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Article

Cosmology due to Thermodynamics of Apparent Horizon

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Abstract: In this paper we propose new entropy of the apparent horizon $S_h = (1/\beta)\arctan(\beta S_{BH})$, where S_{BH} is the Bekenstein–Hawking entropy. As parameter $\beta \to 0$ one comes to the Bekenstein–Hawking entropy. This allows us to consider the generalised Friedmann–Lemaître–Robertson–Walker (FLRW) equations for the barotropic matter fluid with $p = w\rho$ for arbitrary equation of state parameter w. We obtain the matter pressure p and density energy ρ corresponding to the apparent horizon. The modified Friedmann's equations are found. The addition term in the second modified Friedmann's equation plays the role of a dynamical cosmological constant. The dark energy density, pressure and the deceleration parameter are found. It was shown that at some parameters w and β we can have two phases, acceleration and deceleration or the eternal inflation. The model under consideration by using the holographic principle describes the universe inflation. Thus, we consider the holographic dark energy model with the generalised entropy of the apparent horizon. We shown that entropic cosmology with our entropy proposed is equivalent to cosmology based on the teleparallel gravity with the function $F(T)$. New cosmology based on the generalized entropy can be of interest for a description of inflation and late time of the universe evolution.

Keywords: entropy; apparent horizon; Friedmann's equations; barotropic matter fluid; dynamical cosmological constant; dark energy; deceleration parameter; cosmology based on the generalized entropy can be of interest for a description of inflation and late time of the universe evolution

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1. Introduction

It was proven that black holes obey thermodynamic laws where entropy is proportional to the horizon area [1,2] and temperature is connected with the surface gravity so that gravity is related to ordinary thermodynamics [3–6]. The Friedmann equations also can be obtained from the first law of apparent horizon thermodynamics [7–17]. Different forms of entropies were considered which lead to modified Friedmann's equations [18–24]. Some holographic dark energy models were studied in Refs. [25–30]. Entropies are sources of holographic energy densities that can describe the dark energy of the universe [31,32]. Here, we propose new apparent horizon entropy $S_h = (1/\beta)\arctan(\beta S_{BH})$ with S_{BH} being the Bekenstein–Hawking entropy. Our entropy, as well as other viable entropies, vanishes when the Bekenstein–Hawking entropy becomes zero. In addition, the entropy under consideration is the monotonically increasing function of the Bekenstein–Hawking entropy S_{BH} and is positive which is the natural requirement. When parameter $\beta \to 0$ we have the Bekenstein–Hawking entropy, $S_h \to S_{BH}$. It is worth mentioning that the apparent horizon thermodynamics leads to the Friedmann equations within Einstein's gravity only for a particular case when the matter is a perfect fluid with equation of state (EoS) given by $p = -\rho$, where p is the matter pressure and ρ is the density energy of matter [16]. Here, we modify the Bekenstein–Hawking entropy S_{BH} by apparent horizon entropy S_h to consider the general case with arbitrary EoS state parameter for barotropic perfect fluid $w = p/\rho$ including the case $w = -1$. It is known that the long-range gravitational interactions are better described by generalized entropies.

We will show that our entropy results to modified Friedmann's equations and the universe inflation. This approach corresponds to Einstein's equations with dynamical cosmological constant. As

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a result, the universe inflation and dark energy are due to dynamical cosmological constant. It is worth mentioning that the inflation of the universe can be explained, for example, by coupling Einstein's gravity with nonlinear electrodynamics (see [33] and references therein).

2. Thermodynamics of Apparent Horizon

In the following we consider the FLRW flat universe with the metric

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\Omega_2^2), \tag{2.1}$$

where $a(t)$ is a scale factor and $d\Omega_2^2$ denotes the line element of an 2-dimensional unit sphere. For the FLRW universe the radius of the apparent horizon $R_h = a(t)r$ reads

$$R_h = \frac{1}{H}, \tag{2.2}$$

with the Hubble parameter of the universe $H = \dot{a}(t)/a(t)$, where dot over $a(t)$ means the derivative with respect to the cosmological time t. The total energy inside the space is

$$E = \rho V_h = \frac{4\pi}{3}\rho R_h^3, \tag{2.3}$$

while $-dE$ is the change of the energy inside the apparent horizon. Here, ρ denotes the energy density of matter fields. The first law of apparent horizon thermodynamics is given by

