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Dumitru Adam *

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On the Method for Proving the RH Using the Alcantara-Bode Equivalence

Dumitru Adam

No Affiliations. No Competing Interests. E-mail: dumitru_adam@yahoo.ca (home page: www.riemann-hypothesis.ca)
Former Researcher at Inst. of Math. Romanian Acad. (IMAR). Addr: Windsor, On, n9a3a5 Ca; Sacele, Brasov, Ro.

Abstract: The article presents a solution to the Alcantara-Bode equivalent formulation of the Riemann Hypothesis (RH). The RH, a long-standing unsolved problem, posits that the non-trivial zeros of the Riemann Zeta function lie on the vertical line $\sigma = 1/2$. The Alcantara-Bode equivalent states that RH holds if and only if the null space of a specific integral operator on $L^2(0,1)$ contains only the element 0, equivalently the operator is injective. In this paper is provided a proof for this equivalent by investigating the injectivity of linear bounded operators through their positivity properties on dense sets in separable Hilbert spaces. The theory and associated methods involve approximations on finite dimension subspaces whose union is dense.

Keywords: integral operators, hilbert spaces, approximation methods, multi-level discretizations, riemann hypothesis

MSC: 31A10, 45P05, 47G10, 65R99, 11M26

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1. Introduction.

The Riemann Hypothesis (RH) claims (1859) that the Riemann Zeta function defined by the infinite sum $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ has its non trivial zeros on the vertical line $\sigma = 1/2$ ([7]). The Hilbert-Schmidt integral operator T_ρ defined on $L^2(0,1)$ having the kernel function $\rho(y,x) = \{y/x\}$, the fractional part of the quantity between brackets,

$$(T_\rho u)(y) = \int_0^1 \rho(y,x)u(x)dx, \quad u \in L^2(0,1) \tag{1}$$

has been used by Alcantara-Bode ([2], pg. 151) in his theorem of the equivalent formulation of the RH obtained from Beurling equivalent formulation ([4]). Denoting by N_{T_ρ} the null space of this operator, the equivalent formulation consists in:

The Riemann Hypothesis holds if and only if $N_{T_\rho} = \{0\}$.

Its kernel function $\rho \in L^2(0,1)^2$ defined by the fractional part of the ratio (y/x) is continue almost everywhere, the discontinuities in $(0,1)^2$ consisting in a set of numerable one dimensional lines of the form $y = kx, k \in N$, being of Lebesgue measure zero. The integral operator is Hilbert-Schmidt ([2]) and so compact, allowing us to consider its approximations on finite dimension subspaces ([3]). From the strict positivity of an integral operator like in (1) on such subspaces whose union is dense, we will obtain that it is injective, equivalently $N_{T_\rho} = \{0\}$.

2. Two Theorems on Injectivity

Let H be a separable Hilbert space and denote with $\mathcal{L}(H)$ the class of the linear bounded operators on H. If $T \in \mathcal{L}(H)$ is positive on a dense set $S \subset H$, i.e. $\langle Tv, v\rangle > 0 \, \forall v \in S$ then T has no zeros in the dense set. Otherwise, if there exists $w \in S$ such that $Tw = 0$ then $\langle Tw, w\rangle = 0$ contradicting the

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positivity of T on S. Follows: its '*eligible*' zeros are all in the difference set $E := H \setminus S$, i.e. $N_T \subset E$.

An operator linear T and its associated Hermitian T^*T have the same null space. We could switch T with its Hermitian when no information on its positivity there exists. So, our method could be applied to any linear bounded operator, that is the reason we will consider the linear operator be positive definite $\langle Tu, u\rangle > 0$ on a dense set. The next theorem ([11]) is the starting point of our solution.

Theorem 1. *If $T \in \mathcal{L}(H)$ is strict positive on a dense set of a separable Hilbert space then T is injective, equivalently $N_T = \{0\}$.*
Proof.

