frac{|\tau|^{l_0+kl_1}(1+|\tau|^{2k})}{\exp\left(\alpha_D k^2 \Delta_{S_d,k} |\tau|^{2k}\right)} \times \left(\int_0^1 (1-u)^{\frac{l_0}{k}-1} u^{l_1+\frac{1}{k}} \frac{du}{u}\right) \leq M_{S_d,k,l_0,l_1,\alpha_D} \end{aligned} \tag{124}$$

for all $\tau \in S_d \cup D_\rho$.

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Collecting (123), (116) and (124) we land up at

$$\mathcal{B}_{\tau,m} \leq \frac{M_{S_d,k,l_0,l_1,\alpha_D}}{\delta_{S_d,k,\alpha_D}} ||\omega||_{(\nu,\beta,\mu,k,\rho)} \mathbf{A}_{\underline{l}} \frac{\mathfrak{R}_{\underline{l}}}{\mathfrak{R}_D} \frac{|\tau|}{1+|\tau|^{2k}} e^{\nu|\tau|^k} (1+|m|)^{-\mu} e^{-\beta|m|} \mathcal{A}_1(m) \tag{125}$$

where $\mathcal{A}_1(m)$ is given by (119), as long as $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$. Eventually, gathering (125) and (120) gives rise to a constant $C_2 > 0$ (hinging on $\mu, R_D, R_{\underline{l}}, S_d, k, \alpha_D, l_0, l_1$) with

$$\mathcal{B}_{\tau,m} \leq C_2 ||\omega||_{(\nu,\beta,\mu,k,\rho)} \mathbf{A}_{\underline{l}} \frac{|\tau|}{1+|\tau|^{2k}} \exp(\nu|\tau|^k)(1+|m|)^{-\mu} e^{-\beta|m|} \tag{126}$$

for all $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$, which is equivalent to (122). □

The components of the third sum of operators are upper bounded in the following

Lemma 5. *Let $\underline{l} = (l_0, l_1, l_2) \in \mathcal{A}$ with $l_0 = 0$ and $l_2 > 1$. There exists a constant $C_3 > 0$ (relying on $\mu, R_D, R_{\underline{l}}, S_d, k, \alpha_D, l_1, l_2, \rho$) such that*

$$\begin{aligned} || \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1) \Big(\int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}} \xi^{kl_1} \omega(\xi, m_1) \mathbb{D}_{k,\frac{k}{l_2}}(\xi, \tau^{l_2}) d\xi \Big) \\ \times R_{\underline{l}}(\sqrt{-1} m_1) dm_1 ||_{(\nu,\beta,\mu,k,\rho)} \leq C_3 \mathbf{A}_{\underline{l}} ||\omega(\tau,m)||_{(\nu,\beta,\mu,k,\rho)} \end{aligned} \tag{127}$$

for all $\omega \in F^d_{(\nu,\beta,\mu,k,\rho)}$.

Proof. In the first part of the proof, we provide upper bounds for the integral map

$$\mathbb{K}(\tau, m_1) := \int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}} \xi^{kl_1} \omega(\xi, m_1) \mathbb{D}_{k,\frac{k}{l_2}}(\xi, \tau^{l_2}) d\xi \tag{128}$$

for $\tau \in S_d \cup D_\rho$ and $m_1 \in \mathbb{R}$. Indeed, the next auxiliary result holds.

Sublemma 1. *1) For any $\tilde{\rho} \geq \rho$, there exists a constant $\mathbb{K}^1_{\tilde{\rho},k,l_1,l_2} > 0$ with*

$$|\mathbb{K}(\tau, m_1)| \leq \mathbb{K}^1_{\tilde{\rho},k,l_1,l_2} ||\omega||_{(\nu,\beta,\mu,k,\rho)} |\tau|^k (1+|m_1|)^{-\mu} e^{-\beta|m_1|} \tag{129}$$

for all $\tau \in D_{\tilde{\rho}}$, all $m_1 \in \mathbb{R}$.

2) One can pinpoint a constant $\mathbb{K}^2_{\rho,k,l_1,l_2} > 0$ such that

$$|\mathbb{K}(\tau, m_1)| \leq \mathbb{K}^2_{\rho,k,l_1,l_2} ||\omega||_{(\nu,\beta,\mu,k,\rho)} |\tau|^{k+\frac{k}{l_2-1}} \exp\left(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{(\rho/2)^{\frac{k}{l_2-1}}} \right) (1+|m_1|)^{-\mu} e^{-\beta|m_1|} \tag{130}$$

for all $\tau \in S_d$ with $|\tau| \geq (\rho/2)^{1/l_2}$, all $m_1 \in \mathbb{R}$.

Proof. Let us consider $\omega \in F^d_{(\nu,\beta,\mu,k,\rho)}$. We observe that the bounds (112) hold. Owing to our assumption (9) and according to the construction of the integral operator $\hat{\mathcal{D}}_{k,k/l_2}$ discussed in Proposition 6, for any fixed $\tau \in S_d \cup D_{\tilde{\rho}}$, one chooses a direction $\gamma_{\tau^{l_2}} \in \mathbb{R}$ and a positive real number $\Delta_1 > 0$ such that

$$\cos(k(\gamma_{\tau^{l_2}} - \arg(\tau^{l_2}))) > \Delta_1. \tag{131}$$

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From the definition of the Hankel path $\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}$, for any $m_1 \in \mathbb{R}$, one can split the integral $\mathbb{K}(\tau, m_1)$ as a sum of three pieces

$$\mathbb{K}(\tau, m_1) = \mathbb{K}_{\gamma^+_{\tau^{l_2}}}(\tau, m_1) - \mathbb{K}_{\gamma^-_{\tau^{l_2}},\gamma^+_{\tau^{l_2}}}(\tau, m_1) - \mathbb{K}_{\gamma^-_{\tau^{l_2}}}(\tau, m_1) \tag{132}$$

where

$$\mathbb{K}_{\gamma^+_{\tau^{l_2}}}(\tau, m_1) = \int_{L_{[0,\rho/2];\frac{k}{l_2};\gamma^+_{\tau^{l_2}}}} \xi^{kl_1}\omega(\xi, m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi, \tau^{l_2})d\xi \tag{133}$$

whose integration path is the segment $L_{[0,\rho/2];\frac{k}{l_2};\gamma^+_{\tau^{l_2}}} = [0, \rho/2]e^{\sqrt{-1}(\gamma_{\tau^{l_2}} + \frac{\pi l_2}{2k} + \frac{\delta'}{2})}$, for some small $\delta' > 0$ and

$$\mathbb{K}_{\gamma^+_{\tau^{l_2}},\gamma^-_{\tau^{l_2}}}(\tau, m_1) = \int_{C_{\rho/2;\gamma^+_{\tau^{l_2}},\gamma^-_{\tau^{l_2}}}} \xi^{kl_1}\omega(\xi, m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi, \tau^{l_2})d\xi \tag{134}$$

along the arc of circle

$$C_{\rho/2;\gamma^+_{\tau^{l_2}},\gamma^-_{\tau^{l_2}}} = \{\frac{\rho}{2}e^{\sqrt{-1}\theta}/\theta \in [\gamma_{\tau^{l_2}} - \frac{\pi l_2}{2k} - \frac{\delta'}{2}, \gamma_{\tau^{l_2}} + \frac{\pi l_2}{2k} + \frac{\delta'}{2}]\},$$

along with

$$\mathbb{K}_{\gamma^-_{\tau^{l_2}}}(\tau, m_1) = \int_{L_{[0,\rho/2];\frac{k}{l_2};\gamma^-_{\tau^{l_2}}}} \xi^{kl_1}\omega(\xi, m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi, \tau^{l_2})d\xi \tag{135}$$

where $L_{[0,\rho/2];\frac{k}{l_2};\gamma^-_{\tau^{l_2}}} = [0, \rho/2]e^{\sqrt{-1}(\gamma_{\tau^{l_2}} - \frac{\pi l_2}{2k} - \frac{\delta'}{2})}$.

In the next step of the proof, we display bounds for each piece of the splitting (132).

We first provide bounds for $\mathbb{K}_{\gamma^+_{\tau^{l_2}}}(\tau, m_1)$ and $\mathbb{K}_{\gamma^-_{\tau^{l_2}}}(\tau, m_1)$.

According to Proposition 7 2), we can find a constant $M_{k,k/l_2,1,2} > 0$ such that

$$|\mathbb{D}_{k,\frac{k}{l_2}}(\xi, \tau^{l_2})| \leq M_{k,k/l_2,1,2}\frac{l_2}{k}\frac{|\tau|^k}{|\xi|^{\frac{k}{l_2}+1}} \tag{136}$$

provided that $\xi \in L_{[0,\rho/2];\frac{k}{l_2};\gamma^+_{\tau^{l_2}}}$ or $\xi \in L_{[0,\rho/2];\frac{k}{l_2};\gamma^-_{\tau^{l_2}}}$. Then, under the assumption (10) and bearing in mind the bounds (112) together with (136), we are given a constant $K_{\rho,k,l_1,l_2} > 0$ such that

$$\begin{aligned} |\mathbb{K}_{\gamma^+_{\tau^{l_2}}}(\tau, m_1)| &\leq \int_0^{\rho/2} r^{kl_1}||\omega||_{(\nu,\beta,\mu,k,\rho)}(1 + |m_1|)^{-\mu}e^{-\beta|m_1|}re^{\nu r^k}M_{k,k/l_2,1,2}\frac{l_2}{k}\frac{|\tau|^k}{r^{\frac{k}{l_2}+1}}dr \\ &\leq K_{\rho,k,l_1,l_2}||\omega||_{(\nu,\beta,\mu,k,\rho)}|\tau|^k(1 + |m_1|)^{-\mu}e^{-\beta|m_1|}. \end{aligned} \tag{137}$$

together with

$$|\mathbb{K}_{\gamma^-_{\tau^{l_2}}}(\tau, m_1)| \leq K_{\rho,k,l_1,l_2}||\omega||_{(\nu,\beta,\mu,k,\rho)}|\tau|^k(1 + |m_1|)^{-\mu}e^{-\beta|m_1|}. \tag{138}$$

At a second stage, we discuss bounds for $\mathbb{K}_{\gamma^+_{\tau^{l_2}},\gamma^-_{\tau^{l_2}}}(\tau, m_1)$. Two cases arise.

