Brief Report

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Deriving an Explicit Solution for Midsegment Triangles: Extending Pythagorean Principles in Spatial Geometry

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Abstract: In a foundational study, [1] explored how connecting points on a triangle's midsegment to its vertices forms triangles that conform to the Pythagorean theorem in both two- and three-dimensional spaces, unveiling deeper geometric relationships and extending Pythagorean principles to spatial geometry with an implicit solution. Building on this work, I present the derivation of an explicit solution, further elucidating these geometric relationships and enhancing our understanding of their spatial applications.

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1. Introduction

The Pythagorean theorem has long been a cornerstone of mathematical thought, underpinning numerous geometric principles in two-dimensional spaces. Recent advancements, such as the work of [1], have demonstrated that these principles can be extended to three-dimensional geometry, revealing intricate relationships between triangles formed by connecting midsegment points to their respective vertices. This earlier study established a foundation by presenting implicit solutions that adhere to the Pythagorean theorem in both planar and spatial contexts.

Building upon this groundwork, the current paper introduces an explicit solution that provides greater clarity and accessibility to these geometric relationships. By refining the mathematical framework, we extend the practical understanding of these principles in spatial geometry, paving the way for broader applications in mathematics and beyond. This explicit solution serves not only to deepen theoretical insights but also to bridge the gap between abstract formulations and tangible geometric applications.

2. Derivation of an Explicit Equation for the Pythagorean Tetrahedron

Here, we reintroduce the parameters h and l, as originally defined in [1]. Referring to Figure 1, these parameters can be expressed as follows::

$$l = \frac{c^2 - b^2 - a^2}{2a} \tag{1}$$

$$h = \frac{\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}}{2a} \tag{2}$$

Equations (1)–(4)

$$ar(\triangle) = \frac{1}{2}\sqrt{\begin{array}{l} ((x_2y_1) - (x_3y_1) - (x_1y_2) + (x_3y_2) + (x_1y_3) - (x_2y_3))^2 \\ +((x_2z_1) - (x_3z_1) - (x_1z_2) + (x_3z_2) + (x_1z_3) - (x_2z_3))^2 \\ +((y_2z_1) - (y_3z_1) - (y_1z_2) + (y_3z_2) + (y_1z_3) - (y_2z_3))^2 \end{array}} \tag{3}$$

$$\begin{array}{l} ar(\triangle APB) = ar(\triangle BPC) + ar(\triangle APC) \\ ar(\triangle BPC) = ar(\triangle APB) + ar(\triangle APC) \\ ar(\triangle APC) = ar(\triangle APB) + ar(\triangle BPC) \end{array} \tag{4}$$

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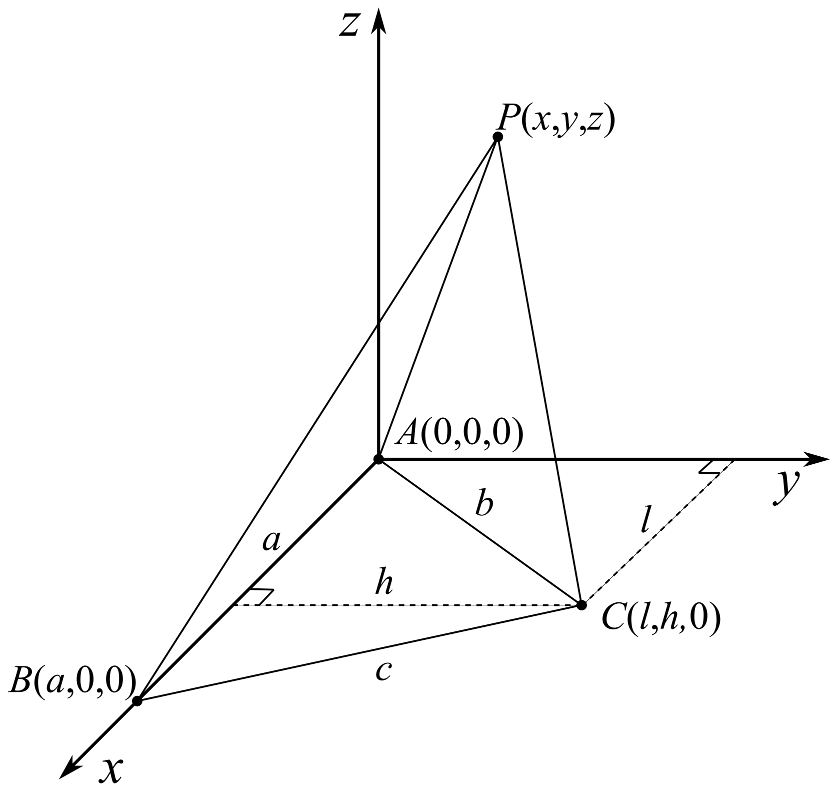

