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The Right-Left WG Inverse Solutions to Quaternion Matrix Equations

Ivan Kyrchei [1,*], Dijana Mosić [2] and Predrag Stanimirović [2]

1 Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of NAS of Ukraine, Lviv, Ukraine
2 Faculty of Sciences and Mathematics, University of Niš, Serbia; dijana@pmf.ni.ac.rs (D.M.); pecko@pmf.ni.ac.rs (P.S.)
* Correspondence: ivankyrchei26@gmail.com

Abstract: This paper studies new characterizations and expressions of the weak group (WG) inverse and its dual over the quaternion skew field. We introduce a dual to the weak group inverse for the first time in literature and give some new characterizations for both the WG inverse and its dual, named the right and left weak group inverses for quaternion matrices. In particular, determinantal representations of the right and left WG inverses are given as direct methods for their constructions. Our other results are related to solving the two-sided constrained quaternion matrix equation $\mathbf{AXB} = \mathbf{C}$ and the according approximation problem that could be expressed in terms of the right and left WG inverse solutions. Within the framework of the theory of noncommutative row-column determinants, we derive Cramer's rules for computing these solutions based on determinantal representations of the right and left WG inverses. A numerical example is given to illustrate the gained results.

Keywords: quaternion matrix; matrix equation; generalized inverse; WG inverse; noncommutative determinant; determinantal representation; approximation matrix problem; Cramer's rule

MSC: 15A09, 15A24,15A15,15B33

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1. Introduction. Preliminaries on Quaternion Matrices and Generalized Inverses

Let $\mathbb{H} = \{\eta_0 + \eta_1\mathbf{i} + \eta_2\mathbf{j} + \eta_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,\ \eta_0, \eta_1, \eta_2, \eta_3 \in \mathbb{R}\}$ be the quaternion skew field. For $\eta = \eta_0 + \eta_1\mathbf{i} + \eta_2\mathbf{j} + \eta_3\mathbf{k} \in \mathbb{H}$, the quaternion $\overline{\eta} = \eta_0 - \eta_1\mathbf{i} - \eta_2\mathbf{j} - \eta_3\mathbf{k}$ and the real number $\|\eta\| = \sqrt{\eta\overline{\eta}} = \sqrt{\overline{\eta}\eta} = \sqrt{\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2}$ are the conjugate and norm of η, respectively.

The symbols $\operatorname{rank}(\mathbf{A})$ and $\mathbf{A}^*$, respectively, stand for the rank and conjugate transpose of $\mathbf{A} \in \mathbb{H}^{m\times n}$, where $\mathbb{H}^{m\times n}$ contains all $m \times n$ matrices on $\mathbb{H}$. The set $\mathbb{H}_r^{m\times n}$ presents the subset of matrices from $\mathbb{H}^{m\times n}$ of rank r. Denote by

$$\mathcal{C}_r(\mathbf{A}) = \{\mathbf{s} \in \mathbb{H}^{m\times 1} : \mathbf{s} = \mathbf{At},\ \mathbf{t} \in \mathbb{H}^{n\times 1}\},\ \mathcal{N}_r(\mathbf{A}) = \{\mathbf{t} \in \mathbb{H}^{n\times 1} :\ \mathbf{At} = \mathbf{0}\},$$

$$\mathcal{R}_l(\mathbf{A}) = \{\mathbf{s} \in \mathbb{H}^{1\times n} :\ \mathbf{s} = \mathbf{tA},\ \mathbf{t} \in \mathbb{H}^{1\times m}\},\ \mathcal{N}_l(\mathbf{A}) = \{\mathbf{t} \in \mathbb{H}^{1\times m} :\ \mathbf{tA} = \mathbf{0}\},$$

the right column space, the right null space, the left row space, and the left null space of $\mathbf{A}$, respectively. It is evident that $\mathcal{R}_l(\mathbf{A}) = \mathcal{C}_r(\mathbf{A}^*)$ and $\mathcal{N}_l(\mathbf{A}) = \mathcal{N}_r(\mathbf{A}^*)$.

The quaternion matrix Frobenius norm for $\mathbf{A} = (a_{il}) \in \mathbb{H}^{m\times n}$ is defined as follows,

$$\|\mathbf{A}\|_F = \sqrt{\operatorname{tr} \mathbf{A}^*\mathbf{A}} = \sqrt{\sum_l \|\mathbf{a}_{.l}\|^2} = \sqrt{\sum_l \sum_i \|a_{il}\|^2}.$$

Generalized inverses extend over the quaternion skew field in the usual way with minor features.

Definition 1. *For $\mathbf{A} \in \mathbb{H}^{m\times n}$, the unique solution $\mathbf{X} \in \mathbb{H}^{n\times m}$ to the system of the four equations*

$$(1)\ \mathbf{AXA} = \mathbf{A};\ (2)\ \mathbf{XAX} = \mathbf{X};\ (3)\ (\mathbf{AX})^* = \mathbf{AX};\ (4)\ (\mathbf{XA})^* = \mathbf{XA}.$$

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is the Moore-Penrose *(or shortly* MP*)* inverse $\mathbf{A}^{\dagger}$ *of* $\mathbf{A}$.

The *index* of $\mathbf{A} \in \mathbb{H}^{n\times n}$ (denoted by $k = \mathrm{Ind}(\mathbf{A})$) is the smallest nonnegative integer such that $\mathrm{rank}(\mathbf{A}^{k+1}) = \mathrm{rank}(\mathbf{A}^{k})$.

Definition 2. *The* Drazin inverse $\mathbf{A}^{\mathrm{D}}$ *of* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *with the index* $k = \mathrm{Ind}(\mathbf{A})$ *is the unique matrix* $\mathbf{X}$ *for which*

$$(1^k)\ \mathbf{A}^k\mathbf{X}\mathbf{A} = \mathbf{A}^k;\ (2)\ \mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X};\ (5)\ \mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{A}.$$

Especially, for $\mathrm{Ind}(\mathbf{A}) \leq 1$, $\mathbf{A}^{\mathrm{D}} = \mathbf{A}^{\#}$ *reduces to the* group inverse *of* $\mathbf{A}$.

A matrix $\mathbf{A}$ satisfying the conditions $(i), (j), \dots, (k)$ is called an $\{i, j, \dots, k\}$-inverse of $\mathbf{A}$, and is denoted by $\mathbf{A}^{(i,j,\dots,k)}$. The set of matrices $\mathbf{A}^{(i,j,\dots,k)}$ is denoted $\mathbf{A}\{i, j, \dots, k\}$. In particular, $\mathbf{A}^{(1)}$ is an inner inverse, $\mathbf{A}^{(2)}$ is an outer inverse, $\mathbf{A}^{(1,2)}$ is a reflexive inverse, $\mathbf{A}^{(1,2,3,4)}$ is the Moore-Penrose inverse, etc.

The concepts of the core inverse and core-EP inverse, introduced for complex matrices in [1,2], have been extended to quaternion matrices [3] as follows.

Definition 3. The core-EP *inverse* $\mathbf{A}^{\textcircled{\dagger}}$ *of* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *presents the distinctive solution to*

$$\mathbf{X} = \mathbf{X}\mathbf{A}\mathbf{X}, \qquad \mathcal{C}_r(\mathbf{X}) = \mathcal{C}_r(\mathbf{A}^{\mathrm{D}}) = \mathcal{C}_r(\mathbf{X}^{*}).$$

When $\mathrm{Ind}(\mathbf{A}) \leq 1$, $\mathbf{A}^{\textcircled{\dagger}} = \mathbf{A}^{\textcircled{\#}}$ *is the* core inverse *of* $\mathbf{A}$.

Definition 4. The dual core-EP *inverse* $\mathbf{A}_{\textcircled{\dagger}}$ *of* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is the unique solution to*

$$\mathbf{X} = \mathbf{X}\mathbf{A}\mathbf{X}, \qquad \mathcal{R}_l(\mathbf{X}) = \mathcal{R}_l(\mathbf{A}^{\mathrm{D}}) = \mathcal{R}_l(\mathbf{X}^{*}).$$

In particular, when $\mathrm{Ind}(\mathbf{A}) \leq 1$, $\mathbf{A}_{\textcircled{\dagger}} = \mathbf{A}_{\textcircled{\#}}$ *is called the* dual core inverse *of* $\mathbf{A}$.

The following representations, obtained in [4] for a ring with involution, are also applicable to quaternion matrices.

Lemma 1. *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *with* $k = \mathrm{Ind}(\mathbf{A})$. *Then for any* $\ell \geq k$,

$$\mathbf{A}^{\textcircled{\dagger}} = \mathbf{A}^{\mathrm{D}}\mathbf{A}^{\ell}\left(\mathbf{A}^{\ell}\right)^{\dagger},\ \ \mathbf{A}_{\textcircled{\dagger}} = \left(\mathbf{A}^{\ell}\right)^{\dagger}\mathbf{A}^{\ell}\mathbf{A}^{\mathrm{D}}.$$

From Lemma 1 and Definition 2, it follows that

$$\mathbf{A}^{\textcircled{\dagger}} = \mathbf{A}^{\mathrm{D}}\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^{\dagger} = \mathbf{A}^{k}(\mathbf{A}^{k+1})^{\dagger}, \tag{1}$$

$$\mathbf{A}_{\textcircled{\dagger}} = \left(\mathbf{A}^{k+1}\right)^{\dagger}\mathbf{A}^{k+1}\mathbf{A}^{\mathrm{D}} = (\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^{k}. \tag{2}$$

Since the quaternion core-EP inverse $\mathbf{A}^{\textcircled{\dagger}}$ is associated with the right space $\mathcal{C}_r(\mathbf{A})$ of $\mathbf{A} \in \mathbb{H}^{n\times n}$, while the quaternion dual core-EP inverse $\mathbf{A}_{\textcircled{\dagger}}$ is linked to the left space $\mathcal{R}_l(\mathbf{A})$, is advisable to refer to these generalized inverses as the right and left core-EP inverses, respectively.

Building on the results related to the core-EP inverse from [2], the quaternion right and left core-EP inverses are characterized by specific restricted equations.

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Lemma 2. [5] (Lemma 6) *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *with* $k = \mathrm{Ind}(\mathbf{A})$. *Then* $\mathbf{X} \in \mathbb{H}^{n\times n}$ *is the right core-EP inverse of* $\mathbf{A}$ *if and only if*

$$\mathbf{X}\mathbf{A}^{k+1} = \mathbf{A}^k,\ \mathbf{A}\mathbf{X}^2 = \mathbf{X},\ (\mathbf{A}\mathbf{X})^* = \mathbf{A}\mathbf{X} \text{ and } \mathcal{C}_r(\mathbf{X}) \subseteq \mathcal{C}_r(\mathbf{A}^k).$$

Lemma 3. [5] (Lemma 8) *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *with* $k = \mathrm{Ind}(\mathbf{A})$. *The left core-EP inverse* $\mathbf{X} \in \mathbb{H}^{n\times n}$ *of* $\mathbf{A}$ *is defined as the solution to*

$$\mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^k,\ \mathbf{X}^2\mathbf{A} = \mathbf{X},\ (\mathbf{X}\mathbf{A})^* = \mathbf{X}\mathbf{A} \text{ and } \mathcal{R}_l(\mathbf{X}) \subseteq \mathcal{R}_l(\mathbf{A}^k).$$

Recently, Wang and Chen [6] introduce the weak group inverse of $\mathbf{A} \in \mathbb{C}^{n\times n}$ that evidently can be expanded to quaternion matrices.

Definition 5. *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *with the index* $k = \mathrm{Ind}(\mathbf{A})$. *The weak group inverse* $\mathbf{A}^{ⓦ}$ *of* $\mathbf{A}$ *is the unique matrix* $\mathbf{X} \in \mathbb{H}^{n\times n}$ *satisfying* $\mathbf{A}\mathbf{X}^2 = \mathbf{X},\ \mathbf{A}\mathbf{X} = \mathbf{A}^{\textcircled{\dagger}}\mathbf{A}$.

Lemma 4. *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *with the index* $k = \mathrm{Ind}(\mathbf{A})$. *Then*

$$\mathbf{A}^{ⓦ} = (\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A})^{\#} = (\mathbf{A}^{\textcircled{\dagger}})^2\mathbf{A} = \left(\mathbf{A}^2\right)^{\textcircled{\dagger}}\mathbf{A}. \tag{3}$$

The study of the weak group inverse and the generalized weak group inverse is garnered significant attention (see, for example, [7–10]). To our knowledge, the dual of the weak group (WG) inverse has not yet been introduced or studied for complex matrices. Furthermore, the WG inverse and its dual have not been explored for matrices over the quaternion skew field. Investigating the characteristic representations of the WG inverse and its dual is particularly important due to their applications in solving matrix equations with constraints on the solution space. In this paper, we extend the concepts of the WG inverse and its dual to quaternion matrices and provide their characterizations.

We pay particular attention to their determinantal representations (abbreviated as $\mathfrak{D}$-repre-sentation), which serves as the unique direct method for computing the quaternionic weak group (WG) inverse and its dual. It is well known that the $\mathfrak{D}$-representation of the ordinary inverse can be obtained as the matrix with cofactors in its entries. However, there are numerous $\mathfrak{D}$-representations of generalized inverses for matrices over complex numbers [11–15], resulting from the search for more applicable explicit expressions. Given the non-commutativity of quaternions, addressing the $\mathfrak{D}$-representation of quaternion generalized inverses becomes more complex, primarily due to the challenge of defining the determinant of a matrix with non-commutative entries, known as a noncommutative determinant (see [16–18] for details). In this paper, we provide $\mathfrak{D}$-representations of the quaternion WG inverse and its dual, based on the theory of noncommutative column-row determinants developed in [19,20].

Solving matrix equations is the main application of generalized inverses. Moreover, using various generalized inverses enables us to select solutions belonging to specific constrained subspaces of a solution space. For instance, the equation $\mathbf{A}\mathbf{X} = \mathbf{B}$ has a solution if and only if $\mathbf{B} \in \mathcal{C}_r(\mathbf{A})$. However, it is often necessary to find a solution that lies within a constrained subspace of $\mathcal{C}_r(\mathbf{A})$ that corresponds to the given matrix $\mathbf{B}$. Various types of generalized inverses serve as essential tools for addressing such problems.