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Case a. We assume that $\tau \in D_{\tilde{\rho}}$. We need upper bounds for the kernel $\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})$ provided that $\xi \in C_{\rho/2,\gamma^{+}_{\tau^{l_2}},\gamma^{-}_{\tau^{l_2}}}$. According to the decomposition (53), we observe the next factorization

$$\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2}) = \frac{l_2}{k}\frac{1}{\xi^{\frac{k}{l_2}+1}}\tau^{k}\mathbf{D}_{l_2}((\frac{\tau^{l_2}}{\xi})^{k/l_2}) \tag{139}$$

where $\mathbf{D}_{l_2}(z)$ is an entire function on $\mathbb{C}$ (as shown in Lemma 1). In particular, the function $\mathbf{D}_{l_2}(z)$ is bounded by some constant $M_{l_2,\tilde{\rho}} > 0$ on the disc $D_{\frac{\tilde{\rho}^k}{(\rho/2)^{k/l_2}}}$. As a result,

$$|\mathbf{D}_{l_2}((\frac{\tau^{l_2}}{\xi})^{k/l_2})| \leq M_{l_2,\tilde{\rho}} \tag{140}$$

provided that $\xi \in C_{\rho/2,\gamma^{+}_{\tau^{l_2}},\gamma^{-}_{\tau^{l_2}}}$ and $\tau \in D_{\tilde{\rho}}$, since in particular $|\xi| = \rho/2$ and $|\tau| \leq \tilde{\rho}$, which yields the bounds

$$|\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})| \leq \frac{l_2}{k}\frac{1}{(\rho/2)^{\frac{k}{l_2}+1}}|\tau|^{k}M_{l_2,\tilde{\rho}} \tag{141}$$

for all $\xi \in C_{\rho/2,\gamma^{+}_{\tau^{l_2}},\gamma^{-}_{\tau^{l_2}}}$ and $\tau \in D_{\tilde{\rho}}$. These latter bounds (141) together with (112) allow us to find some constant $K^{+-}_{\tilde{\rho},k,l_1,l_2} > 0$ such that

$$\begin{aligned}|\mathbb{K}_{\gamma^{+}_{\tau^{l_2}},\gamma^{-}_{\tau^{l_2}}}(\tau,m_1)| &\leq \int_{\gamma_{\tau^{l_2}}-\frac{\pi l_2}{2k}-\frac{\delta'}{2}}^{\gamma_{\tau^{l_2}}+\frac{\pi l_2}{2k}+\frac{\delta'}{2}}(\rho/2)^{kl_1}||\omega||_{(\nu,\beta,\mu,k,\rho)}(1+|m_1|)^{-\mu}e^{-\beta|m_1|}\\ &\times(\rho/2)e^{\nu(\rho/2)^k}\frac{l_2}{k}\frac{1}{(\rho/2)^{\frac{k}{l_2}+1}}|\tau|^{k}M_{l_2,\tilde{\rho}}(\rho/2)d\theta\\ &\leq K^{+-}_{\tilde{\rho},k,l_1,l_2}||\omega||_{(\nu,\beta,\mu,k,\rho)}|\tau|^{k}(1+|m_1|)^{-\mu}e^{-\beta|m_1|}.\end{aligned} \tag{142}$$

for all $m_1 \in \mathbb{R}$.

At last, from the decomposition (132) and the three estimates (137), (138) and (142), we deduce the awaited bounds (129) from the first point 1).

Case b. We assume that $\tau \in S_d$ with $|\tau| \geq (\rho/2)^{1/l_2}$. According to Proposition 7 1), we come up with a constant $M_{k,k/l_2} > 0$ such that

$$\begin{aligned}|\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})| &\leq \frac{M_{k,k/l_2}}{\frac{k}{l_2}|\xi|^{\frac{k}{l_2}+1}}|\tau^{l_2}|^{\frac{k}{l_2}}\left(|\frac{\tau^{l_2}}{\xi}|\right)^{\frac{(k/l_2)^2}{k-\frac{k}{l_2}}} \times \exp\left((|\frac{\tau^{l_2}}{\xi}|)^{\frac{k^2/l_2}{k-\frac{k}{l_2}}}\right)\\ &= \frac{l_2}{k}M_{k,k/l_2}|\tau|^{k+\frac{k}{l_2-1}}\frac{1}{|\xi|^{\frac{k}{l_2}+1+\frac{k}{l_2(l_2-1)}}} \times \exp\left(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{|\xi|^{\frac{k}{l_2-1}}}\right)\end{aligned} \tag{143}$$

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for all $\xi \in C_{\rho/2,\gamma^{+}_{\tau^{l_2}},\gamma^{-}_{\tau^{l_2}}}$. These last bounds (143) combined with (112) beget a constant $K^{+-;2}_{\rho,k,l_1,l_2} > 0$ such that

$$|\mathbb{K}_{\gamma^{+}_{\tau^{l_2}},\gamma^{-}_{\tau^{l_2}}}(\tau,m_1)| \leq \int_{\gamma_{\tau^{l_2}}-\frac{\pi l_2}{2k}-\frac{\delta'}{2}}^{\gamma_{\tau^{l_2}}+\frac{\pi l_2}{2k}+\frac{\delta'}{2}} (\rho/2)^{kl_1}||\omega||_{(\nu,\beta,\mu,k,\rho)}(1+|m_1|)^{-\mu}e^{-\beta|m_1|}$$
$$\times (\rho/2)e^{\nu(\rho/2)^k}\frac{l_2}{k}M_{k,k/l_2}|\tau|^{k+\frac{k}{l_2-1}}\frac{1}{|\rho/2|^{\frac{k}{l_2}+1+\frac{k}{l_2(l_2-1)}}} \times \exp\left(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{|\rho/2|^{\frac{k}{l_2-1}}}\right)(\rho/2)d\theta$$
$$\leq K^{+-;2}_{\rho,k,l_1,l_2}||\omega||_{(\nu,\beta,\mu,k,\rho)}|\tau|^{k+\frac{k}{l_2-1}}\exp\left(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{|\rho/2|^{\frac{k}{l_2-1}}}\right)(1+|m_1|)^{-\mu}e^{-\beta|m_1|} \quad (144)$$

for all $m_1 \in \mathbb{R}$.

In conclusion, the decomposition (132) together with the three bounds (137), (138) and (144) trigger the expected bounds (130) in the second point 2). □

We set

$$\mathcal{C}_{\tau,m} = \left|\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\mathbb{K}(\tau,m_1)R_{\underline{l}}(\sqrt{-1}m_1)dm_1\right| \quad (145)$$

for $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$. Taking heed of the definition $\mathbf{A}_{\underline{l}}$ displayed in (22), the lower bounds (74) stated in Lemma 2 together with the upper bounds (129) for $\tilde{\rho} = \max(\rho,(\rho/2)^{1/l_2})$ along with (130) and the polynomial inequalities (114), we arrive at

$$\mathcal{C}_{\tau,m} \leq \frac{1}{\mathfrak{R}_D(1+|m|)^{\deg(R_D)}}$$
$$\times \frac{\max(\mathbb{K}^1_{\tilde{\rho},k,l_1,l_2}|\tau|^{k-1},\mathbb{K}^2_{\rho,k,l_1,l_2}|\tau|^{k-1+\frac{k}{l_2-1}}\exp\left(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{(\rho/2)^{\frac{k}{l_2-1}}}\right))}{\delta_{S_d,k,\alpha_D}\exp\left(\alpha_D k^2\Delta_{S_d,k}|\tau|^{2k}\right)} \times (1+|\tau|^{2k})e^{-\nu|\tau|^k}$$
$$\times \left[\frac{|\tau|}{1+|\tau|^{2k}}e^{\nu|\tau|^k}||\omega||_{(\nu,\beta,\mu,k,\rho)}\right] \times \int_{-\infty}^{+\infty}\mathbf{A}_{\underline{l}}(1+|m-m_1|)^{-\mu}e^{-\beta|m-m_1|}(1+|m_1|)^{-\mu}e^{-\beta|m_1|}$$
$$\times \mathfrak{R}_{\underline{l}}(1+|m_1|)^{\deg(R_{\underline{l}})}dm_1 \quad (146)$$

provided that $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$.

Two situations ensue. In the case $l_2 = 2$, we observe that $2k = \frac{kl_2}{l_2-1}$ and in the situation $l_2 > 2$, we notice that $2k > \frac{kl_2}{l_2-1}$. In both cases, under the assumption (109), we deduce a constant $\mathbb{M}_{S_d,k,l_1,l_2,\alpha_D,\rho} > 0$ with

$$\frac{\max(\mathbb{K}^1_{\tilde{\rho},k,l_1,l_2}|\tau|^{k-1},\mathbb{K}^2_{\rho,k,l_1,l_2}|\tau|^{k-1+\frac{k}{l_2-1}}\exp\left(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{(\rho/2)^{\frac{k}{l_2-1}}}\right))}{\delta_{S_d,k,\alpha_D}\exp\left(\alpha_D k^2\Delta_{S_d,k}|\tau|^{2k}\right)}$$
$$\times (1+|\tau|^{2k})e^{-\nu|\tau|^k} \leq \mathbb{M}_{S_d,k,l_1,l_2,\alpha_D,\rho} \quad (147)$$

for all $\tau \in S_d \cup D_\rho$.

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The collection of (146), (116) and (147) spawns

$$\mathcal{C}_{\tau,m} \leq \mathbb{M}_{S_d,k,l_1,l_2,\alpha_D,\rho}||\omega||_{(\nu,\beta,\mu,k,\rho)}\mathbf{A}_{\underline{l}}\frac{\mathfrak{R}_{\underline{l}}}{\mathfrak{R}_D}\Big[\frac{|\tau|}{1+|\tau|^{2k}}e^{\nu|\tau|^k}(1+|m|)^{-\mu}e^{-\beta|m|}\Big]\mathcal{A}_1(m) \tag{148}$$

where $\mathcal{A}_1(m)$ is stated in (119), whenever $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$. Finally, the last bounds (148) and (120) foster a constant $C_3 > 0$ (depending on $\mu, R_D, R_{\underline{l}}, S_d, k, \alpha_D, l_1, l_2, \rho$) with

$$\mathcal{C}_{\tau,m} \leq C_3||\omega||_{(\nu,\beta,\mu,k,\rho)}\mathbf{A}_{\underline{l}}\frac{|\tau|}{1+|\tau|^{2k}}\exp(\nu|\tau|^k)(1+|m|)^{-\mu}e^{-\beta|m|} \tag{149}$$

for all $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$, which can be recast as (127). □

The constituents of the fourth sum involved in (108) are evaluated in the next

Lemma 6. *We select $\underline{l} = (l_0, l_1, l_2) \in \mathcal{A}$ with $l_0 \geq 1$ and $l_2 > 1$. Then a constant $C_4 > 0$ (relying on $\mu, R_D, R_{\underline{l}}, S_d, k, \alpha_D, l_0, l_1, l_2, \rho$) can be found such that*

$$||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\Big(\int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}} \mathbf{C}_{k,l_0,l_1}(\omega)(\xi,m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi\Big) \\ \times R_{\underline{l}}(\sqrt{-1}m_1)dm_1||_{(\nu,\beta,\mu,k,\rho)} \leq C_4\mathbf{A}_{\underline{l}}||\omega(\tau,m)||_{(\nu,\beta,\mu,k,\rho)} \tag{150}$$

provided that $\omega \in F^d_{(\nu,\beta,\mu,k,\rho)}$.

Proof. The proof follows closely the one displayed for Lemma 5. The first part of the discussion is devoted to upper bounds for the integral map

$$\tilde{\mathbb{K}}(\tau,m_1) := \int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}} \mathbf{C}_{k,l_0,l_1}(\omega)(\xi,m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi \tag{151}$$

where by definition

$$\mathbf{C}_{k,l_0,l_1}(\omega)(\xi,m_1) := \frac{\xi^k}{\Gamma(l_0/k)}\int_0^{\xi^k}(\xi^k - s)^{\frac{l_0}{k}-1}(ks)^{l_1}\omega(s^{1/k},m_1)\frac{ds}{s} \tag{152}$$

for all $\tau \in S_d \cup D_\rho$ and $m_1 \in \mathbb{R}$. Namely, the following statement holds.

