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On Superization of Nonlinear Integrable Dynamical Systems

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Abstract: We study an interesting superization problem of integrable nonlinear dynamical systems on functional manifolds. As an example, we considered a quantum many-particle Schrödinger-Davydov model on the axis, whose quasi-classical reduction proved to be a completely integrable Hamiltonian system on a smooth functional manifold. We checked that so called "naive" approach, based on the superization of the related phase space variables via extending the corresponding Poisson brackets upon the related functional supermanifold, fails to retain the dynamical system super-integrability. Moreover, we have demonstrated that there exists a wide class of classical Lax type integrable nonlinear dynamical systems on axis regarding which a superization scheme consists in a reasonable superization of the related Lax type representation by means of passing from the basic algebra of pseudo-differential operators on the axis to the corresponding superalgebra of super-pseudodifferential operators on the superaxis.

Keywords: supersymmetry; super-differentiation; Lie superalgebra, algebra of pseudo-differential operators; coadjoint action; Lax integrability; Lie-algebraic approach; gradient-holonomic scheme; Casimir invariants; super-Poisson structure

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1. Introduction

Main modern field theoretic string theories of fundamental interactions are essentially grounded [19,20,36] on supersymmetric extensions both of the space-time variables and canonical field variables, making possible to construct governing evolution systems free of singularities and nonphysical peculiarities. As often from the very beginning there are considered field equations on usual classical phase spaces, an important problem of constructing their corresponding supersymmetric extensions [20,32,33] arises and which during past decades has been solving by means of various mathematical tools and approaches. In particular, within the two-dimensional completely integrable field theories, like Sin-Gordon, Thirring, Nonlinear Schödinger, Born-Infeld and others their supersymmetric integrable extensions were constructed by means of natural supersymmetric generalizations either of physically motivated reasonings [8,15,16,23,29–31,34,35,39–41,58] about the system evolution regarding the energy interaction Hamiltonian structure or the related hidden supersymmetry Lie algebraic structure [1,8,12,24,25,38,43,45,47–50,56,59] responsible for their complete integrability. Being interested in more detailed analysis of these superization schemes, we considered a physically motivated [11,52,55] spatially one-dimensional quantum interacting manyparticle model, described by the Hamiltonian operator

$$H_N = -\sum_{j=\overline{1,N}} \frac{\partial^2}{\partial x_j^2} + 2\sum_{j=\overline{1,N}} n(x_j), \tag{1}$$

of $N \in \mathbb{N}$ charged bose-particles, specified by the position dependent intensities $n(x_j) \in \mathbb{R}$ at points $x_j \in \mathbb{R}$, $j = \overline{1,N}$, and acting on the Hilbert space $L_2(\mathbb{R}^N;\mathbb{C})$ of the corresponding quantum states. In

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case of a medium with the infinite number of particles the Hamiltonian operator should be naturally considered [5,6,13] within the secondary Fock representation

$$\hat{H} = \int_{\mathbb{R}} dx[\psi_x^+(x)\psi_x(x) + 2n(x)\psi^+(x)\psi(x)], \tag{2}$$

acting already on the tensor product Fock space $\Phi \otimes \Theta$, generated by a vacuum state $|0) \in \Phi \otimes \Theta$ and, in part, creation-annihilitation operators $\psi^+(x), \psi(x) : \Phi \to \Phi$ respectively, satisfying the following canonical operator commutation brackets:

$$\begin{aligned} [\psi(x), \psi^+(y)] &= \delta(x-y), \\ [\psi(x), \psi(y)] &= 0 = [\psi^+(x), \psi^+(y)] \end{aligned} \tag{3}$$

supplemented with the operator commutation brackets

$$\begin{aligned} [\psi(x), n(y)] &= 0 = [n(x), \psi^+(y)], \\ [n(x), n(y)] &= \partial\delta(x-y)/\partial x \end{aligned} \tag{4}$$

at arbitrary points $x, y \in \mathbb{R}$ for the intensity operator $n(x) : \Theta \to \Theta$, describing the simplest self-interacting quantum medium, whose quantum states are modeled by the related Fock space Θ. The corresponding Heisenberg evolution in time $t \in \mathbb{R}$ equations [5,6,13] for the dynamical operator varibales $\psi(x), \psi^+(s)$ and $n(x) : \Phi \otimes \Theta \to \Phi \otimes \Theta$ read as

$$\begin{aligned} \partial\psi/\partial t &= \frac{1}{i}[\hat{H}, \psi] = i\psi_{xx} - 2n\psi, \\ \partial n/\partial t &= \frac{1}{i}[\hat{H}, n] = -2n(\psi\,\psi^+)_x, \\ \partial\psi^+/\partial t &= \frac{1}{i}[\hat{H}, \psi^+] = -i\psi_{xx}^+ + 2n\psi^+, \end{aligned} \tag{5}$$

and were before intensively studied in [11,55] as a dynamical model for describing the mechanism of muscle contraction in living tissue. The obtained system of operator Schrödinger-Davydov type equations (5) allows the following quasi-classical Hamiltonian form

$$\begin{aligned} \partial\psi/\partial t &= \{H, \psi\}_P = i\psi_{xx} - 2n\psi, \\ \partial n/\partial t &= \{H, n\}_P = -2n(\psi\,\psi^*)_x, \\ \partial\psi^+/\partial t &= \{H, \psi^+\}_P = -i\psi_{xx}^* + 2n\psi^* \end{aligned} \tag{6}$$

endowed with the following quasi-classical Poisson brackets

$$\begin{aligned} \{\psi(x), \psi^*(y)\}_P = \delta(x-y), \{\psi(x), n(y)\}_P = 0 = \{n(x), \psi^*(y)\}_P, \\ \{\psi(x), \psi(y)\}_P = 0 = \{\psi^*(x), \psi^*(y)\}_P, \{n(x), n(y)\}_P = \partial\delta(x-y)/\partial x \end{aligned} \tag{7}$$

at any points $x, y \in \mathbb{R}$ on a smooth functional manifold $M \subset \{(\psi, n, \psi^*) \in C^2(\mathbb{R}; \mathbb{C} \times \mathbb{R} \times \mathbb{C})\}$, easily following from (3) and (4) within the classical Dirac's correspondence [13] principle.