$$dE = -T_h dS_h + W dV_h, \tag{2.4}$$

where the work density in the cosmology is

$$W = -\frac{1}{2}\mathrm{Tr}(T^{\mu\nu}) = \frac{1}{2}(\rho - p).$$

where p being the matter pressure. The apparent horizon temperature is given by

$$T_h = \frac{H}{2\pi}\left|1 + \frac{\dot{H}}{2H^2}\right|. \tag{2.5}$$

From first law of apparent horizon thermodynamics (2.4), taking into account Equations (2.2), (2.3) and (2.5), we obtain

$$\frac{H}{2\pi}\left|1 + \frac{\dot{H}}{2H^2}\right| dS_h = -\frac{4\pi}{3H^3}d\rho + \frac{2\pi(\rho + p)}{H^4}dH, \tag{2.6}$$

or

$$\frac{H}{2\pi}\left|1 + \frac{\dot{H}}{2H^2}\right|\dot{S}_h = -\frac{4\pi\dot{\rho}}{3H^3}\left(1 + \frac{\dot{H}}{2H^2}\right), \tag{2.7}$$

where we have used the continuity equation (the energy momentum conservation)

$$\dot{\rho} = -3H(\rho + p). \tag{2.8}$$

3. Modified FLRW Equations

From Equations (2.7), (2.8) and assuming that $1 + \dot{H}/(2H^2) > 0$, we obtain

$$\frac{H}{2\pi}\dot{S}_h = \frac{4\pi(\rho + p)}{H^2}. \tag{3.1}$$

From our proposed entropy (see Appendix)

$$S_h = \frac{1}{\beta}\arctan(\beta S_{BH}) \tag{3.2}$$

with $S_{BH} = \pi R_h^2/G = \pi/(GH^2)$, one finds from Equation (3.1) the modified Friedmann equation

$$\frac{\dot{H}}{1+\beta^2\pi^2/(G^2H^4)} = -4\pi G(\rho+p). \tag{3.3}$$

At $\beta \to 0$ equation (3.3) becomes the usual Friedmann equation for flat universe within general relativity. Utilizing Equation (2.8) and after integrating Equation (3.3), we obtain the second modified Friedmann equation

$$H^2 - \frac{\beta\pi}{G}\arctan\left(\frac{GH^2}{\beta\pi}\right) = \frac{8\pi G}{3}\rho. \tag{3.4}$$

At $\beta = 0$ we comes to usual FLRW equation for flat universe within Einstein's gravity. Equation (3.4) can be represented as

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda_{eff}}{3}. \tag{3.5}$$

where $\Lambda_{eff} = \frac{3\beta\pi}{G}\arctan\left(\frac{GH^2}{\beta\pi}\right)$ plays the role of the effective (a dynamical) cosmological constant. From Equation (3.5) we obtain the dark energy density

$$\rho_D = \frac{\Lambda_{eff}}{8\pi G} = \frac{3\beta}{8G^2}\arctan\left(\frac{GH^2}{\beta\pi}\right). \tag{3.6}$$

At small $\pi\beta/(GH^2)$ the effective cosmological constant becomes $\Lambda_{eff} \approx 3\beta\pi^2/2G$ while at small $GH^2/(\pi\beta)$ it reads $\Lambda_{eff} \approx 3H^2$. The plot of Λ_{eff} versus H at different $b = \pi\beta/G$ is given in Figure 1.

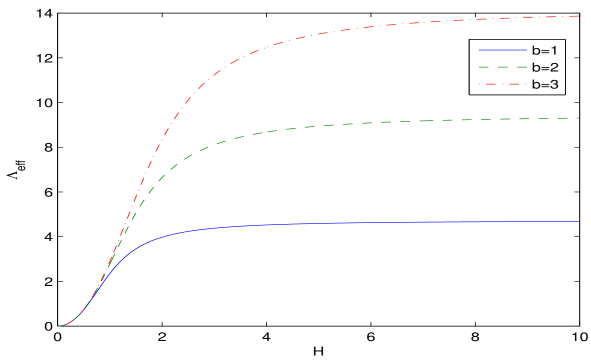

Figure 1. The function Λ_{eff}) versus H at $b = \pi\beta/G = 1,2,3$. Figure 1 shows that Λ_{eff} increases as b increases. The asymptotic of effective cosmological constant is $\Lambda_{eff} \to 3\pi b/2$ as $H \to \infty$ (for early times). For later times $H \to 0$, we have $\Lambda_{eff} \to 0$.

The normalized density parameters are given by $\Omega_m = 8\pi G\rho/(3H^2)$ and $\Omega_D = 8\pi G\rho_D/(3H^2)$. Making use of Equations (3.4) and (3.6) we obtain $\Omega_m + \Omega_D = 1$. From Equation (3.4) one finds the normalized density for the matter

$$\Omega_m = 1 - \frac{b}{H^2}\arctan\left(\frac{H^2}{b}\right), \tag{3.7}$$

where $b = \pi\beta/G$. The plot of Ω_m versus H is depicted in Figure 2.