Sublemma 2. *1) For any prescribed $\tilde{\rho} \geq \rho$, there exists a constant $\mathbb{K}^1_{\tilde{\rho},k,l_0,l_1,l_2} > 0$ with*

$$|\tilde{\mathbb{K}}(\tau,m_1)| \leq \mathbb{K}^1_{\tilde{\rho},k,l_0,l_1,l_2}||\omega||_{(\nu,\beta,\mu,k,\rho)}|\tau|^k(1+|m_1|)^{-\mu}e^{-\beta|m_1|} \tag{153}$$

provided that $\tau \in D_{\tilde{\rho}}$ and $m_1 \in \mathbb{R}$.

2) A constant $\mathbb{K}^2_{\rho,k,l_0,l_1,l_2} > 0$ can be singled out such that

$$|\tilde{\mathbb{K}}(\tau,m_1)| \leq \mathbb{K}^2_{\rho,k,l_0,l_1,l_2}||\omega||_{(\nu,\beta,\mu,k,\rho)}|\tau|^{k+\frac{k}{l_2-1}}\exp\Big(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{(\rho/2)^{\frac{k}{l_2-1}}}\Big)(1+|m_1|)^{-\mu}e^{-\beta|m_1|} \tag{154}$$

as long as $\tau \in S_d$ with $|\tau| \geq (\rho/2)^{1/l_2}$ and $m_1 \in \mathbb{R}$.

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Proof. We set up $\omega \in F^{d}_{(\nu,\beta,\mu,k,\rho)}$ and we take $\tau \in S_d \cup D_{\tilde{\rho}}$ for some given $\tilde{\rho} \geq \rho$. Keeping the same notations as in Lemma 5, we can break up the integral $\tilde{\mathbb{K}}(\tau, m_1)$ in three parts

$$\tilde{\mathbb{K}}(\tau, m_1) = \tilde{\mathbb{K}}_{\gamma^{+}_{\tau^{l_2}}}(\tau, m_1) - \tilde{\mathbb{K}}_{\gamma^{-}_{\tau^{l_2}}, \gamma^{+}_{\tau^{l_2}}}(\tau, m_1) - \tilde{\mathbb{K}}_{\gamma^{-}_{\tau^{l_2}}}(\tau, m_1) \tag{155}$$

where

$$\tilde{\mathbb{K}}_{\gamma^{+}_{\tau^{l_2}}}(\tau, m_1) = \int_{L_{[0,\rho/2];\frac{k}{l_2};\gamma^{+}_{\tau^{l_2}}}} \mathbf{C}_{k,l_0,l_1}(\omega)(\xi, m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi, \tau^{l_2})d\xi \tag{156}$$

and

$$\tilde{\mathbb{K}}_{\gamma^{+}_{\tau^{l_2}}, \gamma^{-}_{\tau^{l_2}}}(\tau, m_1) = \int_{C_{\rho/2;\gamma^{+}_{\tau^{l_2}}, \gamma^{-}_{\tau^{l_2}}}} \mathbf{C}_{k,l_0,l_1}(\omega)(\xi, m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi, \tau^{l_2})d\xi \tag{157}$$

together with

$$\tilde{\mathbb{K}}_{\gamma^{-}_{\tau^{l_2}}}(\tau, m_1) = \int_{L_{[0,\rho/2];\frac{k}{l_2};\gamma^{-}_{\tau^{l_2}}}} \mathbf{C}_{k,l_0,l_1}(\omega)(\xi, m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi, \tau^{l_2})d\xi \tag{158}$$

where the paths of integration are the same as in the integrals (133), (134) and (135).

In the ongoing part of the proof, we disclose bounds for each piece of the decomposition (155). As a preliminary, we rearrange the map (152) by means of the parametrization $s = \xi^k s_1$ where $0 \leq s_1 \leq 1$,

$$\mathbf{C}_{k,l_0,l_1}(\omega)(\xi, m_1) = \frac{\xi^{l_0+kl_1}}{\Gamma(l_0/k)} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} k^{l_1} s_1^{l_1} \omega(\xi s_1^{1/k}, m_1) \frac{ds_1}{s_1} \tag{159}$$

and from the bounds (112), we observe that

$$|\omega(\xi s_1^{1/k}, m_1)| \leq ||\omega||_{(\nu,\beta,\mu,k,\rho)}(1+|m_1|)^{-\mu}e^{-\beta|m_1|} \frac{|\xi| s_1^{1/k}}{1+|\xi s_1^{1/k}|^{2k}} \exp(\nu|\xi|^k s_1) \tag{160}$$

for all $\xi \in \tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}$, $0 \leq s_1 \leq 1$ and $m_1 \in \mathbb{R}$.

Bounds for the integrals $\tilde{\mathbb{K}}_{\gamma^{+}_{\tau^{l_2}}}(\tau, m_1)$ and $\tilde{\mathbb{K}}_{\gamma^{-}_{\tau^{l_2}}}(\tau, m_1)$ along segments are first achieved.

On the ground of the bounds (136), the factorization (159) and the upper estimates (160), under the assumption (10), a constant $K_{\rho,k,l_0,l_1,l_2} > 0$ can be reached with

$$\begin{aligned}|\tilde{\mathbb{K}}_{\gamma^{+}_{\tau^{l_2}}}(\tau, m_1)| &\leq \int_0^{\rho/2} \frac{r^{l_0+kl_1}}{\Gamma(l_0/k)} \times \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} k^{l_1} s_1^{l_1} s_1^{1/k} \frac{ds_1}{s_1} \right) \\ &\quad \times ||\omega||_{(\nu,\beta,\mu,k,\rho)}(1+|m_1|)^{-\mu}e^{-\beta|m_1|} r e^{\nu r^k} M_{k,k/l_2,1,2} \frac{l_2}{k} \frac{|\tau|^k}{r^{\frac{k}{l_2}+1}} dr \\ &\leq K_{\rho,k,l_0,l_1,l_2} ||\omega||_{(\nu,\beta,\mu,k,\rho)} |\tau|^k (1+|m_1|)^{-\mu} e^{-\beta|m_1|}. \end{aligned} \tag{161}$$

together with

$$|\tilde{\mathbb{K}}_{\gamma^{-}_{\tau^{l_2}}}(\tau, m_1)| \leq K_{\rho,k,l_0,l_1,l_2} ||\omega||_{(\nu,\beta,\mu,k,\rho)} |\tau|^k (1+|m_1|)^{-\mu} e^{-\beta|m_1|}. \tag{162}$$

In the next phase, bounds for the integral $\tilde{\mathbb{K}}_{\gamma^{+}_{\tau^{l_2}}, \gamma^{-}_{\tau^{l_2}}}(\tau, m_1)$ along the arc of circle are devised. Two cases are distinguished.

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Case a. The variable τ belongs to $D_{\tilde{\rho}}$. Based on the bounds (141), the factorization (159) and the upper estimates (160), we are given a constant $K^{+-}_{\tilde{\rho},k,l_0,l_1,l_2} > 0$ such that

$$
\begin{aligned}
|\tilde{\mathbb{K}}_{\gamma^{+}_{\tau^{l_2}},\gamma^{-}_{\tau^{l_2}}}(\tau,m_1)| &\leq \int_{\gamma_{\tau^{l_2}}-\frac{\pi l_2}{2k}-\frac{\delta'}{2}}^{\gamma_{\tau^{l_2}}+\frac{\pi l_2}{2k}+\frac{\delta'}{2}} (\rho/2)^{l_0+kl_1}\frac{1}{\Gamma(l_0/k)}\Big(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}k^{l_1}s_1^{l_1}s_1^{1/k}\frac{ds_1}{s_1}\Big) \\
&\times ||\omega||_{(\nu,\beta,\mu,k,\rho)}(1+|m_1|)^{-\mu}e^{-\beta|m_1|} \times (\rho/2)e^{\nu(\rho/2)^k}\frac{l_2}{k}\frac{1}{(\rho/2)^{\frac{k}{l_2}+1}}|\tau|^k M_{l_2,\tilde{\rho}}(\rho/2)d\theta \\
&\leq K^{+-}_{\tilde{\rho},k,l_0,l_1,l_2}||\omega||_{(\nu,\beta,\mu,k,\rho)}|\tau|^k(1+|m_1|)^{-\mu}e^{-\beta|m_1|}. \quad (163)
\end{aligned}
$$

for all $m_1 \in \mathbb{R}$.

Eventually, departing from the splitting (155) and taking heed of the three upper bounds (161), (162) and (163), the due bounds (153) from the first point 1) are established.

Case b. The variable τ is assumed to belong to S_d under the constraint $|\tau| \geq (\rho/2)^{1/l_2}$. Acquired from the bounds (143), the factorization (159) and the upper estimates (160), a constant $K^{+-;2}_{\rho,k,l_0,l_1,l_2} > 0$ exists such that

$$
\begin{aligned}
|\tilde{\mathbb{K}}_{\gamma^{+}_{\tau^{l_2}},\gamma^{-}_{\tau^{l_2}}}(\tau,m_1)| &\leq \int_{\gamma_{\tau^{l_2}}-\frac{\pi l_2}{2k}-\frac{\delta'}{2}}^{\gamma_{\tau^{l_2}}+\frac{\pi l_2}{2k}+\frac{\delta'}{2}} (\rho/2)^{l_0+kl_1}\frac{1}{\Gamma(l_0/k)}\Big(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}k^{l_1}s_1^{l_1}s_1^{1/k}\frac{ds_1}{s_1}\Big) \\
&\times ||\omega||_{(\nu,\beta,\mu,k,\rho)}(1+|m_1|)^{-\mu}e^{-\beta|m_1|} \\
&\times (\rho/2)e^{\nu(\rho/2)^k}\frac{l_2}{k}M_{k,k/l_2}|\tau|^{k+\frac{k}{l_2-1}}\frac{1}{|\rho/2|^{\frac{k}{l_2}+1+\frac{k}{l_2(l_2-1)}}} \times \exp\Big(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{|\rho/2|^{\frac{k}{l_2-1}}}\Big)(\rho/2)d\theta \\
&\leq K^{+-;2}_{\rho,k,l_0,l_1,l_2}||\omega||_{(\nu,\beta,\mu,k,\rho)}|\tau|^{k+\frac{k}{l_2-1}}\exp\Big(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{|\rho/2|^{\frac{k}{l_2-1}}}\Big)(1+|m_1|)^{-\mu}e^{-\beta|m_1|} \quad (164)
\end{aligned}
$$

for all $m_1 \in \mathbb{R}$.