As it was stated in [7,42,52], the derived there naturallySCHEME related to the system (7) hydrodynamic and Boltzmann-Vlasov type kinetic equations proved to be completely integrable Hamiltonian systems. Moreover, as it will be demonstrated below, the derived above nonlinear quasiclassical Schrödinger-Davydov type system (6) proves to be also a completely integrable [42,57] bi-Hamiltonian flow on the functional manifold M and whose possible superization schemes are analyzed in detail in our work under regard.

2. Quasi-Classical Integrability and a Simple Superization Scheme

Let us begin with analyzing the integrability of the derived above quasi-classical Schrödinger-Davydov type nonlinear dynamical system

$$\left.\begin{array}{r}\partial\psi/\partial t = i\psi_{xx} - 2n\psi, \\ \partial n/\partial t = -2n(\psi\,\psi^*)_x \\ \partial\psi^*/\partial t = -i\psi^*_{xx} + 2n\psi^*\end{array}\right\} := K[\psi, n, \psi^*], \tag{8}$$

with respect to the evolution parameter $t \in \mathbb{R}$, considered as a smooth vector field $K : M \to T(M)$ on the functional manifold M, via making use of the gradient-holonomic scheme, devised in [3,42,54]. As a first step we need to demonstrate the existence of an infinite hierarchy of conservation laws and to state their commuting to each other with respect the Poisson bracket (7), presented above. Namely, for any smooth functionals $\gamma, \mu \in \mathcal{D}(M)$ their Poisson bracket is calculated via the expression

$$\{\gamma, \mu\}_P = (\mathrm{grad}\gamma | P\mathrm{grad}\mu), \tag{9}$$

where grad : $\mathcal{D}(M) \to T^*(M)$ denotes the Gateau derivative with respect to the usual bilinear form $(\cdot|\cdot) : T^*(M) \times T(M) \to \mathbb{C}$ and the Poisson operator $P : T^*(M) \to T(M)$ is skew-symmetric, satisfies the following weak functional relationship:

$$\{(\psi(x), n(x), \psi^*(x))^\intercal, (\psi(y), n(y), \psi^*(y))\}_P = P\delta(x-y) \tag{10}$$

for any $x, y \in \mathbb{R}$, $\delta(x-y)$ - the classical generalized Dirac delta-function, acting on an arbitrary continuous function $f \in C(\mathbb{R};\mathbb{C})$ via the symbolic integral operation $f(x) := \int_{\mathbb{R}} \delta(x-y) f(y) dy$, satisfied for all $x \in \mathbb{R}$. To calculate the infinite hierarchy of conservation laws for the vector field (8) it is enough to study special solutions to the governing linear Noether-Lax equation

$$\varphi_t + K'^{,*}\varphi = 0, \tag{11}$$

where $K'^{,*} : T^*(M) \to T^*(M)$ denotes the adjoint to the Frechet derivative operator $K' : T(M) \to T(M)$ of the vector field (8) and a covector $\varphi \in T^*(M)$ can be chosen as

$$\varphi = (1, a, b)^\intercal \exp(-i\lambda^2 t + \partial^{-1}\sigma(x;\lambda)), \;\; \partial/\partial x \cdot \partial^{-1} = 1, \tag{12}$$

and the expressions

$$\begin{aligned}\sigma(x;\lambda) &\sim \sum_{j\in\mathbb{Z}_+\cup\{-2,-1\}} \sigma_j[\psi, n, \psi^*]\lambda^{-j}, a(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} a_j[\psi, n, \psi^*]\lambda^{-j}, \\ b(x;\lambda) &\sim \sum_{j\in\mathbb{Z}_+} b_j[\psi, n, \psi^*]\lambda^{-j},\end{aligned} \tag{13}$$

are considered to be asymptotical with respect to an arbitrary complex parameter $\mathbb{C} \ni \lambda \to \infty$. Taking into account that

$$K'^{,*} = \begin{pmatrix} i\partial^2 - 2in & 2\psi^*\partial & 2in \\ -2i\psi & 0 & 2i\psi^* \\ 0 & 2\psi & 2in - i\partial^2 \end{pmatrix}, \tag{14}$$

one easily obtains a system of recurrent differential-algebraic relationships, giving rise to the following functional expressions:

$$\begin{aligned}&\sigma_{-2} = -i, \sigma_{-1} = 1, \sigma_0 = \frac{1}{2}n, \sigma_1 = \psi^*\psi, \sigma_2 = \frac{1}{2}[n^2 - i(\psi^*\psi_x - \psi\psi^*_x)], \sigma_3 = \psi^*_x\psi_x + 2n\psi^*\psi, \\ &\sigma_4 = \frac{1}{2}n^2 - 6(\psi^*\psi)^2 - \frac{1}{2}n_x^2 + 6in(\psi^*\psi_x - \psi\psi^*_x) - 2i(\psi^*\psi_{3x} - \psi\psi^*_{3x}), ...,\end{aligned} \tag{15}$$

