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On the Method for Proving the RH using the Alcantara-Bode Equivalence

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Abstract: In this article we provided the theory and the methods for the investigation of the linear, bounded operators injectivity on separable Hilbert spaces, extending the solution given in [1]. The Theorem 1 shows that a linear, bounded operator on a separable Hilbert space strict positive on a dense set, is injective. Choosing the dense set the union of including subspaces built on the indicator interval functions in $L^2(0,1)$, in order to compensate the unbounded positivity parameters sequence of the operator on the finite dimension subspaces, we separated the methods in two versions: one, considering the restrictions of the operator involving its adjoint and another, taking the finite rank operator approximations and compensating this time the strict positivity of the operator with its compactness property. The related theorem shows that, an Hilbert-Schmidt integral operator on a separable Hilbert space having the finite rank approximations with the sequence of the positivity parameters inferior bounded, is injective. Using the methods introduced, we proved the injectivity of the integral operator, injectivity that is the equivalent formulation of Alcantara-Bode ([3]) for the Riemann Hypothesis (RH). Follows, RH holds.

Keywords: Integral Operators, Hilbert Spaces, Approximation Methods, Riemann Hypothesis

MSC: 31A10, 45P05, 47G10, 65R99

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1. Introduction

Following the Beurling equivalence ([4], 1955) of the Riemann Hypothesis (RH, 1859), the Alcantara-Bode equivalence ([3], 1993) reduced RH to a concrete problem of the injectivity of a certain Hilbert-Smith integral operator on $L^2(0,1)$.

The Riemann Hypothesis (RH) claims (1859) that the Riemann Zeta function defined by the infinite sum $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ has its non trivial zeros on the vertical line $\sigma = 1/2$. The Hilbert-Schmidt integral operator T_ρ defined on $L^2(0,1)$ having the kernel function $\rho(y,x) = \{\frac{y}{x}\}$, the fractional part of the quantity between brackets,

$$(T_\rho u)(y) = \int_0^1 \rho(y,x)u(x)dx, \quad u \in L^2(0,1) \tag{1}$$

has been used by Alcantara-Bode ([2], pg. 151) in his theorem of the equivalent formulation of the RH. Denoting by N_{T_ρ} the null space of this operator, the equivalent formulation consists in (*Alcantara-Bode's Equivalence of RH*):

The Riemann Hypothesis holds if and only if $N_{T_\rho} = \{0\}$.

Thus, the task is to prove that $N_{T_\rho} = \{0\}$ and for doing it, we provided a solution containing the theory and the associated methods for the investigation of the linear bounded operators on separable Hilbert spaces.

2. An Injectivity Theorem On Separable Hilbert Spaces

Let H be a separable Hilbert space and T be a linear bounded operator on H strict positive on a dense set $S \subset H$, i.e. there exists $\alpha > 0$ such that $\langle Tv, v\rangle \geq \alpha\|v\|^2 \ \forall v \in S$. Then T has no zeros in

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S and its '*eligible*' zeros are all in the difference set $E := H \setminus S$, i.e. $N_T \subset E$. The next theorem (see also an earlier version in [9]) proved that T has no zeros in the difference set either, obtaining $N_T = \{0\}$.

Theorem 1. *A linear bounded operator T strict positive definite on a dense set of a separable Hilbert space is injective, equivalently $N_T = \{0\}$.*
Proof.

Let's take in consideration only the set of eligible zeros that are on the unit sphere without restricting the generality, once an element $w \in H, w \neq 0$ and $w/\|w\|$ are both or are not both in N_T. The set $S \subset H$ is dense if its closure coincides with H. Then, if $w \in E$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that $\|u_{\varepsilon,w} - w\| < \varepsilon$. Then

$$| \|w\| - \|u_{\varepsilon,w}\| | < \varepsilon \tag{2}$$

Consider w an eligible element from the unit sphere, $\|w\| = 1$.
Given ε_n, there exists at least one element in the dense set $u_{\varepsilon_n,w} \in S$ such that $\|u_{\varepsilon_n,w} - w\| < \varepsilon_n$ holds. Follows from (2), taking $\varepsilon_n = 1/n$: $|$ 1 - $\|u_{\varepsilon_n,w}\| \; | < 1/n$ showing that, for any choices of the sequence approximating w, $u_{\varepsilon_n,w} \in S, n \geq 1$, the sequence $\|u_{\varepsilon_n,w}\| \to 1$ with $n \to \infty$.