In conclusion, from the splitting (155) together with the three upper bounds (161), (162) and (164), the forecast bounds (154) from the second point 2) hold. □

The remaining part of the proof is similar to the one of Lemma 5. Namely, we define

$$
\tilde{C}_{\tau,m} = \Big|\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\tilde{\mathbb{K}}(\tau,m_1)R_{\underline{l}}(\sqrt{-1}m_1)dm_1\Big| \quad (165)
$$

provided that $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$. The definition $\mathbf{A}_{\underline{l}}$ displayed in (22), the lower bounds (74) established in Lemma 2, the upper bounds (153) for $\tilde{\rho} = \max(\rho, (\rho/2)^{1/l_2})$ along with (154) and the polynomial inequalities (114), beget the next inequality

$$\tilde{C}_{\tau,m} \leq \frac{1}{\mathfrak{R}_D(1+|m|)^{\deg(R_D)}}$$
$$\times \frac{\max(\mathbb{K}^1_{\tilde{\rho},k,l_0,l_1,l_2}|\tau|^{k-1}, \mathbb{K}^2_{\rho,k,l_0,l_1,l_2}|\tau|^{k-1+\frac{k}{l_2-1}} \exp\left(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{(\rho/2)^{\frac{k}{l_2-1}}}\right))}{\delta_{S_d,k,\alpha_D} \exp\left(\alpha_D k^2 \Delta_{S_d,k}|\tau|^{2k}\right)} \times (1+|\tau|^{2k})e^{-\nu|\tau|^k}$$
$$\times \left[\frac{|\tau|}{1+|\tau|^{2k}} e^{\nu|\tau|^k}||\omega||_{(\nu,\beta,\mu,k,\rho)}\right] \times \int_{-\infty}^{+\infty} \mathbf{A}_{\underline{l}}(1+|m-m_1|)^{-\mu}e^{-\beta|m-m_1|}(1+|m_1|)^{-\mu}e^{-\beta|m_1|}$$
$$\times \mathfrak{R}_{\underline{l}}(1+|m_1|)^{\deg(R_{\underline{l}})}dm_1 \quad (166)$$

as long as $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$.

Two alternative arise. In the case $l_2 = 2$, we observe that $2k = \frac{kl_2}{l_2-1}$ and in the situation $l_2 > 2$, we notice that $2k > \frac{kl_2}{l_2-1}$. Under the assumption (109), needed only in the case $l_2 = 2$, we deduce a constant $\mathbb{M}_{S_d,k,l_0,l_1,l_2,\alpha_D,\rho} > 0$ with

$$\frac{\max(\mathbb{K}^1_{\tilde{\rho},k,l_0,l_1,l_2}|\tau|^{k-1}, \mathbb{K}^2_{\rho,k,l_0,l_1,l_2}|\tau|^{k-1+\frac{k}{l_2-1}} \exp\left(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{(\rho/2)^{\frac{k}{l_2-1}}}\right))}{\delta_{S_d,k,\alpha_D} \exp\left(\alpha_D k^2 \Delta_{S_d,k}|\tau|^{2k}\right)}$$
$$\times (1+|\tau|^{2k})e^{-\nu|\tau|^k} \leq \mathbb{M}_{S_d,k,l_0,l_1,l_2,\alpha_D,\rho} \quad (167)$$

for all $\tau \in S_d \cup D_\rho$.

The gathering of (166), (116) and (167) yields

$$\tilde{C}_{\tau,m} \leq \mathbb{M}_{S_d,k,l_0,l_1,l_2,\alpha_D,\rho}||\omega||_{(\nu,\beta,\mu,k,\rho)}\mathbf{A}_{\underline{l}}\frac{\mathfrak{R}_{\underline{l}}}{\mathfrak{R}_D}\left[\frac{|\tau|}{1+|\tau|^{2k}}e^{\nu|\tau|^k}(1+|m|)^{-\mu}e^{-\beta|m|}\right]\mathcal{A}_1(m) \quad (168)$$

where $\mathcal{A}_1(m)$ is defined in (119), whenever $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$. Finally, the last bounds (168) and (120) trigger a constant $C_4 > 0$ (depending on $\mu, R_D, R_{\underline{l}}, S_d, k, \alpha_D, l_0, l_1, l_2, \rho$) with

$$\tilde{C}_{\tau,m} \leq C_4||\omega||_{(\nu,\beta,\mu,k,\rho)}\mathbf{A}_{\underline{l}}\frac{|\tau|}{1+|\tau|^{2k}}\exp(\nu|\tau|^k)(1+|m|)^{-\mu}e^{-\beta|m|} \quad (169)$$

provided that $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$, which can be rewritten using norms as (150). □

An integral expression related to the fifth building block of (108) is assessed in the next

Lemma 7. *One can single out a constant $C_5 > 0$ (relying on $\mu, R_D, Q_1, Q_2, S_d, k, \alpha_D$) with*

$$||\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\left(\tau^k\int_0^{\tau^k}\omega_1((\tau^k-s)^{1/k}, m-m_1)Q_1(\sqrt{-1}(m-m_1))\right.$$
$$\left.\times\, \omega_2(s^{1/k}, m_1)Q_2(\sqrt{-1}m_1)\frac{1}{(\tau^k-s)s}ds\right)dm_1||_{(\nu,\beta,\mu,k,\rho)} \leq C_5||\omega_1||_{(\nu,\beta,\mu,k,\rho)}||\omega_2||_{(\nu,\beta,\mu,k,\rho)} \quad (170)$$

for all $\omega_1, \omega_2 \in F^d_{(\nu,\beta,\mu,k,\rho)}$.

Proof. Let us take $\omega_1,\omega_2 \in F^d_{(\nu,\beta,\mu,k,\rho)}$. Owing to the bounds (112), we deduce the next upper estimates

$$|\omega_1((\tau^k-s)^{1/k},m-m_1)| \leq ||\omega_1||_{(\nu,\beta,\mu,k,\rho)}(1+|m-m_1|)^{-\mu}e^{-\beta|m-m_1|} \times \frac{|(\tau^k-s)^{1/k}|}{1+|\tau^k-s|^2}\exp(\nu|\tau^k-s|) \quad (171)$$

and

$$|\omega_2(s^{1/k},m_1)| \leq ||\omega_2||_{(\nu,\beta,\mu,k,\rho)}(1+|m_1|)^{-\mu}e^{-\beta|m_1|}\frac{|s^{1/k}|}{1+|s|^2}\exp(\nu|s|) \quad (172)$$

for all $\tau \in S_d \cup D_\rho$, all $s \in [0,\tau^k]$, with $m,m_1 \in \mathbb{R}$. Besides, since $Q_1(X)$ and $Q_2(X)$ are polynomials, two constants $\mathfrak{Q}_1,\mathfrak{Q}_2 > 0$ can be exhibited such that

$$|Q_1(\sqrt{-1}(m-m_1))| \leq \mathfrak{Q}_1(1+|m-m_1|)^{\deg(Q_1)} \ , \ |Q_2(\sqrt{-1}m_1)| \leq \mathfrak{Q}_2(1+|m_1|)^{\deg(Q_2)} \quad (173)$$

for all $m,m_1 \in \mathbb{R}$. From these latter bounds and bearing in mind the lower estimates (74), we come up to

$$\begin{aligned}\mathcal{D}_{\tau,m} &= \Big|\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\Big(\tau^k\int_0^{\tau^k}\omega_1((\tau^k-s)^{1/k},m-m_1)Q_1(\sqrt{-1}(m-m_1)) \\ &\qquad \times \omega_2(s^{1/k},m_1)Q_2(\sqrt{-1}m_1)\frac{1}{(\tau^k-s)s}ds\Big)dm_1\Big| \\ &\leq \frac{1}{\mathfrak{R}_D(1+|m|)^{\deg(R_D)}\delta_{S_d,k,\alpha_D}\exp(\alpha_D k^2\Delta_{S_d,k}|\tau|^{2k})} \\ &\times \int_{-\infty}^{+\infty}|\tau|^k\int_0^{|\tau|^k}||\omega_1||_{(\nu,\beta,\mu,k,\rho)}(1+|m-m_1|)^{-\mu}e^{-\beta|m-m_1|}\frac{(|\tau|^k-h)^{1/k}}{1+(|\tau|^k-h)^2}\exp(\nu(|\tau|^k-h)) \\ &\times \mathfrak{Q}_1(1+|m-m_1|)^{\deg(Q_1)} \times ||\omega_2||_{(\nu,\beta,\mu,k,\rho)}(1+|m_1|)^{-\mu}e^{-\beta|m_1|} \\ &\times \frac{h^{1/k}}{1+h^2}e^{\nu h}\mathfrak{Q}_2(1+|m_1|)^{\deg(Q_2)}\frac{1}{(|\tau|^k-h)h}dhdm_1 \end{aligned} \quad (174)$$

for all $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$. Moreover, by using the change of variable $h = |\tau|^k u$, for $0 \leq u \leq 1$, one can select a constant $\check{M}_{S_d,k,\alpha_D} > 0$ such that

$$\begin{aligned}&\sup_{\tau\in S_d\cup D_\rho}\frac{(1+|\tau|^{2k})|\tau|^{k-1}\int_0^{|\tau|^k}\frac{(|\tau|^k-h)^{1/k}h^{1/k}}{(|\tau|^k-h)h}\frac{1}{1+(|\tau|^k-h)^2}\frac{1}{1+h^2}dh}{\exp(\alpha_D k^2\Delta_{S_d,k}|\tau|^{2k})} \\ &= \sup_{\tau\in S_d\cup D_\rho}\frac{(1+|\tau|^{2k})|\tau|\int_0^1\frac{(1-u)^{\frac{1}{k}-1}}{1+|\tau|^{2k}(1-u)^2}\frac{u^{\frac{1}{k}-1}}{1+|\tau|^{2k}u^2}du}{\exp(\alpha_D k^2\Delta_{S_d,k}|\tau|^{2k})} \leq \check{M}_{S_d,k,\alpha_D}. \end{aligned} \quad (175)$$

Combining (174) and (175) prompts

$$\mathcal{D}_{\tau,m} \leq \frac{\check{M}_{S_d,k,\alpha_D}}{\delta_{S_d,k,\alpha_D}}\frac{\mathfrak{Q}_1\mathfrak{Q}_2}{\mathfrak{R}_D}||\omega_1||_{(\nu,\beta,\mu,k,\rho)}||\omega_2||_{(\nu,\beta,\mu,k,\rho)} \times \Big\{\frac{|\tau|}{1+|\tau|^{2k}}\exp(\nu|\tau|^k)(1+|m|)^{-\mu}e^{-\beta|m|}\Big\}\mathcal{A}_2(m) \quad (176)$$

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provided that $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$, where

$$\mathcal{A}_2(m) = (1+|m|)^{\mu-\deg(R_D)} \int_{-\infty}^{+\infty} \frac{1}{(1+|m-m_1|)^{\mu-\deg(Q_1)}} \frac{1}{(1+|m_1|)^{\mu-\deg(Q_2)}} dm_1. \quad (177)$$

According to Lemma 2.2 of [5] and under the assumptions (12) and (21), a constant $C_{5.1} > 0$ is obtained with

$$\mathcal{A}_2(m) \leq C_{5.1} \quad (178)$$

for all $m \in \mathbb{R}$. Finally, from (176) and (178) we deduce a constant $C_5 > 0$ (depending on $\mu, R_D, Q_1, Q_2, S_d, k, \alpha_D$) with

$$\mathcal{D}_{\tau,m} \leq C_5 ||\omega_1||_{(\nu,\beta,\mu,k,\rho)} ||\omega_2||_{(\nu,\beta,\mu,k,\rho)} \times \left\{ \frac{|\tau|}{1+|\tau|^{2k}} \exp(\nu|\tau|^k)(1+|m|)^{-\mu} e^{-\beta|m|} \right\} \quad (179)$$

for all $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$ which precisely means that (170) holds true. □

In the next lemma, the tail piece of (108) is investigated.