and so on. Since, owing to the representation (12), the quantity $\gamma(\lambda) := \int_{\mathbb{R}} dx\sigma(x;\lambda)$ is conservative with respect to the evolution parameter $t \in \mathbb{R}$ for all $\lambda \in \mathbb{C}$, we find that all functionals

$$H_0 = \frac{1}{2}\int_{\mathbb{R}} ndx, H_1 = \int_{\mathbb{R}} \psi^*\psi dx, H_2 = 1/2\int_{\mathbb{R}}[n^2 - i(\psi^*\psi_x - \psi\psi_x^*)]dx, H_3 = \int_{\mathbb{R}}(\psi_x^*\psi_x + 2n\psi^*\psi)dx, \tag{16}$$

$$H_4 = \frac{1}{2}\int_{\mathbb{R}}[n^2 - 12(\psi^*\psi)^2 - n_x^2 + 12in(\psi^*\psi_x - \psi\psi_x^*) - 4i(\psi^*\psi_{3x} - \psi\psi_{3x}^*)]dx, ...$$

ar also conservative. To confirm now that the vector field (8) on the functional manifold M is Hamiltonian, it is enough within the gradient-holonomic scheme [3] to show that the respectively constructed conservation law

$$H_p = (\xi_p|(\psi_x, n_x, \psi_x^*)^\intercal) \tag{17}$$

for some suitably chosen $p \in \mathbb{N}$ generates the Poisson operator $P : T^*(M) \to T(M)$ for the flow (8) as

$$P = \left(\xi_p^{\prime,*} - \xi_p^\prime\right)^{-1}. \tag{18}$$

For the case $p = 2$ one obtains that

$$\begin{array}{ll} H_2 = 1/2\int_{\mathbb{R}}[n^2 - i(\psi^*\psi_x - \psi\psi_x^*)]dx = & ((-i\psi^*, -\partial^{-1}n, i\psi)^\intercal|(\psi_x, n_x, \psi_x^*)^\intercal) = \\ = (\xi_2|(\psi_x, n_x, \psi_x^*)^\intercal), & \xi_2 := (-i\psi^*, -\partial^{-1}n, i\psi)^\intercal, \end{array} \tag{19}$$

ensuing the following Poisson operator

$$P = (\xi_2^{\prime,*} - \xi_2^\prime)^{-1} = \begin{pmatrix} 0 & 0 & i \\ 0 & \partial & 0 \\ -i & 0 & 0 \end{pmatrix}. \tag{20}$$

Doing a similar way, as above, for the case $p = 4$ one derives the second Poisson operator

$$Q = (\xi_4^{\prime,*} - \xi_4^\prime)^{-1} = \begin{pmatrix} -12\psi\partial^{-1}\psi & 4\partial\psi + 2\psi\partial & 12\psi\partial^{-1}\psi^* - 4i\partial^2 + 8in \\ 4\psi\partial + 2\partial\psi & -\partial^3 + 4n\partial + 4\partial n & 4\psi^*\partial + 2\partial\psi^* \\ -i & 4\partial\psi^* + 2\psi^*\partial & -12\psi^*\partial\psi \end{pmatrix}, \tag{21}$$

where, by definition, $H_4 = (\xi_4|(\psi_x, n_x, \psi_x^*)^\intercal)$. Moreover, one can check that the following recurrent relationships

$$Q\text{grad}H_j = 2P\text{grad}H_{j+2} \tag{22}$$

hold for all $j \in \mathbb{Z}_+$, meaning that the Poisson operators (20) and (21) are compatible, that is the affine sum $\lambda P + Q : T^*(M) \to T(M)$ is also a Posson operator for all $\lambda \in \mathbb{C}$. The latter makes it possible to state that the infinite hierarchy of conservation laws (16) is commuting to each other with respect to the both Poisson brackets

$$\{H_j, H_k\}_P = 0 = \{H_j, H_k\}_Q \tag{23}$$

for all $j, k \in \mathbb{Z}_+$. Since our dynmical system (8) allows the Hamiltonian representation

$$(\psi_t, n_t, \psi_t^*)^\intercal = \{H_3, (\psi, n, \psi^*)^\intercal)\}_P = -P\text{grad } H_1[\psi, n, \psi^*], \tag{24}$$

coinciding with that (6), we can formulate our first proposition.

Proposition 1. *The nonlinear Schrödinger-Davydov dynamical system (8) possesses an infinite hierarchy of commuting to each other c onservation laws (16) and is an integrable bi-Hamiltonian flow on the functional manifold M.*

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Remark 1. *Since there holds the representation $(\psi_t, n_t, \psi_t^*)^\intercal = -Q grad\ H_1[\psi, n, \psi^*]$, one sates that the dynamical system (8) is bi-Hamiltonian with respect to the both Poisson structures (20) and (21) on the functional manifold M.*

