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Article

Trigonometric Polynomial Points in the Plane of a Triangle

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Abstract: It is well known that the four ancient Greek triangle centers and others have homogeneous barycentric coordinates that are polynomials in the sidelengths a, b, c of a triangle ABC. For example, the circumcenter is represented by the polynomial $a(b^2 + c^2 - a^2)$. It is not so well known that triangle centers having barycentric coordinates such as $\tan A : \tan B : \tan C$ are also representable by polynomials, in this case, by $p(a, b, c)$: $p(b, c, a) : p(c, a, b)$, where $p(a, b, c) = a(a^2 + b^2 - c^2)(a^2 + c^2 - b^2)$. This paper presents and discusses polynomial representations of triangle centers that have barycentric coordinates of the form $f(a, b, c) : f(b, c, a) : f(c, a, b)$, where f depends on one or more of the functions in the set $\{\cos, \sin, \tan, \sec, \csc, \cot\}$. The topics discussed include infinite trigonometric orthopoints, the n-Euler line, and symbolic substitution.

Keywords: homogeneous coordinates; barycentric; trilinear; triangle; triangle center; polynomial point; trigonometry; isogonal conjugate; isotomic conjugate; Euler line; Nagel line; symbolic substitution

MSC: Primary 51N20; Secondary 51M05

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1. Introduction

One of the most productive systems of representation for points and lines in the plane of a triangle ABC is a system widely known as homogeneous barycentric coordinates (henceforth simply *barycentrics)*. Serving as the "origin" in this system are the three vertices of ABC, shown here with their barycentrics:

$$A = 1 : 0 : 0 \qquad B = 0 : 1 : 0 \qquad C = 0 : 0 : 1.$$

The lengths of the sides opposite the vertex angles (which, like the vertex points, are denoted by A, B, C) are given the symbols a, b, c, respectively, and may be regarded as variables or algebraic indeterminates. For an excellent introduction to the subject of barycentrics, see Yiu [16].

Many triangle centers (as defined in [14]) have barycentrics that are polynomials. Following [5], we refer to a triangle center X that has barycentrics

$$p(a, b, c) : p(b, c, a) : p(c, a, b),$$

where $p(a, b, c)$ is a polynomial, as a *polycenter*. If X also has barycentrics

$$f(a, b, c) : f(b, c, a) : f(c, a, b),$$

where $f(a, b, c)$ involves trigonometric functions of the angles A, B, C as a *trigonometric polycenter*. Analogously, we have *polylines* and *trigonometric polylines*. Note that if the first barycentric of X is written as $h(a, b, c)$, than the second and third barycentrics are determined (viz. $h(b, c, a)$ and $h(c, a, b)$), so that the shorter notation $h(a, b, c)$ :: is sufficient.

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Important examples of trigonometric polycenters include these:

$$\begin{aligned}
G = &\ \text{centroid} \ = 1:1:1 = 1 :: \\
O = &\ \text{circumcenter} \ = a^2(b^2+c^2-a^2) ::= \sin 2A :: \\
H = &\ \text{orthocenter} \ = (c^2+a^2-b^2)(a^2+b^2-c^2) ::= \tan A :: \\
N = &\ \text{nine-point center} \ = a^2(b^2+c^2)-(b^2-c^2)^2 ::= \sin A\cos(B-C) ::
\end{aligned}$$

As an example of a trigonometric polyline, the Euler line, which passes through the points G, O, H, N, is given in terms of a variable point $x : y : z$ by both of the following equations:

- $(b^2-c^2)(b^2+c^2-a^2)x + (c^2-a^2)(c^2+a^2-b^2)y + (a^2-b^2)(a^2+b^2-c^2)z = 0$
- $(\tan B - \tan C)x + (\tan C - \tan A)y + (\tan A - \tan B)z = 0$

Of great importance in triangle geometry are the following objects:

- isotomic conjugate of X, with barycentrics $1/x : 1/y : 1/z = yz : zx : xy$
- isogonal conjugate of X, with barycentrics $a^2/x : b^2/y : c^2/z = a^2yz : b^2zx : c^2xy$
- the line at infinity, L^∞, with barycentric equation $x+y+z=0$
- Steiner circumellipse, with equation $1/x+1/y+1/z=0$
- circumcircle, with equation $a^2/x+b^2/y+c^2/z=0$

2. Trigonometric Polycenters

In this section, we shall see that for all integers n, the triangle centers $f(nA)$ :: for $f = \cos, \sin, \tan$, and others, are polycenters. We begin with the usual recurrences of Chebyshev polynomials of the first kind, T_n, and the second kind, U_n:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \text{ for } n \geq 2, \qquad T_0(x)=1, \qquad T_1(x)=x; \tag{1}$$
$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \text{ for } n \geq 2, \qquad U_0(x)=1, \qquad U_1(x)=2x, \tag{2}$$

Another well-known type of recurrence relation for these families of polynomials ([12], [13]) depends on complex numbers:

$$T_n(x) = (1/2)\Big((x-\sqrt{x^2-1})^n + (x+\sqrt{x^2-1})^n\Big) \tag{3}$$
$$U_n(x) = (1/(2\sqrt{x^2-1})\Big((x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}\Big). \tag{4}$$

Theorem 1. *Let $\Lambda = b^2+c^2-a^2$. Then*

$$\cos nA :: = a^n\Big((\Lambda-2iS)^n + (\Lambda+2iS)^n\Big) :: \tag{5}$$
$$\sin nA :: = a^n\Big((\Lambda-2iS)^n - (\Lambda+2iS)^n\Big) :: . \tag{6}$$

Proof. We have $\cos A = \Lambda/(2bc)$, and (3) gives

$$\begin{aligned}
T_n(\cos A) &= (1/2)\Big((\cos A - i\sin A)^n + (\cos A + i\sin A)^n\Big) \\
&= (1/2)(\frac{\Lambda}{2bc} - \frac{iS}{bc})^n + (\frac{\Lambda}{2bc} + \frac{iS}{bc})^n,
\end{aligned}$$