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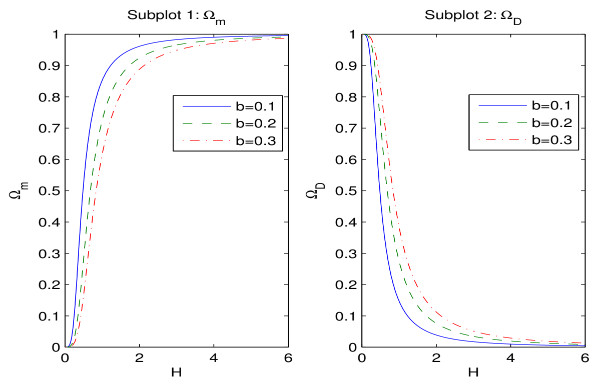

Figure 2. The function Ω_m versus H at $b == 0.1, 0.2, 0.3$. **Left panel**: Figure shows that Ω_m increases as H increases. As $H \to \infty$ ($R_h \to 0$) we have $\Omega_m \to 1$. When parameter b increases Ω_m decreases. **Right panel**: Ω_D decreases as H increases. If parameter b increases Ω_D also increases.

Figure 2 shows that for some parameter b we can find $\Omega_m = 0.26$ for the current era. Implying that dark substance obeys ordinary conservation law, where no mutual interaction between the cosmos components, we find the pressure

$$p_D = -\frac{\dot{\rho}_D}{3H} - \rho_D. \tag{3.8}$$

Making use of Equations (3.6) and (3.8) one finds the pressure

$$p_D = -\frac{\pi\beta^2\dot{H}}{4G(\pi^2\beta^2 + G^2H^4)} - \frac{3\beta}{8G^2}\arctan\left(\frac{GH^2}{\beta\pi}\right). \tag{3.9}$$

From Equations (3.3), (3.4) and (3.9) we obtain

$$p_D = \frac{3b^2(1+w)}{8\pi G H^4}\left(H^2 - b\arctan\left(\frac{H^2}{b}\right)\right) - \rho_D. \tag{3.10}$$

By virtue of Equations (3.6) and (3.10) one finds EoS for dark energy

$$w_D = \frac{p_D}{\rho_D} = \frac{(1+w)b^2}{H^4}\left(\frac{H^2}{b\arctan(H^2/b)} - 1\right) - 1. \tag{3.11}$$

The plot of w_D versus H is given in Figure 3.

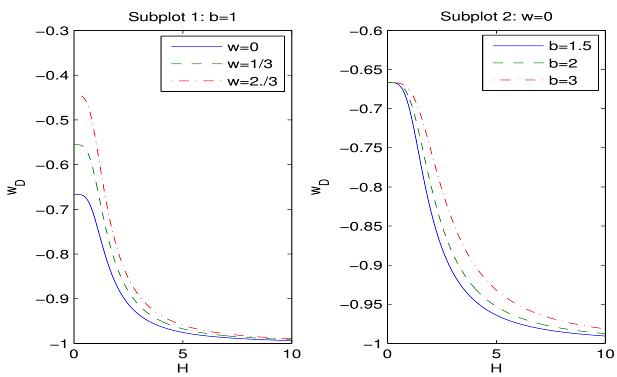

Figure 3. **Left panel**: The function w_D versus H at $b = 1$, $w = 0, 1/3, 2/3$. According to Figure 3 the w_D increases as EoS parameter for the matter w increases, at fixed H. **Right panel**: In accordance with figure when parameter b increases, at fixed H, EoS parameter for dark energy w_D also increases.

According to Figure 3 (Left panel) at $b = 1$ and $w = 0, 1/3, 2/3$ when w increases, at fixed H, EoS parameter for dark energy w_D also increases. Figure 3 shows (Right panel) that at $w = 1$ and $b = 1.5, 2, 3$ if b increases, at fixed H, w_D also increases. It is interesting that at $H = 0$ ($R_h = \infty$) we get

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from Equation (3.11) the finite value $\lim_{H\to 0} w_D = (w-2)/3$ which does not depend on parameter $b = \pi\beta/G$. At the same time, $\lim_{H\to\infty} w_D = -1$ ($R_h = 0$) corresponding to de Sitter space.