Recently, in [21], the minimization problem in the Frobenius norm was examined for the case when the equation $\mathbf{A}\mathbf{X} = \mathbf{C}$ has no solution. It was found that $\mathbf{X} = \mathbf{A}^{ⓦ}\mathbf{C}$ is the uniquely determined solution to the minimization problem.

$$\min \|\mathbf{A}^2\mathbf{X} - \mathbf{A}\mathbf{C}\|_F \quad \text{provided that} \quad \mathcal{R}(\mathbf{X}) \subseteq \mathcal{R}(\mathbf{A}^l).$$

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This paper focuses on the two-sided quaternion matrix equation (TQME), defined as $\mathbf{AXB} = \mathbf{C}$. As a particular case of the Sylvester equation, TQME is applicable in various fields, including the statistics of quaternion random signal [22], quaternion matrix optimization problems [23], signal and color image processing [24], face recognition [25], and more. Exploring and solving matrix equations over quaternion algebras is currently a hot topic in both pure mathematics and applied fields. Studies on matrix equations are emerging not only over Hamilton's quaternion skew field (see, e.g., [26–31]) but also over the quaternion split algebra [32–34], the generalized commutative quaternion algebra [35,36], the algebra of dual quaternions [37], and others. Remarkably, these works in various quaternion algebras are mutually stimulating and complementary, fostering a mutual exchange of ideas. Inspired by previous research on quaternion matrix equations, this paper aims to investigate solutions to two-sided constrained quaternion matrix equations (CQME) and their one-sided partial cases, specifically with constraints involving the WG inverse and its dual, particularly in cases where CQME has a solution. Additionally, we explore approximation problems related to CQME, expressed in terms of the WG inverse and its dual.

The following notation will be useful

$$\begin{aligned}
\mathbb{H}^{(n)(k)} &= \left\{\mathbf{M} \in \mathbb{H}^{n\times n} \mid \operatorname{Ind}(\mathbf{M}) = k\right\},\\
(\mathbf{M}|\mathbf{N}) \in \mathbb{H}^{(n|m)(k|q)} &\iff \mathbf{M} \in \mathbb{H}^{(n)(k)}, \mathbf{N} \in \mathbb{H}^{(m)(q)},\\
\mathbf{Q} \in \mathcal{C}_{r,\subset}(\mathbf{M}) &\iff \mathcal{C}_r(\mathbf{Q}) \subset \mathcal{C}_r(\mathbf{M}),\\
\mathbf{Q} \in \mathcal{R}_{l,\subset}(\mathbf{N}) &\iff \mathcal{R}_l(\mathbf{Q}) \subset \mathcal{R}_l(\mathbf{N}),\\
\mathbf{Q} \in \mathcal{O}_{\subset}(\mathbf{M},\mathbf{N}) &\iff \mathcal{C}_r(\mathbf{Q}) \subset \mathcal{C}_r(\mathbf{M}),\ \mathcal{R}_l(\mathbf{Q}) \subset \mathcal{R}_l(\mathbf{N}).
\end{aligned}$$

This paper continues the discussion of the applications of generalized inverses in solving CQME with specific constraints involving different inverses that started in [38–40].

The remainder of our article is directed as follows. Some new characterizations of the WG inverse and its dual are given in Section 2. $\mathfrak{D}$-representations of the WG inverse and its dual over the quaternion skew field are derived in Section 3. The solvability of constrained quaternion two-sided matrix equations with their partial cases that could be expressed in terms of the right and left WG inverse solutions are considered in Section 4 and the related approximation problem is studied in Section 5. Cramer's rules of obtained solutions are derived in Section 6. A numerical example that illustrates our results is given in Section 7. Concluding comments are stated in Section 8.

2. Characterizations of the WG Inverse and Its Dual

Characterizations of the WG inverse based on the core-EP decomposition were derived in [6]. We give the characterization of the WG inverse based on one of the representations in (3) and the characteristic equations of the Drazin and core-EP inverses.

Theorem 1. *Let $\mathbf{A} \in \mathbb{H}^{(n)(k)}$. The subsequent statements are equivalent:*

(i) $\mathbf{X} = \mathbf{A}^{\circledR} = \left(\mathbf{A}^{\textcircled{\dagger}}\right)^2 \mathbf{A}$ *is the weak group inverse of* $\mathbf{A}$.
(ii) $\mathbf{X}$ *is the unique solution to the equations*

$$\mathbf{A}\mathbf{X}^2 = \mathbf{X}, \quad \mathbf{A}\mathbf{X} = \mathbf{A}^{\textcircled{\dagger}}\mathbf{A}. \tag{4}$$

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Proof. (i) $\mapsto$ (ii) The first step in the proof consists of checking that $\mathbf{X} = (\mathbf{A}^{\oplus})^2\mathbf{A}$ satisfies (4). Since, by (1), $\mathbf{A}^{\oplus} = \mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}$, then we have

$$\begin{aligned}
\mathbf{A}\mathbf{X}^2 =& \mathbf{A}\left[(\mathbf{A}^{\oplus})^2\mathbf{A}\right]^2 = \mathbf{A}\left[\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}\right]^2 \\
=& \mathbf{A}\left[\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^{k+1}(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}\right] \\
=& \mathbf{A}^{k+1}(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\mathbf{A} \\
=& \mathbf{A}^{k+1}(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^{k+1}\mathbf{A}^{\mathrm{D}}(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\mathbf{A} \\
=& \left[\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\right]\mathbf{A} \\
=& (\mathbf{A}^{\oplus})^2\mathbf{A} = \mathbf{X},
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\mathbf{A}\mathbf{X} =& \mathbf{A}(\mathbf{A}^{\oplus})^2\mathbf{A} = \mathbf{A}\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\mathbf{A} \\
=& \mathbf{A}^{k+1}(\mathbf{A}^{k+1})^{\dagger}\mathbf{A}^{k+1}\mathbf{A}^{\mathrm{D}}(\mathbf{A}^{k+1})^{\dagger}\mathbf{A} \\
=& \left[\mathbf{A}^{\mathrm{D}}\mathbf{A}^{k+1}(\mathbf{A}^{k+1})^{\dagger}\right]\mathbf{A} = \left[\mathbf{A}^k(\mathbf{A}^{k+1})^{\dagger}\right]\mathbf{A} \\
=& \mathbf{A}^{\oplus}\mathbf{A}.
\end{aligned} \tag{6}$$

From (5) and (6), it follows that $\mathbf{X} = (\mathbf{A}^{\oplus})^2\mathbf{A}$ is a solution to (4).

Furthermore, assume that both $\mathbf{X}$ and $\mathbf{Y}$ satisfy (4). Then

$$\mathbf{X} = \mathbf{A}\mathbf{X}^2 = \mathbf{A}^{\oplus}\mathbf{A}\mathbf{X} = \mathbf{A}^{\oplus}\mathbf{A}^{\oplus}\mathbf{A} = \mathbf{A}^{\oplus}\mathbf{A}\mathbf{Y} = \mathbf{A}\mathbf{Y}^2 = \mathbf{Y},$$

that is, the system of equations (4) is uniquely solvable.

Denoting $\mathbf{X} = (\mathbf{A}^{\oplus})^2\mathbf{A} =: \mathbf{A}^{ⓦ}$ completes the proof. □

Theorem 2. *Let $\mathbf{A} \in \mathbb{H}^{(n)(k)}$. The following statements are equivalent:*

(i) $\mathbf{X} = \mathbf{A}_{ⓦ} = \mathbf{A}(\mathbf{A}_{\oplus})^2$ *is the dual (or left) weak group inverse of* $\mathbf{A}$.
(ii) $\mathbf{X}$ *is the unique solution to the equations*

$$\mathbf{X}^2\mathbf{A} = \mathbf{X}, \quad \mathbf{X}\mathbf{A} = \mathbf{A}\mathbf{A}_{\oplus}. \tag{7}$$

Proof. The proof is similar to the proof of Theorem 1. □

Remark 1. *Since the WG inverse $\mathbf{A}^{ⓦ} = (\mathbf{A}^{\oplus})^2\mathbf{A}$ is represented by the right core-EP inverse, then it can be called the right WG inverse. Similarly, $\mathbf{A}_{ⓦ}$ is the left WG inverse.*

The following representations for $\mathbf{A}^{ⓦ}$ and $\mathbf{A}_{ⓦ}$ follow as the expression for the weak group inverse of complex matrices proposed in [7].

Lemma 5. *If $\mathbf{A} \in \mathbb{H}^{(n)(k)}$, then*

$$\mathbf{A}^{ⓦ} = \mathbf{A}^k(\mathbf{A}^{k+2})^{\dagger}\mathbf{A}, \tag{8}$$
$$\mathbf{A}_{ⓦ} = \mathbf{A}(\mathbf{A}^{k+2})^{\dagger}\mathbf{A}^k. \tag{9}$$

Proof. The representation (8) of the right weak group inverse can be proved similarly as in the complex case [7] (Theorem 2.1.).

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We prove (9). Applying $\mathbf{A}_{\oplus} = (\mathbf{A}^m)^\dagger \mathbf{A}^m \mathbf{A}^{\mathrm{D}}$ from (2) for all $m \geq k,\ m \in \mathbb{N}$, it follows

$$\begin{aligned}\mathbf{A}_{\circledcirc} &= \mathbf{A}\,(\mathbf{A}_{\oplus})^2 = \mathbf{A}\left[(\mathbf{A}^m)^\dagger \mathbf{A}^m \mathbf{A}^{\mathrm{D}}\right]^2\\ &= \mathbf{A}(\mathbf{A}^m)^\dagger \mathbf{A}^{\mathrm{D}} \mathbf{A}^m (\mathbf{A}^m)^\dagger \mathbf{A}^m \mathbf{A}^{\mathrm{D}} = \mathbf{A}(\mathbf{A}^m)^\dagger \mathbf{A}^{\mathrm{D}} \mathbf{A}^m \mathbf{A}^{\mathrm{D}}\\ &= \mathbf{A}(\mathbf{A}^{k+2})^\dagger \mathbf{A}^{\mathrm{D}} \mathbf{A}^{k+2} \mathbf{A}^{\mathrm{D}} = \mathbf{A}(\mathbf{A}^{k+2})^\dagger \mathbf{A}^{\mathrm{D}} \mathbf{A}^{k+1} = \mathbf{A}(\mathbf{A}^{k+2})^\dagger \mathbf{A}^k.\end{aligned}$$

□

Lemma 6 presents some properties of projections involving $\mathbf{A}^{\circledcirc}$ and $\mathbf{A}_{\circledcirc}$.

Lemma 6. *Let* $\mathbf{A} \in \mathbb{H}^{(n)(k)}$. *Then*

$$\begin{aligned}&\mathcal{C}_r(\mathbf{A}^k) = \mathcal{C}_r(\mathbf{A}^{\circledcirc}\mathbf{A}), \qquad \mathcal{N}_r((\mathbf{A}^k)^*\mathbf{A}^2) = \mathcal{N}_r(\mathbf{A}^{\circledcirc}\mathbf{A}),\\ &\mathcal{R}_l((\mathbf{A}^k)^*\mathbf{A}) = \mathcal{R}_l(\mathbf{A}\mathbf{A}^{\circledcirc}), \qquad \mathcal{N}_l(\mathbf{A}^k) = \mathcal{N}_l(\mathbf{A}\mathbf{A}^{\circledcirc}),\\ &\mathcal{R}_l(\mathbf{A}^k) = \mathcal{R}_l(\mathbf{A}\mathbf{A}_{\circledcirc}), \qquad \mathcal{N}_l(\mathbf{A}^2(\mathbf{A}^k)^*) = \mathcal{N}_l(\mathbf{A}\mathbf{A}_{\circledcirc}),\\ &\mathcal{C}_r(\mathbf{A}(\mathbf{A}^k)^*) = \mathcal{C}_r(\mathbf{A}_{\circledcirc}\mathbf{A}), \qquad \mathcal{N}_r(\mathbf{A}^k) = \mathcal{N}_r(\mathbf{A}_{\circledcirc}\mathbf{A}).\end{aligned}$$

Proof. According to [8], it follows $\mathcal{C}_r(\mathbf{A}^k) = \mathcal{C}_r(\mathbf{A}^{\mathrm{D}}) = \mathcal{C}_r(\mathbf{A}^{\circledcirc}\mathbf{A})$ as well as

$$\begin{aligned}\mathcal{N}_r((\mathbf{A}^k)^*\mathbf{A}^2) &= \mathcal{N}_r((\mathbf{A}^{k+1})^*\mathbf{A}^2) = \mathcal{N}_r((\mathbf{A}^{k+1})^\dagger\mathbf{A}^2) = \mathcal{N}_r(\mathbf{A}^k(\mathbf{A}^{k+1})^\dagger\mathbf{A}^2)\\ &= \mathcal{N}_r(\mathbf{A}^{\oplus}\mathbf{A}^2) = \mathcal{N}_r(\mathbf{A}^{\circledcirc}\mathbf{A}).\end{aligned}$$

Since $\mathbf{A}\mathbf{A}^{\circledcirc} = \mathbf{A}(\mathbf{A}^{\oplus})^2\mathbf{A} = \mathbf{A}^{\oplus}\mathbf{A} = \mathbf{A}^k(\mathbf{A}^{k+1})^\dagger\mathbf{A}$, we obtain

$$\mathcal{R}_l((\mathbf{A}^k)^*\mathbf{A}) = \mathcal{R}_l((\mathbf{A}^{k+1})^*\mathbf{A}) = \mathcal{R}_l((\mathbf{A}^{k+1})^\dagger\mathbf{A}) = \mathcal{R}_l(\mathbf{A}^k(\mathbf{A}^{k+1})^\dagger\mathbf{A}) = \mathcal{R}_l(\mathbf{A}\mathbf{A}^{\circledcirc})$$

and

$$\begin{aligned}\mathcal{N}_l(\mathbf{A}^k) \subseteq \mathcal{N}_l(\mathbf{A}\mathbf{A}^{\circledcirc}) &= \mathcal{N}_l(\mathbf{A}^k(\mathbf{A}^{k+1})^\dagger\mathbf{A}) \subseteq \mathcal{N}_l(\mathbf{A}^k(\mathbf{A}^{k+1})^\dagger\mathbf{A}^{k+1}) = \mathcal{N}_l(\mathbf{A}^k(\mathbf{A}^k)^\dagger\mathbf{A}^k) =\\ &= \mathcal{N}_l(\mathbf{A}^k),\end{aligned}$$

i.e. $\mathcal{N}_l(\mathbf{A}^k) = \mathcal{N}_l(\mathbf{A}^k(\mathbf{A}^{k+1})^\dagger\mathbf{A}) = \mathcal{N}_l(\mathbf{A}\mathbf{A}^{\circledcirc})$.

The rest of the proof follows similarly. □

Necessary and sufficient conditions for a quaternion matrix $\mathbf{X}$ to be the weak group inverse of $\mathbf{A}$ are developed in Theorem 3. Notice that most of the presented conditions are new in the literature.