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where

$$\begin{aligned} S &= 2(\text{ area of } \triangle ABC) \\ &= (1/2)\sqrt{(a+b+c)(b+c-1)(c+a-b)(a+b-c)}, \end{aligned}$$

so that

$$S^2 = (1/4)(-a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2).$$

Now since $\cos nA = T_n(\cos A)$, we have

$$\cos nA = (1/2)\frac{1}{(2bc)^n}\Big((\Lambda - 2iS)^n + (\Lambda + 2iS)^n\Big),$$

so that (5) holds. Similarly, Equation (6) follows from (4) and the well-known fact that $\sin nA = U_{n-1}(\cos A)\sin A$. □

Our main goal in this section is to represent $\cos nA$ :: and $\sin nA$ :: as polycenters. To that end, let

$$u = \Lambda - 2iS \text{ and } v = \Lambda + 2iS, \tag{7}$$

so that expressions in (3) and (4) can be recast in order to define sequences (c_n) and (s_n) as follows:

$$c_n = c_n(a,b,c) = a^n(u^n + v^n) \tag{8}$$
$$s_n = s_n(a,b,c) = a^n(u^n - v^n). \tag{9}$$

Next, we have a lemma about u and v.

Lemma 1.

$$\begin{aligned} u^2 - 2(b^2+c^2-a^2)u + 4b^2c^2 &= v^2 - 2(b^2+c^2-a^2)v + 4b^2c^2 \\ &= 0. \end{aligned}$$

Proof. The imaginary terms cancel, and the real term is

$$-(b^2+c^2-a^2)^2 - 4S^2 + 4b^2c^2 = 0.$$

□

Theorem 2. *Let (c_n) be the sequence given by (8). Then c_n is a polynomial in a, b, c, given by the following three initial terms and 2^{nd}-order recurrence:*

$$\begin{aligned} c_0 &= 1 \\ c_1 &= a(b^2+c^2-a^2) \\ c_2 &= a^2(a^4+b^4+c^4-2a^2b^2-2a^2c^2) \\ c_n &= a(b^2+c^2-a^2)c_{n-1} - a^2b^2c^2c_{n-2} \text{ for } n \geq 3, \end{aligned}$$

and

$$c_n(a,b,c) : c_n(b,c,a) : c_n(c,a,b) = \cos nA : \cos nB : \cos nC. \tag{10}$$

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Proof. It is easy to verify that c_0, c_1, c_2 are polycenters as claimed. Suppose now that $n \geq 3$. Using u and v as in (7) and Lemma 1, we have

$$u^{n-2}\left(u^2 - 2(b^2+c^2-a^2)u + 4b^2c^2\right) = -v^{n-2}\left(v^2 - 2(b^2+c^2-a^2)v + 4b^2c^2\right),$$

so that

$$u^n + v^n = 2(b^2+c^2-a^2)(u^{n-1}+v^{n-1}) - a^2b^2c^2(a/2)^{n-2}(u^{n-2}+v^{n-2})$$
$$(a/2)^n(u^n+v^n) = a(b^2+c^2-a^2)(a/2)^{n-1}(u^{n-1}+v^{n-1}) - 4b^2c^2(u^{n-2}+v^{n-2}).$$

This shows that if $d_k = (a/2)^k(u^k+v^k)$ for $k \geq 3$, then

$$d_n = a(b^2+c^2-a^2)d_{n-1} - a^2b^2c^2d_{n-2} \text{ for } n \geq 3.$$

By Theorem 1, $\cos nA ::= a^n(u^n+v^n) ::$, and since $d_n ::= c_n ::$, we have $c_n ::= \cos nA ::$. □

Theorem 3. *Let (s_n) be the sequence given by (9). Then s_n is a polynomial in a, b, c, given by these two initial terms and 2^{nd}-order recurrence:*

$$s_1 = a$$
$$s_2 = a^2(b^2+c^2-a^2)$$
$$s_n = a(b^2+c^2-a^2)s_{n-1} - a^2b^2c^2s_{n-2} \text{ for } n \geq 2,$$

and

$$s_n(a,b,c) : s_n(b,c,a) : s_n(c,a,b) = \sin nA : \sin nB : \sin nC. \quad (11)$$

Proof. A proof similar to that of Theorem 2 springs from (4), leading, by way of the identity $\sin nA = U_{n-1}(\cos A)\sin A$, to

$$\sin nA = \frac{1}{2}\frac{1}{(2bc)^n}\left((\Lambda - 2iS)^n - (\Lambda + 2iS)^n)\right),$$

The rest of the proof, using Lemma 1, follows in a manner much as in the proof of Theorem 2. □

Example 1. *Polycenter representations for* $\cos 3A ::$ *and* $\cos 4A ::$ *are given by*

$$c_3 = a^3(a^2-b^2-c^2)\left(a^4+b^4+c^4-2a^2(b^2+c^2)-b^2c^2\right);$$
$$c_4 = a^4\Big(a^8+b^8+c^8-4a^6(b^2+c^2)+2a^4(3b^4+3c^4+4b^2c^2)$$
$$-4a^2(b^4+c^4)(b^2+c^2)\Big).$$

Example 2. *Polycenter representations for* $\sin nA ::$, *for* $n = 3, 4, 5, 6$, *are given by*

$$
\begin{aligned}
& s_3 = a^3(a^2 - b^2 - c^2 - bc)(a^2 - b^2 - c^2 + bc); \\
& s_4 = a^4(a^2 - b^2 - c^2)\Big(a^4 + b^4 + c^4 - 2a^2(b^2 + c^2)\Big); \\
& s_5 = a^5 f_1 f_2, \textit{ where} \\
& \quad f_1 = a^4 + b^4 + c^4 + a^2(bc - 2b^2 - 2c^2) + bc(bc - b^2 - c^2) \\
& \quad f_2 = a^4 + b^4 + c^4 + a^2(-bc - 2b^2 - 2c^2) + bc(bc + b^2 + c^2); \\
& s_6 = a^5 g_1 g_2 g_3 g_4, \textit{ where} \\
& \quad g_1 g_2 g_3 = (a^2 - b^2 - c^2)(a^2 - b^2 - c^2 - bc)(a^2 - b^2 - c^2 + bc) \\
& \quad g_4 = a^4 + b^4 + c^4 - 2a^2(b^2 + c^2) - b^2c^2.
\end{aligned}
$$