The second law of thermodynamics leads to the requirement $\dot{S}_h \geq 0$ and from Equation (3.2) we obtain $\dot{S}_{BH}/(1+\beta^2 S_{BH}^2) \geq 0$ or $\dot{S}_{BH} = -2\pi\dot{H}/(GH^3) \geq 0$. Thus, we have the same inequality as for the Bekenstein–Hawking entropy. This requirement for positive Hubble parameter gives $\dot{H} \leq 0$. As a result, from Equation (3.3) one finds $\rho + p \geq 0$ or for positive energy density we have for EoS parameter $w \geq -1$. One can use the redshift $z = a_0/a(t) - 1$ instead of the scale factor $a(t)$, where a_0 is a constant corresponding to a scale factor at the current time. Then from the continuity equation (2.8) and EoS $p = w\rho$ we find the density energy of matter as

$$\rho = \rho_0 \left(\frac{1+z}{a_0}\right)^{3(1+w)}. \tag{3.12}$$

where ρ_0 is the density energy of matter at the present time. Making use of Equations (3.4) and (3.12) we obtain

$$\frac{1}{R_h^2} - \frac{\beta\pi}{G}\arctan\left(\frac{G}{\beta\pi R_h^2}\right) = \frac{8\pi G}{3}\rho_0\left(\frac{1+z}{a_0}\right)^{3(1+w)}. \tag{3.13}$$

In Figure 2 we depicted the function of apparent horizon radius R_h versus redshift z for $G = 1$, $\rho_0 = 1$, $a_0 = 1$.

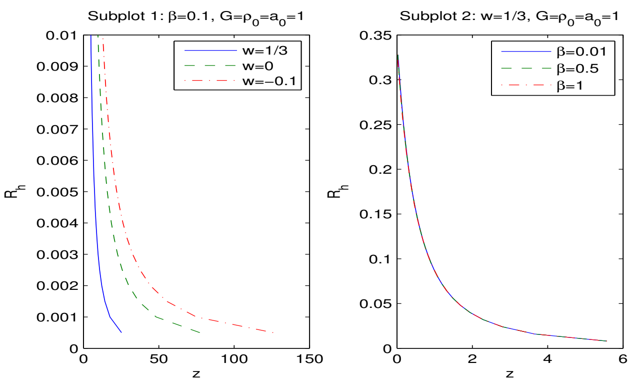

Figure 4. Left panel: The function R_h versus z at $\beta = 0.1$, $w = 1/3, 0, -0.1$, $G = 1$, $\rho_0 = 1$, $a_0 = 1$. Figure 2 shows that R_h decreases as z increases. At fixed R_h, when EoS parameter w increases the redshift z decreases. **Right panel**: According to figure the dependance of the apparent horizon radius on β is very weak.

As redshift increases the apparent horizon radius decreases.

The deceleration parameter is defined as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2}. \tag{3.14}$$

When $q < 0$ the acceleration phase takes place but as $q > 0$ we have the universe deceleration. By virtue of Equations (3.3), (3.12) and (3.14) we obtain

$$q = \frac{4\pi\rho_0(1+w)(G^2H^4 + \pi^2\beta^2)}{GH^6}\left(\frac{1+z}{a_0}\right)^{3(1+w)} - 1. \tag{3.15}$$

Equations (3.4), (3.12) and (3.15) define the function of the deceleration parameter q versus redshift z. Making use of Equations (3.4) and (3.15) one finds also the deceleration parameter q as a function of H at fixed β and EoS parameter w

$$q = \frac{3(1+w)(G^2H^4 + \pi^2\beta^2)}{2G^2H^6}\left(H^2 - \frac{\beta\pi}{G}\arctan\left(\frac{GH^2}{\beta\pi}\right)\right) - 1. \tag{3.16}$$

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In Figure 5 we plotted the function of the deceleration parameter q versus the Hubble parameter H for $G = 1$, $\rho_0 = 1$, $a_0 = 1$.

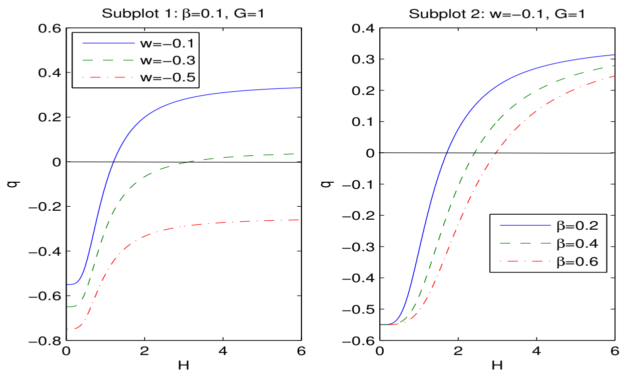

Figure 5. Left panel: The function q versus H at $\beta = 0.1$, $w = -0.1, -0.3$, -0.5, $G = 1$, $\rho_0 = 1$, $a_0 = 1$. Figure 2 shows that q increases as H increases. At fixed β and H, when EoS parameter w increases the deceleration parameter q also increases. At $w = -0.1$ and -0.3 there are two phases, acceleration $q < 0$ and deceleration $q > 0$ but at $w = -0.5$ one has only the acceleration phase (eternal inflation). **Right panel**: According to figure, if parameter β increases at fixed w and H the deceleration parameter q also increases. Here we have two phases: acceleration and deceleration.