Recall now the Poisson brackets (7) on the functional manifold M

$$\begin{gathered}\{\psi(x), \psi^*(y)\}_P = i\delta(x-y), \{\psi(x), n(y)\}_P = 0 = \{n(x), \psi^*(y)\}_P, \\ \{\psi(x), \psi(y)\}_P = 0 = \{\psi^*(x), \psi^*(y)\}_P, \{n(x), n(y)\}_P = \partial\delta(x-y)/\partial x\end{gathered} \tag{25}$$

at all points $x, y \in \mathbb{R}$ and observe that they are canonically ultra-local [6,14] except the field variable $n \in M$, depending on the delta-function derivative. The latter, in particular, means that this variable can not be secondly quantized on some suitably chosen Fock space Θ. Nonetheless, this quantization can be performed, if to superize the functional manifold M by means of the following scheme: $(\psi, n, \psi^*) \ni M \to (\tilde{\psi}, \tilde{n}, \tilde{\psi}^*) \simeq (\tilde{\psi}, \tilde{u}, \tilde{\psi}^*) \in \tilde{M} \in C^2(\mathbb{R}^{1|1}; \Lambda_0 \times \Lambda_1 \times \Lambda_0^*)$, where $\mathbb{R}^{1|1} := (x, \theta) \in \mathbb{R} \times \Lambda_1$, $\Lambda_0 \oplus \Lambda_1 := \Lambda$ is the classical one-dimensional Grassmann algebra over the complex field $\mathbb{C}$. If to assume that the superfield $\tilde{n} := \tilde{u}_\theta = D_\theta \tilde{u}$, where $D_\theta = \partial/\partial\theta + \theta\partial/\partial x$, is the usual supersymetry derivation with respect to a variable $(x, \theta) \in \mathbb{R}^{1|1}$, satisfying the useful relationship $D_\theta^2 = \partial/\partial x$, the Poisson brackets (25) naturally pass into the following ultra-canonical super-Poisson brackets

$$\begin{gathered}\{\tilde{\psi}(x,\theta), \tilde{\psi}^*(y,\eta)\}_{\tilde{P}} = i\delta(x,\theta|y,\eta), \{\tilde{\psi}(x,\theta), \tilde{u}(y,\eta)\}_P = 0 = \{\tilde{u}(x,\theta), \tilde{\psi}^*(y,\eta)\}_P, \\ \{\tilde{\psi}(x,\theta), \tilde{\psi}(y,\eta)\}_{\tilde{P}} = 0 = \{\tilde{\psi}^*(x,\theta), \tilde{\psi}^*(y,\eta)\}_{\tilde{P}}, \{\tilde{u}(x,\theta), \tilde{u}(y,\eta)\}_{\tilde{P}} = \delta(x,\theta|y,\eta)\end{gathered} \tag{26}$$

at all super-points $(x, \theta), (y, \eta) \in \mathbb{R}^{1|1}$ on the functional supermanifold $\tilde{M}$, where

$$\delta(x,\theta|y,\eta) = \delta(x-y-\theta\eta)(\theta-\eta) \tag{27}$$

denotes the supersymmetric Dirac delta-function, satisfying for any continuous super-function $\tilde{f} \in C^0(\mathbb{R}^{1|1}; \Lambda)$ the determining relationship

$$\tilde{f}(x,\theta) := \int_{\mathbb{R}} dy \int d\eta \delta(x,\theta|y,\eta) \tilde{f}(y,\eta) \tag{28}$$

for all $(x, \theta) \in \mathbb{R}^{1|1}$ jointly with the following Berezin integrals [2,32,33], assumed to be fulfilled:

$$\int d\theta = 0, \int \theta d\theta = 1. \tag{29}$$

The introduced above super-variables $(\tilde{\psi}, \tilde{u}, \tilde{\psi}^*) \in \tilde{M}$ possess the following superalgebraic expansions:

$$\begin{aligned}&\tilde{\psi}(x,\theta) = \psi_0(x) + \theta\psi_1(x) \in \Lambda_0, \tilde{\psi}^*(x,\theta) = \psi_0^*(x) + \theta\psi_1^*(x) \in \Lambda_0^*, \\ &\tilde{u}(x,\theta) = u_1(x) + \theta u_0(x) \in \Lambda_0\end{aligned} \tag{30}$$

The corresponding supersymmetric tangent space $T(\tilde{M})$ and cotangent $T^*(\tilde{M})$ spaces can be endowed with the following super-bilinear form $(\cdot|\cdot) : T^*(\tilde{M}) \times T(\tilde{M}) \to \Lambda$, where for any $\tilde{f} \in T^*(\tilde{M}), \tilde{g} \in T(\tilde{M})$:

$$(\tilde{f}|\tilde{g}) := \int_{\mathbb{R}} dx \int d\theta \langle \tilde{f}(x,\theta)|\tilde{g}(x,\theta)\rangle_{\mathbb{E}^3}. \tag{31}$$

Having now applied the super-Poisson operator $\tilde{P} : T^*(\tilde{M}) \to T(\tilde{M})$ brackets (26) to the superized Hamiltonian operator $H_3 \in \mathcal{D}(M)$ in the form

$$\tilde{H}_3 = \int_{\mathbb{R}} dx \int d\theta (\tilde{\psi}^*_{\theta\theta} \tilde{\psi}_{\theta\theta} + 2\tilde{u}_\theta \tilde{\psi}^* \tilde{\psi}), \tag{32}$$

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one derives the following super-Hamiltonian system:

$$\tilde{\psi}_t = \{\tilde{H}_3, \tilde{\psi}\}_{\tilde{P}} = i\tilde{\psi}_{4\theta} - 2\tilde{u}_\theta\tilde{\psi}, \tilde{u}_t = -2(\tilde{\psi}^*\tilde{\psi})_\theta,$$
$$\tilde{\psi}^*_t = \{\tilde{H}_3, \tilde{\psi}^*\}_{\tilde{P}} = -i\tilde{\psi}^*_{4\theta} + 2\tilde{u}_\theta\tilde{\psi}^*, \tag{33}$$

regarding which one poses the following natural question: does it inherit the classical integrability property of the Schrödinger-Davydov dynamical system (8) as considered on the functional supermanifold $\tilde{M}$, and which will be analyzed in the section to follow.