Inductively, c_n and s_n both have degree $3n$ for $n \geq 0$ and both are polynomial multiples of a^n. By Theorems 2 and 3, the sequences (c_n) and (s_n) have the same second-order recurrence signature:

$$\Big(a(b^2 + c^2 - a^2), -a^2b^2c^2\Big).$$

Next, let $t_n = s_n / c_n$, so that $t_n ::= \tan nA ::$. For the sake of brevity, we shall sometimes write a polycenter of the form $f(a, b, c)g(b, c, a)g(c, a, b)$ as a quotient: $f(a, b, c)/g(a, b, c)$. Shown here are representations for polycenters $\tan nA ::$ for $n = 1, 2, 3$.

$$
\begin{aligned}
& t_1 = 1/(a^2 - b^2 - c^2); \\
& t_2 = (a^2 - b^2 - c^2)/\big(a^4 + b^4 + c^4 - 2a^2(b^2 + c^2)\big); \\
& t_3 = 1/\big(a^4 + b^4 + c^4 - 2a^2)(b^2 + c^2) - b^2c^2\big);
\end{aligned}
$$

The sequence (t_n), as well as its equivalent sequence of polynomials, appears—expectedly—to be not linearly recurrent. However, the sequence given by $u_n(a, b, c) = \sin(nA)\cos(nB)\cos(nC)$, is linearly recurrent, since the three sequences $(\sin(nA)), (\cos(nB)), (\cos(nC))$ are linearly recurrent, and, of course,

$$u_n(a, b, c) : u_n(b, c, a) : u_n(c, a, b) = \tan nA : \tan nB : \tan nC.$$

A sequence of associated polycenters derived from on $u_n(a, b, c)$ is considered in Section 7. Likewise the triangle centers $\sec nA ::$, $\csc nA ::$, $\cot nA ::$ are polycenters for all nonzero integers n. Geometrically, these are isotomic conjugates, given by $1/t_n, 1/s_n, c_n/s_n$, respectively. As indicated in Example 3, many geometric and algebraic properties of the specific polycenters mentioned above can be found in the Encyclopedia of Triangle Centers (ETC) [2]:

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Example 3. *A few trigonometric polycenters in ETC [2].*

$\sin A ::= X_1$	$\csc A ::= X_{75}$
$\cos A ::= X_{63}$	$\sec A ::= X_{92}$
$\tan A ::= X_4$	$\cot A ::= X_{69}$
$\sin 2A ::= X_3$	$\csc 2A ::= X_{264}$
$\cos 2A ::= X_{1993}$	$\sec 2A ::= X_{5392}$
$\tan 2A ::= X_{68}$	$\cot 2A ::= X_{317}$
$\sin 3A ::= X_{6149}$	$\csc 3A ::= X_{63759}$
$\cos 3A ::= X_{63760}$	$\sec 3A ::= X_{63764}$
$\tan 3A ::= X_{562}$	$\cot 3A ::= X_{63761}$
$\sin 4A ::= X_{1147}$	$\csc 4A ::= X_{55553}$
$\cos 4A ::= X_{63762}$	$\sec 4A ::= X_{63765}$
$\tan 4A ::= X_{43973}$	$\cot 4A ::= X_{55552}$

Barycentric products and quotients ([16], 99-102), denoted by $*$ and $/$, of the polycenters listed in Example 3 are also trigonometric polycenters; e.g., $X_1 * X_{63} = X_3$ and $X_1/X_{63} = X_4$.

In particular, if f is a trigonometic polycenter, then f^n, where n is any positive integer, is also a trigonometric polycenter, as represented by these squares:

Example 4. *Trigonometric square polycenters in ETC [2]. (See also Section 7.)*

$$\begin{aligned}
\sin^2 A &::= a^2 ::= X_6 \\
\csc^2 A &::= b^2c^2 ::= X_{76} \\
\cos^2 A &::= a^2(b^2+c^2-a^2)^2 ::= X_{394} \\
\sec^2 A &::= b^2c^2(b^2+c^2-a^2)^{-2} ::= X_{2052} \\
\cos^2(B-C) &::= b^2c^2(b^4+c^4-2b^2c^2-a^2b^2-a^2c^2)^2 ::= X_{45793} \\
\sin^2(B-C) &::= b^2c^2(b^2-c^2)^2 ::= X_{338} \\
\csc^2(B-C) &::= a^2(b^2-c^2)^{-2} ::= X_{249}
\end{aligned}$$

3. More Trigonometric Polycenters

In this section, we first present polycenters for triangle centers of the forms $\cos(nB - nC)$ :: and $\sin(nB - nC)$, and follow with a proof-by-computer-code for a recurrence equation for the points $\cos(nB - nC)$ :: as polycenters . Let $M = a^2(b^2 + c^2) - (b^2 - c^2)^2$. Then

$$\begin{aligned}
\cos(B-C) &:: = p_1(a,b,c) ::, \text{ where } p_1(a,b,c) = bcM \\
\cos(2(B-C)) &:: = p_2(a,b,c) ::, \text{ where } p_2(a,b,c) = b^2c^2\Big((b^2-c^2)^4 \\
&\qquad + a^4(b^4+c^4) + 2a^2(-b^6+b^4c^2+b^2c^4-c^6)\Big) :: \\
\cos(n(B-C)) &:: = p_n(a,b,c) ::, \text{ where} \\
&\qquad p_n(a,b,c) = bcMp_{n-1}(a,b,c) - a^4b^4c^4p_{n-2}(a,b,c) \text{ for } n \geq 3. \\
\sin(B-C) &:: = q_1(a,b,c) ::, \text{ where } q_1(a,b,c) = bc(b^2-c^2) \\
\sin(2(B-C)) &:: = q_2(a,b,c) ::, \text{ where } q_2(a,b,c) = b^2c^2(b^2-c^2)M \\
\sin(n(B-C)) &:: = q_n(a,b,c) ::, \text{ where} \\
&\qquad q_n = bcMq_{n-1} - a^4b^4c^4q_{n-2} \text{ for } n \geq 2.
\end{aligned}$$