Lemma 8. *A constant $\mathcal{F}_J > 0$ (depending on $\mathcal{F}_j$, $j \in J$, $R_D, S_d, k, \alpha_D, \nu$) can be singled for which*

$$|| \sum_{j \in J} \frac{\mathcal{F}_j(m)}{P_m(\tau)} \tau^j ||_{(\nu,\beta,\mu,k,\rho)} \leq \mathcal{F}_J. \quad (180)$$

Proof. By definition of (24), we notice that

$$|\mathcal{F}_j(m)| \leq \mathbf{F}_j (1+|m|)^{-\mu} e^{-\beta|m|} \quad (181)$$

for all $m \in \mathbb{R}$. Besides, according to the geometric assumption (14) and the lower bounds (74) reached in Lemma 2, we see that

$$|P_m(\tau)| \geq \min_{m \in \mathbb{R}} |R_D(\sqrt{-1}m)| \delta_{S_d,k,\alpha_D} \exp(\alpha_D k^2 \Delta_{S_d,k} |\tau|^{2k}) \quad (182)$$

for all $\tau \in S_d \cup D_\rho$, all $m \in \mathbb{R}$. As a result, we get

$$\mathcal{E}_{\tau,m} = \left| \sum_{j \in J} \frac{\mathcal{F}_j(m)}{P_m(\tau)} \tau^j \right| \leq \sum_{j \in J} \mathbf{F}_j (1+|m|)^{-\mu} e^{-\beta|m|} \frac{1}{\min_{m \in \mathbb{R}} |R_D(\sqrt{-1}m)| \delta_{S_d,k,\alpha_D}}$$
$$\times |\tau|^{j-1} \exp(-\alpha_D k^2 \Delta_{S_d,k} |\tau|^{2k})(1+|\tau|^{2k}) \exp(-\nu|\tau|^k) \times \left[\frac{|\tau|}{1+|\tau|^{2k}} \exp(\nu|\tau|^k) \right] \quad (183)$$

provided that $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$. Since $J \subset \mathbb{N}^*$ does not contain the origin, a constant $\hat{M}_{j,S_d,k,\nu,\alpha_D} > 0$ can be pinpointed such that

$$\sup_{\tau \in S_d \cup D_\rho} |\tau|^{j-1}(1+|\tau|^{2k}) \exp(-\alpha_D k^2 \Delta_{S_d,k} |\tau|^{2k}) \exp(-\nu|\tau|^k) = \hat{M}_{j,S_d,k,\nu,\alpha_D}. \quad (184)$$

Lastly, adding up (183) and (184), we arrive at

$$\mathcal{E}_{\tau,m} \leq \left\{ \sum_{j \in J} \mathbf{F}_j \frac{\hat{M}_{j,S_d,k,\nu,\alpha_D}}{\min_{m \in \mathbb{R}} |R_D(\sqrt{-1}m)| \delta_{S_d,k,\alpha_D}} \right\} (1+|m|)^{-\mu} e^{-\beta|m|} \frac{|\tau|}{1+|\tau|^{2k}} \exp(\nu|\tau|^k) \quad (185)$$

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as long as $\tau \in S_d \cup D_\rho$, $m \in \mathbb{R}$. Lemma 8 follows. □

Now, we choose the constants $\mathbf{A}_{\underline{l}}$ for $\underline{l} \in \mathcal{A}$ and $c_{Q_1Q_2} \in \mathbb{C}^*$ close enough to 0 in a manner that one can find some radius $\varpi > 0$ fulfilling the next constraint

$$\sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0=0,l_2=1}} \frac{1}{(2\pi)^{1/2}} k^{l_1} C_1 \mathbf{A}_{\underline{l}}\varpi + \sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0\geq 1,l_2=1}} \frac{1}{(2\pi)^{1/2}} \frac{k^{l_1}}{\Gamma(l_0/k)} C_2 \mathbf{A}_{\underline{l}}\varpi$$
$$+ \sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0=0,l_2>1}} \frac{1}{(2\pi)^{1/2}} k^{l_1} \frac{k^2/l_2}{2\pi} C_3 \mathbf{A}_{\underline{l}}\varpi + \sum_{\substack{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}\\ l_0\geq 1,l_2>1}} \frac{1}{(2\pi)^{1/2}} \frac{k^2/l_2}{2\pi} C_4 \mathbf{A}_{\underline{l}}\varpi$$
$$+ |c_{Q_1Q_2}| \frac{1}{(2\pi)^{1/2}} C_5 \varpi^2 + \mathcal{F}_J \leq \varpi \quad (186)$$

for the constants $C_j > 0$, $1 \leq j \leq 5$ and $\mathcal{F}_J > 0$ appearing in the above lemmas. Eventually, the appliance of the bounds recorded in the lemmas 3, 4, 5, 6, 7 and 8 under the condition (186) yields the expected inclusion (110).

We turn to the second item 2. Let us fix the radius $\varpi > 0$ as above and select $\omega_1, \omega_2 \in B_\varpi \subset F^d_{(\nu,\beta,\mu,k,\rho)}$. In the next list of lemmas, we discuss bounds for each piece of the difference $\mathcal{H}(\omega_1) - \mathcal{H}(\omega_2)$.

A direct issue of Lemma 3 gives rise to

Lemma 9. *Take $\underline{l} = (l_0, l_1, l_2) \in \mathcal{A}$ with $l_0 = 0$ and $l_2 = 1$. Then,*

$$||\frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\tau^{kl_1}[\omega_1(\tau,m_1) - \omega_2(\tau,m_1)] R_{\underline{l}}(\sqrt{-1}m_1)dm_1||_{(\nu,\beta,\mu,k,\rho)}$$
$$\leq C_1 \mathbf{A}_{\underline{l}} ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\rho)} \quad (187)$$

holds for the constant $C_1 > 0$ disclosed in Lemma 3.

As a consequence of Lemma 4, we obtain

Lemma 10. *Let $\underline{l} = (l_0, l_1, l_2) \in \mathcal{A}$ with $l_0 \geq 1$ and $l_2 = 1$. Then,*

$$||\frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\left[\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{l_0}{k}-1} s^{l_1} [\omega_1(s^{1/k},m_1) - \omega_2(s^{1/k},m_1)] \frac{ds}{s}\right]$$
$$\times R_{\underline{l}}(\sqrt{-1}m_1)dm_1||_{(\nu,\beta,\mu,k,\rho)} \leq C_2 \mathbf{A}_{\underline{l}} ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\rho)} \quad (188)$$

holds for the constant $C_2 > 0$ croping up in Lemma 4.

An application of Lemma 5 yields

Lemma 11. *Let $\underline{l} = (l_0, l_1, l_2) \in \mathcal{A}$ with $l_0 = 0$ and $l_2 > 1$. The next inequality*

$$||\frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\left(\int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}} \xi^{kl_1}[\omega_1(\xi,m_1) - \omega_2(\xi,m_1)] \mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi\right)$$
$$\times R_{\underline{l}}(\sqrt{-1}m_1)dm_1||_{(\nu,\beta,\mu,k,\rho)} \leq C_3 \mathbf{A}_{\underline{l}} ||\omega_1(\tau,m) - \omega_2(\tau,m)||_{(\nu,\beta,\mu,k,\rho)} \quad (189)$$

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holds for the constant $C_3 > 0$ appearing in Lemma 5.

Lemma 6 enables to set up the next

Lemma 12. *We choose $\underline{l} = (l_0, l_1, l_2) \in \mathcal{A}$ with $l_0 \geq 1$ and $l_2 > 1$. Then,*

$$||\frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} A_{\underline{l}}(m - m_1) \Big(\int_{\tilde{\gamma}_{\frac{k}{l_2}, \tau^{l_2}}} \mathbf{C}_{k,l_0,l_1}(\omega_1 - \omega_2)(\xi, m_1) \mathbb{D}_{k,\frac{k}{l_2}}(\xi, \tau^{l_2}) d\xi \Big) \times R_{\underline{l}}(\sqrt{-1}m_1) dm_1 ||_{(\nu,\beta,\mu,k,\rho)} \leq C_4 \mathbf{A}_{\underline{l}} ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\rho)} \quad (190)$$

holds true for the constant $C_4 > 0$ showing up in Lemma 6.

In order to control the norm of the nonlinear terms of the difference $\mathcal{H}(\omega_1) - \mathcal{H}(\omega_2)$, we rewrite the next difference as a sum

$$\begin{aligned} &\omega_1((\tau^k - s)^{1/k}, m - m_1) Q_1(\sqrt{-1}(m - m_1)) \omega_1(s^{1/k}, m_1) Q_2(\sqrt{-1}m_1) \\ &\quad - \omega_2((\tau^k - s)^{1/k}, m - m_1) Q_1(\sqrt{-1}(m - m_1)) \omega_2(s^{1/k}, m_1) Q_2(\sqrt{-1}m_1) \\ &= [\omega_1((\tau^k - s)^{1/k}, m - m_1) - \omega_2((\tau^k - s)^{1/k}, m - m_1)] Q_1(\sqrt{-1}(m - m_1)) \times \omega_1(s^{1/k}, m_1) Q_2(\sqrt{-1}m_1) \\ &\quad + \omega_2((\tau^k - s)^{1/k}, m - m_1) Q_1(\sqrt{-1}(m - m_1)) \times [\omega_1(s^{1/k}, m_1) - \omega_2(s^{1/k}, m_1)] Q_2(\sqrt{-1}m_1). \quad (191) \end{aligned}$$

As a result of Lemma 7 and the above reordering (191), we come up with the next

Lemma 13. *The following inequality*

$$\begin{aligned} &||\frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \Big(\tau^k \int_0^{\tau^k} \omega_1((\tau^k - s)^{1/k}, m - m_1) Q_1(\sqrt{-1}(m - m_1)) \\ &\qquad \times \omega_1(s^{1/k}, m_1) Q_2(\sqrt{-1}m_1) \frac{1}{(\tau^k - s)s} ds \Big) dm_1 \\ &\quad - \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \Big(\tau^k \int_0^{\tau^k} \omega_2((\tau^k - s)^{1/k}, m - m_1) Q_1(\sqrt{-1}(m - m_1)) \\ &\quad \times \omega_2(s^{1/k}, m_1) Q_2(\sqrt{-1}m_1) \frac{1}{(\tau^k - s)s} ds \Big) dm_1 || \leq C_5 ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\rho)} \\ &\qquad \times \big[||\omega_1||_{(\nu,\beta,\mu,k,\rho)} + ||\omega_2||_{(\nu,\beta,\mu,k,\rho)} \big] \quad (192) \end{aligned}$$

holds where $C_5 > 0$ is the constant arising in Lemma 7.