3. Superintegrability Analysis

To analyze the super-integrability problem regarding the super-Hamiltonian system (33) we will present it, as the vector superfield

$$\left.\begin{array}{c} \partial\tilde{\psi}/\partial t = i\tilde{\psi}_{4\theta} - 2\tilde{u}_\theta\tilde{\psi}, \\ \partial\tilde{u}/\partial t = -2(\tilde{\psi}^*\tilde{\psi})_\theta \\ \partial\tilde{\psi}^*/\partial t = -i\tilde{\psi}^*_{4\theta} + 2\tilde{u}_\theta\tilde{\psi}^* \end{array}\right\} := \tilde{K}[\tilde{\psi}, \tilde{u}, \tilde{\psi}^*], \tag{34}$$

on the superized functional supermanifold $\tilde{M}$, and look for special solutions [3,53] to the corresponding Noether-Lax equation

$$\tilde{\varphi}_t + \tilde{K}'^{,*}\tilde{\varphi} = 0 \tag{35}$$

in the following asymptotical as $\mathbb{C} \ni \lambda \to \infty$ form:

$$\tilde{\varphi} = (1, \tilde{a}, \tilde{b})^\intercal \exp[-i\lambda^2 t + D_\theta^{-1}\tilde{\sigma}(x, \theta)], \tag{36}$$

where

$$\tilde{\sigma}(x, \theta; \lambda) \sim \sum_{j\in\mathbb{Z}_+\cup\{-2,-1\}} \tilde{\sigma}_j[\tilde{\psi}, \tilde{u}, \tilde{\psi}^*]\lambda^{-j}, \tilde{a}(x, \theta; \lambda) \sim \sum_{j\in\mathbb{Z}_+} \tilde{a}_j[\tilde{\psi}, \tilde{u}, \tilde{\psi}^*]\lambda^{-j},$$
$$\tilde{b}(x, \theta; \lambda) \sim \sum_{j\in\mathbb{Z}_+} \tilde{b}_j[\tilde{\psi}, \tilde{u}, \tilde{\psi}^*]\lambda^{-j}, \tag{37}$$

at arbitrary point $(x, \theta) \in \mathbb{R}^{1|1}$. Taking into account that the adjoint operator $\tilde{K}'^{,*} : T^*(\tilde{M}) \to T^*(\tilde{M})$ -is given by the expression

$$\tilde{K}'^{,*} = \begin{pmatrix} iD_\theta^4 - 2i\tilde{u}_\theta & -2\tilde{\psi}^*D_\theta & 0 \\ 2iD_\theta\tilde{\psi} & 0 & -2iD_\theta\tilde{\psi}^* \\ 0 & -2\tilde{\psi}D_\theta - & 2i\tilde{u}_\theta - iD_\theta^4 \end{pmatrix}, \tag{38}$$

one obtains easily the following infinite recurrent system:

$$\begin{array}{c} -i\delta_{j,-2} + D_\theta^{-1}\tilde{\sigma}_{j-k,\theta}\tilde{\sigma}_{k,\theta} + i\tilde{\sigma}_{j,x\theta} - 2i\tilde{u}_\theta\delta_{j,0} - 2\tilde{\psi}^*\tilde{a}_{j,\theta} + 2\tilde{\psi}\tilde{a}_{j-k}\tilde{a}_k = 0, \\ \tilde{a}_{j,\theta} - i\tilde{a}_{j+2} + \tilde{a}_{j-k,\theta}D_\theta^{-1}\tilde{\sigma}_{k,t} + 2i\tilde{\psi}_\theta\delta_{j,0} + 2i\tilde{\psi}\tilde{\sigma}_j - 2i\tilde{\psi}^*_\theta\tilde{b}_j - 2i\tilde{\psi}^*\tilde{b}_{j,\theta} - 2i\tilde{\psi}^*\tilde{b}_{j-k}\,\tilde{\sigma}_k = 0, \\ \tilde{b}_{j,\theta} - i\tilde{b}_{j+2} + \tilde{b}_{j-k,\theta}D_\theta^{-1}\tilde{\sigma}_{k,t} - 2\tilde{\psi}\tilde{a}_{j,\theta} + 2\tilde{\psi}\tilde{a}_{j-k,}\tilde{\sigma}_k + 2i\tilde{u}_\theta\tilde{b}_j - \\ -i(\tilde{b}_{j,xx} + 2\tilde{b}_{j-k}\tilde{\sigma}_{k,\theta} + \tilde{b}_{j-k}\tilde{\sigma}_{k,\theta x} + \tilde{b}_{j-k}\tilde{\sigma}_{k-s,\theta}\tilde{\sigma}_{s,\theta}) = 0 \end{array} \tag{39}$$

for all $j \in \mathbb{Z}_+ \cup \{-2, -1\}$. Trying to dissolve recurrently the above system (39), we obtain that first its coefficients are equal to