For some parameters w and β there are two eras: inflation and deceleration but in some w and β we have only eternal university acceleration (inflation). From Equation (3.16) we obtain the asymptotic

$$\lim_{H\to\infty} q = \frac{3w+1}{2}. \tag{3.17}$$

Thus, the asymptotic of the deceleration parameter does not depend on the entropy parameter β. At $\beta = 0$ we obtain from Equation (3.16) that $q = (3w+1)/2$. Figure 3 is in accordance with the formula (3.17). Making use of Equation (3.17), we obtain the condition when two phases, acceleration and deceleration, take place: $w > -1/3$ ($q > 0$). When $w < -1/3$ the eternal inflation is realised. With the help of Equation (3.4) and (3.12) we obtain the redshift

$$z = \tag{3.18}$$

$$a_0\left(\frac{3}{8\pi\rho_0 G}\left(H^2 - \frac{\beta\pi}{G}\arctan\left(\frac{GH^2}{\beta\pi}\right)\right)\right)^{1/(3(1+w))} - 1. \tag{3.19}$$

The approximate real and positive solutions to Equation (3.16) for the transition redshifts z_t when $q = 0$, $G = 1$, $w = -0.1$ are given in Table 1.

Table 1. The approximate solutions to Equations (3.13) and (3.15) for the transition redshifts $q = 0$ at $G = 1$, $a_0 = \rho_0 = 1$, $w = -0.1$.

β	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
H	3.20	3.42	3.63	3.82	4.01	4.19	4.36	4.53	4.68	4.84	4.99
z_t	-0.05	-0.004	0.04	0.08	0.12	0.16	0.19	0.23	0.26	0.29	0.32

Table 1 shows that when the entropy parameter β increases the Hubble parameter H and reshift z also increase (at fixed w) for a divided point $q = 0$ between two pases, universe acceleration and deceleration. One can calculate deceleration parameter q for matter dominated era ($w = 0$) and for the current era ($z = 0$), from Equations (3.16) and (3.18). We obtain from Equation (3.18) for the current

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era, when $z = 0$, solutions for the Habble parameter H and deceleration parameter q from Equation (3.16) for different entropy parameters β, presented in Table 2.

Table 2. The approximate solutions to Equations (3.13) and (3.15) for the current era $z = 0$ at $G = 1$, $a_0 = \rho_0 = 1$, $w = -2/3$.

β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
H	2.977	3.053	3.125	3.193	3.258	3.320	3.378	3.435	3.489
$-q$	0.527	0.549	0.567	0.583	0.597	0.609	0.619	0.629	0.637

Negative value of the deceleration parameter q in Table 2 indicates on the acceleration phase in the current time. According to [35] the deceleration parameter at the current time is $q_0 \approx -0.6$. Table 2 shows that there is entropy parameter $\beta \approx 0.5$ which can produce that result. Making use of (3.15) we depicted the dependence of Habble parameter H on redshift z in Figure 6.

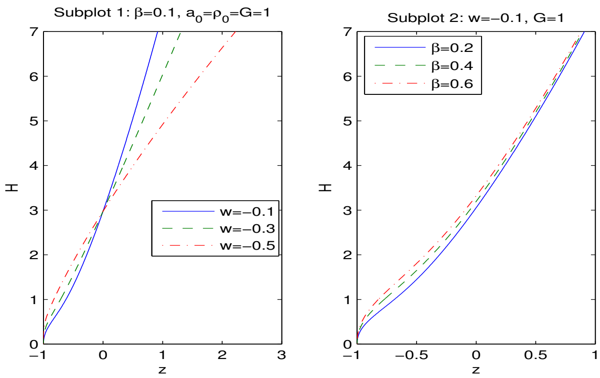

Figure 6. Left panel: The function H versus z at $\beta = 0.1$, $w = -0.1, -0.3$, -0.5, $G = 1$, $\rho_0 = a_0 = 1$. According to Figure 4, when z increases, H also increases. At fixed β, if EoS parameter w increases the Habble parameter H also increases. **Right panel**: In accordance with figure, if parameter β increases at fixed w the Habble parameter H also increases.