Let's take in consideration only the set of eligible zeros that are on the unit sphere without restricting the generality, once for an element $w \in H, w \neq 0$ and for $w/\|w\|$ both are or both are not in N_T. The set $S \subset H$ is dense if its closure coincides with H. Then, if $w \in E := H \setminus S$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that $\|w - u_{\varepsilon,w}\| < \varepsilon$. Now, (2) results as follows. If $\|w\| \geq \|u_{\varepsilon,w}\|$:
$0 \leq \|w\| - \|u_{\varepsilon,w}\| = \|w - u_{\varepsilon,w} + u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| \leq \|w - u_{\varepsilon,w}\| + \|u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| < \varepsilon$.
If $\|u_{\varepsilon,w}\| \geq \|w\|$ instead, then:
$0 \leq \|u_{\varepsilon,w}\| - \|w\| = \|u_{\varepsilon,w} - w + w\| - \|w\| \leq \|w - u_{\varepsilon,w}\| < \varepsilon$.
So, given $w \in E$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that

$$||w\| - \|u_{\varepsilon,w}|| < \varepsilon \tag{2}$$

Let w be an eligible element from the unit sphere, $\|w\| = 1$ and take $\varepsilon_n = 1/n$.
Then there exists at least one element $u_{\varepsilon_n,w} \in S$ such that $\|u_{\varepsilon_n,w} - w\| < \varepsilon_n$ holds. Follows from (2), | 1 - $\|u_{\varepsilon_n,w}\|$ | $< 1/n$ showing that, for any choices of a sequence approximating w, $u_{\varepsilon_n,w} \in S, n \geq 1$, it verifies $\|u_{\varepsilon_n,w}\| \to 1$.

If $T \in \mathcal{L}(H)$ is strict positive on S, then there exists $\alpha > 0$ such that $\forall u \in S$, $\langle Tu, u\rangle \geq \alpha\|u\|^2$.
Suppose that there exists $w \in E, \|w\| = 1$ a zero of T, i.e. $w \in N_T$ and consider a sequence of approximations of w, $u_{\varepsilon_n,w} \in S, n \geq 1$ that, as we showed, has its normed sequence converging in norm to 1. From the positivity of T on the dense set S, follows:

$$\alpha\|u_{\varepsilon_n,w}\|^2 \leq \langle Tu_{\varepsilon_n,w}, u_{\varepsilon_n,w}\rangle = \langle T(u_{\varepsilon_n,w} - w), u_{\varepsilon_n,w}\rangle < \varepsilon_n\|T\|\|u_{\varepsilon_n,w}\| \tag{3}$$

With c=$\|T\|/\alpha$, we obtain $\|u_{\varepsilon_n,w}\| \leq c/n$. Then, $\|u_{\varepsilon_n,w}\| \to 0$ with $n \to \infty$, in contradiction with its convergence $\|u_{\varepsilon_n,w}\| \to 1$ with $n \to \infty$.
Or, this happen for any choice of the sequence of approximations of w, verifying $\|w - u_{\varepsilon_n,w}\| < \varepsilon_n, n \geq 1$, when $Tw = 0$.
Thus $w \notin N_T$, valid for any $w \in E, \|w\| = 1$, proving the theorem because no zeros of T there are in S either. □

From now on, we will suppose that the dense set S is the result of an union of finite dimension subspaces of a family F: $S = \cup_{n\geq 1} S_n, \overline{S} = H$.
It is not mandatory but will ease our proofs considering that the subspaces are including: $S_n \subset S_{n+1}, n \geq 1$. There are such approximation subspaces having the including property. For example one in which the subspaces are spanned by functions indicator of semi-open disjoint intervals of the same length $h, nh = 1$. In this case, by halving the intervals we obtain the including property.

Observation 1. Let $\beta_n(u) := \|u - u_n\|$ be the normed residuum of the eligible element $u \in E$ after its orthogonal projection on S_n. Then, $\beta_n(u) \to 0$ with $n \to \infty$.
Proof.