Figure 1. Pythagorean Tetrahedron: 3D Pythagorean composition from $\triangle ABC$.

yields three distinct implicit solutions for the *Pythagorean Tetrahedron*, depicted in Figure 1, namely:

$$\frac{1}{2}\left(a\sqrt{z^2+y^2}-\sqrt{(ly-hx)^2+z^2(l^2+h^2)}-\right.$$
$$\left.\sqrt{(ah-hx-ay+ly)^2+z^2(h^2+(l-a)^2)}\right)=0 \tag{5}$$

$$\frac{1}{2}\left(-a\sqrt{z^2+y^2}+\sqrt{(ly-hx)^2+z^2(l^2+h^2)}-\right.$$
$$\left.\sqrt{(ah-hx-ay+ly)^2+z^2(h^2+(l-a)^2)}\right)=0 \tag{6}$$

$$\frac{1}{2}\left(-a\sqrt{z^2+y^2}-\sqrt{(ly-hx)^2+z^2(l^2+h^2)}+\right.$$
$$\left.\sqrt{(ah-hx-ay+ly)^2+z^2(h^2+(l-a)^2)}\right)=0 \tag{7}$$

To solve Equation (5) for z, we first multiply both sides by 2 to simplify the expression, thereby isolating some radicals on the left-hand side.

$$a\sqrt{z^2+y^2}-\sqrt{(ly-hx)^2+z^2(l^2+h^2)}=$$
$$\sqrt{(ah-hx-ay+ly)^2+z^2(h^2+(l-a)^2)} \tag{8}$$

Both sides of the equation are squared initially to minimize the number of square root terms.

$$(ly-hx)^2+z^2\left(l^2+h^2\right)+a^2\left(z^2+y^2\right)-$$
$$2a\sqrt{(z^2+y^2)((hx-ly)^2+z^2(l^2+h^2))}=$$
$$(ah-hx-ay+ly)^2+z^2\left(h^2+(l-a)^2\right) \tag{9}$$

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Any remaining radical expressions are then isolated on the left-hand side.

$$-2a\sqrt{(z^2+y^2)((hx-ly)^2+z^2(l^2+h^2))} = -(ly-hx)^2 + \\ (ah-hx-ay+ly)^2 - z^2\left(l^2+h^2\right) + z^2\left(h^2+(l-a)^2\right) - a^2\left(z^2+y^2\right) \tag{10}$$

The sides are squared a second time to completely eliminate the square roots.

$$4a^2\left(z^2+y^2\right)\left((hx-ly)^2+z^2\left(l^2+h^2\right)\right) = \\ \left(-(ly-hx)^2+(ah-hx-ay+ly)^2-z^2\left(l^2+h^2\right)+ \right. \\ \left. z^2\left(h^2+(l-a)^2\right)-a^2\left(z^2+y^2\right)\right)^2 \tag{11}$$

The equation is subsequently transformed into a quartic polynomial, which is rearranged to ensure it is expressed in standard form.