According to Figure 6, when z increases, H also increases. At fixed β, if EoS parameter w increases the Habble parameter H also increases. In accordance with figure, if parameter β increases at fixed w the Habble parameter H also increases. With the help of Equations (3.16) and (3.18) we plotted the deceleration parameter q versus redshift z in Figure 7.

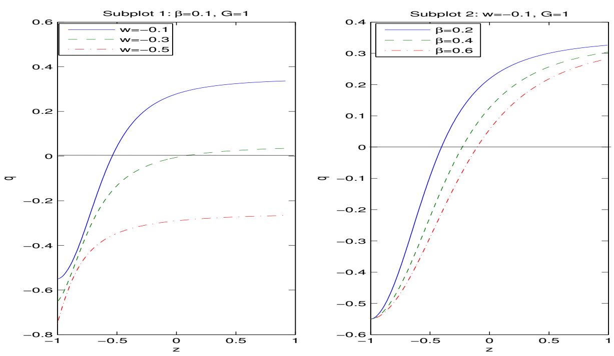

Figure 7. Left panel: The function q versus z at $\beta = 0.1$, $w = -0.1, -0.3$, -0.5, $G = 1$, $\rho_0 = a_0 = 1$. According to Figure 5, when z increases q also increases. At fixed β, if EoS parameter w increases the deceleration parameter q also increases. At $w = -0.1$ and -0.3 there are two phases, acceleration $q < 0$ and deceleration $q > 0$ but at $w = -0.5$ one has only the acceleration phase (eternal inflation). **Right panel**: In accordance with figure, if parameter β increases at fixed w the deceleration parameter q also increases. Here we have two phases: acceleration and deceleration.

According to Figure 7, when z increases q also increases. At fixed β (Left panel), if EoS parameter w increases the deceleration parameter q also increases. At $w = -0.1$ and -0.3 there are two phases, acceleration $q < 0$ and deceleration $q > 0$ but at $w = -0.5$ one has only the acceleration phase (eternal inflation). In accordance with Figure 5 (Right panel), if parameter β increases at fixed w the deceleration parameter q also increases. Here we have two phases: acceleration and deceleration.

4. F(T)-Gravity from Generalized Entropy

The torsion T in teleparallel gravity is a field analogues to the curvature R in Einstein's general relativity theory [36]. The inflationary era and the current Universe accelerating expansion cam be described by a Lagrangian in the form of $F(T)$. The torsion scalar field T is defined as [37,38]

$$T = S_\rho^{\ \mu\nu} T^\rho_{\ \mu\nu}, \tag{4.1}$$

and tensors $S_\rho^{\ \mu\nu}$ and $T^\rho_{\ \mu\nu}$ are given by

$$\begin{aligned} S_\rho^{\ \mu\nu} &= \frac{1}{2}\left(K^{\mu\nu}_{\ \ \ \rho} + \delta^\mu_\rho T^{\alpha\nu}_{\ \ \ \alpha} - \delta^\nu_\rho T^{\alpha\mu}_{\ \ \ \alpha}\right), \\ K^{\mu\nu}_{\ \ \ \rho} &= -\frac{1}{2}\left(T^{\mu\nu}_{\ \ \ \rho} - T^{\nu\mu}_{\ \ \ \rho} - T_\rho^{\ \mu\nu}\right), \\ T^\rho_{\ \mu\nu} &= e^\rho_i\left(\partial_\mu e^i_\nu - \partial_\nu e^i_\mu\right), \end{aligned} \tag{4.2}$$

where e^i_ν ($i = 0, 1, 2, 3$) is a vierbein. The metric tensor is given by $g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu$ and η_{ij} is the flat metric $\eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$). We have $e^i_\mu = \mathrm{diag}(1, a, a, a)$ for FLRW metric (2.1) and the torsion scalar field is $T = -6H^2$. By variation of the action with respect to e^i_μ, with the Lagrangian $F(T)$, we find [39]

$$\frac{1}{6}\left[F(T) - 2TF'(T)\right]|_{T=-6H^2} = \left(\frac{8\pi G}{3}\right)\rho. \tag{4.3}$$

By virtue of equations (3.4) and (4.3) we obtain

$$\left[F(T) - 2TF'(T)\right] = -T + 6b \arctan\left(\frac{T}{6b}\right). \tag{4.4}$$

After integration of Equation (4.4), one finds the function $F(T)$:

$$F(T) = T - \arctan\left(\frac{T}{b}\right) + \frac{b}{2T}\ln\left(1 + \frac{T^2}{b^2}\right). \tag{4.5}$$

As a result, the teleparallel gravity with the function (4.5) leads to entropic cosmology with entropy (3.2) under consideration.