We adjust the constants $\mathbf{A}_{\underline{l}}$, $\underline{l} \in \mathcal{A}$ and $c_{Q_1 Q_2} \in \mathbb{C}^*$ nearby the origin in a way that the next restriction

$$\begin{aligned} &\sum_{\substack{\underline{l}=(l_0,l_1,l_2) \in \mathcal{A} \\ l_0 = 0, l_2 = 1}} \frac{1}{(2\pi)^{1/2}} k^{l_1} C_1 \mathbf{A}_{\underline{l}} + \sum_{\substack{\underline{l}=(l_0,l_1,l_2) \in \mathcal{A} \\ l_0 \geq 1, l_2 = 1}} \frac{1}{(2\pi)^{1/2}} \frac{k^{l_1}}{\Gamma(l_0/k)} C_2 \mathbf{A}_{\underline{l}} \\ &\quad + \sum_{\substack{\underline{l}=(l_0,l_1,l_2) \in \mathcal{A} \\ l_0 = 0, l_2 > 1}} \frac{1}{(2\pi)^{1/2}} k^{l_1} \frac{k^2/l_2}{2\pi} C_3 \mathbf{A}_{\underline{l}} + \sum_{\substack{\underline{l}=(l_0,l_1,l_2) \in \mathcal{A} \\ l_0 \geq 1, l_2 > 1}} \frac{1}{(2\pi)^{1/2}} \frac{k^2/l_2}{2\pi} C_4 \mathbf{A}_{\underline{l}} \\ &\qquad + |c_{Q_1 Q_2}| \frac{1}{(2\pi)^{1/2}} C_5 2\varpi \leq 1/2 \quad (193) \end{aligned}$$

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holds. The collection of lemmas 9,10,11,12 and 13, accounting of the above condition (193) yields the contraction property (111).

At the end, we choose the constants $\mathbf{A}_{\underline{l}}$ for $\underline{l} \in \mathcal{A}$ and $c_{Q_1Q_2} \in \mathbb{C}^*$ appropriately close to 0, along with a radius $\varpi > 0$ in a manner that both conditions (186) and (193) hold at once. It follows that the map $\mathcal{H}$ obeys both features (110) and (111). Proposition 9 follows. □

The next proposition provides sufficient conditions for which the auxiliary equation (72) is endowed with a solution in the Banach space described in Definition 5.

Proposition 10. *Under the assumptions made in the statement of Proposition 9, we can find a constant $\varpi > 0$ for which the auxiliary equation (72) hosts a unique solution ω^d which belongs to the space $F^d_{(\nu,\beta,\mu,k,\rho)}$ and is subjected to the bounds*

$$||\omega^d||_{(\nu,\beta,\mu,k,\rho)} \leq \varpi. \tag{194}$$

Proof. For $\varpi > 0$ suitably chosen as in Proposition 9, we observe that the map $\mathcal{H}$ induces a contractive application from the metric space (B_ϖ, d) into itself, where B_ϖ stands for the closed ball of radius $\varpi > 0$ centered at 0 in $F^d_{(\nu,\beta,\mu,k,\rho)}$ and the distance d is induced from the norm $||.||_{(\nu,\beta,\mu,k,\rho)}$ by the expression $d(x,y) = ||x-y||_{(\nu,\beta,\mu,k,\rho)}$. Since $(F^d_{(\nu,\beta,\mu,k,\rho)}, ||.||_{(\nu,\beta,\mu,k,\rho)})$ is a Banach space, the metric space (B_ϖ, d) is complete. Then, according to the classical contractive mapping theorem, the map $\mathcal{H}$ has a fixed point we denote ω^d in B_ϖ, meaning that

$$\mathcal{H}(\omega^d) = \omega^d, \tag{195}$$

which implies in particular that the analytic map ω^d solves the equation (72). Proposition 10 ensues. □

5. Statement of the Main Results

We are in position to state the first prominent result of our work.

Theorem 1. *Let us take for granted that the assumptions (9), (10), (11), (12), (14), (21), (23), (25), (26) hold for the shape of the main problem (8) with vanishing initial condition $u(0,z) \equiv 0$.*

We assume furthermore that the sector S_{Q,R_D} defined in (13) obeys the requirements asked in Lemma 2. We select

- *an unbounded sector S_d edged at 0, with bisecting direction d belonging to the set Θ_{Q,R_D} (introduced in Lemma 2) fulfilling the two conditions (86), (87),*
- *a disc D_ρ whose radius $\rho > 0$ fits the restrictions (88), (89).*

Then, provided that

- *the constant α_D appearing in the leading operator (15) of infinite order in (8) is chosen in agreement with (109),*
- *the constants $\mathbf{A}_{\underline{l}}$, for $\underline{l} \in \mathcal{A}$, set up in (22) and the coefficient $c_{Q_1Q_2}$ of the nonlinear term of (8) are taken close enough to 0*

there exist a formal power series $\hat{u}(t,z) = \sum_{n\geq 1} u_n(z)t^n$ solution to (8) with $\hat{u}(0,z) \equiv 0$,

- *whose coefficients u_n belong to the Banach space $\mathcal{O}_b(H_{\beta'})$ of bounded holomorphic functions on the strip $H_{\beta'}$ (given in (18)) for any prescribed $0 < \beta' < \beta$ endowed with the sup norm $||.||_\infty$,*
- *which is m_k−summable in any direction d chosen as above in the set Θ_{Q,R_D} as a series with coefficients in $(\mathcal{O}_b(H_{\beta'}), ||.||_\infty)$ (see Definition 3).*

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Proof. Under the assumptions made in Theorem 1, we observe that Proposition 10 holds. For the given sector S_d with $d \in \Theta_{Q,R_D}$ and disc D_ρ as constructed in Lemma 2, for any given real number $\nu > 0$ and for the constants $\beta, \mu, k > 0$ prescribed in Section 2, we depart from the solution ω^d of the auxiliary equation (72) that belongs to the Banach space $F^d_{(\nu,\beta,\mu,k,\rho)}$ under the condition

$$||\omega^d||_{(\nu,\beta,\mu,k,\rho)} \leq \varpi \tag{196}$$

for some well chosen constant $\varpi > 0$. By construction, since the partial map $\tau \mapsto \omega^d(\tau, m)$ is holomorphic on the disc D_ρ it has a convergent power series expansion

$$\omega^d(\tau, m) = \hat{\omega}(\tau, m) := \sum_{n \geq 1} \omega_n(m)\tau^n \tag{197}$$

on the disc $D_{\rho/2}$, where the coefficients $\omega_n(m)$ can be expressed in integral form

$$\omega_n(m) = \frac{1}{2\pi\sqrt{-1}} \int_{C_{\rho/2}} \frac{\omega^d(\xi, m)}{\xi^{n+1}} d\xi \tag{198}$$

along the positively oriented circle $C_{\rho/2}$ centered at 0 with radius $\rho/2$. According to (196), a constant $C_{\varpi,k,\nu,\rho} > 0$ (relying on ϖ, k, ν, ρ) can be deduced with

$$|\omega_n(m)| \leq C_{\varpi,k,\nu,\rho}(2/\rho)^n(1 + |m|)^{-\mu}e^{-\beta|m|} \tag{199}$$

for all $m \in \mathbb{R}$. In particular, each coefficient $m \mapsto \omega_n(m)$ belongs to the Banach space $(E_{(\beta,\mu)}, ||.||_{(\beta,\mu)})$ and

$$\hat{\omega}(\tau, m) \in E_{(\beta,\mu)}\{\tau\}. \tag{200}$$

Furthermore, owing to (196), we notice that the partial map $\tau \mapsto \hat{\omega}(\tau, m)$, seen as a holomorphic map on $D_{\rho/2}$ in the Banach space $E_{(\beta,\mu)}$ can be extended to a holomorphic map denoted $\tau \mapsto \omega^d(\tau, m)$ on the sector S_d with bounds

$$||\omega^d(\tau, m)||_{(\beta,\mu)} \leq \varpi \frac{|\tau|}{1 + |\tau|^{2k}} \exp(\nu|\tau|^k) \tag{201}$$

for all $\tau \in S_d$.

Let us define the formal power series

$$\hat{U}(t, m) = \sum_{n \geq 1} U_n(m)t^n \tag{202}$$

where the coefficients $U_n(m)$ are defined by

$$U_n(m) = \omega_n(m)\Gamma(n/k) \tag{203}$$

for all $n \geq 1$. By construction, the series $\hat{\omega}(\tau, m)$ given by (197) represents the m_k-Borel transform of the formal power series (202). From (200) and (201), we deduce that the formal series $\hat{U}(t, m)$ is m_k-summable in direction d, viewed as series with coefficients in $(E_{(\beta,\mu)}, ||.||_{(\beta,\mu)})$, see Definition 3.

According to Proposition 10, the convergent series $\hat{\omega}(\tau, m)$ matches the auxiliary equation (72). Taking heed of Proposition 8, we deduce that the formal series (202) solves the differential/convolution equation (30). Let us introduce the formal power series

$$\hat{u}(t,z) = \sum_{n \geq 1} u_n(z) t^n \tag{204}$$

where the coefficients $u_n(z)$ are defined as the inverse Fourier transform

$$u_n(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} U_n(m) e^{\sqrt{-1}zm} dm \tag{205}$$

for all integers $n \geq 1$, $z \in H_{\beta'}$, for any given $0 < \beta' < \beta$. As claimed by Proposition 1, it follows that $\hat{u}(t,z)$ formally solves the main equation (8).

Bearing in mind (199) and (203), the next upper bounds

$$|u_n(z)| \leq \frac{1}{(2\pi)^{1/2}} C_{\varpi,k,\nu,\rho} \Gamma(n/k)(2/\rho)^n \int_{-\infty}^{+\infty} (1+|m|)^{-\mu} e^{-(\beta-\beta')|m|} dm \tag{206}$$

hold provided that $n \geq 1$ and $z \in H_{\beta'}$, with prescribed $0 < \beta' < \beta$. In particular, we observe that each coefficient u_n belongs to $(\mathcal{O}_b(H_{\beta'}), ||.||_\infty)$, for $n \geq 1$. It ensues that the series

$$\sum_{n \geq 0} \frac{\sup_{z \in H_{\beta'}} |u_n(z)|}{\Gamma(n/k)} (\rho')^n$$

is convergent for any $0 < \rho' < \rho/2$. As a result, the m_k-Borel transform of $\hat{u}$ given by

$$\mathcal{B}_{m_k}(\hat{u})(\tau) = \sum_{n \geq 1} \frac{u_n(z)}{\Gamma(n/k)} \tau^n \tag{207}$$

is convergent on $D_{\rho'}$ as a series in coefficients in the Banach space $(\mathcal{O}_b(H_{\beta'}), ||.||_\infty)$, meaning that

$$\mathcal{B}_{m_k}(\hat{u}) \in \tau \mathcal{O}_b(H_{\beta'})\{\tau\}. \tag{208}$$

Besides, the expansion (197) allows the m_k-Borel transform $\mathcal{B}_{m_k}(\hat{u})$ to be expressed in integral form

$$\mathcal{B}_{m_k}(\hat{u})(\tau) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \omega^d(\tau, m) e^{\sqrt{-1}zm} dm \tag{209}$$

for all $z \in D_{\rho'}$ with $0 < \rho' < \rho/2$. Based on the bounds (201), it follows that the map $\tau \mapsto \mathcal{B}_{m_k}(\hat{u})(\tau)$ viewed as a function from $D_{\rho'}$ into $(\mathcal{O}_b(H_{\beta'}), ||.||_\infty)$ can be analytically continued along the unbounded sector S_d and is subjected to the bounds

$$\sup_{z \in H_{\beta'}} |\mathcal{B}_{m_k}(\hat{u})(\tau)| \leq \frac{\varpi}{(2\pi)^{1/2}} \frac{|\tau|}{1+|\tau|^{2k}} \exp(\nu|\tau|^k) \Big(\int_{-\infty}^{+\infty} (1+|m|)^{-\mu} e^{-(\beta-\beta')|m|} dm \Big) \tag{210}$$

as long as $\tau \in S_d$. Finally, bearing in mind Definition 3, on the ground of the two above features (208) and (210), we deduce that the formal solution $\hat{u}(t,z)$ to (8) with $\hat{u}(0,z) \equiv 0$ given by (204) is m_k-summable in direction d. □

In the second foremost outcome of the paper (Theorem 2), we disclose some functional equations satisfied by the m_k-sum of the formal solution $\hat{u}(t,z)$ to (8) built up in Theorem 1.