$$\tilde{\sigma}_{-1} = \theta, \tilde{\sigma}_0 = 0, \tilde{\sigma}_1 = \tilde{u}, \tilde{a}_0 = 0, \tilde{a}_1 = 2\tilde{\psi}\theta,$$
$$\tilde{a}_2 = 2\tilde{\psi}_\theta, \tilde{b}_0 = 0, \tilde{b}_1 = 0, b_2 = 0, \tag{40}$$

but the second $\tilde{\sigma}$- coefficient satisfies the locally unsolvable differential-algebraic relationship

$$\tilde{\sigma}_{2,\theta} = -\frac{1}{2}\tilde{u}_{x\theta} + 3\tilde{\psi}^*\tilde{\psi} \tag{41}$$

meaning that the recurrent system (39) fails to be infinitely continued. As an inference from this failure we need to state that our naively constructed super-Hamiltonian system (33) does not possess an infinite hierarchy of conservation laws and suitably is not super-integrable on the superized functional supermanifold $\tilde{M}$. This negative result is also teachable, per say, informing us that a simple naive a priori superization of a classical integrable nonlinear dynamical system generally loses its integrability, or in other words, *"Der Irrtum ist eine ebenso wichtige Lebensbedingung wie die Wahrheit.", i.e. "Error is as important a condition for the progress of life as truth." (C.G. Jung*)

In order to construct a more feasible and in some sense natural superization of the nonlinear dynamical Schrödinger-Davydov system (8), we first proceed to presenting its classical Lax-type operator representation, and then to its suitably superized generalization that will generate a priori integrable super-Hamiltonian flows, which we are interested in finding.

4. The Lax Type Representation Scheme

We will start from the infinite hierarchy of gradient relationships (22) and observe that it can be rewritten as

$$Q\text{grad}\,\gamma(\lambda) = 2\lambda^2 P\text{grad}\,\gamma(\lambda), \tag{42}$$

where, by definition,

$$\gamma(\lambda) := \int_{\mathbb{R}} dx\sigma(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \lambda^{-j} \int_{\mathbb{R}} H_j[\psi, n, \psi^*]dx \tag{43}$$

is a generating as $\mathbb{C} \ni \lambda \to \infty$ function of conservation laws for the dynamical system (8), which can be identified [3,14,42] with the trace-functional of the monodromy matrix $S(x;\lambda) \in \text{End}\,\mathbb{E}^m, x \in \mathbb{R}$, naturally assigned to a matrix Lax type "spectral" problem

$$\partial f/\partial x = l[\psi, n, \psi^*; \lambda]f, \tag{44}$$

where $l[\psi, n, \psi^*; \lambda] \in \text{End}\,\mathbb{E}^m$ for some finite $m \in \mathbb{N}$ is considered, for brevity, 2π- periodic in $x \in \mathbb{R}$ and $f \in L_\infty(\mathbb{R}; \mathbb{E}^m)$. Namely, if to put

$$\gamma(\lambda) := \text{tr}S(x;\lambda), \tag{45}$$

where, by definition, $S(x;\lambda) := F(x + 2\pi\pi, x; \lambda)$ and $F(x, y; \lambda) \in \text{End}\,\mathbb{E}^m$, $F(x, x; \lambda) = I, x \in \mathbb{R}$, denotes the fundamental matrix to the linear problem (44), depending on a point $(\psi, n, \psi^*) \in M$. Taking into account that the gradient element $\varphi(x;\lambda) := \text{grad}\gamma(\lambda) \in T^*(M)$ for all $(x;\lambda) \in \mathbb{R} \times \mathbb{C}$ satisfies the gradient relationship (42) and can be simultaneously represented as

$$\varphi(x;\lambda) = \text{tr}\{l[\psi, n, \psi^*; \lambda]^{\prime,*} S(x;\lambda)), \tag{46}$$

where the monodromy matrix $S(x;\lambda) \in \text{End}\,\mathbb{E}^m$ solves [46] on the axis $\mathbb{R}$ the linear Novikov equation

$$\partial S/\partial x = [l, S], \tag{47}$$

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one can construct within the gradient-holonomuc scheme [3,42] a finite set of differential algebraic matrix relationships in $l[\psi, n, \psi^*; \lambda] \in \mathrm{End}\, \mathbb{E}^m$, $\mathrm{tr} l[\psi, n, \psi^*; \lambda] = 0$, whose solution gives rise via simple enough but cumbersome calculations to the following result: $m = \dim l[\psi, n, \psi^*; \lambda] = 3$ and

$$l[\psi, n, \psi^*; \lambda] = \begin{pmatrix} 2i\lambda - \frac{in}{2\lambda} & \psi^* & -\frac{in}{2\lambda} \\ \frac{\psi}{2\lambda} & 0 & \frac{\psi}{2\lambda} \\ \frac{in}{2\lambda} & \psi^* & \frac{in}{2\lambda} - 2i\lambda \end{pmatrix}. \tag{48}$$

It is now easy to observe that the linear Lax type spectral problem (44) reduces to the next pseudo-differential form:

$$-\partial^2 f/\partial x^2 + 2nf - 2i\psi^* \partial^{-1} \psi f = 4\lambda^2 f, \tag{49}$$

where $f \in W \subset L_\infty(\mathbb{R}; \mathbb{C})$ is a scalar function and $\lambda \in \mathbb{C}$ serves as a true spectral parameter.