If T is a linear bounded operator on H strict positive on S, then there exists $\alpha > 0$ such that $\forall u \in S$, $\langle Tu, u\rangle \geq \alpha\|u\|^2$.
Suppose that there exists $w \in E, \|w\| = 1$ a zero of T, i.e. $w \in N_T$ and consider a sequence of approximations of w, $u_{\varepsilon_n,w} \in S, n \geq 1$ that, as we showed, is converging in norm to 1. From the positivity of T on the dense set S, follows:

$$\alpha\|u_{\varepsilon_n,w}\|^2 \leq \langle Tu_{\varepsilon_n,w}, u_{\varepsilon_n,w}\rangle = \langle T(u_{\varepsilon_n,w} - w), u_{\varepsilon_n,w}\rangle < \varepsilon_n\|T\|\|u_{\varepsilon_n,w}\| \tag{3}$$

With c=$\|T\|/\alpha$, we obtain $\|u_{\varepsilon_n,w}\| \leq c/n$. Then, $\|u_{\varepsilon_n,w}\| \to 0$ with $n \to \infty$, in contradiction with its convergence to 1. Or, this happen for any choice of the sequence of approximations of w, verifying $\|u_{\varepsilon_n,w} - w\| < \varepsilon_n, n \geq 1$, when $Tw = 0$.
Thus $w \notin N_T$, valid for any $w \in E, \|w\| = 1$, proving the theorem because no zeros of T there are in S either.

For using the theorem for practical purposes, we have to ease the terms in which it works considering the positivity only of the operator on a dense set. Then additional requests should be imposed in order to remain in terms of the theorem.

Suppose that the dense set S is the result of a union of finite dimension subspaces: $S = \cup_{n\geq 2} S_n, \overline{S} = H$.

Observation 1. Let $\beta_n(u) := \|u - u_n\|$ be the normed residuum of the eligible element $u \in E$ after its orthogonal projection on S_n. Then, $\beta_n(u) \to 0$ with $n \to \infty$.
Proof.

Given $\epsilon > 0$, from the density of the set S in H there exists $u_\epsilon \in S$ verifying $\|u - u_\epsilon\| < \epsilon$, as per the observations made above in the proof of the Theorem 1. Let S_{n_ϵ} be the coarsest subspace, i.e. with the smallest dimension, from the family of subspaces containing u_ϵ. Because the best approximation of u in S_{n_ϵ} is its orthogonal projection, we obtain

$\beta_{n_\epsilon}(u) := \|u - P_{n_\epsilon}u\| \leq \|u - u_\epsilon\| < \epsilon$,
inequality valid for every $\epsilon > 0$, proving our assertion.

Let T be a linear bounded operator strict positive on each subspace S_n, $\langle Tv, v\rangle \geq \alpha_n\|v\|^2 \; \forall v \in S_n$, $\alpha_n > 0, n \geq 1$. Being strict positive definite on every finite dimension subspace of the family, the operator has no zeros in the family subspaces and, its zeros if there are, are in the difference set $E := H \setminus S$. Denoting the orthogonal projection of an eligible u onto S_n by $u_n := P_n u$ there exists

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$n_0 := n_0(u)$ due to the density of S, such that $u_n \neq 0$ for any n $\geq n_0$.
The relationship between the residuum on finite dimension subspaces of an eligible zero of an linear bounded operator and the operator positivity parameters on these subspaces is exploited in defining the methods for investigation of its injectivity.

Supposing $u \in N_T$ is an eligible zero of T with $\|u\| = 1$, the relation in (3) on S_n involving its not null projection $u_n, n \geq n_0 := n_0(u)$, defines the inequalities corresponding to the methods for investigation of the injectivity:

a) If $\alpha_n \geq \alpha > 0$ then

Corollary. The following inequality holds for $n \geq n_0 := n_0(u)$:

$$\alpha_n \|u_n\|^2 \leq \langle Tu_n, u_n \rangle = \langle T(u_n - u), u_n \rangle \leq \beta_n \|T\| \|u_n\|. \tag{4}$$

$\alpha / \|T\| \leq \alpha_n / \|T\| \leq \beta_n / \|u_n\| = \beta_n / \sqrt{1 - \beta_n^2}$
This is a relationship in which the right side converges to zero and so, there exists a range $n_0 := n_0(u)$ such that for $n \geq n_0$ the inequality is violated for any potential zero of unitary norm of the operator, making the operator injective.

b) If $\alpha_n \to 0$ then we can use:

Injectivity Criteria. From (4) this time we evaluate the parameters from:
$\langle T(u_n - u), u_n \rangle = \langle (u_n - u), T^* u_n \rangle$. Then, $\alpha_n \|u_n\|^2 \leq \beta_n \omega_n \|u_n\|$ where ω_n is closest to the norm of the adjoint operator: $\|T^* v\| \leq \omega_n \|v\|$ for every $v \in S_n$.
Defining the injectivity parameter $\mu_n = \alpha_n / \omega_n$ the following relation holds:

$$\mu_n \leq \beta_n \|u_n\| := \beta_n / \sqrt{1 - \beta_n^2} \tag{5}$$

Lemma 1. *If one from the sequences of the positivity parameters $\{\alpha_n, n \geq 1\}$ and the injectivity parameters $\{\mu_n, n \geq 1\}$ is inferior bounded by a not null positive constant, then T is injective on H.*
Proof.

a) (*Corollary*) Suppose that $\alpha_n \geq \alpha > 0$ for $n \geq 1$. From (4), $\alpha / \|T\| \leq \beta_n \|u_n\| \to 0$ with $n \to \infty$ that is a contradiction. Hence, $u \in E$ is not a zero of T. Because T has no zeros in S either, its null space contains only 0: $N_T = \{0\}$.

b) (*Injectivity Criteria*) If the sequence α_n is unbounded inferior, from (5), $\mu_n \leq \beta_n \|u_n\|$. If there exists a constant $C > 0$ for which $\mu_n \geq C$ for every $n \geq 1$ then from $0 < C \leq \mu_n \leq \beta_n \|u_n\|$ we obtain $u \notin N_T$, valid for any zero of T eligible, so $N_T = \{0\}$ because T has no zeros from the dense set either.

For now on, $H = L^2(0, 1)$. The integral operator T_ρ defined in (1) having the kernel function $\rho(y, x)$ continuous everywhere on $(0,1)^2$ excepting a set of measure Lebesgue zero - it is countable set of unidimensional lines in $(0,1)^2$ of the form y = kx, k$\in N$ - is a linear, bounded, Hilbert-Schmidt operator ([2]) and so, compact. Then, it could be approximated by projections on the finite dimension subspaces (see also [3]) of the $L^2(0, 1)$.

3. Algebraic - Functional Approach

Let $h := 2^{-m}, m \geq 1$, nh =1 and define the set of disjoint and open intervals of length h, $\Delta_h^k = ((k-1)/2^m, k/2^m)$ that are the supports of the interval indicator functions:

$$I_h^k(t) = 1 \text{ for } t \in \Delta_h^k \text{ and 0 otherwise.} \tag{6}$$

The union S of the family of spans $S_h := span\{I_h^k, k = 1, n, nh = 1, n \geq 2\}$ is dense in H, well known in literature. Then by Observation 1, for any $u \in E := H \setminus S$,

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$\beta_h(u) := \|u - u_h\| \to 0$ with u_h the orthogonal projection of u on S_h.
Having the convergence to zero of the residuum, see Observation 1, next step is to compute the positivity parameters of our operator on the approximation subspaces. For this, we have to evaluate the inner product between $(T_\rho u_h) \notin S_h$ and $u_h \in S_h$ where u_h is the not null orthogonal projection of an eligible $u \in E := H \setminus S$.

Denoting with $P_h : E \subset H \to S_h \subset H$ the orthogonal projection onto S_h, then:
$E \ni u \to u_h := P_h u = \sum_{k=1,n} I_h^k \frac{1}{|\Delta_h^k|} \int_{\Delta_h^k} u(s)ds \ \in S_h$ holds. Or,

$$u_h(t) := (P_h u)(t) = h^{-1} \sum_{k=1,n} \Big(\int_{\Delta_h^k} u(s)ds \Big) I_h^k(t) \tag{7}$$