Theorem 3. *The following statements are equivalent for* $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ *and* $\mathbf{X} \in \mathbb{H}^{n\times n}$:

(i) $\mathbf{X} = \mathbf{A}^{\circledcirc}$.
(ii) $\mathbf{A}^{\mathrm{D}}\mathbf{A}\mathbf{X} = \mathbf{X},\ \mathbf{A}\mathbf{X} = \mathbf{A}^{\oplus}\mathbf{A}$.
(iii) $\mathbf{A}^{\mathrm{D}}\mathbf{A}\mathbf{X} = \mathbf{X},\ \ \mathbf{A}^{\mathrm{D}}\mathbf{X} = (\mathbf{A}^{\oplus})^3\mathbf{A}$.
(iv) $\mathbf{A}^{\oplus}\mathbf{A}\mathbf{X} = \mathbf{X},\ \ \mathbf{A}\mathbf{X} = \mathbf{A}^{\oplus}\mathbf{A}$.
(v) $\mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X},\ \ \mathbf{A}\mathbf{X} = \mathbf{A}^{\oplus}\mathbf{A}, \mathbf{X}\mathbf{A}^{\oplus} = \mathbf{A}^{\oplus}\mathbf{A}^{\oplus}$.
(vi) $\mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X},\ \mathbf{A}\mathbf{X} = \mathbf{A}^{\oplus}\mathbf{A},\ \mathbf{X}\mathbf{A}^{\mathrm{D}} = \mathbf{A}^{\mathrm{D}}\mathbf{A}^{\mathrm{D}}$.
(vii) $\mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X},\ \mathbf{A}\mathbf{X} = \mathbf{A}^{\oplus}\mathbf{A}, \mathbf{X}\mathbf{A} = (\mathbf{A}^{\oplus})^2\mathbf{A}^2$.
(viii) $\mathbf{A}\mathbf{X}\mathbf{A} = \mathbf{A}^{\oplus}\mathbf{A}^2, \mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X}, \mathbf{X}\mathbf{A} = (\mathbf{A}^{\oplus})^2\mathbf{A}^2,\ \mathbf{A}\mathbf{X} = \mathbf{A}^{\oplus}\mathbf{A}$.
(ix) $\mathbf{X}\mathbf{A}^{\oplus}\mathbf{A} = \mathbf{X},\ \ \mathbf{X}\mathbf{A}^{\oplus} = \mathbf{A}^{\oplus}\mathbf{A}^{\oplus}$.
(x) $\mathbf{X}\mathbf{A}^{\oplus}\mathbf{A} = \mathbf{X},\ \mathbf{X}\mathbf{A}^{\mathrm{D}} = \mathbf{A}^{\mathrm{D}}\mathbf{A}^{\mathrm{D}}$.
(xi) $\mathbf{X}\mathbf{A}^{\oplus}\mathbf{A} = \mathbf{X},\ \ \mathbf{X}\mathbf{A} = (\mathbf{A}^{\oplus})^2\mathbf{A}^2$.
(xii) $\mathbf{X}\mathbf{A}^{\oplus}\mathbf{A} = \mathbf{X},\ \ \mathbf{X}\mathbf{A}^{k+1} = \mathbf{A}^k$.

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(xiii) $\mathbf{X}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}^2\mathbf{X} = \mathbf{X},\ \mathbf{A}^{\textcircled{\dagger}}\mathbf{A}^2\mathbf{X} = \mathbf{A}^{\textcircled{\dagger}}\mathbf{A},\ \mathbf{X}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}^2 = (\mathbf{A}^{\textcircled{\dagger}})^2\mathbf{A}^2.$
(xiv) $\mathbf{X}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}^2\mathbf{X} = \mathbf{X},\ \mathbf{A}^{\textcircled{\dagger}}\mathbf{A}^2\mathbf{X}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}^2 = \mathbf{A}^{\textcircled{\dagger}}\mathbf{A}^2,\ \ \mathbf{A}^{\textcircled{\dagger}}\mathbf{A}^2\mathbf{X} = \mathbf{A}^{\textcircled{\dagger}}\mathbf{A},\ \mathbf{X}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}^2 = (\mathbf{A}^{\textcircled{\dagger}})^2\mathbf{A}^2.$
(xv) $(\mathbf{A}^{\textcircled{\dagger}})^2\mathbf{A}^2\mathbf{X} = \mathbf{X},\ (\mathbf{A}^k)^*\mathbf{A}^2\mathbf{X} = (\mathbf{A}^k)^*\mathbf{A}.$

Proof. (i) $\mapsto$ (ii) The equalities $\mathbf{X} = \mathbf{A}^{\textcircled{w}} = (\mathbf{A}^{\textcircled{\dagger}})^2\,\mathbf{A}$ and $\mathbf{A}^{\textcircled{\dagger}} = \mathbf{A}^{\mathrm{D}}\mathbf{A}^k\left(\mathbf{A}^k\right)^{\dagger}$ give this implication.
(ii) $\mapsto$ (iii) Notice that $(\mathbf{A}^{\textcircled{\dagger}})^3 = (\mathbf{A}^{\mathrm{D}}\mathbf{A}^k\left(\mathbf{A}^k\right)^{\dagger})^3 = (\mathbf{A}^{\mathrm{D}})^3\mathbf{A}^k\left(\mathbf{A}^k\right)^{\dagger} = (\mathbf{A}^{\mathrm{D}})^2\mathbf{A}^{\textcircled{\dagger}}$. The use of $\mathbf{A}\mathbf{X} = \mathbf{A}^{\textcircled{\dagger}}\mathbf{A}$ leads to

$$\mathbf{A}^{\mathrm{D}}\mathbf{X} = (\mathbf{A}^{\mathrm{D}})^2(\mathbf{A}\mathbf{X}) = (\mathbf{A}^{\mathrm{D}})^2\mathbf{A}^{\textcircled{\dagger}}\mathbf{A} = (\mathbf{A}^{\textcircled{\dagger}})^3\mathbf{A}$$

(iii) $\mapsto$ (i) The assumptions $\mathbf{A}^{\mathrm{D}}\mathbf{A}\mathbf{X} = \mathbf{X}$ and $\mathbf{A}^{\mathrm{D}}\mathbf{X} = (\mathbf{A}^{\textcircled{\dagger}})^3\mathbf{A}$ imply

$$\mathbf{X} = \mathbf{A}(\mathbf{A}^{\mathrm{D}}\mathbf{X}) = \mathbf{A}(\mathbf{A}^{\textcircled{\dagger}})^3\mathbf{A} = (\mathbf{A}^{\textcircled{\dagger}})^2\mathbf{A}.$$

(i) $\leftrightarrow$ (iv) $-$ (vi) These equivalences follow due to [8].
(i) $\mapsto$ (vii) $\mapsto$ (viii) It is evident.
(viii) $\mapsto$ (i) It can be noticed

$$\mathbf{X} = (\mathbf{X}\mathbf{A})\mathbf{X} = (\mathbf{A}^{\textcircled{\dagger}})^2\mathbf{A}(\mathbf{A}\mathbf{X}) = (\mathbf{A}^{\textcircled{\dagger}})^2\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A} = (\mathbf{A}^{\textcircled{\dagger}})^2\mathbf{A}.$$

The rest of the proof follows similarly. □

Dual characterizations for $\mathbf{A}_{\textcircled{w}}$ can be proved by analogy to above.

Theorem 4. *The subsequent statements are equivalent for* $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ *and* $\mathbf{X} \in \mathbb{H}^{n\times n}$*:*

(i) $\mathbf{X} = \mathbf{A}_{\textcircled{w}}.$
(ii) $\mathbf{X}\mathbf{A}\mathbf{A}^{\mathrm{D}} = \mathbf{X},\ \ \mathbf{X}\mathbf{A} = \mathbf{A}\mathbf{A}_{\textcircled{\dagger}}.$
(iii) $\mathbf{X}\mathbf{A}\mathbf{A}^{\mathrm{D}} = \mathbf{X},\ \ \mathbf{X}\mathbf{A}^{\mathrm{D}} = \mathbf{A}(\mathbf{A}_{\textcircled{\dagger}})^3.$
(iv) $\mathbf{X}\mathbf{A}\mathbf{A}_{\textcircled{\dagger}} = \mathbf{X},\ \ \mathbf{X}\mathbf{A} = \mathbf{A}\mathbf{A}_{\textcircled{\dagger}}.$
(v) $\mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X},\ \mathbf{X}\mathbf{A} = \mathbf{A}\mathbf{A}_{\textcircled{\dagger}},\ \mathbf{A}_{\textcircled{\dagger}}\mathbf{X} = \mathbf{A}_{\textcircled{\dagger}}\mathbf{A}_{\textcircled{\dagger}}.$
(vi) $\mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X},\ \ \mathbf{X}\mathbf{A} = \mathbf{A}\mathbf{A}_{\textcircled{\dagger}},\ \mathbf{A}^{\mathrm{D}}\mathbf{X} = \mathbf{A}^{\mathrm{D}}\mathbf{A}^{\mathrm{D}}.$
(vii) $\mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X},\ \ \mathbf{X}\mathbf{A} = \mathbf{A}\mathbf{A}_{\textcircled{\dagger}},\ \mathbf{A}\mathbf{X} = \mathbf{A}^2(\mathbf{A}_{\textcircled{\dagger}})^2.$
(vii) $\mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X},\ \mathbf{A}\mathbf{X}\mathbf{A} = \mathbf{A}^2\mathbf{A}_{\textcircled{\dagger}},\ \ \mathbf{X}\mathbf{A} = \mathbf{A}\mathbf{A}_{\textcircled{\dagger}},\ \mathbf{A}\mathbf{X} = \mathbf{A}^2(\mathbf{A}_{\textcircled{\dagger}})^2.$
(ix) $\mathbf{A}\mathbf{A}_{\textcircled{\dagger}}\mathbf{X} = \mathbf{X},\ \ \mathbf{A}_{\textcircled{\dagger}}\mathbf{X} = \mathbf{A}_{\textcircled{\dagger}}\mathbf{A}_{\textcircled{\dagger}}.$
(x) $\mathbf{A}\mathbf{A}_{\textcircled{\dagger}}\mathbf{X} = \mathbf{X},\ \ \mathbf{A}^{\mathrm{D}}\mathbf{X} = \mathbf{A}^{\mathrm{D}}\mathbf{A}^{\mathrm{D}}.$
(xi) $\mathbf{A}\mathbf{A}_{\textcircled{\dagger}}\mathbf{X} = \mathbf{X},\ \ \mathbf{A}\mathbf{X} = \mathbf{A}^2(\mathbf{A}_{\textcircled{\dagger}})^2.$
(xii) $\mathbf{A}\mathbf{A}_{\textcircled{\dagger}}\mathbf{X} = \mathbf{X},\ \ \mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^k.$
(xiii) $\mathbf{X}\mathbf{A}^2\mathbf{A}_{\textcircled{\dagger}}\mathbf{X} = \mathbf{X},\ \ \mathbf{A}^2\mathbf{A}_{\textcircled{\dagger}}\mathbf{X} = \mathbf{A}^2(\mathbf{A}_{\textcircled{\dagger}})^2,\ \mathbf{X}\mathbf{A}^2\mathbf{A}_{\textcircled{\dagger}} = \mathbf{A}\mathbf{A}_{\textcircled{\dagger}}.$
(xiv) $\mathbf{X}\mathbf{A}^2\mathbf{A}_{\textcircled{\dagger}}\mathbf{X} = \mathbf{X},\ \mathbf{A}^2\mathbf{A}_{\textcircled{\dagger}}\mathbf{X}\mathbf{A}^2\mathbf{A}_{\textcircled{\dagger}} = \mathbf{A}^2\mathbf{A}_{\textcircled{\dagger}},\ \mathbf{A}^2\mathbf{A}_{\textcircled{\dagger}}\mathbf{X} = \mathbf{A}^2(\mathbf{A}_{\textcircled{\dagger}})^2,\ \mathbf{X}\mathbf{A}^2\mathbf{A}_{\textcircled{\dagger}} = \mathbf{A}\mathbf{A}_{\textcircled{\dagger}}.$
(xv) $\mathbf{X}\mathbf{A}^2(\mathbf{A}_{\textcircled{\dagger}})^2 = \mathbf{X},\ \mathbf{X}\mathbf{A}^2(\mathbf{A}^k)^* = \mathbf{A}(\mathbf{A}^k)^*.$

Recall that, by [41], an outer inverse of $\mathbf{A} \in \mathbb{H}^{n\times n}$ with predefined right column space T_1 and right null space S_1 is a solution to the constrained equation

$$\mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X}, \quad \mathcal{C}_r(\mathbf{X}) = T_1, \quad \mathcal{N}_r(\mathbf{X}) = S_1.$$

Recall that, by [41], an outer inverse of $\mathbf{A} \in \mathbb{H}^{n\times n}$ with predefined the right column space T_1, and the right null space S_1 is a solution to the constrained equation

$$\mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X}, \quad \mathcal{R}_l(\mathbf{X}) = T_2, \quad \mathcal{N}_l(\mathbf{X}) = S_2.$$

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It is unique (if it exists) and denoted by $\mathbf{X} = \mathbf{A}^{(2)}_{l_{T_2,S_2}}$. An outer inverse of $\mathbf{A} \in \mathbb{H}^{n\times n}$ with prescribed the right column space T_1, the right null space S_1, the left row space T_2, and the left null space S_2 is a solution to the constrained equation

$$\mathbf{XAX} = \mathbf{X}, \quad \mathcal{C}_r(\mathbf{X}) = T_1, \quad \mathcal{N}_r(\mathbf{X}) = S_1, \quad \mathcal{R}_l(\mathbf{X}) = T_2, \quad \mathcal{N}_l(\mathbf{X}) = S_2.$$

It is unique (if it exists) and denoted by $\mathbf{X} = \mathbf{A}^{(2)}_{(T_1,T_2),(S_1,S_2)}$. If some of the above mentioned outer inverses $\mathbf{A}^{(2)}_*$ satisfy $\mathbf{AA}^{(2)}_*\mathbf{A} = \mathbf{A}$, we use the notation $\mathbf{A}^{(1,2)}_*$.

The results obtained in Theorem 3 and Theorem 4 lead to the following representations and characterizations.