Given $\epsilon > 0$, from the density of the set S in H there exists $u_\epsilon \in S$ verifying $\|u - u_\epsilon\| < \epsilon$, as per the observations made in the proof of the Theorem 1. Let S_{n_ϵ} be the coarsest subspace, i.e. with the

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smallest dimension, from the family of subspaces containing u_ϵ. Because the best approximation of u in S_{n_ϵ} is its orthogonal projection, we obtain

$\beta_{n_\epsilon}(u) := \|u - P_{n_\epsilon}u\| \leq \|u - u_\epsilon\| < \epsilon,$

inequality valid for every $\epsilon > 0$, proving our assertion. □

Pointing out. Let $T \in \mathcal{L}(H)$ be positive on a dense set S. Consider the operator restricted to the subspaces $S_n \in F, n \geq 1$ and define its positivity parameters by $\alpha_n(T)$ verifying $\langle Tv, v\rangle \geq \alpha_n(T)\|v\|^2$, $\forall v \in S_n, n \geq 1$. We distinguish three cases:

1.) $\alpha_n(T) \geq \alpha > 0, \forall n \geq 1$. Then T is strict positive on the dense set and by Theorem 1, $N_T = \{0\}$.

2.) $\alpha_n(T) \to 0$ then we have to provide a criteria for the investigation of the operator injectivity.

3.) If T is not positive on the dense set, we could replace it with its associated Hermitian operator T^*T. We could consider this case to fit in one of the previous cases provided that the Hermitian is positive definite on the dense set (thus it has not zeros in the dense set). Thus, we have to provide a criteria only for the case when the positivity parameters sequence of the operator restrictions on the approximation subspaces S_n converges to 0. A such criteria, used in Appendix involving the adjoint operator is given in [1]. The following result shows that, if there exists a dense set on which the operator has a sequence of operator approximations positive with the corresponding positivity parameters bounded inferior then it is injective. More, the operator is strict positive on the dense set.

Given $T \in \mathcal{L}(H)$, let $T_n, n \geq 1$ be a sequence of operator approximations on $S_n, n \geq 1$ having the property $\epsilon_n := \|T - T_n\| \to 0$.
Suppose that for every $n \geq 1$, the operator approximation T_n is positive on S_n and denote with $\alpha_n := \alpha_n(T_n)$ its positivity parameter.

Theorem 2. *Let T be a linear bounded operator on H, positive on a dense set S, $\langle Tv, v\rangle > 0$ for every $v \in S$. If S is a result of the union of including finite dimension subspaces of a family F on which the sequence $\{T_n, n \geq 1\}$ of its approximations on the family F verifies:*

i) $\langle T_n v, v\rangle \geq \alpha_n\|v\|^2, \forall v \in S_n, S_n \in F,$

ii) $\epsilon_n := \|T - T_n\| \to 0$ with $n \to \infty$,

iii) $\alpha_n \geq \alpha > 0, n \geq 1$

then $N_T = \{0\}$.

Proof.

Being positive on S, the operator has no zeros in the dense set. Showing now, that T has no zeros in the difference set. Next observation is a consequence of the fact that if a not null $u \in N_T$ then $u/\|u\| \in N_T$. For $u \in E := H \setminus S$, $\|u\| = 1$ having the not null orthogonal projections $P_n u, n \geq n_0 := n_0(u)$, we have on any subspace S_n, $\|u\|^2 = \|P_n u\|^2 + \beta_n^2(u)$ where $\beta_n(u) = \|u - P_n u\|$ is its residuum. If there exists $u \in N_T \cap E$, i.e. $Tu = 0$, then for it:

$\alpha_n\|P_n u\|^2 \leq \langle T_n P_n u, P_n u\rangle \leq \|T_n P_n u\|\|P_n u\|$
$= (\|T_n P_n u - TP_n u + TP_n u - Tu\|)\|P_n u\|$
$\leq (\|T - T_n\|\|P_n u\| + \|T\|\|u - P_n u\|)\|P_n u\|,$

obtaining from Observation 1 and iii):