Before stating the main result, we need to introduce some integral operators acting on Fourier-Laplace transforms that are described in the next

Proposition 11. *We consider an unbounded sector S_d edged at 0 with bisecting direction $d \in \Theta_{Q,R_D}$ and a small disc D_ρ centered at 0 with radius $\rho > 0$ chosen as in Lemma 2. We take for granted that the constant $\alpha_D > 0$ appearing in (15) fulfills the condition (109). We fix some real number $\nu > 0$ and prescribe the constants $\beta, \mu, k > 0$ as in Section 2.*

For any given $\boldsymbol{\omega}^d$ in $F^d_{(\nu,\beta,\mu,k,\rho)}$, we define the Fourier-Laplace transform

$$\mathbf{u}^d(t,z) = \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_d} \boldsymbol{\omega}^d(\tau,m) \exp\big(-(\frac{\tau}{t})^k\big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \tag{211}$$

along the halfline $L_d = [0,+\infty)e^{\sqrt{-1}d}$. According to Definitions 2 and 3, we know that the function $\mathbf{u}^d$ represents a bounded holomorphic map on the product $S_{d,\vartheta,R} \times H_{\beta'}$, where $S_{d,\vartheta,R}$ is a bounded sector of the form (35) for an angle ϑ satisfying $\frac{\pi}{k} < \vartheta < \frac{\pi}{k} + \mathrm{Op}(S_d)$, with $\mathrm{Op}(S_d)$ standing for the opening of S_d and for a small enough radius $R > 0$, where $H_{\beta'}$ is the strip displayed in (18) for any given $0 < \beta' < \beta$.

We distinguish two different situations.

- *We assume that the constant $\Delta_{S_d,k} > 0$ appearing in the lower bounds (74) satisfies the requirement*

$$\cos(2k\theta) > \Delta_{S_d,k} \tag{212}$$

for all $\theta \in \mathbb{R}$ with $e^{\sqrt{-1}\theta} \in S_d$. We introduce the next integral operator defined by its action on $\mathbf{u}^d$ as follows

$$\exp\big(-\alpha_D(t^{k+1}\partial_t)^2\big)\mathbf{u}^d(t,z) := \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_d} \exp\big(-\alpha_D(k\tau^k)^2\big)\boldsymbol{\omega}^d(\tau,m) \times \exp\big(-(\frac{\tau}{t})^k\big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm. \tag{213}$$

Let $\underline{l} = (l_0,l_1,l_2) \in \mathcal{A}$, where $\mathcal{A}$ is depicted in Section 2. According to the notation (70), we set

$$\mathbf{C}_{k,l_0,l_1}(\boldsymbol{\omega}^d)(\tau,m) := \frac{\tau^k}{\Gamma(l_0/k)} \int_0^{\tau^k} (\tau^k - s)^{\frac{l_0}{k}-1} (ks)^{l_1} \boldsymbol{\omega}^d(s^{1/k},m) \frac{ds}{s} \tag{214}$$

for all integers $l_0 \geq 1$, $l_1 \geq 0$. Besides, when $l_0 = 0$, we denote

$$\mathbf{C}_{k,0,l_1}(\boldsymbol{\omega}^d)(\tau,m) := (k\tau^k)^{l_1}\boldsymbol{\omega}^d(\tau,m). \tag{215}$$

We define the next integral operator through its action on $\mathbf{u}^d$ by

$$\mathbb{G}^-_{l_0,l_1,l_2,k,\alpha_D}(\mathbf{u}^d)(t,z) := \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_d} \exp\big(-\alpha_D(k\tau^k)^2\big) \times \Big(-\frac{k^2/l_2}{2\sqrt{-1}\pi} \int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}} \mathbf{C}_{k,l_0,l_1}(\boldsymbol{\omega}^d)(\xi,m) \mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2}) d\xi\Big) \times \exp\big(-(\frac{\tau}{t})^k\big) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm. \tag{216}$$

Then, both functions (213) and (216) are well defined and bounded holomorphic on the product $S_{d,\vartheta,R} \times H_{\beta'}$.

- *The constant $\Delta_{S_d,k} > 0$ that arises in the lower bounds (74) is assumed to obey the condition*

$$\cos(2k\theta) < -\Delta_{S_d,k} \tag{217}$$

for all $\theta \in \mathbb{R}$ with $e^{\sqrt{-1}\theta} \in S_d$. We set up the following two integral operators acting on $\mathbf{u}^d$ by means of

$$\exp\big(\alpha_D(t^{k+1}\partial_t)^2\big)\mathbf{u}^d(t,z) := \frac{k}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_d}\exp\big(\alpha_D(k\tau^k)^2\big)\boldsymbol{\omega}^d(\tau,m) \times \exp\big(-(\frac{\tau}{t})^k\big)e^{\sqrt{-1}zm}\frac{d\tau}{\tau}dm \quad (218)$$

and for any $\underline{l} = (l_0, l_1, l_2) \in \mathcal{A}$,

$$\mathbb{G}^{+}_{l_0,l_1,l_2,k,\alpha_D}(\mathbf{u}^d)(t,z) := \frac{k}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_d}\exp\big(\alpha_D(k\tau^k)^2\big) \times\Big(-\frac{k^2/l_2}{2\sqrt{-1}\pi}\int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}}\mathbf{C}_{k,l_0,l_1}(\boldsymbol{\omega}^d)(\xi,m)\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi\Big)\times\exp\big(-(\frac{\tau}{t})^k\big)e^{\sqrt{-1}zm}\frac{d\tau}{\tau}dm, \quad (219)$$

keeping the notations (214) and (215). As a result, the two expressions (218) and (219) represent bounded holomorphic maps on the product $S_{d,\vartheta,R} \times H_{\beta'}$.

Proof. We focus on the first item. Under the restriction (212), we remind from (98) that the inequality

$$|\exp(-\alpha_D k^2\tau^{2k})| \leq \exp\big(-\alpha_D k^2 \Delta_{S_d,k}|\tau|^{2k}\big) \quad (220)$$

holds provided that $\tau \in S_d$. It follows that the expression (213) is well defined and bounded holomorphic on $S_{d,\vartheta,R} \times H_{\beta'}$. For $\underline{l} = (l_0, l_1, l_2) \in \mathcal{A}$, one sets

$$\mathbf{K}(\tau,m) = \int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}}\mathbf{C}_{k,l_0,l_1}(\boldsymbol{\omega}^d)(\xi,m)\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi. \quad (221)$$

According to Sublemma 1 and 2, we get that

1. For any given $\tilde{\rho} \geq \rho$, there exists a constant $\mathbf{K}^1_{\tilde{\rho},k,l_0,l_1,l_2} > 0$ with

$$|\mathbf{K}(\tau,m)| \leq \mathbf{K}^1_{\tilde{\rho},k,l_0,l_1,l_2}||\boldsymbol{\omega}^d||_{(\nu,\beta,\mu,k,\rho)}|\tau|^k(1+|m|)^{-\mu}e^{-\beta|m|} \quad (222)$$

   provided that $\tau \in D_{\tilde{\rho}}$ and $m \in \mathbb{R}$.
2. A constant $\mathbf{K}^2_{\rho,k,l_0,l_1,l_2} > 0$ can be reached such that

$$|\mathbf{K}(\tau,m)| \leq \mathbf{K}^2_{\rho,k,l_0,l_1,l_2}||\boldsymbol{\omega}^d||_{(\nu,\beta,\mu,k,\rho)}|\tau|^{k+\frac{k}{l_2-1}}\exp\Big(\frac{|\tau|^{\frac{kl_2}{l_2-1}}}{(\rho/2)^{\frac{k}{l_2-1}}}\Big)(1+|m|)^{-\mu}e^{-\beta|m|} \quad (223)$$

   as long as $\tau \in S_d$ with $|\tau| \geq (\rho/2)^{1/l_2}$ and $m \in \mathbb{R}$.

Since $2k = \frac{kl_2}{l_2-1}$ when $l_2 = 2$ and $2k > \frac{kl_2}{l_2-1}$ for $l_2 > 2$, from the bounds (220), (222), (223), under the constraint (109), we deduce that the integral (216) is well defined and represents a bounded holomorphic function on $S_{d,\vartheta,R} \times H_{\beta'}$.

In the second part of the proof, the second item is discussed. It follows from the condition (217) and the lower bounds (105) showing that

$$|\exp(\alpha_D k^2\tau^{2k})| \leq \exp\big(-\alpha_D k^2 \Delta_{S_d,k}|\tau|^{2k}\big) \quad (224)$$

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for all $\tau \in S_d$. Hence, the expression (218) turns out to be well defined and bounded holomorphic on $S_{d,\vartheta,R} \times H_{\beta'}$.

At last, taking heed of the above upper bounds (222), (223), (224) and the restriction (109), we observe that the integral (219) represents a bounded holomorphic function on $S_{d,\vartheta,R} \times H_{\beta'}$. □

The second principal result of this paper is disclosed in the next

Theorem 2. *Let us assume that the hypotheses formulated in Theorem 1 hold. Let $d \in \mathbb{R}$ be a direction chosen in the set Θ_{Q,R_D} (discussed in Lemma 2). We consider the formal power series $\hat{u}(t,z)$ solution of our main equation (8) with initial vanishing data $\hat{u}(0,z) \equiv 0$. From Theorem 1, we know that $\hat{u}(t,z)$ is m_k-summable in the given direction d. We denote $u^d(t,z)$ its m_k-sum in the direction d. The map $u^d(t,z)$ defines a bounded holomorphic function on the product $S_{d,\vartheta,R} \times H_{\beta'}$, where $S_{d,\vartheta,R}$ denotes a bounded sector shaped as (35) for an angle ϑ satisfying $\frac{\pi}{k} < \vartheta < \frac{\pi}{k} + \mathrm{Op}(S_d)$, with $\mathrm{Op}(S_d)$ representing the opening of S_d and for a small enough radius $R > 0$, where $H_{\beta'}$ is the strip given in (18) for any given $0 < \beta' < \beta$.*

Two alternatives arise.