Corollary 1. *The right and left WG inverses of $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ can be represented as*

$$\begin{aligned}
\mathbf{A}^{\circledW} =&\mathbf{A}^{(2)}_{r_{\mathcal{C}_r(\mathbf{A}^k),\mathcal{N}_r((\mathbf{A}^k)^*\mathbf{A})}} = \mathbf{A}^{(2)}_{l_{\mathcal{R}_l((\mathbf{A}^k)^*\mathbf{A}),\mathcal{N}_l(\mathbf{A}^k)}}\\
=&\mathbf{A}^{(2)}_{(\mathcal{C}_r(\mathbf{A}^k),\mathcal{R}_l((\mathbf{A}^k)^*\mathbf{A})),(\mathcal{N}_r((\mathbf{A}^k)^*\mathbf{A}),\mathcal{N}_l(\mathbf{A}^k))}\\
=&(\mathbf{A}^{\oplus}\mathbf{A}^2)^{(1,2)}_{r_{\mathcal{C}_r(\mathbf{A}^k),\mathcal{N}_r((\mathbf{A}^k)^*\mathbf{A})}} = (\mathbf{A}^{\oplus}\mathbf{A}^2)^{(2)}_{l_{\mathcal{R}_l((\mathbf{A}^k)^*\mathbf{A}),\mathcal{N}_l(\mathbf{A}^k)}}\\
=&(\mathbf{A}^{\oplus}\mathbf{A}^2)^{(1,2)}_{(\mathcal{C}_r(\mathbf{A}^k),\mathcal{R}_l((\mathbf{A}^k)^*\mathbf{A})),(\mathcal{N}_r((\mathbf{A}^k)^*\mathbf{A}),\mathcal{N}_l(\mathbf{A}^k))},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A}_{\circledW} &= \mathbf{A}^{(2)}_{r_{\mathcal{C}_r(\mathbf{A}(\mathbf{A}^k)^*),\mathcal{N}_r(\mathbf{A}^k)}} = \mathbf{A}^{(2)}_{l_{\mathcal{R}_l(\mathbf{A}^k),\mathcal{N}_l(\mathbf{A}(\mathbf{A}^k)^*)}}\\
&= \mathbf{A}^{(2)}_{(\mathcal{C}_r(\mathbf{A}(\mathbf{A}^k)^*),\mathcal{R}_l(\mathbf{A}^k)),(\mathcal{N}_r(\mathbf{A}^k),\mathcal{N}_l(\mathbf{A}(\mathbf{A}^k)^*)}\\
&= (\mathbf{A}^2\mathbf{A}_{\oplus})^{(1,2)}_{r_{\mathcal{C}_r(\mathbf{A}(\mathbf{A}^k)^*),\mathcal{N}_r(\mathbf{A}^k)}} = (\mathbf{A}^2\mathbf{A}_{\oplus})^{(1,2)}_{l_{\mathcal{R}_l(\mathbf{A}^k),\mathcal{N}_l(\mathbf{A}(\mathbf{A}^k)^*)}}\\
&= (\mathbf{A}^2\mathbf{A}_{\oplus})^{(1,2)}_{(\mathcal{C}_r(\mathbf{A}(\mathbf{A}^k)^*),\mathcal{R}_l(\mathbf{A}^k)),(\mathcal{N}_r(\mathbf{A}^k),\mathcal{N}_l(\mathbf{A}(\mathbf{A}^k)^*)}.
\end{aligned}$$

Proof. Theorem 3 and Theorem 4 imply that $\mathbf{A}^{\circledW}$ and $\mathbf{A}_{\circledW}$ are outer inverses of $\mathbf{A}$, $\mathbf{A}^{\circledW}$ is both inner and outer inverse of $\mathbf{A}^{\oplus}\mathbf{A}^2$, and $\mathbf{A}_{\circledW}$ is both inner and outer inverse of $\mathbf{A}^2\mathbf{A}_{\oplus}$. The proof can be completed by Lemma 5. □

3. Determinantal Representations of the WG Inverse and Its Dual

The problem of $\mathfrak{D}$-representing quaternion generalized inverses can be successfully resolved based on the theory of noncommutative column-row determinants developed in [19,20].

3.1. Preliminaries on Quaternion Determinantal Representations

For $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$, there exists a method to generate n row ($\mathfrak{R}$-)determinants and n column ($\mathfrak{C}$-)determinants by generating a certain order of factors in each term.

- *The ith $\mathfrak{R}$-determinant* of $\mathbf{A}$, for an arbitrary row index $i \in I_n = \{1,\ldots,n\}$, is given by

$$\mathrm{rdet}_i\mathbf{A} := \sum_{\sigma\in S_n} (-1)^{n-r} (a_{i\,i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i}) \ldots (a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}}),$$

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whereat S_n denotes the symmetric group on I_n, while the permutation σ is defined as a product of mutually disjoint subsets ordered from the left to right by the rules

$$\sigma = \left(i\, i_{k_1} i_{k_1+1} \dots i_{k_1+l_1}\right)\left(i_{k_2} i_{k_2+1} \dots i_{k_2+l_2}\right) \dots \left(i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}\right),$$
$$i_{k_t} < i_{k_t+s},\ \ i_{k_2} < i_{k_3} < \dots < i_{k_r},\ \ \forall\ t = 2, \dots, r,\ \ s = 1, \dots, l_t.$$

- For an arbitrary column index $j \in I_n$, *the jth* $\mathfrak{C}$*-determinant* of $\mathbf{A}$ is defined as

$$\mathrm{cdet}_j \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-r} (a_{j_{k_r} j_{k_r+l_r}} \cdots a_{j_{k_r+1} j_{k_r}}) \cdots (a_{j j_{k_1+l_1}} \cdots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}),$$

in which a permutation τ is ordered from the right to left in the following way:

$$\tau = \left(j_{k_r+l_r} \cdots j_{k_r+1} j_{k_r}\right) \cdots \left(j_{k_2+l_2} \cdots j_{k_2+1} j_{k_2}\right)\left(j_{k_1+l_1} \cdots j_{k_1+1} j_{k_1} j\right),$$
$$j_{k_t} < j_{k_t+s},\ \ j_{k_2} < j_{k_3} < \dots < j_{k_r}.$$

The non-commutativity of quaternion operations causes all $\mathfrak{R}$- and $\mathfrak{C}$-determinants in general to be different, except if $\mathbf{A}$ is a Hermitian matrix, then the following equalities hold [19]:

$$\mathrm{rdet}_1 \mathbf{A} = \dots = \mathrm{rdet}_n \mathbf{A} = \mathrm{cdet}_1 \mathbf{A} = \dots = \mathrm{cdet}_n \mathbf{A} = \alpha \in \mathbb{R}.$$

This property allows to define the unique determinant of a Hermitian matrix $\mathbf{A}$ by putting $\det\mathbf{A} = \alpha$. The denotation $|\mathbf{A}| := \det\mathbf{A}$ will also be used.

The next symbols related to $\mathfrak{D}$-representations will be used. Let $\mathbf{a}_{i.}$ and $\mathbf{a}_{.j}$ denote the ith row and jth column of $\mathbf{A}$, respectively. Further, $\mathbf{A}_{.j}(\mathbf{c})$ (resp. $\mathbf{A}_{i.}(\mathbf{b})$) stand for the matrices formed by replacing jth column (resp. ith row) of $\mathbf{A}$ by the column vector $\mathbf{c}$ (resp. by the row vector $\mathbf{b}$). Suppose $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$ and $\beta := \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\}$ are subsets with $1 \leq k \leq \min\{m, n\}$. For $\mathbf{A} \in \mathbb{H}^{m\times n}$, the notation $\mathbf{A}_\beta^\alpha$ stands for a submatrix with rows and columns indexed by α and β, respectively. When $\mathbf{A} \in \mathbb{H}^{n\times n}$ is Hermitian, $\mathbf{A}_\alpha^\alpha$ and $|\mathbf{A}|_\alpha^\alpha$ denote a principal submatrix and a principal minor of $\mathbf{A}$, respectively. The usual notation $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \dots, \alpha_k),\ 1 \leq \alpha_1 < \dots < \alpha_k \leq n\}$ is the set of strictly increasing sequences of $k \in \{1, \dots, n\}$ integers elected from $\{1, \dots, n\}$. For some selected $i \in \alpha$ and $j \in \beta$, it is usual to write $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$, $J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$.

Lemma 7. [42] (Theorem 4.5.) *If* $\mathbf{A} \in \mathbb{H}_s^{m\times n}$*, then its Moore-Penrose inverse* $\mathbf{A}^\dagger = \left(a_{ij}^\dagger\right) \in \mathbb{H}^{n\times m}$ *possesses the* $\mathfrak{D}$*-representations*

$$a_{ij}^\dagger = \frac{\sum_{\beta \in J_{s,n}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{a}_{.j}^*\right)\right)_\beta^\beta}{\sum_{\beta \in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta} \tag{10}$$

$$= \frac{\sum_{\alpha \in I_{s,m}\{j\}} \mathrm{rdet}_j \left((\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{a}_{i.}^*)\right)_\alpha^\alpha}{\sum_{\alpha \in I_{s,m}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha}, \tag{11}$$

where $\mathbf{a}_{.j}^*$ *and* $\mathbf{a}_{i.}^*$ *stand for the jth column and ith row of* $\mathbf{A}^*$*.*

The $\mathfrak{D}$-representations of the right and left quaternion core-EP inverses are derived in [3].

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3.2. $\mathfrak{D}$-representations of quaternion right and left WG inverses

Theorem 5. *If $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ with $\operatorname{rank}(\mathbf{A}^k) = s$, then the right WG inverse $\mathbf{A}^{ⓦ} = \left(a_{ij}^{ⓦ,r}\right)$ possesses the subsequent $\mathfrak{D}$-representation*

$$a_{ij}^{ⓦ,r} = \frac{\sum\limits_{t=1}^{n} \sum\limits_{\alpha \in I_{s,n}\{t\}} \operatorname{rdet}_t \left(\left[\mathbf{A}^{k+2} \left(\mathbf{A}^{k+2} \right)^* \right]_{t.} \left(\hat{\mathbf{a}}_{i.}^{(2)} \right) \right)_{\alpha}^{\alpha} a_{tj}}{\sum\limits_{\alpha \in I_{s,n}} \left| \mathbf{A}^{k+2} \left(\mathbf{A}^{k+2} \right)^* \right|_{\alpha}^{\alpha}}, \tag{12}$$

where $\hat{\mathbf{a}}_{i.}^{(2)}$ is the ith row of $\hat{\mathbf{A}}_2 = \mathbf{A}^k(\mathbf{A}^{k+2})^*$.

Proof. Since $\operatorname{rank}(\mathbf{A}^{k+2}) = \operatorname{rank}(\mathbf{A}^{k+1}) = \operatorname{rank}(\mathbf{A}^k) = s$, then from Equation (8) it follows

$$a_{ij}^{ⓦ,r} = \sum_{t=1}^{n} \sum_{m=1}^{n} a_{im}^{(k)} \left(a_{mt}^{(k+2)} \right)^{\dagger} a_{tj}.$$

By Equation (11),

$$\left(a_{mt}^{(k+2)} \right)^{\dagger} = \frac{\sum\limits_{\alpha \in I_{s_2,n}\{t\}} \operatorname{rdet}_t \left(\left[\mathbf{A}^{(k+2)} \left(\mathbf{A}^{(k+2)} \right)^* \right]_{t.} \left(\mathbf{a}_{m.}^{(k+2),*} \right) \right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{s_2,n}} \left| \mathbf{A}^{(k+2)} \left(\mathbf{A}^{(k+2)} \right)^* \right|_{\alpha}^{\alpha}},$$

where $\mathbf{a}_{m.}^{(k+2),*}$ is the mth row of $\left(\mathbf{A}^{(k+2)} \right)^*$.

Denote $\hat{\mathbf{A}}_2 = \mathbf{A}^k(\mathbf{A}^{k+2})^*$. Then, Equation (12) holds because of $\sum\limits_{m=1}^{n} a_{im}^{(k)} \mathbf{a}_{m.}^{(k+2),*} = \hat{\mathbf{a}}_{i.}^{(2)}$. □

Theorem 6. *If $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ with $\operatorname{rank}(\mathbf{A}^k) = s$, then the left WG inverse $\mathbf{A}_{ⓦ} = \left(a_{ij}^{ⓦ,l}\right)$ possesses the determinantal representation*

$$a_{ij}^{ⓦ,l} = \frac{\sum\limits_{t=1}^{n} a_{it} \sum\limits_{\beta \in J_{s,n}\{t\}} \operatorname{cdet}_t \left(\left(\left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right)_{.t} \left(\check{\mathbf{a}}_{.j}^{(2)} \right) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{s,n}} \left| \left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right|_{\beta}^{\beta}}, \tag{13}$$

where $\check{\mathbf{a}}_{.j}^{(2)}$ is the jth column of $\check{\mathbf{A}}_2 = (\mathbf{A}^{k+2})^* \mathbf{A}^k$.

Proof. The proof is analogous to the proof of Theorem 5.
□

4. WG–Dual-WG Solutions of CQMEs

In this section, we consider the solvability of CQMEs

$$(\mathbf{A}^k)^* \mathbf{A}^2 \mathbf{X} \mathbf{B}^2 (\mathbf{B}^q)^* = (\mathbf{A}^k)^* \mathbf{A} \mathbf{C} \mathbf{B} (\mathbf{B}^q)^*, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}^k, \mathbf{B}^q) \tag{14}$$

$$\mathbf{A}^{k+1} \mathbf{X} \mathbf{B}^{q+1} = \mathbf{A}^k \mathbf{C} \mathbf{B}^q, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}(\mathbf{A}^k)^*, (\mathbf{B}^q)^* \mathbf{B}). \tag{15}$$

Applying the WG inverse of **A** and the dual WG inverse of **B**, the unique solution to the CQME (14) is presented in the first result of this section.

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Theorem 7. *The CQME (14) has a uniquely determined solution represented by*

$$\mathbf{X} = \mathbf{A}^{\textcircled{w}}\mathbf{C}\mathbf{B}_{\textcircled{w}}. \tag{16}$$

Proof. The known equalities $\mathbf{A}^{\textcircled{\dagger}} = \mathbf{A}^{\mathrm{D}}\mathbf{A}^k(\mathbf{A}^k)^{\dagger}$ and $\mathbf{B}_{\textcircled{\dagger}} = (\mathbf{B}^q)^{\dagger}\mathbf{B}^q\mathbf{B}^{\mathrm{D}}$ give $\mathbf{A}^{\textcircled{w}} = (\mathbf{A}^{\textcircled{\dagger}})^2\mathbf{A} = (\mathbf{A}^{\mathrm{D}})^2\mathbf{A}^k(\mathbf{A}^k)^{\dagger}$ and $\mathbf{B}_{\textcircled{w}} = \mathbf{B}(\mathbf{B}_{\textcircled{\dagger}})^2 = (\mathbf{B}^q)^{\dagger}\mathbf{B}^q(\mathbf{B}^{\mathrm{D}})^2$, respectively. Hence,

$$\mathbf{X} = \mathbf{A}^{\textcircled{w}}\mathbf{C}\mathbf{B}_{\textcircled{w}} = (\mathbf{A}^{\mathrm{D}})^2\mathbf{A}^k(\mathbf{A}^k)^{\dagger}\mathbf{A}\mathbf{C}\mathbf{B}(\mathbf{B}^q)^{\dagger}\mathbf{B}^q(\mathbf{B}^{\mathrm{D}})^2 \in \mathcal{O}_{\subset}(\mathbf{A}^k, \mathbf{B}^q),$$

and

$$\begin{aligned}
(\mathbf{A}^k)^*\mathbf{A}^2\mathbf{X}\mathbf{B}^2(\mathbf{B}^q)^* &= (\mathbf{A}^k)^*\mathbf{A}^2(\mathbf{A}^{\mathrm{D}})^2\mathbf{A}^k(\mathbf{A}^k)^{\dagger}\mathbf{A}\mathbf{C}\mathbf{B}(\mathbf{B}^q)^{\dagger}\mathbf{B}^q(\mathbf{B}^{\mathrm{D}})^2\mathbf{B}^2(\mathbf{B}^q)^* \\
&= (\mathbf{A}^k)^*\mathbf{A}\mathbf{A}^{\mathrm{D}}\mathbf{A}^k(\mathbf{A}^k)^{\dagger}\mathbf{A}\mathbf{C}\mathbf{B}(\mathbf{B}^q)^{\dagger}\mathbf{B}^q\mathbf{B}^{\mathrm{D}}\mathbf{B}(\mathbf{B}^q)^* \\
&= (\mathbf{A}^k)^*\mathbf{A}^k(\mathbf{A}^k)^{\dagger}\mathbf{A}\mathbf{C}\mathbf{B}(\mathbf{B}^q)^{\dagger}\mathbf{B}^q(\mathbf{B}^q)^* \\
&= (\mathbf{A}^k)^*\mathbf{A}\mathbf{C}\mathbf{B}(\mathbf{B}^q)^*,
\end{aligned}$$

imply that $\mathbf{X} = \mathbf{A}^{\textcircled{w}}\mathbf{C}\mathbf{B}_{\textcircled{w}}$ is a solution of (14).