5. Summary

Thus, we have proposed entropy $S_h = (1/\beta)\arctan(\beta S_{BH})$ which shares similar property as the Bekenstein–Hawking entropy S_{BH}: it vanishes when the apparent horizon radius R_h vanishes; S_h monotonically increases as the apparent horizon radius R_h increases and it is positive. We consider the barotropic perfect fluid and flat FLRW universe. From first law of apparent horizon thermodynamics we obtained the modified Friedmann's equations. The addition term in the second Friedmann's equation is treated as a dynamical cosmological constant. We showed that the universe inflation is due to holographic dark energy. It is worth noting that Barrow's and Tsallis's entropies also lead to Einsten's equations with the dynamical cosmological constant [34]. By analysing the deceleration parameter we find that for some parameters our model can describe inflation and deceleration phases or only eternal inflation. We have calculated the transition redshifts when $q = 0$, presented in Table

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I for some parameters w and β. Table 2 shows that at $\beta \approx 0.5$ and $w = -2/3$ we obtain the current deceleration parameter $q_0 \approx -0.6$. It has been proven that entropic cosmology with our entropy proposed is equivalent to cosmology based on the teleparallel gravity with the function $F(T)$ (Equation (4.4)). Cosmology based on the modified Friedmann equations obtained may be of interest for a description of inflation and late time of the universe evolution.

Appendix A

Let us consider new nonadditive entropy

$$S_h = -\frac{1}{\beta}\sum_{i=1}^{W} p_i \arctan(\ln(p_i^{\beta})), \tag{A1}$$

where W is a number of states, each with a probability p_i (the probability distribution $\{p_i\}$) and β is a free parameter (a constant describing some statistics). We explore units with the Boltzmann constant $k_B = 1$. The summation in Equation (1) is performed over all possible microstates of the system. At $\beta = 0$ our entropy becomes the additive Gibbs entropy

$$S_G = -\sum_{i=1}^{W} p_i \ln(p_i). \tag{A2}$$

If probabilities are all equal (each microstate is assumed to be populated with equal probability), we have $1/p_i = W$ ($i = 1, 2, ..., W$) and Equation (2) is converted into the Boltzmann entropy $S_B = \ln(W)$. By using $1/p_i = W$, Equation (1) becomes

$$S_h = \frac{1}{\beta}\arctan(\beta \ln(W)). \tag{A3}$$

At the same time the Bekenstein–Hawking entropy $S_{BH} = \ln(W)$. By virtue of Equation (3) we obtain

$$S_h = \frac{1}{\beta}\arctan(\beta S_{BH}).) \tag{A4}$$

From Equation (4), at $\beta = 0$, one obtains the Bekenstein–Hawking entropy $S_h = S_{BH}$. For probabilistically independent systems A and B we have entropy $S_h(A+B) \neq S_h(A) + S_h(B)$. Thus, parameter β can be considered as a measure of entropy nonadditivity.

References

1. J. D. Bekenstein, Black Holes and Entropy, Phys. Rev. D **7** (1973), 2333-2346.
2. S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43** (1975), 199-220; Erratum: ibid. **46** (1976), 206.
3. T. Jacobson, Thermodynamics of Spacetime: The Einstein Equation of State, Phys. Rev. Lett. **75** (1995), 1260.
4. T. Padmanabhan, Gravity and the Thermodynamics of Horizons, Phys. Rept. **406** (2005), 49.
5. T. Padmanabhan, Thermodynamical Aspects of Gravity: New insights, Rept. Prog. Phys. **73** (2010), 046901.
6. S. A. Hayward, Unified first law of black-hole dynamics and relativistic thermodynamics, Class. Quant. Grav. **15** (1998), 3147-3162.
7. M. Akbar and R. G. Cai, Thermodynamic Behavior of Friedmann Equation at Apparent Horizon of FRW Universe, Phys. Rev. D **75** (2007), 084003.
8. R. G. Cai and L. M. Cao, Unified First Law and Thermodynamics of Apparent Horizon in FRW Universe, Phys. Rev. D **75** (2007), 064008.
9. A. Paranjape, S. Sarkar and T. Padmanabhan, Thermodynamic route to Field equations in Lanczos-Lovelock Gravity, Phys. Rev. D **74** (2006), 104015.
10. A. Sheykhi, B. Wang and R. G. Cai, Thermodynamical Properties of Apparent Horizon in Warped DGP Braneworld, Nucl. Phys. B **779**, (2007) 1.