$\alpha \leq \left(\epsilon_n + \|T\|\beta_n(u)/\sqrt{1 - \beta_n^2(u)}\right) \to 0.$

The inequality is violated from an $n_1 \geq n_0$, involving $u \notin N_T$, valid for any supposed zero of T in E. Once T has no zeros in the dense set, $N_T = \{0\}$. □

When $T_n, n \geq 1$ are the restrictions of T onto the subspaces $S_n \in F$, both i) and ii) are fulfilled and, the condition iii) could be avoided by using the Injectivity Criteria. We will deal with the special case of the approximations of the Hilbert-Schmidt integral operators that, being compact operators could be approximated on finite dimension subspaces such that the condition ii) is satisfied ([3]).

Let $T := T_\varphi$ be an Hilbert-Schmidt integral operator. A method for obtaining the approximations for an integral operator is explained and used in [5]. Thus, ii) in the Theorem 2 is fulfilled when

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$T_n, n \geq 1$ are finite rank approximations on the subspaces of the family F obtained by orthogonal projection integral operators like in [5]: $T_n := P_n^r(T)$. Then:

$\|Tu - T_n u\| = \|(I - P_n^r)Tu\| \leq \|I - P_n^r\|\|Tu\| \to 0$, for every $u \in H$.

Moreover, the bounding of the positivity parameter sequence of the finite rank operator approximations involves the strict positivity of the operator on the dense set.

Lemma 1. (Criteria for finite rank approximations). *If the finite rank approximations of a positive Hilbert-Schmidt operator on a dense set S are positive on the family of approximation subspaces F and the sequence of the positivity parameters are inferior bounded, then it is injective.*

Proof.

The requests i), ii) in the Theorem 2 are fulfilled by the previous observations. From the convergence to zero of the sequence $\epsilon_n, n \geq 1$ there exists ϵ_0 a 'compactness' parameter verifying $\epsilon_0 := max_n\{\epsilon_n; \epsilon_n < \alpha\}$ corresponding to a subspace $S_{n_0}, n_0 < \infty$. The parameter ϵ_0 in independent from any $v \in S$ and,

$\alpha_n \geq \alpha > \epsilon_0 \geq \epsilon_n$, for $n \geq 1$.

For an arbitrary $v \in S$ there exists a coarser subspace (i.e. with a smaller dimension) $S_n, n \geq n_0 := n_0(v)$, for which $v \in S_n$. For it, we have:

$\langle Tv, v\rangle = \langle T_n v, v\rangle - \langle (T_n - T)v, v\rangle > 0$.

Because T and T_n are positive on S both inner products in the right side of the equality being real valued. So, if $\langle (T_n - T)v, v\rangle \geq 0$, then $\langle (T_n - T)v, v\rangle \leq \epsilon_n\|v\|^2 \leq \alpha_n\|v\|^2$. Follows,

$\langle Tv, v\rangle \geq (\alpha_n - \epsilon_n)\|v\|^2 \geq (\alpha - \epsilon_0)\|v\|^2$.

Now, if $\langle (T_n - T)v, v\rangle < 0$, $\langle Tv, v\rangle \geq \alpha_n\|v\|^2 \geq \alpha\|v\|^2 \geq (\alpha - \epsilon_0)\|v\|^2$ proving the condition iii) in the Theorem 2 with $\alpha(T) = \alpha - \epsilon_0 > 0$ and so, $N_T = \{0\}$. □

The following lemma is dealing with the cases in which no finite rank approximations could be defined (see [1], [11]).