In order to prove that (14) has unique solution, let $\mathbf{X}_1$ and $\mathbf{X}$ be two solutions to (14). Then $(\mathbf{A}^k)^*\mathbf{A}^2(\mathbf{X}_1 - \mathbf{X})\mathbf{B}^2(\mathbf{B}^q)^* = \mathbf{0}$, $\mathcal{C}_r(\mathbf{X}_1) \subset \mathcal{C}_r(\mathbf{A}^k)$ and $\mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^k)$ yield

$$(\mathbf{X}_1 - \mathbf{X})\mathbf{B}^2(\mathbf{B}^q)^* \in \mathcal{N}_r((\mathbf{A}^k)^*\mathbf{A}^2) \cap \mathcal{C}_r(\mathbf{A}^k) = \mathcal{N}_r(\mathbf{A}^{\textcircled{w}}\mathbf{A}) \cap \mathcal{C}_r(\mathbf{A}^{\textcircled{w}}\mathbf{A}) = \{\mathbf{0}\}.$$

Now, $\mathcal{R}_l(\mathbf{X}_1) \subset \mathcal{R}_l(\mathbf{B}^q)$, $\mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^q)$ and $(\mathbf{X}_1 - \mathbf{X})\mathbf{B}^2(\mathbf{B}^q)^* = \mathbf{0}$ give

$$\mathbf{X}_1 - \mathbf{X} \in \mathcal{N}_l(\mathbf{B}^2(\mathbf{B}^q)^*) \cap \mathcal{R}_l(\mathbf{B}^q) = \mathcal{N}_l(\mathbf{B}\mathbf{B}_{\textcircled{w}}) \cap \mathcal{R}_l(\mathbf{B}\mathbf{B}_{\textcircled{w}}) = \{\mathbf{0}\},$$

i.e. $\mathbf{X}_1 = \mathbf{X}$. □

As a consequence of Theorem 7, we solve the CQMEs when $\mathbf{A} = \mathbf{I}_n$ or $\mathbf{B} = \mathbf{I}_m$.

Corollary 2. *Let* $\mathbf{C} \in \mathbb{H}^{n\times m}$.

(a) *If* $\mathbf{A} \in \mathbb{H}^{(n)(k)}$, *then*
$$\mathbf{X} = \mathbf{A}^{\textcircled{w}}\mathbf{C} \tag{17}$$
is unique solution to $(\mathbf{A}^k)^*\mathbf{A}^2\mathbf{X} = (\mathbf{A}^k)^*\mathbf{A}\mathbf{C}, \quad \mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^k)$.

(b) *If* $\mathbf{B} \in \mathbb{H}^{(m)(q)}$, *then*
$$\mathbf{X} = \mathbf{C}\mathbf{B}_{\textcircled{w}} \tag{18}$$
is unique solution to $\mathbf{X}\mathbf{B}^2(\mathbf{B}^q)^* = \mathbf{C}\mathbf{B}(\mathbf{B}^q)^*, \quad \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^q)$.

Similarly as Theorem 7, we can prove the solvability of the next CQME.

Corollary 3. *Let* $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(n|m)(k|q)}$ *and* $\mathbf{C} \in \mathbb{H}^{n\times m}$. *Then* (16) *is the unique solution to*

$$\mathbf{A}^2\mathbf{X}\mathbf{B}^2 = \mathbf{A}^k(\mathbf{A}^k)^{\dagger}\mathbf{A}\mathbf{C}\mathbf{B}(\mathbf{B}^q)^{\dagger}\mathbf{B}^q, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}^k, \mathbf{B}^q).$$

When $\mathrm{Ind}(\mathbf{A}) = \mathrm{Ind}(\mathbf{B}) = 1$ in Theorem 7 and Corollary 3, we get the following result.

Corollary 4. *Let* $\mathbf{C} \in \mathbb{H}^{n\times m}$, $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\mathbf{B} \in \mathbb{H}^{m\times m}$ *with* $\mathrm{Ind}(\mathbf{A}) = 1$ *and* $\mathrm{Ind}(\mathbf{B}) = 1$. *Then* $\mathbf{X} = \mathbf{A}^{\#}\mathbf{C}\mathbf{B}^{\#}$ *is the unique solution to*

(a) $\mathbf{A}^*\mathbf{A}^2\mathbf{X}\mathbf{B}^2\mathbf{B}^* = \mathbf{A}^*\mathbf{A}\mathbf{C}\mathbf{B}\mathbf{B}^*, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}, \mathbf{B})$;
(b) $\mathbf{A}^2\mathbf{X}\mathbf{B}^2 = \mathbf{A}\mathbf{C}\mathbf{B}, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}, \mathbf{B})$.

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Theorem 8. *The CQME (15) has a uniquely determined solution that can be represented as follows*

$$\mathbf{X} = \mathbf{A}_{\textcircled{w}}\mathbf{C}\mathbf{B}^{\textcircled{w}}. \tag{19}$$

Proof. Since $\mathcal{C}_r((\mathbf{A}^k)^\dagger) = \mathcal{C}_r((\mathbf{A}^k)^*)$ and $\mathcal{R}_l((\mathbf{B}^q)^\dagger) = \mathcal{R}_l((\mathbf{B}^q)^*)$, we have that

$$\mathbf{X} = \mathbf{A}_{\textcircled{w}}\mathbf{C}\mathbf{B}^{\textcircled{w}} = \mathbf{A}(\mathbf{A}^k)^\dagger\mathbf{A}^k(\mathbf{A}^{\mathrm{D}})^2\mathbf{C}(\mathbf{B}^{\mathrm{D}})^2\mathbf{B}^q(\mathbf{B}^q)^\dagger\mathbf{B}$$

satisfies $\mathbf{X} \in \mathcal{O}_\subset(\mathbf{A}(\mathbf{A}^k)^\dagger, (\mathbf{B}^q)^\dagger\mathbf{B}) = \mathcal{O}_\subset(\mathbf{A}(\mathbf{A}^k)^*, (\mathbf{B}^q)^*\mathbf{B})$ and

$$\begin{aligned} \mathbf{A}^{k+1}\mathbf{X}\mathbf{B}^{q+1} &= \mathbf{A}^{k+2}(\mathbf{A}^k)^\dagger\mathbf{A}^k(\mathbf{A}^{\mathrm{D}})^2\mathbf{C}(\mathbf{B}^{\mathrm{D}})^2\mathbf{B}^q(\mathbf{B}^q)^\dagger\mathbf{B}^{q+2} \\ &= \mathbf{A}^{k+2}(\mathbf{A}^{\mathrm{D}})^2\mathbf{C}(\mathbf{B}^{\mathrm{D}})^2\mathbf{B}^{q+2} \\ &= \mathbf{A}^k\mathbf{C}\mathbf{B}^q. \end{aligned}$$

It means that (19) is a solution to the CQME (15).

To check that CQME (15) has unique solution, assume that $\mathbf{X}_1$ and $\mathbf{X}$ are its two solutions. Further, by $\mathbf{A}^{k+1}(\mathbf{X}_1 - \mathbf{X})\mathbf{B}^{q+1} = \mathbf{0}$, $\mathcal{C}_r(\mathbf{X}_1) \subset \mathcal{C}_r(\mathbf{A}(\mathbf{A}^k)^*)$ and $\mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}(\mathbf{A}^k)^*)$, it follows that

$$(\mathbf{X}_1 - \mathbf{X})\mathbf{B}^{q+1} \in \mathcal{N}_r(\mathbf{A}^{k+1}) \cap \mathcal{C}_r(\mathbf{A}(\mathbf{A}^k)^*) = \mathcal{N}_r(\mathbf{A}_{\textcircled{w}}\mathbf{A}) \cap \mathcal{C}_r(\mathbf{A}_{\textcircled{w}}\mathbf{A}) = \{\mathbf{0}\}.$$

Using $\mathcal{R}_l(\mathbf{X}_1) \subset \mathcal{R}_l((\mathbf{B}^q)^*\mathbf{B})$, $\mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l((\mathbf{B}^q)^*\mathbf{B})$ and $(\mathbf{X}_1 - \mathbf{X})\mathbf{B}^{q+1} = \mathbf{0}$, we obtain

$$\mathbf{X}_1 - \mathbf{X} \in \mathcal{N}_l(\mathbf{B}^{q+1}) \cap \mathcal{R}_l((\mathbf{B}^q)^*\mathbf{B}) = \mathcal{N}_l(\mathbf{B}\mathbf{B}^{\textcircled{w}}) \cap \mathcal{R}_l(\mathbf{B}\mathbf{B}^{\textcircled{w}}) = \{\mathbf{0}\}.$$

So, $\mathbf{X}_1 = \mathbf{X}$. □

In special cases, Theorem 8 implies the solvability of the next CQMEs.

Corollary 5. *Let* $\mathbf{C} \in \mathbb{H}^{n\times m}$.

(a) *If* $\mathbf{A} \in \mathbb{H}^{(n)(k)}$, *then*

$$\mathbf{X} = \mathbf{A}_{\textcircled{w}}\mathbf{C} \tag{20}$$

is the unique solution to $\mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^l\mathbf{C}, \quad \mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}(\mathbf{A}^k)^*)$.

(b) *If* $\mathbf{B} \in \mathbb{H}^{(m)(q)}$, *then*

$$\mathbf{X} = \mathbf{C}\mathbf{B}^{\textcircled{w}} \tag{21}$$

is the unique solution to $\mathbf{X}\mathbf{B}^{q+1} = \mathbf{C}\mathbf{B}^q, \quad \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l((\mathbf{B}^q)^*\mathbf{B})$.

Combining the WG inverses of $\mathbf{A}$ and $\mathbf{B}$ or the dual WG inverses of $\mathbf{A}$ and $\mathbf{B}$, we can solve some more CQMEs.

Theorem 9. *Let* $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(n|m)(k|q)}$ *and* $\mathbf{C} \in \mathbb{H}^{n\times m}$.

(a) *Then*

$$\mathbf{X} = \mathbf{A}^{\textcircled{w}}\mathbf{C}\mathbf{B}^{\textcircled{w}} \tag{22}$$

is the unique solution to $(\mathbf{A}^k)^*\mathbf{A}^2\mathbf{X}\mathbf{B}^{q+1} = (\mathbf{A}^k)^*\mathbf{A}\mathbf{C}\mathbf{B}^q$, $\mathbf{X} \in \mathcal{O}_\subset(\mathbf{A}^k, (\mathbf{B}^q)^*\mathbf{B})$.

(b) *Then*

$$\mathbf{X} = \mathbf{A}_{\textcircled{w}}\mathbf{C}\mathbf{B}_{\textcircled{w}} \tag{23}$$

is the unique solution to $\mathbf{A}^{k+1}\mathbf{X}\mathbf{B}^2(\mathbf{B}^q)^* = \mathbf{A}^k\mathbf{C}\mathbf{B}(\mathbf{B}^q)^*$, $\mathbf{X} \in \mathcal{O}_\subset(\mathbf{A}(\mathbf{A}^k)^*, \mathbf{B}^q)$.

Especially, Theorem 9 gives the next result.

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Corollary 6. *Let* $\mathbf{C} \in \mathbb{H}^{n\times m}$, $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\mathbf{B} \in \mathbb{H}^{m\times m}$ *with* $\mathrm{Ind}(\mathbf{A}) = 1$ *and* $\mathrm{Ind}(\mathbf{B}) = 1$. *Then* $\mathbf{X} = \mathbf{A}^{\#}\mathbf{C}\mathbf{B}^{\#}$ *is unique solution to*

(a) $\mathbf{A}^*\mathbf{A}^2\mathbf{X}\mathbf{B}^2 = \mathbf{A}^*\mathbf{A}\mathbf{C}\mathbf{B}, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A},\mathbf{B})$;
(b) $\mathbf{A}^2\mathbf{X}\mathbf{B}^2\mathbf{B}^* = \mathbf{A}\mathbf{C}\mathbf{B}\mathbf{B}^*, \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A},\mathbf{B})$.

5. Constrained Quaternion Matrix Minimization Problems

Let $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(n|m)(k|q)}$ and $\mathbf{C} \in \mathbb{H}^{n\times m}$. The main goal of this section is to present solution of the constrained quaternion matrix minimization problem (CQMMP)

$$\|\mathbf{A}^2\mathbf{X}\mathbf{B}^2 - \mathbf{A}\mathbf{C}\mathbf{B}\|_F = \min \quad \text{subject to} \quad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}^k,\mathbf{B}^q), \tag{24}$$

and its particular kinds. The following result regarding the decomposition of $\mathbf{A}$ and its WG inverse $\mathbf{A}^{ⓦ}$, obtained for complex matrices in [43], is extended to quaternion matrices by analogy to the method used in [38].

Lemma 8. *If* $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ *and* $r = \mathrm{rank}(\mathbf{A}^k)$, *then*

$$\mathbf{A} = \mathbf{Q}\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ 0 & \mathbf{A}_3 \end{bmatrix}\mathbf{Q}^*, \tag{25}$$

where $\mathbf{Q} \in \mathbb{H}^{n\times n}$ *is unitary,* $\mathbf{A}_1 \in \mathbb{H}^{r\times r}$ *is nonsingular and* $\mathbf{A}_3 \in \mathbb{H}^{(n-r)\times(n-r)}$ *is nilpotent of index* l. *Moreover,*

$$\mathbf{A}^{ⓦ} = \mathbf{Q}\begin{bmatrix} \mathbf{A}_1^{-1} & \mathbf{A}_1^{-2}\mathbf{A}_2 \\ 0 & 0 \end{bmatrix}\mathbf{Q}^*. \tag{26}$$

Based on $\mathbf{A}^{ⓦ}$ and $\mathbf{B}_{ⓦ}$, we solve now the CQMMP (24).