Instead of a formal proof of the above recurrence equation for $p_n(a,b,c)$, we quote the Mathematica code, which is essentially a proof with the added advantage of usefulness for further explorations.

```
(* Step 1: trig functions in terms of a, b, c & S*)
   trigRules={Cos[A]->(-a^2+b^2+c^2)/(2 b c),
   Cos[B]->(a^2-b^2+c^2)/(2 a c),
   Cos[C]->(a^2+b^2-c^2)/(2 a b),
   Sin[A]->S/(b c),Sin[B]->S/(a c),Sin[C]->S/(a b)};

(* Step 2: double area powers in terms of a, b, c & S*)
   SRules={S->S,S^x_?EvenQ->2^-x ((a+b-c) (a-b+c)
   (-a+b+c) (a+b+c))^(x/2),
   S^x_?OddQ->2^(1-x)((a+b-c)(a-b+c)(-a+b+c)(a+b+c))^(1/2 (-1+x)) S};

(* Step 3: cyclic permutations of a,b,c *)
   cyclic[coord_]:=Apply[coord/. {a->#1,b->#2,c->#3,A->#4,B->#5,C->#6}&,
   Flatten/@NestList[RotateLeft/@#1&,{{a,b,c},{A,B,C}},2],{1}];

(* Step 4: removal of symmetric factors *)
   removeSym:=(Factor[#1/PolynomialGCD@@#1]&)[Factor[#]]&;

(* Step 5: application of Steps 1-4 *)
   polys = Map[(TrigExpand[cyclic[Cos[#(B-C)]]]//.trigRules
   //.SRules//removeSym//removeSym)[[1]]&,Range[7]]

(* Step 6: find signature of 2nd order recurrence *)
   Factor[FindLinearRecurrence[polys]]
```

The output of the code is the following signature for the recurrence:

$$\left(bc(a^2b^2 - b^4 + a^2c^2 + 2b^2c^2 - c^4), -a^4b^4c^4\right).$$

A proof of the recurrence equation, or more precisely, the signature of the recurrence, for $\sin(n(B - C))$ :: as a polycenter is found in much the same way.

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As an alternative to representing the family $\cos(nB - nC)$ :: by polynomials, there are relatively simpler representations using quotients of polynomials. We begin with

$$u = \cos(B - C) = \cos B \cos C + \sin B \sin C \tag{12}$$
$$= \frac{(b^2 - c^2)^2 - a^2b^2 - a^2c^2}{2a^2bc} \tag{13}$$
$$v = \sin(B - C) = \sin B \cos C + \sin C \cos B \tag{14}$$
$$= \frac{S(b^2 - c^2)}{2a^2bc}, \text{ where} \tag{15}$$
$$S = (1/2)\sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}. \tag{16}$$

Let

$$f_n = f_n(u, v) = (1/2)\Big((u - iv)^n + (u + iv)^n\Big) = \cos(nB - nC).$$

Then by the binomial theorem,

$$f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} u^{n-2k} v^{2k},$$

which satisfies the recurrence

$$f_n = 2u f_{n-1} - (u^2 + v^2) f_{n-2},$$

with $f_0 = 1, f_1 = u$. Since f_n depends only on u and v^2, it is a rational function; i.e., a radical-free quotient of polynomials. Similary, letting $g_n = g_n(u, v) = \sin(nB - nC)/\sin(B - C)$, we find that

$$g_n = 2u g_{n-1} - (u^2 + v^2) g_{n-2},$$

with $g_0 = 0, g_1 = 1$.

Example 5. *A few more trigonometric polycenters in ETC [2].*

$$\sin(B - C) :: = X_{1577} \qquad \csc(B - C) :: = X_{662}$$
$$\cos(B - C) :: = X_{14213} \qquad \sec(B - C) :: = X_{2167}$$
$$\tan(B - C) :: = X_{15412} \qquad \cot(B - C) :: = X_{14570}$$
$$\sin(2B - 2C) :: = X_{18314} \qquad \csc(2B - 2C) :: = X_{18315}$$
$$\cos(2B - 2C) :: = X_{63763} \qquad \sec(2B - 2C) :: = X_{63766}$$

4. Half-Angle Trigonometric Polycenters

The next list shows half-angle functions that involve polynomials (viz. they are "radical multiples of polycenters"). Let

$$P = \sqrt{\Big((b + c)^2 - a^2\Big)/(bc)} \text{ and } Q = \sqrt{\Big(a^2 - (b - c)^2\Big)/(bc)}.$$

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$$\begin{aligned}
\cos(A/2) &:: = p_1(a,b,c) ::, \text{ where } p_1(a,b,c) = P \\
\cos(3A/2) &:: = p_3(a,b,c) ::, \text{ where } p_3(a,b,c) = P(a^2-b^2-c^2+bc) \\
\cos(nA/2) &:: = p_n(a,b,c) ::, \text{ where } p_n(a,b,c) = (-a^2+b^2+c^2)/(bc)p_{n-2} \\
&\quad - a^4b^4c^4 p_{n-4} ::, \text{ for odd } n \geq 5; \\
\sin(A/2) &:: = q_1(a,b,c) ::, \text{ where } q_1(a,b,c) = Q :: \\
\sin(3A/2) &:: = q_3(a,b,c) ::, \text{ where } q_3(a,b,c) = Q(a^2-b^2-c^2-bc) :: \\
\sin(nA/2) &:: = q_n(a,b,c) ::, \text{ where } q_n(a,b,c) = (-a^2+b^2+c^2)/(bc)q_{n-2} \\
&\quad - a^4b^4c^4 q_{n-4}, ::, \text{ for odd } n \geq 5.
\end{aligned}$$