With the basis $\{I_h^k : k = 1, n, nh = 1\}$ on the linear span S_h the matrix representation of the integral operator T_ρ is sparse diagonal because the orthogonality of the basis elements is a consequence of the disjoint supports between any pair $I_h^k, I_h^j, k \neq j$: $I_h^k(y)\rho(y,x)I_h^j(x) = 0$ for $k \neq j$.
So, the matrix $M_h(T_\rho)$ has the diagonal entries $d_{kk}^h, k = 1, n,\ nh = 1$ given by:

$$d_{11}^h = h^2(3 - 2\gamma)/4 \tag{8}$$

$$\begin{aligned} d_{kk}^h &= \int_{(k-1)h}^{kh} \Big[\int_{(k-1)h}^{kh} (y/x)dx - \int_{(k-1)h}^{y} dx \Big] dy \\ &= \frac{h^2}{2}\Big(-1 + \frac{2k-1}{k-1} ln\Big(\frac{k}{k-1}\Big)^{k-1}\Big) \end{aligned} \tag{9}$$

for $k \geq 2$, where γ is Euler-Mascheroni constant, values taken from [1].
In fact, $\langle T_\rho I_h^k, I_h^k \rangle := \langle P_h T_\rho I_h^k, I_h^k \rangle$ applying (7), as per
$P_h T_\rho I_h^k = h^{-1} \sum_{j=1,n} I_h^j(y) \int_{\Delta_h^j} \big[\int_{\Delta_h^k} \rho(y,x)dx \big] dy =$
$h^{-1} \big[\int_{\Delta_h^k} \int_{\Delta_h^k} \rho(y,x)dxdy \big] I_h^k(y) = h^{-1} d_{kk}^h I_h^k(y)$. So,

$$d_{kk}^h = \int_{\Delta_h^k} \int_{\Delta_h^k} I_h^k(y)\rho(y,x)I_h^k(x)dxdy := \int_{\Delta_h^k} \int_{\Delta_h^k} \rho(y,x)dxdy \tag{10}$$

and their values for our operator of interest are in (9). Then:
$\langle T_\rho I_h^k, I_h^k \rangle = \langle h^{-1} d_{kk}^h I_h^k, I_h^k \rangle = h^{-1} d_{kk}^h \|I_h^k\|^2 = d_{kk}^h$ and,
$\alpha_h(T_\rho) = h^{-1} d_{11}^h = h(3 - 2\gamma)/4 \to 0$

The positivity parameter value is the same on S_h as per estimating $P_h(T_\rho u_h)$ with $u_h = \sum_{k=1,n} c_k I_h^k$ then applying (7) as we will do now:
$(P_h T_\rho \sum_{k=1,n} c_k I_h^k)(y) = h^{-1} \sum_{k=1,n} c_k d_{kk}^h I_h^k(y)$.
So, we are able now, to compute the inner product $\langle T_\rho u_h, u_h \rangle$ in (4):
$\langle (P_h T_\rho \sum_{k=1,n} c_k I_h^k)(y), \sum_{k=1,n} c_k I_h^k(y) \rangle = h^{-1} \sum_{k=1,n} d_{kk}^h c_k \overline{c_k} \|I_h^k\|^2$
$\geq h^{-1} d_{11}^h \big(\sum_{k=1,n} c_k \overline{c_k} \|I_h^k\|^2 \big) = h^{-1} d_{11}^h \|v_h\|^2$,
in order to obtain the sequence of the positivity parameter α_h valued like before.
We could observe that T_ρ is strict positive definite on any S_h but, is only positive definite on S. For investigating the injectivity, we should involve the adjoint operator T_ρ^* whose kernel function is defined by
$\rho^*(y,x) = \overline{\rho(x,y)} = \rho(x,y)$.
So, for $v_h = \sum_{k=1,n} c_k I_h^k \in S_h$,
$T_\rho^* v_h = \sum_{k=1,n} c_k \int_{\Delta_h^k} \rho(x,y) I_h^k(y)dy$

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$= \sum_{k=1,n} c_k \int_{\Delta_h^k} (\sum_{j=1,n} I_h^j(x))\rho(x,y) I_h^k(y) dy$
$= \sum_{k=1,n} c_k \int_{\Delta_h^k} \rho_h^k(x,y) dy,$
a formula obtained from the property $I_h^j(x)\rho(x,y)I_h^k(y) = 0$ for $k \neq j$.
Then
$\|T_\rho^* v_h\|^2 = \langle \sum_{k=1,n} c_k \int_{\Delta_h^k} I_h^k(x)\rho(x,y) I_h^k(y) dy, \sum_{j=1,n} c_j \int_{\Delta_h^j} \rho(x,y) I_h^j(y) dy \rangle$
$= \sum_{k=1,n} c_k \overline{c_k} \big(\int_{\Delta_h^k} \big[\int_{\Delta_h^k} \rho(x,y) I_h^k(y) dy \big]^2 I_h^k(x) dx \big).$
Because $\rho(x,y)$ is valued in [0,1),
$\|T_\rho^* v_h\|^2 \leq \sum_{k=1,n} c_k \overline{c_k} h^3 = h^2 \|v_h\|^2$ and,
$\|T_\rho^* v_h\| \leq h\|v_h\|$. Taking $\omega_h(T_\rho^*) = h$, follows:

$$\mu_h = (3 - 2\gamma)/4, \quad \text{a mesh independent constant } \forall n, nh = 1 \tag{11}$$