Lemma 2. (Criteria for operator restrictions.) *Let $T \in \mathcal{L}(H)$ positive on the subspaces $S_n, n \geq 1$ whose union S is a dense set S, verifying: $\langle Tv, v\rangle \geq \alpha_n\|v\|^2$ for every $v \in S_n$, where $\alpha_n \to 0$ with $n \to \infty$. Consider now the parameter*:

$\mu_n := \alpha_n(T)/\omega_n$ *where ω_n verifies* $\|T^*v\| \leq \omega_n\|v\|, \forall v \in S_n, n \geq 1$.

If there exists $C > 0$ such that $\mu_n \geq C$ for every $n \geq 1$, *then* $N_T = \{0\}$.

Proof.

Suppose that there exists $u \in (H \setminus S) \cap N_T$, $\|u\| = 1$ and let u_n its orthogonal projection on $S_n, n \geq 1$. Then, from the (strict) positivity of T on the subspaces $S_n, n \geq 1$ (see (3)):

$\alpha_n(T)\|u_n\|^2 \leq \langle Tu_n, u_n\rangle = \langle T(u_n - u), u_n\rangle = \langle (u_n - u), T^*u_n\rangle \leq \beta_n\omega_n\|u_n\|$

Then, from

$C \leq \mu_n \leq \beta_n/\sqrt{1 - \beta_n^2} \to 0$ where $\beta_n := \beta_n(u) = \|u - u_n\|$, we obtain a contradiction. Thus, $u \notin N_T$ for any $u \in H \setminus S$. Follows: $N_T = \{0\}$. □

3. Dense Sets in $L^2(0,1)$.

Let $H := L^2(0,1)$. The semi-open intervals of equal lengths $h = 2^{-m}, m \in N$, nh = 1, $\Delta_{h,k} = ((k-1)/2^m, k/2^m]$, $k = 1, n-1$ together with the open $\Delta_{h,n}$ are defining for $m \geq 1$ a partition of (0,1), k=1,n, $n = 2^m, nh = 1$. Consider the interval indicator functions having the supports these intervals (k=1,n), nh=1:

$$I_{h,k}(t) = 1 \text{ for } t \in \Delta_{h,k} \text{ and } 0 \text{ otherwise} \tag{4}$$

The family F of finite dimensional subspaces S_h that are the linear spans of interval indicator functions of the h-partitions defined by (4) with disjoint supports, $S_h = span\{I_{h,k}; k = 1, n, nh = 1\}$, built on a multi-level structure, are including $S_h \subset S_{h/2}$ by halving the mesh h. In fact, the property is obtained from (4) observing that $S_h \ni I_{h,i} = I_{h/2,2i-1} + I_{h/2,2i} \in S_{h/2}, i = 1, n$.

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The set $S = \cup_{n\geq 1} S_h, nh = 1$ is dense in H well known in literature.

Citing [5], (pg 986), the integral operator $P_h^r, n \geq 1$ having the kernel function:

$$r_h(y,x) = h^{-1} \sum_{k=1,n} I_{h,k}(y) I_{h,k}(x) \tag{5}$$

is an finite rank integral operator orthogonal projection having the spectrum ($\{0,1\}$) with the eigenvalue 1 of the multiplicity n (nh=1) corresponding to the orthogonal eigenfunctions $I_{h,k}, k = 1,n$. We will show it, by proving that $\forall u \in H$, $P_h^r u \in S_h$ and $(P_h^r)^2 = P_h^r$ for $n \geq 2, nh = 1$. For any $u \in H$,
$P_h^r u = \int_{x\in(0,1)} \left(h^{-1} \sum_{k=1.n} I_{h,k}(y) I_{h,k}(x)\right) u(x)dx$
$= h^{-1} \sum_{k=1,n} c_k I_{h,k}(y),\ c_k := \int_{\Delta_{h,k}} u(x)dx,$
that is the standard orthogonal projection onto S_h up to the variable switches that come from the definition of an integral operator ($(T_\varphi u)(y) = \int_{x\in(0,1)} \varphi(y,x)u(x)dx$).
Now, if f = $\sum_{k=1,n} c_k I_{h,k} \in S_h$,
$P_h^r(f) = h^{-1} \sum_{j=1,n} \int_{\Delta_{h,j}} I_{h,j}(y)(\sum_{k=1,n} c_k I_{h,k}(y)) I_{h,j}(x)dy$
$= h^{-1} \sum_{j=1,n} I_{h,j}(x) \int_{\Delta_{h,j}} c_j I_{h,j}(y)dy$
$= h^{-1} \sum_{j=1,n} c_j I_{h,j}(x) \int_{\Delta_{h,j}} I_{h,j}(y)dy \ = \sum_{j=1,n} c_j I_{h,j} = f,$
i.e. $P_h^r f = f$ and so, $(P_h^r)^2 u = P_h^r u$ for any $u \in H$. Thus,
$\|I - P_h^r\| \to 0$ for $n \to \infty, nh = 1$. So, ii) in Theorem 2 is fulfilled.