Remark 2. *If to denote the pseudo-differential expression from (49) as*

$$L := -\partial^2/\partial x^2 + 2n - 2i\psi^* \partial^{-1} \psi, \tag{50}$$

it allows to construct [3,4,44,54] the same infinite hierarchy of conservation laws as (16) by means of the operator traces

$$H_j = \mathrm{Tr}\left(L^{1/2} L^j\right), \tag{51}$$

where $L, L^{j/2} \in \Psi OP, j \in \mathbb{Z}_+$, and $\mathrm{Tr} : \Psi OP \to \mathbb{C}$ is the trace operation on the algebra ΨOP of pseudo-differential operators on the axis and coinciding with the integral over $\mathbb{R}$ of the functional coefficient at the inverse differentiation ∂^{-1}.

The spectral problem (49) looks very interesting and represents [17,18,36,51] the Backlund type operator transformation

$$DOP \ni L_0 \to L_0 + \alpha \psi^* \partial^{-1} \psi \in \Psi OP \tag{52}$$

from the algebra DOP of differential operators to that ΨOP of pseudo-differential operators, where, by definition, ψ and $\psi^* \in W$ serve, respectively, as the eigenfunctions of the spectral problem

$$L_0 \psi = \mu \psi \tag{53}$$

for some $\mu \in \mathbb{C}$ and its adjoint:

$$L_0^* \psi^* = \nu^* \psi^* \tag{54}$$

for some $\nu^* \in \mathbb{C}$.

Remark 3. *More details of this Backlund type operator transformation (52) can be found in [18]. Mention here only that the found before compatible pair of Poisson operators (20) and (21) follows from thje canonical Poisson bracket on the space $\Psi OP \times W \times W^*$ via the operator mapping (52).*

Since the obtained above pseudo-differential operator (50) is a shifted classical Sturm-Liouville operator on the axis $\mathbb{R}$ of the second order, whose natural superization was first studied in [28], we can logically proceed to generalyzing this result on the subject of the corresponding superization of the completely integrable Schrödinger-Davydov dynamical system under regard.

5. Spectral Operator Problem and Related Superization Scheme

Let us consider the classical Sturm-Liouville operator expression

$$L_0 := -\partial^2/\partial x^2 + 2n(x) \tag{55}$$

on the real axis with a real potential $n(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$ on the functional space W and its super-differential analog

$$\tilde{L}_0 := -D_\theta^3 + 2\tilde{n}(x,\theta) \tag{56}$$

on the super-axis, constructed in [28], where $\tilde{n}(x,\theta) \in \Lambda_1$ for all $(x,\theta) \in \mathbb{R}^{1|1}$. The related to (56) super-differential spectral problem

$$\tilde{L}_0\tilde{\psi} = (-D_\theta^3 + 2\tilde{n})\tilde{\psi} = \mu\tilde{\psi}, \tag{57}$$

where $\mu \in \Lambda_1$ and $\tilde{\psi} \in \tilde{W} \subset L_\infty(\mathbb{R}^{1|1};\Lambda_0)$, and its adjoint problem

$$\tilde{L}_0^*\tilde{\psi}^* = (D_\theta^3 + 2\tilde{n})\tilde{\psi} = \nu^*\tilde{\psi}, \tag{58}$$

where $\nu^* \in \Lambda_1^*$ and $\tilde{\psi}^* \in \tilde{W}^* \subset L_\infty(\mathbb{R}^{1|1};\Lambda_0^*)$ make it possible to superize the super-differential operator (56) as

$$\tilde{L}_0 \to \tilde{L} := -D_\theta^3 + 2\tilde{n} - 2i\tilde{\psi}^* D_\theta^{-1}\tilde{\psi} \tag{59}$$

by shifting on the Backlund transformed term $-2i\tilde{\psi}^* D_\theta^{-1}\tilde{\psi} \in s\Psi OP$ from the algebra $s\Psi OP$ of super-pseudo-differential operators. Based on the super pseudo-differential expression (59), one can calculate [30,31,40,41,45,47**?** –50] the corresponding conserved super-laws as the following Casimir invariant functionals

$$H_j = \mathrm{sTr}\left(L^{j/3}\right), \tag{60}$$

$j \in \mathbb{Z}_+$, of the related Lie superalgebra $\mathrm{Lie}(s\Psi OP)$, where the super-trace operation $\mathrm{sTr} : s\Psi OP \to \Lambda$ is defined as the super-integral over the super-axis $\mathbb{R}^{1|1}$ of the coefficient at the inverse super-differentiation D_θ^{-1}. In particular, taking into account that

$$\begin{aligned} \tilde{L}^{1/3} \sim -D_\theta + \tilde{w}_0^1 + (\tilde{w}_1^1 + \tilde{w}_1^0 D_\theta)\partial^{-1} + (\tilde{w}_2^1 + \tilde{w}_2^0 D_\theta)\partial^{-2} + \\ + (\tilde{w}_3^1 + \tilde{w}_3^0 D_\theta)\partial^{-3} + (\tilde{w}_4^1 + \tilde{w}_4^0 D_\theta)\partial^{-4}) + \ldots, \end{aligned} \tag{61}$$

whith the coefficients satisfying the conditions $\tilde{w}_0^1(\tilde{\psi}^*\tilde{\psi} - \tilde{w}_{0,\theta}^1) = 2\tilde{n}$, $-D_\theta(\tilde{w}_1^1\tilde{w}_1^0) = \tilde{\psi}^*\tilde{\psi}, \ldots,$ and so on, we can easily calculate the super-conservation laws