Theorem 10. *The CQMMP (24) has a uniquely determined solution represented by (16).*

Proof. The hypothesis $\mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}^k,\mathbf{B}^q)$ implies that $\mathbf{X} = \mathbf{A}^k\mathbf{Z}\mathbf{B}^q$, for some $\mathbf{Z} \in \mathbb{H}^{n\times m}$. By Lemma 8, we can write

$$\mathbf{A} = \mathbf{Q}_1\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{A}_3 \end{bmatrix}\mathbf{Q}_1^*, \quad \mathbf{B}^* = \mathbf{Q}_2\begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{0} & \mathbf{B}_3 \end{bmatrix}\mathbf{Q}_2^*,$$

where $\mathrm{rank}(\mathbf{A}^k) = r_1$, $\mathrm{rank}(\mathbf{B}^q) = r_2$, $\mathbf{A}_1 \in \mathbb{H}^{r_1\times r_1}$ and $\mathbf{B}_1 \in \mathbb{H}^{r_2\times r_2}$ are nonsingular, $\mathbf{A}_3 \in \mathbb{H}^{(n-r_1)\times(n-r_1)}$ and $\mathbf{B}_1 \in \mathbb{H}^{(m-r_2)\times(m-r_2)}$ are nilpotent matrices of indexes k and q, respectively. Using [4] (Remark 1.3), we have $\mathbf{B}_{㊉} = [(\mathbf{B}^*)^{㊉}]^*$, which gives

$$\mathbf{B}_{ⓦ} = \mathbf{B}(\mathbf{B}_{㊉})^2 = (\mathbf{B}^*)^*\left([(\mathbf{B}^*)^{㊉}]^*\right)^2 = \left([(\mathbf{B}^*)^{㊉}]^2\mathbf{B}^*\right)^* = [(\mathbf{B}^*)^{ⓦ}]^*.$$

According to Lemma 8, it follows

$$\mathbf{A}^{ⓦ} = \mathbf{Q}_1\begin{bmatrix} \mathbf{A}_1^{-1} & \mathbf{A}_1^{-2}\mathbf{A}_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{Q}_1^*, \quad \mathbf{B}_{ⓦ} = [(\mathbf{B}^*)^{ⓦ}]^* = \mathbf{Q}_2\begin{bmatrix} (\mathbf{B}_1^{-1})^* & \mathbf{0} \\ \mathbf{B}_2^*(\mathbf{B}_1^{-2})^* & \mathbf{0} \end{bmatrix}\mathbf{Q}_2^*.$$

Set

$$\mathbf{Q}_1^*\mathbf{Z}\mathbf{Q}_2 = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{bmatrix}, \quad \mathbf{Q}_1^*\mathbf{C}\mathbf{Q}_2 = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & \mathbf{C}_4 \end{bmatrix},$$

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where $\mathbf{Z}_1, \mathbf{C}_1 \in \mathbb{H}^{r_1 \times r_2}$, $\mathbf{Z}_2, \mathbf{C}_2 \in \mathbb{H}^{r_1 \times (m-r_2)}$, $\mathbf{Z}_3, \mathbf{C}_3 \in \mathbb{H}^{(n-r_1) \times r_2}$, and $\mathbf{Z}_4$, $\mathbf{C}_4 \in \mathbb{H}^{(n-r_1)\times(m-r_2)}$. For appropriate matrices $\mathbf{S}_1 \in \mathbb{H}^{(n-r_1)\times r_1}$ and $\mathbf{S}_2 \in \mathbb{H}^{(m-r_2)\times r_2}$, we obtain

$$
\begin{aligned}
\mathbf{A}^2\mathbf{X}\mathbf{B}^2 &= \mathbf{A}^{k+2}\mathbf{Z}\mathbf{B}^{q+2} \\
&= \mathbf{Q}_1 \begin{bmatrix} \mathbf{A}_1^2 & \mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_3 \\ \mathbf{0} & \mathbf{A}_3^2 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1^k & \mathbf{S}_1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}_1^*\mathbf{Z}\mathbf{Q}_2 \\
&\times \begin{bmatrix} (\mathbf{B}_1^*)^q & \mathbf{0} \\ \mathbf{S}_2 & \mathbf{0} \end{bmatrix} \begin{bmatrix} (\mathbf{B}_1^*)^2 & \mathbf{0} \\ \mathbf{B}_2^*\mathbf{B}_1^* + \mathbf{B}_3^*\mathbf{B}_2^* & (\mathbf{B}_3^*)^2 \end{bmatrix} \mathbf{Q}_2^* \\
&= \mathbf{Q}_1 \begin{bmatrix} \mathbf{A}_1^{k+2} & \mathbf{A}_1^2\mathbf{S}_1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{bmatrix} \begin{bmatrix} (\mathbf{B}_1^*)^{q+2} & \mathbf{0} \\ \mathbf{S}_2(\mathbf{B}_1^*)^2 & \mathbf{0} \end{bmatrix} \mathbf{Q}_2^* \\
&= \mathbf{Q}_1 \begin{bmatrix} \mathbf{A}_1^{k+2}\mathbf{Z}_1(\mathbf{B}_1^*)^{q+2} + \mathbf{A}_1^2\mathbf{S}_1\mathbf{Z}_3(\mathbf{B}_1^*)^{q+2} + \mathbf{A}_1^{k+2}\mathbf{Z}_2\mathbf{S}_2(\mathbf{B}_1^*)^2 + \mathbf{A}_1^2\mathbf{S}_1\mathbf{Z}_4\mathbf{S}_2(\mathbf{B}_1^*)^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}_2^*
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{A}\mathbf{C}\mathbf{B} &= \mathbf{Q}_1 \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{A}_3 \end{bmatrix} \mathbf{Q}_1^*\mathbf{C}\mathbf{Q}_2 \begin{bmatrix} \mathbf{B}_1^* & \mathbf{0} \\ \mathbf{B}_2^* & \mathbf{B}_3^* \end{bmatrix} \mathbf{Q}_2^* \\
&= \mathbf{Q}_1 \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{A}_3 \end{bmatrix} \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & \mathbf{C}_4 \end{bmatrix} \begin{bmatrix} \mathbf{B}_1^* & \mathbf{0} \\ \mathbf{B}_2^* & \mathbf{B}_3^* \end{bmatrix} \mathbf{Q}_2^* \\
&= \mathbf{Q}_1 \begin{bmatrix} \mathbf{G} & \mathbf{A}_1\mathbf{C}_2\mathbf{B}_3^* + \mathbf{A}_2\mathbf{C}_4\mathbf{B}_3^* \\ \mathbf{A}_3\mathbf{C}_3\mathbf{B}_1^* + \mathbf{A}_3\mathbf{C}_4\mathbf{B}_2^* & \mathbf{A}_3\mathbf{C}_4\mathbf{B}_3^* \end{bmatrix} \mathbf{Q}_2^*,
\end{aligned}
$$

where $\mathbf{G} = \mathbf{A}_1\mathbf{C}_1\mathbf{B}_1^* + \mathbf{A}_2\mathbf{C}_3\mathbf{B}_1^* + \mathbf{A}_1\mathbf{C}_2\mathbf{B}_2^* + \mathbf{A}_2\mathbf{C}_4\mathbf{B}_2^*$. Therefore,

$$
\begin{aligned}
&\|\mathbf{A}^2\mathbf{X}\mathbf{B}^2 - \mathbf{A}\mathbf{C}\mathbf{B}\|_F^2 = \\
&= \|\mathbf{A}_1^{k+2}\mathbf{Z}_1(\mathbf{B}_1^*)^{q+2} + \mathbf{A}_1^2\mathbf{S}_1\mathbf{Z}_3(\mathbf{B}_1^*)^{q+2} + \mathbf{A}_1^{k+2}\mathbf{Z}_2\mathbf{S}_2(\mathbf{B}_1^*)^2 + \mathbf{A}_1^2\mathbf{S}_1\mathbf{Z}_4\mathbf{S}_2(\mathbf{B}_1^*)^2 - \mathbf{G}\|_F^2 \\
&+ \|\mathbf{A}_1\mathbf{C}_2\mathbf{B}_3^* + \mathbf{A}_2\mathbf{C}_4\mathbf{B}_3^*\|_F^2 + \|\mathbf{A}_3\mathbf{C}_3\mathbf{B}_1^* + \mathbf{A}_3\mathbf{C}_4\mathbf{B}_2^*\|_F^2 + \|\mathbf{A}_3\mathbf{C}_4\mathbf{B}_3^*\|_F^2.
\end{aligned}
$$

Because of $\mathbf{X}$ is a solution to (24) if and only if $\mathbf{Z}$ is a solution to $\|\mathbf{A}^{k+2}\mathbf{Z}\mathbf{B}^{q+2} - \mathbf{A}\mathbf{C}\mathbf{B}\|_F = \min$, one can see

$$
\min_{\mathbf{Z}_i \text{ for } i=1,\dots,4} \|\mathbf{A}_1^{k+2}\mathbf{Z}_1(\mathbf{B}_1^*)^{q+2} + \mathbf{A}_1^2\mathbf{S}_1\mathbf{Z}_3(\mathbf{B}_1^*)^{q+2} + \mathbf{A}_1^{k+2}\mathbf{Z}_2\mathbf{S}_2(\mathbf{B}_1^*)^2 + \mathbf{A}_1^2\mathbf{S}_1\mathbf{Z}_4\mathbf{S}_2(\mathbf{B}_1^*)^2 - \mathbf{G}\|_F^2 = 0,
$$

i.e.,

$$
\|\mathbf{A}^{k+2}\mathbf{Z}\mathbf{B}^{q+2} - \mathbf{A}\mathbf{C}\mathbf{B}\|_F = \min \|\mathbf{A}_1\mathbf{C}_2\mathbf{B}_3^* + \mathbf{A}_2\mathbf{C}_4\mathbf{B}_3^*\|_F^2 + \|\mathbf{A}_3\mathbf{C}_3\mathbf{B}_1^* + \mathbf{A}_3\mathbf{C}_4\mathbf{B}_2^*\|_F^2 + \|\mathbf{A}_3\mathbf{C}_4\mathbf{B}_3^*\|_F^2
$$

for arbitrary $\mathbf{Z}_i$ for $i = 2, 3, 4$ and

$$
\mathbf{Z}_1 = \mathbf{A}_1^{-(k+2)}\mathbf{G}(\mathbf{B}_1^*)^{-(q+2)} - \mathbf{A}_1^{-k}\mathbf{S}_1\mathbf{Z}_3 - \mathbf{Z}_2\mathbf{S}_2(\mathbf{B}_1^*)^{-q} - \mathbf{A}_1^{-k}\mathbf{S}_1\mathbf{Z}_4\mathbf{S}_2(\mathbf{B}_1^*)^{-q}. \tag{27}
$$

If we substitute (27) into

$$
\begin{aligned}
\mathbf{X} = \mathbf{A}^k\mathbf{Z}\mathbf{B}^q &= \mathbf{Q}_1 \begin{bmatrix} \mathbf{A}_1^k & \mathbf{S}_1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{bmatrix} \begin{bmatrix} (\mathbf{B}_1^*)^q & \mathbf{0} \\ \mathbf{S}_2 & \mathbf{0} \end{bmatrix} \mathbf{Q}_2^* = \\
&= \mathbf{Q}_1 \begin{bmatrix} \mathbf{A}_1^k\mathbf{Z}_1(\mathbf{B}_1^*)^q + \mathbf{S}_1\mathbf{Z}_3(\mathbf{B}_1^*)^q + \mathbf{A}_1^k\mathbf{Z}_2\mathbf{S}_2 + \mathbf{S}_1\mathbf{Z}_4\mathbf{S}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}_2^*, \qquad (28)
\end{aligned}
$$

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we get

$$\mathbf{X} = \mathbf{Q}_1 \begin{bmatrix} \mathbf{A}_1^{-2}\mathbf{G}(\mathbf{B}_1^*)^{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}_2^* = \mathbf{Q}_1 \begin{bmatrix} \mathbf{A}_1^{-1} & \mathbf{A}_1^{-2}\mathbf{A}_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & \mathbf{C}_4 \end{bmatrix} \begin{bmatrix} (\mathbf{B}_1^{-1})^* & \mathbf{0} \\ \mathbf{B}_2^*(\mathbf{B}_1^{-2})^* & \mathbf{0} \end{bmatrix} \mathbf{Q}_2^*$$
$$= \mathbf{A}^{\text{ⓦ}}\mathbf{C}\mathbf{B}_{\text{ⓦ}}$$

is a uniquely determined solution of CQMMP (24). □

For $\mathrm{Ind}(\mathbf{A}) = 1$ or $\mathrm{Ind}(\mathbf{B}) = 1$ in Theorem 10, the next consequence follows.

Corollary 7. *Let* $\mathbf{C} \in \mathbb{H}^{n\times m}$, $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\mathbf{B} \in \mathbb{H}^{m\times m}$ *with* $\mathrm{Ind}(\mathbf{A}) = 1$ *and* $\mathrm{Ind}(\mathbf{B}) = 1$. *Then* $\mathbf{X} = \mathbf{A}^{\#}\mathbf{C}\mathbf{B}^{\#}$ *is unique solution to CQMMP*

$$\|\mathbf{A}^2\mathbf{X}\mathbf{B}^2 - \mathbf{A}\mathbf{C}\mathbf{B}\|_F = \min \quad \textit{subject to} \qquad \mathbf{X} \in \mathcal{O}_{\subset}(\mathbf{A}, \mathbf{B}).$$

As special types of CQMMP (24), we can consider CQMMPs:

$$\|\mathbf{A}^2\mathbf{X} - \mathbf{A}\mathbf{C}\|_F = \min \quad \text{subject to} \qquad \mathcal{C}_r(\mathbf{X}) \subset \mathcal{C}_r(\mathbf{A}^k), \tag{29}$$

where $\mathbf{C} \in \mathbb{H}^{n\times m}$ and $\mathbf{A} \in \mathbb{H}^{(n)(k)}$; and

$$\|\mathbf{X}\mathbf{B}^2 - \mathbf{C}\mathbf{B}\|_F = \min \quad \text{subject to} \qquad \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{B}^q), \tag{30}$$

where $\mathbf{C} \in \mathbb{H}^{n\times m}$ and $\mathbf{B} \in \mathbb{H}^{(m)(q)}$.

Corollary 8. (a) *The CQMMP (29) has a uniquely determined solution by (17).*
(b) *The CQMMP (30) has a uniquely determined solution by (18).*

6. Cramer's-Type Representations of Derived Solutions

Firstly, we get the $\mathfrak{D}$-representations for the solution (16) and its special cases (17) and (18).