- *Assume that the unbounded sector S_d (displayed in the first item of Theorem 1) and the constant $\Delta_{S_d,k} > 0$ stemming from the lower bounds (74) conform the condition (212). Then, the m_k-sum $u^d(t,z)$ solves the next functional equation involving the integral operators (213) and (216) given by*

$$\exp(-\alpha_D(t^{k+1}\partial_t)^2)Q(\partial_z)u^d(t,z) = [\frac{1}{2} + \frac{1}{2}\exp(-2\alpha_D(t^{k+1}\partial_t)^2)]R_D(\partial_z)u^d(t,z)$$
$$+ \sum_{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A};l_2=1} a_{\underline{l}}(z)R_{\underline{l}}(\partial_z)\exp(-\alpha_D(t^{k+1}\partial_t)^2)[t^{l_0}(t^{k+1}\partial_t)^{l_1}u^d(t,z)]$$
$$+ \sum_{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A};l_2>1} a_{\underline{l}}(z)R_{\underline{l}}(\partial_z)\mathbb{G}^{-}_{l_0,l_1,l_2,k,\alpha_D}(u^d)(t,z)$$
$$+ c_{Q_1Q_2}\exp(-\alpha_D(t^{k+1}\partial_t)^2)[Q_1(\partial_z)u^d(t,z)Q_2(\partial_z)u^d(t,z)]$$
$$+ \exp(-\alpha_D(t^{k+1}\partial_t)^2)f(t,z) \quad (225)$$

  on the domain $S_{d,\vartheta,R} \times H_{\beta'}$ provided that the radius $R > 0$ is taken small enough.
- *Take for granted that the sector S_d and the constant $\Delta_{S_d,k} > 0$ obey the condition (217). Then, the m_k-sum $u^d(t,z)$ is a solution of the following functional equation which comprises the integral operators (218) and (219) displayed as*

$$\exp(\alpha_D(t^{k+1}\partial_t)^2)Q(\partial_z)u^d(t,z) = [\frac{1}{2} + \frac{1}{2}\exp(2\alpha_D(t^{k+1}\partial_t)^2)]R_D(\partial_z)u^d(t,z)$$
$$+ \sum_{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A};l_2=1} a_{\underline{l}}(z)R_{\underline{l}}(\partial_z)\exp(\alpha_D(t^{k+1}\partial_t)^2)[t^{l_0}(t^{k+1}\partial_t)^{l_1}u^d(t,z)]$$
$$+ \sum_{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A};l_2>1} a_{\underline{l}}(z)R_{\underline{l}}(\partial_z)\mathbb{G}^{+}_{l_0,l_1,l_2,k,\alpha_D}(u^d)(t,z)$$
$$+ c_{Q_1Q_2}\exp(\alpha_D(t^{k+1}\partial_t)^2)[Q_1(\partial_z)u^d(t,z)Q_2(\partial_z)u^d(t,z)]$$
$$+ \exp(\alpha_D(t^{k+1}\partial_t)^2)f(t,z) \quad (226)$$

  on the product $S_{d,\vartheta,R} \times H_{\beta'}$ as long as the radius $R > 0$ is chosen nearby the origin.

Proof. The assumptions made in Theorem 1 enable Proposition 10 to be applied. For some prescribed sector S_d with $d \in \Theta_{Q,R_D}$ and disc D_ρ as put forward in Lemma 2, for any given real number $\nu > 0$ and for the constants $\beta, \mu, k > 0$ chosen in Section 2, we depart from the solution

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ω^d of the auxiliary equation (72) that belongs to the Banach space $F^d_{(\nu,\beta,\mu,k,\rho)}$ under the condition (196) for some well chosen constant $\varpi > 0$. Owing to the integral representation (209), we know from Definition 3 and the expression (34) of the Laplace transform of order k, that the m_k-sum $u^d(t,z)$ of the formal solution $\hat{u}(t,z)$ of (8) given by (204) is expressed as the next Fourier-Laplace transform

$$u^d(t,z) = \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_d} \omega^d(\tau,m) \exp\left(-(\frac{\tau}{t})^k\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \tag{227}$$

which defines a bounded holomorphic map on the product $S_{d,\vartheta,R} \times H_{\beta'}$, where $S_{d,\vartheta,R}$ stands for a bounded sector shaped as (35) with an angle ϑ subjected to $\frac{\pi}{k} < \vartheta < \frac{\pi}{k} + \mathrm{Op}(S_d)$, where $\mathrm{Op}(S_d)$ represents the opening of S_d and for a small enough radius $R > 0$, where $H_{\beta'}$ represents the strip given in (18) for any given $0 < \beta' < \beta$.

We first assume the condition (212) imposed in the first item of Theorem 2. We multiply the auxiliary equation (72) fulfilled by $\omega^d(\tau,m)$ by the function $\exp(-\alpha_D(k\tau^k)^2)$ which yields the next equality

$$\begin{aligned}
&Q(\sqrt{-1}m)\exp(-\alpha_D(k\tau^k)^2)\omega^d(\tau,m) = [\frac{1}{2} + \frac{1}{2}\exp(-2\alpha_D(k\tau^k)^2)] R_D(\sqrt{-1}m)\omega^d(\tau,m)\\
&+ \sum_{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}; l_2=1} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\exp(-\alpha_D(k\tau^k)^2)\mathbf{C}_{k,l_0,l_1}(\omega^d)(\tau,m_1)R_{\underline{l}}(\sqrt{-1}m_1)dm_1\\
&\quad + \sum_{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}; l_2>1} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A_{\underline{l}}(m-m_1)\exp(-\alpha_D(k\tau^k)^2)\\
&\quad \times \left(-\frac{k^2/l_2}{2\sqrt{-1}\pi}\int_{\gamma_{\frac{k}{l_2},\tau^{l_2}}} \mathbf{C}_{k,l_0,l_1}(\omega^d)(\xi,m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi\right) \times R_{\underline{l}}(\sqrt{-1}m_1)dm_1\\
&+ c_{Q_1Q_2}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} \exp(-\alpha_D(k\tau^k)^2)\left(\tau^k \int_0^{\tau^k} \omega^d((\tau^k-s)^{1/k}, m-m_1)Q_1(\sqrt{-1}(m-m_1))\right.\\
&\quad \left. \times \omega^d(s^{1/k},m_1)Q_2(\sqrt{-1}m_1)\frac{1}{(\tau^k-s)s}ds\right)dm_1 + \exp(-\alpha_D(k\tau^k)^2) \times \sum_{j\in J}\mathcal{F}_j(m)\tau^j
\end{aligned} \tag{228}$$

for all $\tau \in S_d \cup D_\rho$ and all $m \in \mathbb{R}$. We apply the Laplace transform $\mathcal{L}_{m_k}$ of order k in direction d displayed by the formula (34) and the inverse Fourier transform (17) to the left and right handside of the above equality (228). From the first item of Proposition 11, together with the identities (19), (20) in Definition 2 and the formula (38) disclosed in Proposition 4, we deduce that the m_k-sum $u^d(t,z)$ given by (227) solves the functional equation (225) on the domain $S_{d,\vartheta,R} \times H_{\beta'}$, provided that the radius $R > 0$ is chosen small enough.

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Assume the condition (217) holds as asked in the second item of Theorem 2. Each side of the equation (72) satisfied by $\omega^d(\tau,m)$ is then multiplied by the function $\exp(\alpha_D(k\tau^k)^2)$ which is recast in the form

$$
\begin{aligned}
Q(\sqrt{-1}m)\exp(\alpha_D(k\tau^k)^2)\omega^d(\tau,m) &= \Big[\frac{1}{2}+\frac{1}{2}\exp(2\alpha_D(k\tau^k)^2)\Big]R_D(\sqrt{-1}m)\omega^d(\tau,m)\\
&+\sum_{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A};l_2=1}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_{\underline{l}}(m-m_1)\exp(\alpha_D(k\tau^k)^2)\mathbf{C}_{k,l_0,l_1}(\omega^d)(\tau,m_1)R_{\underline{l}}(\sqrt{-1}m_1)dm_1\\
&+\sum_{\underline{l}=(l_0,l_1,l_2)\in\mathcal{A};l_2>1}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_{\underline{l}}(m-m_1)\exp(\alpha_D(k\tau^k)^2)\\
&\times\Big(-\frac{k^2/l_2}{2\sqrt{-1}\pi}\int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}}\mathbf{C}_{k,l_0,l_1}(\omega^d)(\xi,m_1)\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi\Big)\times R_{\underline{l}}(\sqrt{-1}m_1)dm_1\\
&+c_{Q_1Q_2}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\exp(\alpha_D(k\tau^k)^2)\Big(\tau^k\int_0^{\tau^k}\omega^d((\tau^k-s)^{1/k},m-m_1)Q_1(\sqrt{-1}(m-m_1))\\
&\times\omega^d(s^{1/k},m_1)Q_2(\sqrt{-1}m_1)\frac{1}{(\tau^k-s)s}ds\Big)dm_1+\exp(\alpha_D(k\tau^k)^2)\times\sum_{j\in J}\mathcal{F}_j(m)\tau^j
\end{aligned}
\tag{229}
$$

whenever $\tau\in S_d\cup D_\rho$ and $m\in\mathbb{R}$. The Laplace transform $\mathcal{L}_{m_k}$ of order k in direction d given by the formula (34) and the inverse Fourier transform (17) are applied to the left and right handside of the above equality (229). According to the second item of Proposition 11 and the identities discussed in Definition 2 and Proposition 4, we deduce that the m_k-sum $u^d(t,z)$ expressed as a double integral (227) obeys the functional equation (226) on the product $S_{d,\vartheta,R}\times H_{\beta'}$, on the condition that the radius $R>0$ is chosen small enough. □

At last, we justify the statement of Theorem 2 with the next

Remark Observe that the m_k-sum $u^d(t,z)$ of the formal solution of (8) given by the expression (227) does not (in general) fulfill the same equation (8). There are two reasons for that.

- The action of the infinite order differential operator

$$\cosh(\alpha_D(t^{k+1}\partial_t)^2)=\frac{1}{2}\exp(\alpha_D(t^{k+1}\partial_t)^2)+\frac{1}{2}\exp(-\alpha_D(t^{k+1}\partial_t)^2)$$

  given in (15) is not well defined on $u^d(t,z)$ since the map $\tau\mapsto\exp(\alpha_D(k\tau^k)^2)$ or $\tau\mapsto\exp(-\alpha_D(k\tau^k)^2)$ has an exponential growth of order $2k$ where a growth rate of at most order k is required on the sector S_d.
- The action of the Mahler operator $t\mapsto t^{l_2}$ is not properly settled on $t^{l_0}(t^{k+1}\partial_t)^{l_1}u^d(t,z)$ for any given $\underline{l}=(l_0,l_1,l_2)\in\mathcal{A}$ since the analytic map

$$\tau\mapsto\int_{\tilde{\gamma}_{\frac{k}{l_2},\tau^{l_2}}}\mathbf{C}_{k,l_0,l_1}(\omega^d)(\xi,m)\mathbb{D}_{k,\frac{k}{l_2}}(\xi,\tau^{l_2})d\xi$$

  endows (at most) an exponential growth of order $\frac{kl_2}{l_2-1}$ which exceeds the admissible order k on the sector S_d.

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