Remark 1. Because P_h^r is an orthogonal projection, the inner product on the subspace S_h between $u \notin S_h$ and $v \in S_h$ is a result between the orthogonal projection of u and v is like between two step functions in S_h: $\langle u, v\rangle := \langle P_h^r u, v\rangle$.

Let T_ρ be an Hilbert-Schmidt integral operator on H. Its integral operator approximation of T_ρ on S_h is an finite rank operator approximation, T_{ρ_h}, having the kernel function ([5])

$$\rho_h(y,x) = h^{-1} \sum_{k=1,n} I_{h,k}(y)\rho(y,x) I_{h,k}(x) := h^{-1} \sum_{k=1,n} \rho_h^k(y,x) \tag{6}$$

where the pieces of the kernel function ρ_h in the sum have disjoint supports in $L^2(0,1)^2$ namely $\Delta_{h,k} \times \Delta_{h,k}, k = 1,n, nh = 1$.
This observation is crucial in obtaining diagonal matrix representations of the integral operators onto the finite dimension subspaces $S_h, n \geq 2, nh = 1$.
So, for $v \in S_h, v = \sum c_k I_{h,k}$,
$T_{\rho_h} v = h^{-1} \sum_{k=1,n} c_k I_{h,k}(y) \sum_{j=1,n} \int_{\Delta_{h,j}} I_{h,j}(y)\rho(y,x) I_{h,j}(x)dx$
$= h^{-1} \sum_{k=1,n} c_k \left[\int_{\Delta_{h,k}} \rho(y,x) I_{h,k}(x)dx\right] I_{h,k}(y).$
Then, for $v = I_{h,i}$, $(T_{\rho_h} I_{h,i})(y) = h^{-1}\left[\int_{\Delta_{h,i}} \rho(y,x) I_{h,i}(x)dx\right] I_{h,i}(y).$
So, $\langle T_{\rho_h} I_{h.i}, I_{h,j}\rangle = 0$ for $i \neq j$ and the matrix representation of the finite rank operator $P_h^r(T_\rho) := T_{\rho_h}$, $M_h^r(T_\rho) = h^{-1}\left[d_{ij}^h\right]_{i,j=1,n}$ is sparse diagonal i.e. $d_{ij}^h = 0, i \neq j$ with the diagonal entries given by:

$$d_{kk}^h = \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} I_{h,k}(y)\rho(y,x) I_{h,k}(x)dxdy := \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \rho dxdy, k = 1,n, nh = 1 \tag{7}$$