In computing the values of the diagonal matrix entries, we used the observation made by Beurling ([4]): for $0 < a < b < min\{2a, 1\}$ the fractional part $\{b/a\} = (b/a - 1)$. The sequence $(-1 + \frac{2k-1}{k-1} ln(\frac{k}{k-1})^{k-1})/2$ is monotone decreasing for $k \geq 2$ and converges to 0.5 for k $\to \infty$. Follows $\forall h, d_{kk}^h > h^2/2 > d_{11}^h, k \geq 2$. With these estimations of the positivity and injectivity parameters, we have the following result applying the Injectivity Criteria.

Theorem 2.1 (Injectivity Criteria.) The Hilbert-Schmidt operator T_ρ defined in (1) is injective and, consequently RH holds.
Proof.
Because μ_h is a constant for any $h, nh = 1$, there exists an $n1 := n1(u)$ such that for $n \geq n1, nh = 1$, $\beta_h(u)\sqrt{1 - \beta_h(u)} < \mu_h$ that contradicts the relationship in (5) meaning, Lemma 1 could be applied for obtaining $u \notin N_{T_\rho}$, valid for any eligible zero of T_ρ. Because T_ρ has no zeros in the dense set S either, we obtain

$$N_{T_\rho} = \{0\} \tag{12}$$

Now, from (12) half from equivalent formulation of the Alcantara-Bode is true so, the other half is true obtaining: the Riemann Hypothesis holds.

The involvement of the adjoint operator could be avoided by using the finite rank operator approximations together with the compactness property of the integral operator.

4. Finite Rank Operator Approximations

It is well known that on a Hilbert space, every compact operator is a limit of finite-rank operators. The converse is always true. The integral operator in (1) is Hilbert-Schmidt ([2]) and then, compact. Thus, it can be approximated by finite rank integral operators. If P_n^r is a such projection operator, then the finite rank approximations of T_ρ verifies $\|P_n^r(T_\rho) - T_\rho\| \to 0$ for $n \to \infty$. Let be $T_n := P_n^r(T_\rho)$.

Theorem 2. An Hilbert-Schmidt integral operator on a separable Hilbert space T_φ with the finite rank approximations on a family of subspaces whose union is dense having the positivity parameters inferior bounded, is injective.
Proof.
Suppose $S_n \subset S_{n+1}$. Let $\alpha_n \geq \alpha > 0$ be the positivity parameters of the finite rank approximations $T_n, n \geq 1$, satisfying $\alpha_n \|u_h\|^2 \leq \langle T_n u_n, u_n \rangle$
Because $T := T_\varphi$ is Hilbert-Schmidt operator, it is compact, so, $\|T_n - T\| \to 0$ with $n \to \infty$. Now, for $u \in H \setminus U_{n \geq 1} S_n$, $\|u\| = 1$, for the not null orthogonal projections $P_n u$ of u, verifies $\beta_n = \|u - P_n u\| \to 0$.
If $u \in N_T$ then:
$\alpha \|P_n u\|^2 \leq \langle T_n P_n u, P_n u \rangle \leq \|T_n P_n u\| \|P_n u\|$

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$\leq (\|T_nP_nu - TP_nu + TP_nu - Tu\|)\|P_nu\|$.

So:

$\alpha \leq \left(\epsilon_n + \|T\|\beta_n/\sqrt{1-\beta_n^2}\right)$ where: $\epsilon_n := \|T - T_n\| \to 0$ and $\beta_n \to 0$. Then, the inequality is violated from an n_o, $n \geq n_0$ meaning, $u \notin N_T$, for any zero of T in E. Now, if there exists a zero of T in S' then it should be in one of the subspaces of F' i.e. there exists $u_n \in S_n'$ such that $Tu_n = 0$. Or, from:

$\alpha_n\|u_n\|^2 \leq \langle T_nu_n, u_n\rangle = \langle (T_n - T)u_n, u_n\rangle \leq \|T - T_n\|\|u_n\|^2$, obtaining $\alpha \leq \|T_n - T\| \to 0$ with $n \to \infty$ that is a contradiction, shoving that the operator has no zeros in the dense set. Thus, $N_T = \{0\}$ and so injective.