Next we show Mathematica code for obtaining trigonometric rational functions (quotients of polynomials) for $\cos((nB-nC)/2)$ :: .

```
lr = FindLinearRecurrence[
   Map[TrigExpand[Cos[# (B - C)/2]] &, Range[1, 11, 2]]];
cyclic[coord_] :=
  Apply[coord /. {a -> #1, b -> #2, c -> #3, A -> #4, B -> #5,
      C -> #6} &, Flatten /@
    NestList[RotateLeft /@ #1 &, {{a, b, c}, {A, B, C}}, 2], {1}];
trigRules =
  Flatten[{Map[
     cyclic, {Cos[A] -> (-a^2 + b^2 + c^2)/(2  b  c),
      Sin[A] -> S/(b  c),
      Cos[A/2] -> 1/2  Sqrt[((-a + b + c)  (a + b + c))/(b  c)],
      Sin[A/2] -> 1/2  Sqrt[((a + b - c)  (a - b + c))/(b  c)],
      Cos[B/2]*Cos[C/2]*Sin[B/2]*
        Sin[C/2] -> ((-a + b + c)  (a + b - c)  (a - b + c)
        (a + b + c))/(16  a^2  b  c)}]}];
Factor[lr /. trigRules]
```

This code confirms that $\cos((nB-nC)/2)$ :: is a rational function with signature

$$((a^2b^2-b^4+a^2c^2+2b^2c^2-c^4)/(a^2bc), -1).$$

(These rational functions can be transformed into polynomials using a technique developed in Section 6.)

Example 6. *A few half-angle trigonometric polycenters in ETC [2].*

$\cos(A/2) :: = X_{188}$	$\sec(A/2) :: = X_{4146}$
$\sin(A/2) :: = X_{174}$	$\csc(A/2) :: = X_{556}$
$\cos(3A/2) :: = X_{63779}$	$\sin(3A/2) :: = X_{63780}$

5. Sums Involving mB+nC and nB+mC

Proofs of the next two theorems can be obtained by adapting the codes in Sections 3 and 4.

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Theorem 4. *Let* $f(m,n) = f(m,n,a,b,c) = \sin(mB+nC) + \sin(nB+mC)$ *and let* $p(m,n) = p(m,n,a,b,c)$ *be the polycenters given by these recurrences:*

$$p(m,n) = a(a^2-b^2-c^2)p(m-1,n) - a^2b^2c^2p(m-2,n);$$
$$p(m,n) = a(a^2-b^2-c^2)p(m,n-1) - a^2b^2c^2p(m,n-2),$$

where $p(0,0) = 0$, *and* $p(0,1) = p(1,0) = b+c$. *Then*

$$p(m,n,a,b,c) ::= \sin(mB+nC) + \sin(nB+mC) :: \quad .$$

Example 7. *The appearance of* (m,n,X_k) *in the following list signifies that* $\sin(mB+nC) + \sin(nB+mC) ::= X_k$.

$$(0,1,X_{10}), \quad (0,2,X_5), \quad (0,3,X_{63803}), \quad (0,4,X_{5449})$$
$$(1,1,X_1), \quad (1,2,X_{63802}), \quad (1,3,X_{44707})$$
$$(2,2,X_3), \quad (2,3,X_{63801}), \quad (2,4,X_{1154})$$
$$(3,3,X_{6149}), \quad (3,5,X_{63801})$$

Theorem 5. *Let* $g(m,n) = g(m,n,a,b,c) = \cos(mB+nC) + \cos(nB+mC)$ *and let* $q(m,n) = q(m,n,a,b,c)$ *be the polycenters given by the same recurrences as for* $p(m,n)$, *where*

$$q(0,1) = q(1,0) = (b+c)(c+a-b)(a+b-c)$$
$$q(0,2) = (b^2+c^2-a^2)(a^2(b^2+c^2)-(b^2-c^2)^2).$$

Then

$$q(m,n,a,b,c) ::= \cos(mB+nC) + \cos(nB+mC) :: \quad .$$

Example 8. *The appearance of* (m,n,X_k) *here means that* $\cos(mB+nC) + \cos(nB+mC) ::= X_k$.

$$(0,1,X_{226}), \quad (0,2,X_{343}), \quad (0,4,X_{63806})$$
$$(1,1,X_{63}), \quad (1,2,X_{16577}), \quad (1,3,X_{63808})$$
$$(2,2,X_{1993}), \quad (2,3,X_{73802})$$
$$(3,3,X_{63760}), \quad (3,5,X_{63762})$$

6. Polycenters j+k cos(nA)::

Here, we find a sequence $p_n = p_n(a,b,c)$ of polycenters satisfying

$$p_n(a,b,c) : p_n(b,c,a) : p_n(c,a,b)$$
$$= j + k\cos(nA) : j + k\cos(nB) : j + k\cos(nC), \tag{17}$$

where j and k are nonzero real numbers. The strategy is to determine rational functions $u_n = u_n(a,b,c)$ that can be transformed into the polynomials p_n. We begin with the following Mathematica code:

```
f[a_, b_, c_] := f[a, b, c] = ArcCos[(b^2 + c^2 - a^2)/(2 b  c)];
{a1, b1, c1} = {f[a, b, c], f[b, c, a], f[c, a, b]};
a2[n_] := Collect[Factor[TrigExpand[j + k*Cos[n  a1]]], {j, k}];
```