For finding the positivity parameter of T_{ρ_h}, let $v_h = \sum_{k=1,n} c_k I_{h,k} \in S_h$. Because $\|v\|^2 = h \sum_{k=1,n} c_k \overline{c_k}$, the inner product becomes
$\langle T_{\rho_h} v_h, v_h\rangle = h^{-1} \sum_{k=1,n} c_k \overline{c_k} d_{kk}^h.$
Thus, if $d_{kk}^h > 0$, $\forall k = 1,n, nh = 1$, $\langle T_{\rho_h} v_h, v_h\rangle \geq \alpha_h(T_{\rho_h})\|v_h\|^2$ where

$$\alpha_h(T_{\rho_h}) = h^{-2} min_{(k=1,n)} d_{kk}^h \tag{8}$$

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is the positivity parameter of the finite rank operator approximation T_{ρ_h}. In order to meet the requests of the Lemma 1 (or Theorem 2), we need:

i.)- to show that the integral operator T_ρ is positive on the dense set and,

ii.)- the sequence of the positivity parameters $\alpha_h(T_{\rho_h})$ is inferior bounded.

i.) The kernel function ρ is positive valued almost everywhere in $((0,1)$, then:

$\int_{(0,1)}\int_{(0,1)}\rho(y,x)dxdy > 0$ and on any of the subintervals on a h-partition, $\langle T_\rho I_{h,k}, I_{h,j}\rangle = 0$ for $k \neq j$ because the functions of the indicator of intervals have disjoint supports. Now, for $v = \sum_{k=1,n} c_k I_{h,k} \in S_h$:

$\langle T_\rho v, v\rangle = \sum_{k=1,n} c_k\overline{c_k}\int_{\Delta_{h,k}}\int_{\Delta_{h,k}}\rho(y,x)dxdy = \sum_{k=1,n} c_k\overline{c_k}d^h_{kk} > 0$ proving the positivity of T_ρ on every subspace in F and so, on the dense set S.

Let remark that d^h_{kk} has the same formula like in (7) and so, we could define the positivity parameter of the restriction of T_ρ onto S_h:

$$\alpha_h(T_\rho) = h^{-1}min_{(k=1,n)}(d^h_{kk}) := h\alpha_h(T_{\rho_h}), nh = 1 \tag{9}$$

ii.) At this point, for proving the Alcantara-Bode equivalent we have only to show that the sequence of the positivity parameters of the finite rank approximations $\alpha_h(T_{\rho_h})$ is inferior bounded.

4. The Proof of the RH Equivalent

The entries in the diagonal matrix representation $M_h(T_{\rho_h})$ of the finite rank integral operator T_{ρ_h} where the kernel function is $\rho(y.x) = \{y/x\}$ the fractional part of the ratio (y/x), $x, y \in (0,1)$ are given by:

$d^h_{kk} = \int_{\Delta_{h,k}}\int_{\Delta_{h,k}}\rho(y,x)dxdy$, and valued (see also [1]) as follows:

$$d^h_{11} = h^2(3-2\gamma)/4$$

$$d^h_{kk} = \int\limits_{\Delta_{h,k}}\Big[\int\limits_{\Delta_{h,k}}(y/x)dx - \int\limits_{(k-1)h}^{y}dx\Big]dy = \frac{h^2}{2}(-1+\frac{2k-1}{k-1}ln(\frac{k}{k-1})^{k-1}) \tag{10}$$

for $k \geq 2$, where γ is the Euler-Mascheroni constant ($\simeq 0.5772156$). The expression in (10) for computing the fractional part of the ratio $\{y/x\}$ is suggested in [4]: for $0 < a < b < 2a$, $\{b/a\} = (b/a) - 1$. The sequence

$f(k) := h^{-2}d^h_{kk} = (-1 + \frac{2k-1}{k-1}ln(\frac{k}{k-1})^{k-1})/2$ is monotone decreasing for $k \geq 2$ and converges to 0.5 for k $\to \infty$.

Then $\forall h$, we have with $k \geq 2 : d^h_{kk} > 0.5h^2 > d^h_{11}$. So,

$$\alpha_h(T_{\rho_h}) = h^{-2}d^h_{11} = (3-2\gamma)/4 > 0, \; n \geq 2, nh = 1. \tag{11}$$

showing that the finite rank approximations of the integral operator have the sequence of the positivity parameters inferior bounded.

Theorem 3. (*Finite Rank Approximations*): *The Alcantara-Bode equivalent holds and, the Riemann Hypothesis is true.*

Proof.