Theorem 11. *Let* $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(n|m)(k|q)}$, $\mathrm{rank}(\mathbf{A}^k) = s_1$, *and* $\mathrm{rank}(\mathbf{B}^q) = s_2$. *The unique solution* $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{n\times m}$ *from (16) can be expressed componentwise by*

$$x_{ij} = \frac{c_{ij}^{(1)}}{\sum\limits_{\alpha\in I_{s_1,n}} \left|\mathbf{A}^{k+2}\left(\mathbf{A}^{k+2}\right)^*\right|_{\alpha}^{\alpha} \sum\limits_{\beta\in J_{s_2,m}} \left|\left(\mathbf{B}^{q+2}\right)^*\mathbf{B}^{q+2}\right|_{\beta}^{\beta}}, \tag{31}$$

where $\mathbf{C}_1 = (c_{ij}^{(1)}) = \mathbf{\Phi}\mathbf{A}\mathbf{C}\mathbf{B}\mathbf{\Psi}$. *Here* $\mathbf{\Phi} = (\phi_{ij})$ *and* $\mathbf{\Psi} = (\psi_{ij})$ *are determined, respectively, by*

$$\phi_{it} = \sum_{\alpha\in I_{s_1,n}\{t\}} \mathrm{rdet}_t\left(\left[\mathbf{A}^{k+2}\left(\mathbf{A}^{k+2}\right)^*\right]_{t.}(\hat{\mathbf{a}}_{i.}^{(2)})\right)_{\alpha}^{\alpha}, \tag{32}$$

$$\psi_{fj} = \sum_{\beta\in J_{s_2,m}\{f\}} \mathrm{cdet}_f\left(\left[\left(\mathbf{B}^{q+2}\right)^*\mathbf{B}^{q+2}\right]_{.f}(\check{\mathbf{b}}_{.j}^{(2)})\right)_{\beta}^{\beta}, \tag{33}$$

where $\hat{\mathbf{a}}_{i.}^{(2)}$ *is the ith row of* $\hat{\mathbf{A}}_2 = \mathbf{A}^k(\mathbf{A}^{k+2})^*$ *and* $\check{\mathbf{b}}_{.j}^{(2)}$ *is the jth column of* $\check{\mathbf{B}}_2 = (\mathbf{B}^{q+2})^*\mathbf{B}^q$.

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Proof. According to (16) and determinantal representations (12) of the right WG inverse $\mathbf{A}^{ⓦ} = \left(a_{ij}^{ⓦ,r}\right)$ and (13) of the left WG inverse $\mathbf{A}_{ⓦ} = \left(a_{ij}^{ⓦ,l}\right)$, respectively, we have

$$x_{ij} = \sum_{g=1}^{n}\sum_{p=1}^{m} a_{ig}^{ⓦ,r} c_{gp} b_{pj}^{ⓦ,l} = \sum_{g=1}^{n}\sum_{p=1}^{m} \frac{\sum\limits_{t=1}^{n}\sum\limits_{\alpha\in I_{s_1,n}\{t\}} \operatorname{rdet}_t\left(\left[\mathbf{A}^{k+2}\left(\mathbf{A}^{k+2}\right)^*\right]_{t.}\left(\hat{\mathbf{a}}_{i.}^{(2)}\right)\right)_{\alpha}^{\alpha} a_{tg}c_{gp}}{\sum\limits_{\alpha\in I_{s_1,n}}\left|\mathbf{A}^{k+2}\left(\mathbf{A}^{k+2}\right)^*\right|_{\alpha}^{\alpha}}$$
$$\times \frac{\sum\limits_{f=1}^{m} b_{pf}\sum\limits_{\beta\in J_{s_2,m}\{f\}} \operatorname{cdet}_f\left(\left[\left(\mathbf{B}^{q+2}\right)^*\mathbf{B}^{q+2}\right]_{.f}\left(\check{\mathbf{b}}_{.j}^{(2)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{s_2,m}}\left|\left(\mathbf{B}^{q+2}\right)^*\mathbf{B}^{q+2}\right|_{\beta}^{\beta}},$$

where $\hat{\mathbf{a}}_{i.}^{(2)}$ is the ith row of $\hat{\mathbf{A}}_2 = \mathbf{A}^k(\mathbf{A}^{k+2})^*$ and $\check{\mathbf{b}}_{.j}^{(2)}$ is the jth column of $\check{\mathbf{B}}_2 = (\mathbf{B}^{q+2})^*\mathbf{B}^q$.

Suppose that $c_{tf} = \sum\limits_{g=1}^{n}\sum\limits_{p=1}^{m} a_{tg}c_{gp}b_{pf}$ and $\widetilde{\mathbf{C}} = (c_{tf}) = \mathbf{ACB} \in \mathbb{H}^{n\times m}$. If we construct the matrices $\mathbf{\Phi} = (\phi_{it})$ and $\mathbf{\Psi} = (\psi_{fj})$ determined by (32) and (33), respectively, then, by putting $\mathbf{C}_1 = \mathbf{\Phi}\widetilde{\mathbf{C}}\mathbf{\Psi}$, it follows (31). □

The next corollaries evidently follow from Theorem 11.

Corollary 9. *Under assumptions $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ with $\operatorname{rank}(\mathbf{A}^k) = s_1$, and $\mathbf{C} \in \mathbb{H}^{n\times m}$, the matrix $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{n\times m}$ defined in (20) is represented as*

$$x_{ij} = \frac{\tilde{c}_{ij}^{(1)}}{\sum\limits_{\alpha\in I_{s_1,n}}\left|\mathbf{A}^{k+2}\left(\mathbf{A}^{k+2}\right)^*\right|_{\alpha}^{\alpha}},$$

where $\widetilde{\mathbf{C}}_1 = (\tilde{c}_{ij}^{(1)}) = \mathbf{\Phi AC}$ and $\mathbf{\Phi}$ is determined by (32).

Corollary 10. *Assume that $\mathbf{B} \in \mathbb{H}^{(m)(q)}$ with $\operatorname{rank}(\mathbf{B}^q) = s_2$, and $\mathbf{C} \in \mathbb{H}^{n\times m}$, the matrix $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{n\times m}$ defined in (18) is represented as*

$$x_{ij} = \frac{\hat{c}_{ij}^{(1)}}{\sum\limits_{\beta\in J_{s_2,m}}\left|\left(\mathbf{B}^{q+2}\right)^*\mathbf{B}^{q+2}\right|_{\beta}^{\beta}},$$

where $\widehat{\mathbf{C}}_1 = (\hat{c}_{ij}^{(1)}) = \mathbf{CB\Psi}$, and $\mathbf{\Psi} = (\psi_{ij})$ is determined by (33).

Theorem 12. *Suppose that $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(n|m)(k|q)}$, $\operatorname{rank}(\mathbf{A}^k) = s_1$, and $\operatorname{rank}(\mathbf{B}^q) = s_2$. The unique solution $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{n\times m}$ from (19) is expressible elementwise in two possible ways.*

(1)

$$x_{ij} = \frac{\sum\limits_{f=1}^{m}\sum\limits_{\alpha\in I_{s_2,m}\{f\}} \operatorname{rdet}_f\left(\left[\mathbf{B}^{q+2}\left(\mathbf{B}^{q+2}\right)^*\right]_{f.}\left(\hat{\boldsymbol{\phi}}_{i.}\right)\right)_{\alpha}^{\alpha} b_{fj}}{\sum\limits_{\beta\in J_{s_1,n}}\left|\left(\mathbf{A}^{k+2}\right)^*\mathbf{A}^{k+2}\right|_{\beta}^{\beta}\sum\limits_{\alpha\in I_{s_2,m}}\left|\mathbf{B}^{q+2}\left(\mathbf{B}^{q+2}\right)^*\right|_{\alpha}^{\alpha}}, \tag{34}$$

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where $\hat{\boldsymbol{\phi}}_{i.}$ is the ith row of $\hat{\boldsymbol{\Phi}} = \boldsymbol{\Phi}_1 \mathbf{B}^q (\mathbf{B}^{q+2})^*$, and $\boldsymbol{\Phi}_1 = (\phi^{(1)}_{ip})$ is determined by

$$\phi^{(1)}_{ip} = \sum_{t=1}^{n} a_{it} \sum_{\beta \in J_{s_1,n}\{t\}} \operatorname{cdet}_t \left(\left[\left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right]_{.t} (\check{\mathbf{c}}_{.p}) \right)^{\beta}_{\beta}, \tag{35}$$

where $\check{\mathbf{c}}_{.p}$ is the pth column of $\check{\mathbf{C}} = (\mathbf{A}^{k+2})^* \mathbf{A}^k \mathbf{C}$.

(2)

$$x_{ij} = \frac{\sum\limits_{t=1}^{n} a_{it} \sum\limits_{\beta \in J_{s_1,n}\{t\}} \operatorname{cdet}_t \left(\left[\left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right]_{.t} (\check{\boldsymbol{\psi}}_{.j}) \right)^{\beta}_{\beta}}{\sum\limits_{\beta \in J_{s_1,n}} \left| \left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right|^{\beta}_{\beta} \sum\limits_{\alpha \in I_{s_2,m}} \left| \mathbf{B}^{q+2} \left(\mathbf{B}^{q+2} \right)^* \right|^{\alpha}_{\alpha}}, \tag{36}$$

where $\check{\boldsymbol{\psi}}_{.j}$ is the jth column of $\check{\boldsymbol{\Psi}} = (\mathbf{A}^{k+2})^* \mathbf{A}^k \boldsymbol{\Psi}_1$, and $\boldsymbol{\Psi}_1 = (\phi^{(1)}_{gj})$ is determined by

$$\psi^{(1)}_{gj} = \sum_{f=1}^{m} \sum_{\alpha \in I_{s_2,m}\{f\}} \operatorname{rdet}_f \left(\left[\mathbf{B}^{q+2} \left(\mathbf{B}^{q+2} \right)^* \right]_{f.} (\hat{\mathbf{c}}_{g.}) \right)^{\alpha}_{\alpha} b_{fj}, \tag{37}$$

where $\hat{\mathbf{c}}_{g.}$ is the gth row of $\hat{\mathbf{C}} = \mathbf{C}\mathbf{B}^q (\mathbf{B}^{q+2})^*$.

Proof. According to (19) and determinantal representations (13) and (12) of the left WG inverse $\mathbf{A}_{ⓦ} = \left(a^{ⓦ,l}_{ig} \right)$ and the right WG inverse $\mathbf{B}^{ⓦ} = \left(b^{ⓦ,r}_{pj} \right)$, respectively, we have

$$x_{ij} = \sum_{g=1}^{n} \sum_{p=1}^{m} a^{ⓦ,l}_{ig} c_{gp} b^{ⓦ,r}_{pj} = \sum_{g=1}^{n} \sum_{p=1}^{m} \frac{\sum\limits_{t=1}^{n} a_{it} \sum\limits_{\beta \in J_{s_1,n}\{t\}} \operatorname{cdet}_t \left(\left[\left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right]_{.t} (\check{\mathbf{a}}^{(2)}_{.g}) \right)^{\beta}_{\beta} c_{gp}}{\sum\limits_{\beta \in J_{s_1,n}} \left| \left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right|^{\beta}_{\beta}}$$

$$\times \frac{\sum\limits_{f=1}^{m} \sum\limits_{\alpha \in I_{s_2,m}\{f\}} \operatorname{rdet}_f \left(\left[\mathbf{B}^{q+2} \left(\mathbf{B}^{q+2} \right)^* \right]_{f.} (\hat{\mathbf{b}}^{(2)}_{p.}) \right)^{\alpha}_{\alpha} b_{fj}}{\sum\limits_{\alpha \in I_{s_2,m}} \left| \mathbf{B}^{q+2} \left(\mathbf{B}^{q+2} \right)^* \right|^{\alpha}_{\alpha}}, \tag{38}$$

where $\check{\mathbf{a}}^{(2)}_{.g}$ is the gth column of $\check{\mathbf{A}}_2 = (\mathbf{A}^{k+2})^* \mathbf{A}^k$ and $\hat{\mathbf{b}}^{(2)}_{p.}$ is the pth row of $\hat{\mathbf{B}}_2 = \mathbf{B}^q (\mathbf{B}^{q+2})^*$.

To obtain expressive formulas, we make some convolutions of (38).

Denote $\check{\mathbf{C}} = (\check{c}_{ij}) = \check{\mathbf{A}}_2 \mathbf{C}$. Then

$$\sum_{g=1}^{n} \sum_{\beta \in J_{s_1,n}\{t\}} \operatorname{cdet}_t \left(\left[\left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right]_{.t} (\check{\mathbf{a}}^{(2)}_{.g}) \right)^{\beta}_{\beta} c_{gp} =$$
$$= \sum_{\beta \in J_{s_1,n}\{t\}} \operatorname{cdet}_t \left(\left[\left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right]_{.t} (\check{\mathbf{c}}_{.p}) \right)^{\beta}_{\beta}.$$

Further, put

$$\phi^{(1)}_{ip} = \sum_{t=1}^{n} a_{it} \sum_{\beta \in J_{s_1,n}\{t\}} \operatorname{cdet}_t \left(\left[\left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right]_{.t} (\check{\mathbf{c}}_{.p}) \right)^{\beta}_{\beta}$$

and construct the matrix $\mathbf{\Phi}_1 = (\phi_{ip}^{(1)}) \in \mathbb{H}^{n\times m}$. Finally, Equation (34) follows from the setting

$$\sum_{p=1}^{m} \phi_{ip}^{(1)} \hat{\mathbf{b}}_{p.}^{(2)} =: \hat{\boldsymbol{\phi}}_{i.}$$

Now, denote $\hat{\mathbf{C}} = (\hat{c}_{ij}) = \mathbf{C}\hat{\mathbf{B}}_2$. Then

$$\sum_{p=1}^{m} c_{gp} \sum_{\alpha\in I_{s_2,m}\{f\}} \mathrm{rdet}_f \left(\left[\mathbf{B}^{q+2} \left(\mathbf{B}^{q+2} \right)^* \right]_{f.} (\hat{\mathbf{b}}_{p.}^{(2)}) \right)_{\alpha}^{\alpha} =$$
$$= \sum_{\alpha\in I_{s_2,m}\{f\}} \mathrm{rdet}_f \left(\left[\mathbf{B}^{q+2} \left(\mathbf{B}^{q+2} \right)^* \right]_{f.} (\hat{\mathbf{c}}_{g.}) \right)_{\alpha}^{\alpha}.$$

We put

$$\psi_{gj}^{(1)} = \sum_{f=1}^{m} \sum_{\alpha\in I_{s_2,m}\{f\}} \mathrm{rdet}_f \left(\left[\mathbf{B}^{q+2} \left(\mathbf{B}^{q+2} \right)^* \right]_{f.} (\hat{\mathbf{c}}_{g.}) \right)_{\alpha}^{\alpha} b_{fj}.$$

to determine the matrix $\mathbf{\Psi}_1 = (\psi_{gj}^{(1)})$. Then, Equation (36) holds because of $\sum_{g=1}^{n} \check{\mathbf{a}}_{.g}^{(2)} \psi_{gj}^{(1)} = \check{\boldsymbol{\psi}}_{.j}$. $\square$

We possess the next results when, respectively, $\mathbf{A} = \mathbf{I}_n$ or $\mathbf{B} = \mathbf{I}_m$ in Theorem 12.