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```
b2[n_] := Collect[Factor[TrigExpand[j + k*Cos[n  b1]]], {j, k}];
c2[n_] := Collect[Factor[TrigExpand[j + k*Cos[n  c1]]], {j, k}];
t = Table[a2[n], {n, 0, 10}]; Take[t, 4]
FindLinearRecurrence[t]
t = Table[b2[n], {n, 0, 10}]; Take[t, 4]
FindLinearRecurrence[t]
t = Table[c2[n], {n, 0, 10}]; Take[t, 4]
FindLinearRecurrence[t]
```

The code gives

$$u_0 = j + k \tag{18}$$

$$u_1 = j + \frac{-a^2 + b^2 + c^2}{2bc}k \tag{19}$$

$$u_2 = j + \frac{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2}{2b^2c^2}k \tag{20}$$

and recurrence signature $(\tau, -\tau, 1)$, where

$$\tau = \frac{-a^2 + b^2 + c^2 + bc}{bc}. \tag{21}$$

The transformation is simply to multiply, where appropriate, by $2a^n b^n c^n$, obtaining

$$p_0 = 2j + 2k \tag{22}$$

$$p_1 = 2abcj + a(b^2 + c^2 - a^2)k \tag{23}$$

$$p_2 = 2a^2b^2c^2 j + \Big(a^6 - a^4(b^2 + c^2) + a^2(b^4 + c^4)\Big)k, \tag{24}$$

with third-order recurrence signature

$$(-aQ_a, a^2bcQ_a, a^3b^3c^3), \tag{25}$$

where $Q_a = Q_a(a, b, c) = a^2 - b^2 - c^2 - bc$.

Finally, we apply the substitutions $(a, b, c) \to (b, c, a)$ and $(a, b, c) \to (c, a, b)$ to (18)-(25) and thereby obtain (17).

7. Applications of the Technique in Section 6

The procedure in Section 6 applies to other families of trigonometric polycenters. Among them are the families $\cos^2(nB - nC)$ :: and $\sin^2(nB - nC)$ ::. Both $\cos^2(nB - nC)$ and $\sin^2(nB - nC)$ have third-order recurrence with signature $(k(a, b, c), -k(a, b, c), 1)$, where

$$k = \frac{(b^2 - c^2)^4 + a^4(b^4 + c^4 + b^2c^2) + 2a^2(b^4c^2 + b^2c^4 - b^6 - c^6)}{a^4b^2c^2}.$$

The resulting polycenters are too long for display here. We do observe, however, that for every integer $p \geq 2$, the polycenters $\cos^p(nB - nC)$ :: and $\sin^p(nB - nC)$ :: have recurrence order $p + 1$.

Other families to which the procedure applies are represented by

$$\tan nA ::, \quad \sec nA ::, \quad \csc nA ::, \quad \cot nA :: .$$

Here we consider only the rational-function recurrence for $\tan nA$. The most direct approach appears to be to use the identity

$$\tan nA ::= \sin nA \cos nB \cos nC :: .$$

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The resulting sixth-degree recurrence signature for $\sin nA \cos nB \cos nC$ is more efficiently expressed with Conway notation [15] than with a, b, c:

$$(k_1, k_2, k_3, k_2, k_1, 1),$$

where

$$k_1 = \frac{-2(3S_AS_BS_C + S^2S_\omega)}{D}$$
$$k_2 = -\frac{16S^6 + 15S_A^2S_B^2S_C^2 + 2S^2S_AS_BS_CS_\omega - S^4S_\omega^2}{D}$$
$$k_3 = -\frac{-4(-8S^6 + 5S_A^2S_B^2S_C^2 + 2S^2S_aS_BS_CS_\omega + S^4S_\omega^2)}{D},$$

where $D = (S_AS_BS_CS^2S_\omega)^2$.

8. Infinite Trigonometric Orthopoints

The line at infinity consists of all points $X = x : y : z$ satisfying the linear equation

$$x + y + z = 0. \tag{26}$$

Most of the named points on L^∞ are polycenters. Among the simplest are

$$X_{513} = a(b - c) ::$$
$$X_{514} = b - c ::$$
$$X_{30} = \text{cos A - 2 cos B cos C} :: = 2a^4 - (b^2 - c^2)2 - a^2(b^2 + c^2) :: ,$$

these being the points in which the lines $X_1X_{75}, X_1X_2, X_2X_3$ meet L^∞, respectively. Of special importance is X_{30}, as this is the infinite point on the Euler line, X_2X_3.

If $X = x : y : z$ is on L^∞, then X can be regarded as a direction in the plane of $\triangle ABC$, since for every point P not on L^∞, every line parallel to the line PX intersects L^∞ in X. The line through P orthogonal to PX meets L^∞ in a point called the *orthopoint* (or, in [8], the *orthogonal conjugate*) of X. We denote the orthopoint of X by $X^\perp$. Barycentrics for $X^\perp$ are given by

$$X^\perp = y \cos B - z \cos C :: . \tag{27}$$

Thus if X is a polycenter represented by a polynomial $x(a, b, c)$ as first barycentric, then $X^\perp$ is the polycenter

$$X^\perp = x(b, c, a)b(b^2 - a^2 - c^2) - x(c, a, b)c(c^2 - b^2 - a^2) :: . \tag{28}$$

Now for any point $X = x : y : z$, not necessarily a polycenter and not necessarily on L^∞, the points

$$y - z : z - x : x - y \quad \text{and} \quad 2x - y - z : 2y - z - x : 2z - x - y$$

are clearly on L^∞, as are their orthopoints,

$$b(z - x)(b^2 - a^2 - c^2) - c(x - y)(c^2 - b^2 - a^2) :: \tag{29}$$

and

$$b(2y - z - x)(b^2 - a^2 - c^2) - c(2z - x - y)(c^2 - b^2 - a^2) :: , \tag{30}$$

respectively. Moreover, if X is a polycenter, then the orthopoints (29) and (30) are polycenters on L^∞.

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Example 9. *Let*
$$X = x : y : z = \cos^2 A ::= a^2(a^2 - b^2 - c^2)^2 ::= X_{394}.$$

Then

$$y - z :: = \cos^2 B - \cos^2 C ::= X_{523}$$
$$2x - y - z :: = 2\cos^2 A - \cos^2 B - \cos^2 C ::= X_{527},$$

and the two corresponding orthopoints (29) and (30) are respectively

$$X_{30} = 2a^4 - (b^2 - c^2)^2 - a^2(b^2 + c^2) ::$$
$$X_{28292} = bc/(h(a,b,c)h(a,c,b)) ::, \text{ where}$$