We will use this theorem as support for proving the injectivity of our integral operator T_ρ in (1) on $L^2(0,1)$.

The intervals of equal lengths $h = 2^{-m}, m \geq 1$, nh = 1, $\Delta_{h,k}' = ((k-1)/2^m, k/2^m], k = 1, n-1$ together with Δ_h^n define for $m \in N$ a partition of (0,1), k=1,n, $n = 2^m, nh = 1$.

Consider the interval indicator functions having the supports these intervals:

$$I_{h,k}'(t) = 1 \text{ for } t \in \Delta_{h,k}' \text{ and } 0 \text{ otherwise} \tag{13}$$

The family F' of finite dimensional S_h' linear spans of interval indicator functions of the h-partitions defined by (13) with disjoint supports, $S_h' = span\{I_{h,k}'; k = 1, n, nh = 1\}$, built on a multi-level structure, are including $S_h' \subset S_{h/2}'$ by halving the mesh h.

Let observe that the pairs of indicator open interval functions defined in (6) and the indicator semi-open interval functions defined in (13) I_h^j and $I_{h,j}'$ having the supports of size h, as functions in $L^2(0,1)$ differ as values only in the right endpoint $(k+1)h \in (0,1)$ of the support of the semi-open interval and so, being different on a set of measure Lebesgue zero containing only the right endpoint, verifies on $L^2(0,1)$: $\|I_h^j - I_{h,j}'\| = 0$, for j=1,n, nh=1, $n \geq 2$.

Now, suppose that there exists $f \in L^2(0.1)$ orthogonal to any $I_{h,j}'$, j=1,n, nh=1, $n \geq 2$. Then from

$|\langle f, I_h^j\rangle| = |\langle f, I_h^j - I_{h,j}'\rangle| \leq \|f\|\|I_h^j - I_{h,j}'\| = 0$

we obtain that f is orthogonal to $I_h^j, j = 1, n, nh = 1, n \geq 2$ and so, because S is dense, then f=0. Because f has been considered orthogonal in $L^2(0,1)$ to any $I_{h,j}' \in S'$ follows the density of S'. In the same way, is true that if S' is dense then S is dense.

Moreover, because the pairs of indicator interval functions I_h^j and $I_{h,j}'$ j=1,n, nh=1, differ only on the endpoints of the h-partitions of the domain (0,1), the entries in the matrix representations of the integral operator restrictions to S_h' coincide with the corresponding entries in the matrix representations of the integral operator restrictions to S_h, $n \geq 2, nh = 1$ and so, both having the same diagonal entries $d_{kk}^h, k = 1, n, nh = 1$, i.e.

$d_{kk}^h = \int_{\Delta_h^k}\int_{\Delta_h^k}\rho(y,x)dxdy = \int_{\Delta_{h.k}'}\int_{\Delta_{h,k}'}\rho(y,x)dxdy$ valued in (8), (9).

Below, we follow the steps from [5] for defining the orthogonal projection of the integral operator T_ρ by finite rank integral operators as follows.

The orthogonal projections of the kernel function is performed by the orthogonal projections of the integral operator through the finite rank integral operators ([5], pg. 986) whose kernel functions are:

$$r_h(y,x) = h^{-1}\sum_{k=1,n} I_{h,k}'(y)I_{h,k}'(x) \tag{14}$$

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As result, the integral operator projection on $S_h^{'}$, T_{ρ_h} has the kernel function ([5])

$$\rho_h(y,x) = h^{-1} \sum_{k=1,n} I_{h,k}^{'}(y)\rho(y,x)I_{h,k}^{'}(x) := h^{-1} \sum_{k=1,n} \rho_h^k(y,x) \tag{15}$$

that is a sum of kernel pieces with disjoint supports in which $I_{h,k}^{'}(y)\rho(y,x)I_{h,j}^{'}(x)$ are taken to be 0 for $k \neq j$. Due to the observations made between the indicator functions having the supports open intervals and semi-open intervals, on $S_h^{'}$ we have:
$\langle T_{\rho_h} I_{h,k}^{'}, I_{h,k}^{'} \rangle = h^{-1} \int_{\Delta_{h,k}^{'}} \int_{\Delta_{h,k}^{'}} \rho_h^k(y,x)dxdy$
$= h^{-1}d_{kk}^h = h^{-2}d_{kk}^h \|I_{h,k}^{'}\|^2$, k=1,n, nh=1, $n \geq 2$
meaning, the matrix representation of the finite rank integral operator T_{ρ_h} differs by the normalisation factor h^{-1} from the matrix representation of the restriction of the integral operator to $S_h^{'}$ that, in turn coincides with the matrix representation of the restriction of the integral operator to S_h.