Corollary 11. *Suppose that $\mathbf{A} \in \mathbb{H}^{(n)(k)}$ with $\mathrm{rank}(\mathbf{A}^k) = s_1$, and $\mathbf{C} \in \mathbb{H}^{n\times m}$. The matrix $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{n\times m}$ defined in (17) can be represented as*

$$x_{ij} = \frac{\sum_{t=1}^{n} a_{it} \sum_{\beta\in J_{s_1,n}\{t\}} \mathrm{cdet}_t \left(\left[\left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right]_{.t} (\check{c}_{.j}) \right)_{\beta}^{\beta}}{\sum_{\beta\in J_{s_1,n}} \left| \left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right|_{\beta}^{\beta}},$$

where $\check{c}_{.j}$ is the jth column of $\check{\mathbf{C}} = (\mathbf{A}^{k+2})^* \mathbf{A}^k \mathbf{C}$.

Corollary 12. *Suppose that $\mathbf{B} \in \mathbb{H}^{(m)(q)}$ with $\mathrm{rank}(\mathbf{B}^q) = s_2$, and $\mathbf{C} \in \mathbb{H}^{n\times m}$. The matrix $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{n\times m}$ defined in (18) is represented as*

$$x_{ij} = \frac{\sum_{f=1}^{m} \sum_{\alpha\in I_{s_2,m}\{f\}} \mathrm{rdet}_f \left(\left[\mathbf{B}^{q+2} (\mathbf{B}^{q+2})^* \right]_{f.} (\hat{\mathbf{c}}_{i.}) \right)_{\alpha}^{\alpha} b_{fj}}{\sum_{\beta\in J_{s_1,n}} \left| \left(\mathbf{A}^{k+2} \right)^* \mathbf{A}^{k+2} \right|_{\beta}^{\beta} \sum_{\alpha\in I_{s_2,m}} \left| \mathbf{B}^{q+2} (\mathbf{B}^{q+2})^* \right|_{\alpha}^{\alpha}},$$

where $\hat{\mathbf{c}}_{i.}$ is the gth row of $\hat{\mathbf{C}} = \mathbf{C}\mathbf{B}^q(\mathbf{B}^{q+2})^*$.

The rest two theorems can be proved similar to the proofs of Theorems 13 and 14.

Theorem 13. *Let $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(n|m)(k|q)}$, $\mathrm{rank}(\mathbf{A}^k) = s_1$, and $\mathrm{rank}(\mathbf{B}^q) = s_2$. The unique solution $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{n\times m}$ from (22) can be expressed componentwise by*

$$x_{ij} = \frac{\sum_{f=1}^{m} \sum_{\alpha\in I_{s_2,m}\{f\}} \mathrm{rdet}_f \left(\left[\mathbf{B}^{q+2} (\mathbf{B}^{q+2})^* \right]_{f.} (\boldsymbol{\phi}_{i.}^{(2)}) \right)_{\alpha}^{\alpha} b_{fj}}{\sum_{\alpha\in I_{s_1,n}} \left| \mathbf{A}^{k+2} (\mathbf{A}^{k+2})^* \right|_{\alpha}^{\alpha} \sum_{\alpha\in I_{s_2,m}} \left| \mathbf{B}^{q+2} (\mathbf{B}^{q+2})^* \right|_{\alpha}^{\alpha}}, \quad (39)$$

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where $\boldsymbol{\phi}_{i.}^{(2)}$ *is the ith row of* $\boldsymbol{\Phi}_2 = (\phi_{ij}^{(2)}) = \boldsymbol{\Phi}\mathbf{A}\mathbf{C}\mathbf{B}^q\left(\mathbf{B}^{q+2}\right)^*$, *where* $\boldsymbol{\Phi}$ *is determined by* (32).

Theorem 14. *Let* $(\mathbf{A}|\mathbf{B}) \in \mathbb{H}^{(n|m)(k|q)}$, $\operatorname{rank}(\mathbf{A}^k) = s_1$, *and* $\operatorname{rank}(\mathbf{B}^q) = s_2$. *The unique solution* $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{n\times m}$ *from* (23) *can be expressed componentwise by*

$$x_{ij} = \frac{\sum\limits_{t=1}^{n} a_{it} \sum\limits_{\beta\in J_{s_1,n}\{t\}} \operatorname{cdet}_t\left(\left(\left[\left(\mathbf{A}^{k+2}\right)^* \mathbf{A}^{k+2}\right]_{.t}\left(\boldsymbol{\psi}_{.j}^{(2)}\right)\right)_{\beta}^{\beta}\right)}{\sum\limits_{\beta\in J_{s_1,n}} \left|\left(\mathbf{A}^{k+2}\right)^* \mathbf{A}^{k+2}\right|_{\beta}^{\beta} \sum\limits_{\beta\in J_{s_2,m}} \left|\left(\mathbf{B}^{q+2}\right)^* \mathbf{B}^{q+2}\right|_{\beta}^{\beta}}, \tag{40}$$

where $\boldsymbol{\psi}_{.j}^{(2)}$ *is the jth column of* $\boldsymbol{\Psi}_2 = (\mathbf{A}^{k+2})^*\mathbf{A}^k\mathbf{C}\mathbf{B}\boldsymbol{\Psi}$ *and* $\boldsymbol{\Psi}$ *is determined by* (33).

7. An Illustrative Example

Let's use corresponding Cramer's rules to find the WG-dual-WG solutions to CQMEs with the given matrices

$$\mathbf{A} = \begin{bmatrix} -\mathbf{k} & -\mathbf{j} & 0 & \mathbf{i} \\ -1-\mathbf{j} & \mathbf{i}+\mathbf{k} & \mathbf{j} & 1+\mathbf{j} \\ \mathbf{k} & 0 & \mathbf{i} & 0 \\ -\mathbf{i}+\mathbf{k} & 1-\mathbf{j} & \mathbf{i} & \mathbf{i}-\mathbf{k} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4\mathbf{k} & 4\mathbf{j} & -5\mathbf{i} \\ -2\mathbf{j} & 2\mathbf{k} & 3 \\ \mathbf{i} & -1 & \mathbf{k} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} -\mathbf{i} & 0 & -1 \\ 0 & -\mathbf{k} & 0 \\ \mathbf{k} & 0 & \mathbf{j} \\ 0 & -\mathbf{j} & 0 \end{bmatrix}.$$

We have $\operatorname{rank}(\mathbf{A}) = 3$, $\operatorname{rank}(\mathbf{A}^3) = \operatorname{rank}(\mathbf{A}^2) = 2$, $\operatorname{rank}(\mathbf{B}) = 2$, $\operatorname{rank}(\mathbf{B}^2) = \operatorname{rank}(\mathbf{B}^3) = 1$. So, $k = \operatorname{Ind}(\mathbf{A}) = 2$ and $q = \operatorname{Ind}(\mathbf{B}) = 2$. In view of Theorem 11, the Cramer's rule to the solution (31) is computed as follows.

1. Compute $\boldsymbol{\Phi}$ by (32) and $\boldsymbol{\Psi}$ by (33).

$$\boldsymbol{\Phi} = \begin{bmatrix} -15 & -5\mathbf{i} & -10\mathbf{j} & 5\mathbf{j} \\ -10\mathbf{i}-15\mathbf{k} & -5-5\mathbf{j} & 10\mathbf{i}-15\mathbf{k} & -5\mathbf{i}-5\mathbf{k} \\ -15+25\mathbf{j} & -5\mathbf{i} & -25-10\mathbf{j} & 5\mathbf{j} \\ -15+10\mathbf{j} & -5\mathbf{i}+5\mathbf{k} & -15-10\mathbf{j} & 5+5\mathbf{j} \end{bmatrix}, \ \boldsymbol{\Psi} = \begin{bmatrix} -11 & -11\mathbf{i} & -11\mathbf{j} \\ 11\mathbf{i} & -11 & -11\mathbf{k} \\ 11\mathbf{j} & -11\mathbf{k} & -11 \end{bmatrix}.$$

2. Taking into account that $\sum\limits_{\alpha\in I_{2,4}} \left|\mathbf{A}^4\left(\mathbf{A}^4\right)^*\right|_{\alpha}^{\alpha} = 25$ and $\sum\limits_{\beta\in I_{1,3}} \left|\left(\mathbf{B}^4\right)^*\mathbf{B}^4\right|_{\beta}^{\beta} = 33$, from (31) it follows

$$\mathbf{X} = \frac{1}{15}\begin{bmatrix} -7-5\mathbf{i}+2\mathbf{j} & 5-7\mathbf{i}-2\mathbf{k} & -2-7\mathbf{j}-5\mathbf{k} \\ -5\mathbf{j}-9\mathbf{k} & -9\mathbf{j}+5\mathbf{k} & 5+9\mathbf{i} \\ -7-5\mathbf{i}+7\mathbf{j}-5\mathbf{k} & 5-7\mathbf{i}-5\mathbf{j}-7\mathbf{k} & -7+5\mathbf{i}-7\mathbf{j}-5\mathbf{k} \\ -9-5\mathbf{i} & 5-9\mathbf{i} & -9\mathbf{j}-5\mathbf{k} \end{bmatrix}.$$

Due to Theorem 12, the Cramer rule to the solution (19) can be founded as follows.

1. Compute $\boldsymbol{\Phi}_1$ by (35).

$$\boldsymbol{\Phi}_1 = \begin{bmatrix} 25\mathbf{j} & -25\mathbf{i}+25\mathbf{k} & -25\mathbf{k} \\ -55\mathbf{i}+10\mathbf{k} & -45-65\mathbf{j} & 55+10\mathbf{j} \\ 25+25\mathbf{j} & -50\mathbf{i} & 25\mathbf{i}-25\mathbf{k} \\ 10+55\mathbf{j} & -65\mathbf{i}+45\mathbf{k} & 10\mathbf{i}-55\mathbf{k} \end{bmatrix}.$$

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Then,

$$\hat{\mathbf{\Phi}} = \mathbf{\Phi}_1 \mathbf{B}^q (\mathbf{B}^{q+2})^* =$$
$$= \begin{bmatrix} -225 - 225\mathbf{i} - 900\mathbf{j} & -75 + 75\mathbf{i} - 300\mathbf{k} & -300 + 75\mathbf{j} + 75\mathbf{k} \\ 90 + 1890\mathbf{i} - 495\mathbf{j} - 855\mathbf{k} & 630 - 30\mathbf{i} + 285\mathbf{j} - 165\mathbf{k} & -165 - 285\mathbf{i} - 30\mathbf{j} - 630\mathbf{k} \\ -1125 - 225\mathbf{i} - 675\mathbf{j} - 225\mathbf{k} & -75 + 375\mathbf{i} + 75\mathbf{j} - 225\mathbf{k} & -225 - 75\mathbf{i} + 375\mathbf{j} + 75\mathbf{k} \\ -885 - 495\mathbf{i} - 1890\mathbf{j} - 90\mathbf{k} & -165 + 285\mathbf{i} + 30\mathbf{j} - 630\mathbf{k} & -630 - 30\mathbf{i} + 285\mathbf{j} + 165\mathbf{k} \end{bmatrix},$$

and finally by (34),

$$\mathbf{X} = \frac{1}{11}\begin{bmatrix} -60\mathbf{i} + 15\mathbf{j} + 6\mathbf{k} & 60 - 15\mathbf{j} - 15\mathbf{k} & -19 + 19\mathbf{i} - 76\mathbf{k} \\ 57 - 33\mathbf{i} - 126\mathbf{j} + 6\mathbf{k} & 33 + 57\mathbf{i} + 6\mathbf{j} + 126\mathbf{k} & 159.6 - 7.6\mathbf{i} + 72.2\mathbf{j} - 41.8\mathbf{k} \\ 15 - 45\mathbf{i} + 15\mathbf{j} - 75\mathbf{k} & 45 + 15\mathbf{i} - 75\mathbf{j} - 15\mathbf{k} & -19 + 95\mathbf{i} + 19\mathbf{j} - 75\mathbf{k} \\ 6 - 126\mathbf{i} + 33\mathbf{j} - 57\mathbf{k} & 126 + 6\mathbf{i} - 57\mathbf{j} - 33\mathbf{k} & -41.8 + 72.2\mathbf{i} + 7.6\mathbf{j} - 159.6\mathbf{k} \end{bmatrix}.$$

Similarly, based on Theorem 13, Cramer's rule applied to the solution (39) yields

$$\mathbf{X} = \frac{1}{11}\begin{bmatrix} -54\mathbf{i} + 15\mathbf{j} - 21\mathbf{k} & 54 - 21\mathbf{j} - 15\mathbf{k} & -19 + 26.6\mathbf{i} - 68.4\mathbf{k} \\ 27 - 15\mathbf{i} - 60\mathbf{j} & 15 + 27\mathbf{i} + 60\mathbf{k} & 76 + 34.2\mathbf{j} - 19\mathbf{k} \\ 15 - 39\mathbf{i} + 15\mathbf{j} - 81\mathbf{k} & 39 + 15\mathbf{i} - 81\mathbf{j} - 15\mathbf{k} & -19 + 102.6\mathbf{i} + 19\mathbf{j} - 49.4\mathbf{k} \\ -60\mathbf{i} + 15\mathbf{j} - 27\mathbf{k} & 60 - 27\mathbf{j} - 15\mathbf{k} & -19 + 34.2\mathbf{i} - 76\mathbf{k} \end{bmatrix}.$$

By utilizing Cramer's rule from Theorem 14, the solution (40) is

$$\mathbf{X} = \frac{1}{15}\begin{bmatrix} -100\mathbf{i} + 15\mathbf{j} - 35\mathbf{k} & 100 - 35\mathbf{j} - 15\mathbf{k} & -15 + 35\mathbf{i} - 100\mathbf{k} \\ 117 - 33\mathbf{i} - 206\mathbf{j} + 6\mathbf{k} & 33 + 117\mathbf{i} + 6\mathbf{j} + 206\mathbf{k} & 206 - 6\mathbf{i} + 117\mathbf{j} - 33\mathbf{k} \\ 15 - 65\mathbf{i} + 15\mathbf{j} - 135\mathbf{k} & 65 + 15\mathbf{i} - 135\mathbf{j} - 15\mathbf{k} & -15 + 135\mathbf{i} + 15\mathbf{j} - 65\mathbf{k} \\ 6 - 206\mathbf{i} + 33\mathbf{j} - 117\mathbf{k} & 206 + 6\mathbf{i} - 117\mathbf{j} - 33\mathbf{k} & -33 + 117\mathbf{i} + 6\mathbf{j} - 206\mathbf{k} \end{bmatrix}.$$

8. Conclusions

Initial goals in current research are related to characterizations and expressions of the weak group (WG) inverse and its dual over the quaternion skew field. We introduce a dual to the weak group inverse for the first time in literature and give some new characterizations for both the WG inverse and its dual, named as the right and left weak group inverses for quaternion matrices, respectively. In particular, determinantal representations of the right and left WG inverses are given as direct methods for their constructions. Our other results are related to solving the quaternion restricted two-sided matrix equation $\mathbf{AXB} = \mathbf{C}$ and the according approximation problem that could be expressed in terms of the right and left WG inverse solutions. Within the framework of the theory of noncommutative row-column determinants, we derive Cramer's rules for computing these solutions based on determinantal representations of the right and left WG inverses. A numerical example is given to illustrate the gained results.

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