$$h(a,b,c) = a^3 + 3b^3 + c^3 + a^2(b - c) - 5b^2(a + c) + c^2(b - a) + 2abc.$$

Example 9 typifies infinite polycenters of the forms $f(B) - f(C)$ :: and $f(2A) - f(B) - f(C)$ ::. Such polycenters, for which many algebraic and geometric properties are presented in ETC [2], occupy the lists in the next two examples.

Example 10. *Pairs of trigonometric orthopoints.*

$$\cos B - \cos C :: = X_{522}, \quad X_{522}^{\perp} = X_{515}$$
$$\sin B - \sin C :: = X_{514}, \quad X_{514}^{\perp} = X_{516}$$
$$\tan B - \tan C :: = X_{525}, \quad X_{525}^{\perp} = X_{1503}$$
$$\sec B - \sec C :: = X_{521}, \quad X_{521}^{\perp} = X_{6001}$$
$$\csc B - \csc C :: = X_{513}, \quad X_{513}^{\perp} = X_{517}$$
$$\cot B - \cot C :: = X_{523}, \quad X_{523}^{\perp} = X_{30}$$
$$\cos^2 B - \cos^2 C :: = X_{523}, \quad X_{523}^{\perp} = X_{30}$$
$$\sin^2 B - \sin^2 C :: = X_{523}, \quad X_{523}^{\perp} = X_{30}$$

Example 11 continued:

$$\tan^2 B - \tan^2 C :: = X_{520}, \quad X_{520}^{\perp} = X_{6000}$$
$$\sec^2 B - \sec^2 C :: = X_{520}, \quad X_{520}^{\perp} = X_{6000}$$
$$\csc^2 B - \csc^2 C :: = X_{512}, \quad X_{512}^{\perp} = X_{511}$$
$$\cot^2 B - \cot^2 C :: = X_{512}, \quad X_{512}^{\perp} = X_{511}$$
$$\cos 2B - \cos 2C :: = X_{523}, \quad X_{523}^{\perp} = X_{30}$$
$$\sin 2B - \sin 2C :: = X_{525}, \quad X_{525}^{\perp} = X_{1503}$$
$$\sec 2B - \sec 2C :: = X_{924}, \quad X_{924}^{\perp} = X_{13754}$$
$$\cot 2B - \cot 2C :: = X_{6368}, \quad X_{6368}^{\perp} = X_{18400}$$

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Example 11. *More pairs of trigonometric orthopoints.*

$$
\begin{aligned}
2\cos A - \cos B - \cos C :: &= X_{527}, \quad X_{527}^{\perp} = X_{28292} \\
2\sin A - \sin B - \sin C :: &= X_{519}, \quad X_{527}^{\perp} = X_{3667} \\
2\tan A - \tan B - \tan C :: &= X_{30}, \quad X_{30}^{\perp} = X_{523} \\
2\csc A - \csc B - \csc C :: &= X_{536}, \quad X_{536}^{\perp} = X_{28475} \\
2\cot A - \cot B - \cot C :: &= X_{524}, \quad X_{524}^{\perp} = X_{1499} \\
2\cos^2 A - \cos^2 B - \cos^2 C :: &= X_{524}, \quad X_{524}^{\perp} = X_{1499} \\
2\sin^2 A - \sin^2 B - \sin^2 C :: &= X_{524}, \quad X_{524}^{\perp} = X_{1499} \\
2\csc^2 A - \csc^2 B - \csc^2 C :: &= X_{538}, \quad X_{538}^{\perp} = X_{32472} \\
2\cot^2 A - \cot^2 B - \cot^2 C :: &= X_{538}, \quad X_{538}^{\perp} = X_{32472} \\
2\cos 2A - \cos 2B - \cos 2C :: &= X_{524}, \quad X_{524}^{\perp} = X_{1499} \\
2\sin 2A - \sin 2B - \sin 2C :: &= X_{30}, \quad X_{30}^{\perp} = X_{523} \\
2\tan 2A - \tan 2B - \tan 2C :: &= X_{539}, \quad X_{539}^{\perp} = X_{20184}
\end{aligned}
$$

9. Trigonometric Infinity Bisectors

Let O denote the circumcenter, (O) the circumcircle, and L^{∞} the line at infinity. Suppose that $P = p : q : r$ and $U = u : v : w$ are points on (O) and that P, O, U are noncollinear. Let L_1 be the tangent to (O) at P and L_2 the tangent to (O) at U. Let $D = L_1 \cap L_2$ and $M = OD \cap L^{\infty}$. As the line OM bisects the angle between L_1 and L_2, the point M is called the (P, U)-infinity bisector. We denote this point by $M(P, U)$. Its barycentrics are given by

$$M(P,U) = (a^2 - b^2 + c^2)(qu - pv) - (a^2 + b^2 - c^2)(ru - pw) - 2a^2(rv - qw) :: \tag{31}$$

If P and U are trigonometric polycenters, then (31) is also a trigonometric polycenter, since

$$b^2 + c^2 - a^2 : c^2 + a^2 - b^2 : a^2 + b^2 - c^2 = \cot A : \cot B : \cot C.$$

Note that the $M(P, U)$ is the orthopoint of the $PU \cap L^{\infty}$. A few examples follow:

$$
\begin{aligned}
M(X_{74}, X_{477}) &= X_{30} = X_{523}^{\perp} \\
M(X_{110}, X_{476}) &= X_{30} = X_{523}^{\perp} \\
M(X_{74}, X_{476}) &= X_{523} = X_{30}^{\perp} \\
M(X_{110}, X_{477}) &= X_{523} = X_{30}^{\perp} \\
M(X_{74}, X_{98}) &= X_{542} = X_{690}^{\perp} \\
M(X_{99}, X_{110}) &= X_{542} = X_{690}^{\perp} \\
M(X_{74}, X_{99}) &= X_{690} = X_{542}^{\perp} \\
M(X_{98}, X_{110}) &= X_{690} = X_{542}^{\perp} \\
M(X_{74}, X_{110}) &= X_{526} = X_{5663}^{\perp} \\
M(X_{98}, X_{99}) &= X_{804} = X_{2782}^{\perp}
\end{aligned}
$$

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10. Trigonometric Polylines

Among the many central lines [9] of interest in triangle geometry are the trigonometic polylines n-Euler line and n-Nagel line. The Euler line itself is given by the following barycentric equations:

$$\begin{aligned} 0 &= (\tan B - \tan C)x + (\tan C - \tan A)y + (\tan A - \tan B)z && (32)\\ &= (\sin 2B - \sin 2C)x + (\sin 2C - \sin 2A)y + (\sin 2A - \sin 2B)z && (33)\\ &= \cos A \sin(B-C)x + \cos B \sin(C-A)y + \cos C \sin(A-B)z && (34)\\ &= (b^2-c^2)(b^2+c^2-a^2)x + (c^2-a^2)(c^2+a^2-b^2)y \\ &\quad + (a^2-b^2)(a^2+b^2-c^2)z \end{aligned}$$

The n-Euler line is defined by substituting nA, nB, nC for A, B, C, respectively, in (32), (33), or (34). The n-Euler line passes through the following n-polycenters, which, for $n = 1$ are indexed in ETC [2] as X_2, X_3, X_4, X_5, respectively:

$$\begin{aligned} n\text{-centroid} &= c_0 = 1:1:1 \\ n\text{-circumcenter} &= \cos nA :: \\ n\text{-orthocenter} &= \tan nA :: \\ n\text{-nine-point center} &= \sin nA \cos(nB - nC) :: . \end{aligned}$$

These points appear in a little-known paper [7] in a discussion of "layers" in triangle geometry—without mention of the fact that the n-points and n-lines have polynomial representations.

The Nagel line is given by the equations

$$\begin{aligned} 0 &= (\sin B - \sin C)x + (\sin C - \sin A)y + (\sin A - \sin B)z. && (35)\\ &= (b-c)x + (c-a)y + (a-b)z, \end{aligned}$$

and the n-Nagel line by

$$0 = (\sin nB - \sin nC)x + (\sin nC - \sin nA)y + (\sin nA - \sin nB)z. \qquad (36)$$

The Nagel line passes through the incenter, $X_1 = \sin A : \sin B : \sin C$ and the centroid, $X_2 = 1:1:1$, so that the n-Nagel line passes through the centroid and the point $\sin nA : \sin nB : \sin nC$. Thus, for every n, the n-Euler line and n-Nagel line meet in the centroid. Moreover, by (33) and (36), the $2n$-Nagel line and n-Euler line are identical for every positive integer n. Among the notable trigonometric polycenters on the 2-Euler line, alias 4-Nagel line, are the following:

$$\begin{aligned} X_2 &= 1:1:1 \\ X_{54} &= \text{ Kosnita point} = \sin A \sec(B-C) :: \\ &= \text{ isogonal conjugate of the nine-point center, } X_5 \\ X_{68} &= \text{ Prasolov point} = \tan 2A :: \\ X_{1147} &= \sin 4A :: \\ X_{5449} &= \text{2nd Hatzipolakis-Moses point} = \sin 4B + \sin 4C :: \\ &= \text{midpoint of} X_{68} \text{ and } X_{1147} \\ X_{6193} &= \tan B + \tan C - \tan A :: \\ X_{16032} &= \csc A \sec(B-C) :: \end{aligned}$$

Each of these points, and others on the 2-Euler line, has a list of properties in ETC [2], involving many other trigonometric polycenters and their interrelationships.

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11. Concluding Remarks

The notion of trigonometric polycenter extends to various subjects, other than those mentioned above. Several examples follow:

- Triangle centers whose barycentrics depend on angles of the form
$$nA + r\pi, nB + r\pi, nC + r\pi$$
for some nonzero number r, such as the Fermat point,
$$X_{13} = \sin A \csc(A + \pi/3) : \sin B \csc(B + \pi/3) : \sin C \csc(C + \pi/3)$$
$$= a^4 - 2(b^2 - c^2)^2 + a^2(b^2 + c^2 + \Psi) ::,$$
where $\Psi = 4\sqrt{3}\times$(area of triangle ABC), and related points X_i for $i = 14, \ldots, 18$ in ETC [2].
- Bicentric pairs [1] of points, such as the Brocard points,
$$1/b^2 : 1/c^2 : 1/a^2 = \sin^2 C \sin^2 A : \sin^2 A \sin^2 B : \sin^2 B \sin^2 C$$
$$1/c^2 : 1/a^2 : 1/b^2 = \sin^2 B \sin^2 A : \sin^2 C \sin^2 B : \sin^2 A \sin^2 C,$$
leading to Brocard n-points by substituting nA, nB, NC for A, B, C, respectively.
- Cubic curves such as those indexed and elegantly described by Bernard Gibert [3]. Here we sample just one of more than one thousand: K007, the Lucas cubic, consisting of all points $x : y : z$ that satisfy
$$(b^2 + c^2 - a^2)x(y^2 - z^2) + (c^2 + a^2 - b^2)y(z^2 - x^2) + (a^2 + b^2 - c^2)z(x^2 - y^2) = 0.$$
For every n, the symbolic substitution
$$(a \to \cos nA, b \to \cos nB, c \to \cos nC)$$
transforms this "polynomial cubic" into a "trigonometric cubic", and likewise for the substitution
$$(a \to \sin nA, b \to \sin nB, c \to \sin nC),$$
etc. For details regarding symbolic substitutions see [6].
- Triangle centers that result from unary operations on trigonometric polycenters, such as
$$(y - z)/x : (z - x)/y : (x - y)/z,$$
where $x : y : z$ is a trigonometric polycenter. See [10].
- For specific numbers a, b, c, such as $(a, b, c) = (2, 3, 4)$, representing the smallest integer-sided isosceles triangle, we have integer sequences such as given by
$$a_n = 2^{2^n+1} \cos nA,$$
where A, as usual, is the angle opposite side BC in a triangle ABC having sidelengths $|BC| = a, |CA| = b, |AB| = c$. Such sequences have interesting divisibility properties, such as the fact that if p is a prime that divides a term, then the indices n such that p divides n comprise an arithmetic sequence. For this sequence and access to related sequences, see A375880.

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- A final comment may be loosely summarized by the observation that, throughout this paper, the role of homogeneous coordinates can be taken by trilinear coordinates [4], but with different results. For example, in trilinear coordinates, we have

$$X_2 = a : b : c = \sin A : \sin B : \sin C$$
$$X_3 = a(b^2 + c^2 - a^2) ::= \cos A : \cos B \cos C :: ,$$

which lead to trigonometric polycenters by substituting nA, nB, nC for A, B, C. The resulting trilinear representations are equivalent to the barycentric representations $\sin A \sin nA$ :: and $\sin A \cos nA$ ::, these being trigonometric polycenters not previously mentioned in this article.

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