Then, the positivity parameter of the finite rank operator approximations verifies
$\alpha_h(T_{\rho_h}) = h^{-2}min_{k=1,n}d_{kk}^h = h^{-1}\alpha_h(T_\rho|_{S_h^{'}}) = h^{-1}\alpha_h(T_\rho|_{S_h}),\ n \geq 2, nh = 1.$
If we proceed like in [1], we could verify that the positivity parameter $\alpha_h(T_{\rho_h})$ is preserved on the whole family $F^{'}$. So, for $v_h = \sum_{k=1,n} c_k I_{h,k}^{'} \in S_h^{'}$, let compute

$$\begin{aligned}
\langle T_{\rho_h} v_h, v_h \rangle &= h^{-1} \int_0^1 [\int_0^1 \sum_{k=1,n} I_{h,k}^{'}(y)\rho(y,x)I_{h,k}^{'}(x)v_h(x)dx]\overline{v_h}(y)dy \\
&= h^{-1} \sum_{k=1,n} \int_{\Delta_{h,k}^{'}} [\int_{\Delta_{h,k}^{'}} I_{h,k}^{'}(y)\rho(y,x)I_{h,k}^{'}(x)v_h(x)dx]\overline{v_h}(y)dy \\
&= h^{-1} \sum_{k=1,n} c_k\overline{c_k} \int_{\Delta_{h,k}^{'}} \int_{\Delta_{h,k}^{'}} I_{h,k}^{'}\rho(y,x)I_{h,k}^{'}dxdy = h^{-1} \sum_{k=1,n} c_k\overline{c_k}d_{kk}^h \\
&\geq h^{-2}(min_{k=1,n}d_{kk}^h)\|v_h\|^2 \geq h^{-2}d_{11}^h\|v_h\|^2
\end{aligned} \tag{16}$$

The computed values of the entries $d_{kk}^h, k = 1,n,\ nh = 1$ of the diagonal matrix representation $M_h(T_{\rho_h}) = h^{-1}M_h(T_\rho)$ are given by ([1]) and are identical with their values from the previous paragraph.

Follows $\forall h, d_{kk}^h > h^2/2 > d_{11}^h, k \geq 2$ with the positivity parameter

$$\alpha_h(T_{\rho_h}) := h^{-2}d_{11}^h = (3 - 2\gamma)/4, \tag{17}$$

a constant mesh independent, for any $n \geq 2, nh = 1$.

From the convergence in norm of the finite rank approximations to the compact integral operator, $\epsilon_h := \|T_{\rho_h} - T_\rho\| \to 0$ for $n \to \infty, nh = 1$, we have for $v \in S^{'}$:
$|\langle (T_{\rho_h} - T_\rho)v, v\rangle| \leq \epsilon_h\|v\|^2$ with $\epsilon_h \to 0$.
Let observe that the kernel ρ is positive valued on $(0,1)^2$. Then T_ρ is positive definite on $S^{'}$ and has no zeros in the dense set as follows from:
$\langle T_\rho I_{h,k}^{'}, I_{h,k}^{'} \rangle = \int_0^1 (\int_0^1 \rho(y,x)I_{h,k}^{'}(x)dx) I_{h,k}^{'}(y)dy$
$= \int_{\Delta_{h,k}^{'}} \int_{\Delta_{h,k}^{'}} \rho(y,x)dxdy := d_{kk}^h > 0$. Then, $u_h \notin N_{T_\rho}$ because from $0 = \langle T_\rho u_h.u_h \rangle = \sum_{k=1,n} c_k\overline{c_k}d_{kk}^h$ results: $c_k\overline{c_k} = 0, k = 1,n$, i.e. $u_h = 0$ for every $u_h = \sum_{k=1,n} c_k I_{h.k}^{'}$.
Then $\langle T_\rho u_h, u_h \rangle > 0,\ n \geq 2, nh = 1$, showing that he compact operator T_ρ is positive definite on the dense family $F^{'}$ and without zeros in $S^{'}$.