$$\tilde{H}_1 = \int dx \int d\theta \tilde{\psi}^*\tilde{\psi}, \quad \tilde{H}_2 = \int dx \int d\theta (\tilde{\psi}_\theta^*\tilde{\psi}_{\theta\theta} - \tilde{\psi}_{\theta\theta}^*\tilde{\psi}_\theta + 2\tilde{\psi}^*\tilde{\psi}\tilde{n} - \tilde{\psi}_\theta^*\psi), \ldots., \tag{62}$$

and so on, invariant with respect to the super-evolution flow on $\tilde{M}$, equivalently represented as the following Lax type dynamical super-operator flow

$$\partial\tilde{L}/\partial t = [\tilde{L}, \left(\tilde{L}^2\right)_+], \tag{63}$$

where the sign "+" denotes the strictly nonnegative super-differential part of an expression in the bracket (...) above.

Remark 4. *One needs here to mention that the flow (63) is naturally interpreted [3,4,14,44,54] from the Lie-algebraic point of view as the coadjoint action of the operator Lie susperalgebra element* $\left(\tilde{L}^{2/3}\right)_+ \in \mathrm{Lie}(s\Psi OP_+)$ *on the element* $\tilde{L} \in \mathrm{Lie}(s\Psi OP)^*$, *where* $\mathrm{Lie}(s\Psi OP_+)$ *denotes the nonnegative part of the natural direct sum spliting* $\mathrm{Lie}(s\Psi OP) = \mathrm{Lie}(s\Psi OP_+) \oplus \mathrm{Lie}(s\Psi OP_-)$.

Having recalculated the flow (63) regarding the superized variables $(\tilde{\psi}, \tilde{n}, \tilde{\psi}^*) \in \tilde{M}$, one obtains the following Schrödinger-Davydov evolution flow

$$
\begin{aligned}
\partial\tilde{\psi}/\partial t &= i\tilde{\psi}_{\theta\theta\theta} - 2i\tilde{n}\tilde{\psi}^*\tilde{\psi} + i\tilde{\varpi}^1_{0,\theta}\tilde{\varpi}^1_0, \\
\partial\tilde{n}/\partial t &= -2(\tilde{\psi}^*\tilde{\psi}\tilde{\varpi}^0_1 - (\tilde{\psi}^*_\theta\tilde{\psi} - \tilde{\psi}^*\tilde{\psi}_\theta)\tilde{\varpi}^1_0), \\
\partial\tilde{\psi}^*/\partial t &= -i\tilde{\psi}_{\theta\theta\theta} + 2i\tilde{n}\tilde{\psi}^*\tilde{\psi} + i\tilde{\varpi}^1_{0,\theta}\tilde{\varpi}^1_0,
\end{aligned} \tag{64}
$$

which is a super-Hamiltonian system with respect to the following super-Poisson structure

$$
\begin{aligned}
&\{\tilde{\psi}(x,\theta), \tilde{\psi}^*(y,\eta)\}_{\tilde{P}} = i\delta(x,\theta;y,\eta), \{\tilde{\psi}(x,\theta), \tilde{n}(y,\eta)\}_P = \tilde{\psi}^*(x,\theta)\delta(x,\theta;y,\eta), \\
&\{\tilde{n}(x,\theta), \tilde{\psi}^*(y,\eta)\}_P = \tilde{\psi}(x,\theta)\delta(x,\theta;y,\eta), \{\tilde{n}(x,\theta), \tilde{n}(y,\eta)\}_{\tilde{P}} = D_\theta\delta(x,\theta;y,\eta), \\
&\{\tilde{\psi}(x,\theta), \tilde{\psi}(y,\eta)\}_{\tilde{P}} = 0 = \{\tilde{\psi}^*(x,\theta), \tilde{\psi}^*(y,\eta)\}_{\tilde{P}}
\end{aligned}
$$

on the functional supermanifold $\tilde{M}$, that is $(\tilde{\psi}_t, \tilde{n}_t, \tilde{\psi}^*_t)^\intercal = \{\tilde{H}_3, (\tilde{\psi}, \tilde{n}, \tilde{\psi}^*)^\intercal\}_{\tilde{P}}$, coinciding with that of (64), where the evolution parameter $t \in \Lambda_1$ is considered to be odd. The supersymmetric integrable flow (64) presents a suitable superization regarding the classical integrable Schrödinger-Davydov dynamical system on the functional manifold M. It is worth to note that in some cases one can anticipate that such super-evolution vector field $d/dt : \tilde{M} \to T(\tilde{M})$ on the functional supermanifold $\tilde{M}$ can be represented as the supersymmetric super-differentiation $D_\xi = \partial/\partial\theta + \theta\partial/\partial t$ with respect to the super-variable $\theta \in \Lambda_1$ and the real evolution parameter $t \in \mathbb{R}$.

6. Conclusion

We have studied two interesting examples of the superization scheme regarding the classical Schrödinger-Davydov integrable nonlinear dynamical system on functional manifold. In particular, we checked that so called "naive" approach, based on the superization of the phase space variables and extending the corresponding Poisson brackets upon the related functional supermanifold, fails to retain the dynamical system super-integrability. Nonetheless, for a wide class of classical Lax type integrable nonlinear dynamical systems on functional manifolds a possible superization scheme consists in a reasonable superization of the related Lax type repreentation by means of transition from the basic algebra of pseudo-differential operators on the axis to the corresponding superalgebra of super-pseudo-differential operators on the superaxis.

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