11. R. G. Cai and N. Ohta, Horizon Thermodynamics and Gravitational Field Equations in Horava-Lifshitz Gravity, Phys. Rev. D **81** (2010), 084061.
12. M. Jamil, E. N. Saridakis and M. R. Setare, The generalized second law of thermodynamics in Horava-Lifshitz cosmology, JCAP **1011** (2010), 032.
13. Y. Gim, W. Kim and S. H. Yi, The first law of thermodynamics in Lifshitz black holes revisited, JHEP **1407** (2014), 002.
14. Z. Y. Fan and H. Lu, Thermodynamical First Laws of Black Holes in Quadratically-Extended Gravities, Phys. Rev. D **91** (2015), 064009.
15. R. D'Agostino, Holographic dark energy from nonadditive entropy: cosmological perturbations and observational constraints, Phys. Rev. D **99** (2019), 103524.
16. L. M. Sanchez and H. Quevedo, Thermodynamics of the FLRW apparent horizon, Phys. Lett B **839** (2023), 137778.
17. S. Wang, Y. Wang and M. Li, Holographic Dark Energy, Phys. Rept. **696** (2017) 1.
18. C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J. Stat. Phys., **52** (1-2) (1988), 479-487; C. Tsallis, The Nonadditive Entropy S_q and Its Applications in Physics and Elsewhere: Some Remarks, Entropy 13, 1765 (2011).
19. A. R´enyi, Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability, University of California Press (1960), 547-56.
20. A. Sayahian Jahromi et al, Generalized entropy formalism and a new holographic dark energy model, Phys. Lett. B **780** (2018), 21-24.
21. J. D. Barrow, The Area of a Rough Black Hole, Phys. Lett. B **808** (2020), 135643.
22. G. Kaniadakis, Statistical mechanics in the context of special relativity II, Phys. Rev. E **72** (2005), 036108.
23. Marco Masi, A step beyond Tsallis and Rényi entropies, Phys. Lett. A **338** (2005), 217-224.
24. V. G. Czinner and H. Iguchi, Rényi entropy and the thermodynamic stability of black holes, Phys. Lett. B 752 (2016), 306-310.
25. J. Ren, Analytic critical points of charged Renyi entropies from hyperbolic black holes, JHEP **05** (2021), 080.
26. K. Mejrhit and S. E. Ennadifi, Thermodynamics, stability and Hawking–Page transition of black holes from non-extensive statistical mechanics in quantum geometry, Phys. Lett. B **794** (2019), 45-49.
27. Saeed Noori Gashti, Behnam Pourhassan, İzzet Sakallı and Aram Bahroz Brzo, Thermodynamic Topology and Photon Spheres of Dirty Black Holes within Non-Extensive Entropy, Phys. Dark Univ. **47** (2025), 101833.
28. J. Sadeghi, B. Pourhassan, and Z. Abbaspour Moghaddam, Interacting Entropy-Corrected Holographic Dark Energy and IR Cut-Off Length, Int. J. Theor. Phys. **53** (2014), 125–135.
29. B. Pourhassan, Alexander Bonilla, Mir Faizal, and Everton M. C. Abreu, Holographic Dark Energy from Fluid/Gravity Duality Constraint by Cosmological Observations, Phys. Dark Univ. **20** (2018), 41.
30. S. Nojiri, S. D. Odintsov and Tanmoy Paul, Different aspects of entropic cosmology, Phys. Lett. B **835** (2022), 137553.
31. D. Pavon and W. Zimdahl, Holographic dark energy and cosmic coincidence, Phys. Lett. B **628** (2005) 206.
32. R. C. G. Landim, Holographic dark energy from minimal supergravity, Int. J. Mod. Phys. D **25** (2016), 1650050.
33. S. I. Kruglov, Universe inflation and nonlinear electrodynamics, Eur. Phys. J. C **84** (2024), 205.
34. Sofia Di Gennaro, Hao Xu, Yen Chin Ong, How barrow entropy modifies gravity: with comments on Tsallis entropy, Eur. Phys. J. C **82** (2022), 1066.
35. M. Roos, Introduction to Cosmology (John Wiley and Sons, UK, 2003).
36. F. W. Hehl, P. Von Der Heyde, G. D. Kerlick and J. M. Nester, Rev. Mod. Phys. **48** (1976), 393-416.
37. Weitzenböck R., Invarianten Theorie, (Nordhoff, Groningen, 1923).
38. J. W. Maluf, Hamiltonian formulation of the teleparallel description of general relativity, J. Math. Phys. 35, (1994) 335.
39. G. R. Bengochea and R. Ferraro, Dark torsion as the cosmic speed-up, Phys. Rev. D **79** (2009), 124019.