Moreover:

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because ϵ_h converges to 0 with $h \to 0, nh = 1$, there exists ϵ_0 closest to α such that $\epsilon_0 < \alpha$. Let h_0 be the mesh corresponding to ϵ_0.
Then for $n \geq n_0(\epsilon_0), nh = 1$:
$\langle T_\rho u_h, u_h \rangle \geq |\langle T_{\rho_h} u_h, u_h \rangle - |(\langle (T_{\rho_h} - T_\rho) u_h, u_h \rangle)|| \geq \alpha_h(T_\rho) \|u_h\|^2 > 0$
where $\alpha_h(T_\rho) \geq \alpha - \epsilon_0 > 0$ a constant and so, T_ρ is strict positive definite on the dense family $F^{'}$ excepting a finite number of subspaces (see Lemma 2 in [9]).

Thus, at this point, we have:
- the strict positivity of the of the integral operator obtained before through its compactness, will be used as a corollary of Theorem 1. (Theorem 3.1).
- finite rank approximations of T_ρ are strict positive on the dense set and, their sequence of the positive parameters is inferior bounded. This property will be used for a shorter proof (Theorem 3.2) of the injectivity bypassing the request for the strict positivity of the operator on the dense set in the Theorem 1.

Theorem 3.1 (Corollary of Theorem 1). The Hilbert-Schmidt operator T_ρ defined in (1) is injective and, consequently RH holds.
Proof.
Supposing $u \in E \cap N_{T_\rho}, \|u\| = 1$, then (4) holds with $\alpha_h(T_\rho) \geq \alpha - \epsilon_0, nh = 1, n \geq n(\epsilon_0)$ that is the inferior bound of the positivity parameters. Then:
$(\alpha - \epsilon_0)/\|T_\rho\| \leq \beta_h/\sqrt{1-\beta_h^2}$ where right hand converges to zero, we obtain a contradiction. So $u \notin N_{T_\rho}$. Because the operator has no zeros in $S^{'}$ either, it is injective, equivalently $N_{T_\rho} = \{0\}$.

In the hypothesis of the inferior bounded of the positivity parameters of the finite rank operators approximations (17) of the Hilbert-Schmidt integral operator T_ρ we obtain again the injectivity of the operator in (1) through its compactness. The following theorem is a particular case of the generic Theorem 2.
Let the finite rank approximations of the integral operator T_ρ on the separable Hilbert space $L^2(0,1)$ on the dense family of the finite dimension subspaces built on the indicator interval functions of the domain is partitioned of size h, $S_h^{'} \subset S_{h/2}^{'}, nh = 1, n \geq 2$.

Theorem 3.2 (Corollary). If the sequence of the positivity parameters of the finite rank operator approximations of the Hilbert-Schmidt integral operator is bounded inferior, then the operator is injective.
Proof.
T_ρ has its finite rank approximations strict positive definite with positivity parameters mesh independent, $\alpha_h := \alpha = (3-2\gamma)/4$. On $S_h^{'}$:
$\alpha\|u_h\|^2 \leq \langle T_{\rho_h} u_h, u_h \rangle \leq \|T_{\rho_h} u_h\| \|u_h\|$.
Follows for $u \in E \cap N_{T_\rho}, \|u\| = 1$, reminding that $\beta_h := \beta_h(u)$ is the normed residuum of $u \in E$ on $S_h^{'}$ converging to zero (see Observation 1):
$\|T_{\rho_h} u_h\| = \|(T_{\rho_h} - T_\rho)u_h + T_\rho(u_h - u)\| \leq \|T_{\rho_h} - T_\rho\| \|u_h\| + \|T_\rho\|\beta_h$
$\leq \epsilon_h \|u_h\| + \|T_\rho\|\beta_h$
where $\epsilon_h \to 0, \beta_h \to 0, \|u_h\| \to 1$. Thus,

$$(3-2\gamma)/4 \leq \left(\epsilon_h + \|T_\rho\|\beta_h/\sqrt{1-\beta_h^2}\right) \to 0 \tag{18}$$

for any $n \geq n_0 := n_0(u), nh = 1$, from an index $n_0 < \infty$. But, this is a contradiction because both terms in the right side sum converges to 0 from a range $n_1 \geq n_0$. The only supposition we made, has been $T_\rho u = 0$, so, $u \notin N_{T_\rho}$ valid for any zero of T_ρ if there are in E. Then $N_{T_\rho} = \{0\}$ once T_ρ has no zeros in $S^{'}$ either.

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