The sequence of the positivity parameters corresponding to the finite rank operator approximations T_{ρ_h} on the dense family F is inferior bounded, see (11), obtaining from Lemma 1 that $N_{T_\rho} = \{0\}$. Now, having $N_{T_\rho} = \{0\}$ half from equivalent formulation of the Alcantara-Bode holds. Then, the other half should be true i.e.: the Riemann Hypothesis is true. □

5. Appendix

We consider now, the subspaces spanned by the functions indicator open interval where the intervals differ only by ending points kh of the $\Delta_{h,k}$, $k = 1, (n-1)$, then the set of these points is of measure Lebesgue zero on every level $h, nh = 1$. The new set naming it S^o used in the Injectivity Criteria previously, is dense like S, easy to show by taking any function orthogonal to all indicator interval functions in S^o and showing that it is orthogonal to all indicator interval functions in S that is dense so, the function should be zero, involving the density of S^o ([11]). Moreover, any estimation made on S_h is valid on S_h^o too. In [6], taking the closed intervals from $\Delta_{h,k}, k = 1, n, nh = 1$ we find another dense set, S^c on which, the restrictions of the operator to the corresponding subspaces coincide with the ones we used. The dense sets S and S^c built on indicator interval functions have been used in [5] as well in [6] for obtaining optimal evaluations of the decay ratio of convergence to zero of the integral operators eigenvalues having Mercer like kernels ([9]).

The integral operator T_ρ is (strict) positive on $S_h \in F, n \geq 2, nh = 1$ with the parameter valued from (9) and (11),

$\alpha_h(T_\rho) = h\alpha_h(T_{\rho_h}) = h(3-2\gamma)/4 \to 0$ with $n \to \infty$.

In order to apply Lemma 2, we should invoke the adjoint operator of T_ρ whose kernel function is $\rho^*(y,x) = \overline{\rho(x,y)} = \rho(x,y)$. For $v_h = \sum_{k=1,n} c_k I_h^k \in S_h$

$T_\rho^* v_h = \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho(x,y) I_h^k(y)dy = \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho_{h,k}(x,y)dy,$

where $\rho_{h,k} = I_{h,k}(x)\rho(x,y)I_{h,k}(y)$. Follows:

$\|T_\rho^* v_h\|^2 = \langle \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho_{h,k}(x,y)dy, \sum_{j=1,n} c_j \int_{\Delta_{h,j}} \rho_{h,j}(x,y)dy \rangle$

$= \sum_{k=1,n} c_k \overline{c_k} \big(\int_{\Delta_{h,k}} \big[\int_{\Delta_{h,k}} \rho(x,y) I_{h,k}(y)dy \big]^2 I_{h,k}(x)dx \big).$

Because $\rho(x,y)$ is valued in [0,1), $\rho(y,x) < 1$ for every $x, y \in (0,1)$, we obtain:

$\|T_\rho^* v_h\|^2 \leq \sum_{k=1,n} c_k \overline{c_k} h^3 = h^2 \|v_h\|^2$ and, $\|T_\rho^* v_h\| \leq h\|v_h\|$.

Taking $\omega_h(T_\rho^*) = h$, the injectivity parameter of T on S_h is given by:

$$\mu_h = (3-2\gamma)/4, \quad \text{a mesh independent constant } \forall n, nh = 1 \tag{12}$$

Theorem 4. (*Injectivity Criteria*): *The Alcantara-Bode equivalent of RH holds. Consequently, Riemann Hypothesis is true.*

Proof.

Because μ_h is a constant (see (12)) for any $h, nh = 1$, applying Lemma 2 we obtain $N_{T_\rho} = \{0\}$. Consequently, the Riemann Hypothesis is true. □

The references [13-16] are related to other RH equivalents, [8] to exotic integrals and [12] to multi-level discretizations on separable Hilbert spaces.

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