$$4a^2h^2x^2y^2 - 8a^2hlxy^3 + 4a^2l^2y^4 + \\ z^2\left(4a^2h^2x^2 - 8a^2hlxy + 4a^2h^2y^2 + 8a^2l^2y^2\right) + z^4\left(4a^2h^2+4a^2l^2\right) = \\ a^4h^4 - 4a^3h^4x + 4a^2h^4x^2 - 4a^4h^3y + 4a^3h^3ly + 12a^3h^3xy - \\ 8a^2h^3lxy - 8a^2h^3x^2y + 4a^4h^2y^2 - 12a^3h^2ly^2 + \\ 4a^2h^2l^2y^2 - 8a^3h^2xy^2 + 16a^2h^2lxy^2 + 4a^2h^2x^2y^2 + \\ 8a^3hly^3 - 8a^2hl^2y^3 - 8a^2hlxy^3 + 4a^2l^2y^4 + \\ z^2\left(-4a^3h^2l + 8a^2h^2lx + 8a^3hly - 8a^2hl^2y - 8a^2hlxy + 8a^2l^2y^2\right) + \\ 4a^2l^2z^4 \tag{12}$$

$$-a^4h^4 + 4a^3h^4x - 4a^2h^4x^2 + 4a^4h^3y - 4a^3h^3ly - \\ 12a^3h^3xy + 8a^2h^3lxy + 8a^2h^3x^2y - 4a^4h^2y^2 + 12a^3h^2ly^2 - \\ 4a^2h^2l^2y^2 + 8a^3h^2xy^2 - 16a^2h^2lxy^2 - 8a^3hly^3 + 8a^2hl^2y^3 + \\ z^2\left(4a^3h^2l - 8a^2h^2lx + 4a^2h^2x^2 - 8a^3hly + 8a^2hl^2y + 4a^2h^2y^2\right) + \\ 4a^2h^2z^4 = 0 \tag{13}$$

The quartic equation is solved for z, with simplifications applied to powers and products as needed, ultimately yielding four potential solutions for z.

$$z = \pm\sqrt{\frac{\pm\sqrt{\begin{array}{c}\left(4ahl-8aly-8hlx+4hx^2+4hy^2+8l^2y\right)^2 \\ -16h\left(-a^2h^3+4a^2h^2y-4a^2hy^2+4ah^3x-4ah^2ly\right. \\ -12ah^2xy+12ahly^2+8ahxy^2-8aly^3-4h^3x^2 \\ \left.+8h^2lxy+8h^2x^2y-4hl^2y^2-16hlxy^2+8l^2y^3\right)\end{array}}}{8h} + \frac{aly-l^2y}{h} - \frac{al}{2} + lx - \frac{x^2-y2}{2}} \tag{14}$$

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Among these solutions, two yield real numbers (with no imaginary components).

$$z = \pm\sqrt{\frac{\sqrt{\begin{array}{c}(4ahl - 8aly - 8hlx + 4hx^2 + 4hy^2 + 8l^2y)^2 \\ -16h\Big(- a^2h^3 + 4a^2h^2y - 4a^2hy^2 + 4ah^3x - 4ah^2ly \\ -12ah^2xy + 12ahly^2 + 8ahxy^2 - 8aly^3 - 4h^3x^2 \\ +8h^2lxy + 8h^2x^2y - 4hl^2y^2 - 16hlxy^2 + 8l^2y^3\Big)\end{array}}}{8h} + \frac{aly - l^2y}{h} - \frac{al}{2} + lx - \frac{x^2 - y2}{2}} \quad (15)$$

3. Conclusions

Building upon the foundational work of Bernardes (2024), this study derives an explicit solution for the geometric relationships governed by the Pythagorean theorem in both two- and three-dimensional spaces. By reintroducing the parameters h and l and systematically solving the implicit equations through a series of algebraic transformations, including the formulation and resolution of a quartic polynomial, we identify two distinct real solutions. These solutions offer enhanced clarity in understanding the spatial applications of the Pythagorean theorem and extend its utility in three-dimensional geometric contexts. This work not only deepens our comprehension of fundamental geometric principles but also opens avenues for further exploration in spatial geometry and its practical implications.

References

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