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The Riemann Hypothesis: An Approach via Mellin and Widder-Lambert Type Transforms

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Abstract: In this paper we derive a Plancherel theorem specific for a Widder-Lambert type integral transform by employing the corresponding Plancherel theorem for the classical Mellin transform. These findings lead us to explore a class of functions in connection with Salem's equivalence to the Riemann hypothesis.

Keywords: Mellin transform; Widder-Lambert type transform; Plancherel's theorem; Salem's equivalence; Riemann hypothesis

1. Introduction and Preliminaries

The Riemann hypothesis stands as a pivotal issue in mathematics, inspiring numerous related hypotheses and concepts (refer to [1] for an in-depth examination of the Riemann hypothesis). In 1953, the Greek mathematician Raphaël Salem identified a necessary and sufficient condition involving an integral equation that is equivalent to the Riemann hypothesis [2]. This equivalence allows the problem to be approached within the framework of integral transforms. Building on this perspective, this paper presents Plancherel's theorem specific to a Widder-Lambert type integral transform. Consequently, we explore a class of functions in relation to Salem's equivalence with the Riemann hypothesis.

For $\operatorname{Re} s > 1$, the Riemann zeta-function $\zeta(s)$ (see [1]) is expressed as a convergent series:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

Also, denote by $\eta(s)$ the Dirichlet eta-function represented for $\operatorname{Re} s > 0$ by the convergent series:

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}.$$

For $\operatorname{Re} s > 1$ following relation holds

$$\eta(s) = \left(1 - 2^{1-s}\right)\zeta(s),$$

which extends $\zeta(s)$ to $0 < \operatorname{Re} s < 1$.

It is well known that $\zeta(s)$ (and thus $\eta(s)$) has no zeros in $\operatorname{Re} s > 1$.

The Salem equivalence to the Riemann hypothesis (Salem [2] and Broughan [3] (§8.4, p. 139–142)) assures that the Riemann zeta-function is free of zeros in the strip $0 < \operatorname{Re} s < 1$, $\operatorname{Re} s \neq \frac{1}{2}$, (or equivalently, from the symmetry of the zeros, in the strip $\frac{1}{2} < \operatorname{Re} s < 1$), if and only if for $\frac{1}{2} < \delta < 1$, then any bounded measurable function f on $\mathbb{R}_+ = (0, \infty)$ satisfying the integral equation

$$\int_0^\infty \frac{t^{\delta-1}}{e^{xt}+1} f(t)\,dt = 0, \quad \text{for all} \quad x > 0, \tag{1.1}$$

is zero almost everywhere on $\mathbb{R}_+$ (see [4] (section 4), [5–7]).
The Mellin transform of a suitable function f is given as [1]

$$(\mathcal{M}[f])(s) = \int_0^\infty t^{s-1} f(t)dt, \; s \in \mathbb{C}. \tag{1.2}$$

In this paper, motivated by the integral equation (1.1), we consider for each $\delta > 0$ the Widder-Lambert type integral transform (cf. [8] (eq. (4.1)) and [4] (eq. (2.15)))

$$(\mathcal{L}_\delta[f])(x) = \int_0^\infty \frac{t^{\delta-1}}{e^{xt}+1} f(t)dt, \quad x > 0, \tag{1.3}$$

where f is a suitable complex-valued function such that the integral converges.

In section 2 we establish an interesting connection between the Mellin transform (1.2) and the Widder-Lambert type transform (1.3) for some specific class of functions.

In section 3, motivated by the connection established in section 2, we obtain a Plancherel theorem for the Widder-Lambert type transform (1.3) (via the Mellin transform). As a consequence of this result we prove that if f is a bounded measurable function such that $f(t) = O(t^{1/2})$ as $t \to 0^+$, and $\mathcal{L}_\delta[f] = 0$, $\frac{1}{2} < \delta < 1$, then $f = 0$ almost everywhere on $\mathbb{R}_+$. This is a significant approach to the Salem criterion and thus to the Riemann hypothesis.

For other class of functions a corresponding result was obtained by Yakubovich in [4] (section 4, Corollary 1) by means of a change of variables into the Widder-Lambert transform (which becomes (1.3)) and using an inversion formula for this transform (see [4] (Theorem 4, formulae (2.15) and (2.16))).

We denote by $L_{a,p}(\mathbb{R}_+)$, $a \in \mathbb{R}$, $1 \le p < \infty$, the Banach space of the complex-valued functions f on $\mathbb{R}_+$ with the norm

$$\|f\|_{a,p} = \left(\int_0^\infty |f(t)|^p t^{ap-1} dt \right)^{1/p}.$$

2. Mellin Versus Widder-Lambert Type Transforms

Set $s \in \mathbb{C}$ with $\Re s < \delta$, $t > 0$. It follows

$$\int_0^\infty x^{-s} \frac{x^{\delta-1}}{e^{xt}+1} dx = \int_0^\infty x^{-s+\delta-1} \left(\sum_{k=1}^\infty (-1)^{k-1} e^{-kxt} \right) dx$$

$$= \int_0^\infty \lim_{n\to\infty} \sum_{k=1}^n (-1)^{k-1} x^{-s+\delta-1} e^{-kxt} dx. \tag{2.1}$$

Now, for all $n \in \mathbb{N}$

$$\sum_{k=1}^n \left| (-1)^{k-1} x^{-s+\delta-1} e^{-kxt} \right| = \sum_{k=1}^n x^{-\Re s+\delta-1} e^{-kxt} \le \frac{x^{-\Re s+\delta-1}}{e^{xt}-1}.$$

Also, observing that $\frac{xt}{e^{xt}-1} \le 1$, the function of x given by $\frac{x^{-\Re s+\delta-1}}{e^{xt}-1}$ is integrable on $\mathbb{R}_+$ for $\Re s < \delta - 1$. Then, for the dominated convergence theorem the integral in (2.1) is equal to

$$\sum_{k=1}^\infty (-1)^{k-1} \int_0^\infty x^{-s+\delta-1} e^{-kxt} dx, \quad \text{for } \Re s < \delta - 1.$$

Making $u = kxt$ one obtains that this expression is equal to

$$\sum_{k=1}^\infty (-1)^{k-1} \int_0^\infty \frac{u^{-s+\delta-1}}{(kt)^{-s+\delta}} e^{-u} du = \sum_{k=1}^\infty (-1)^{k-1} \frac{t^{s-\delta}}{k^{s-\delta}} \Gamma(\delta - s)$$

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$$= t^{s-\delta}\Gamma(\delta - s) \cdot \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{\delta-s}} \right) = t^{s-\delta}\Gamma(\delta - s)\eta(\delta - s), \quad \text{for } \Re s < \delta - 1.$$

Considering $f \in L_{\Re s,1}(\mathbb{R}_+)$, $\Re s < \delta - 1$, and using Fubini's theorem one has

$$\int_0^\infty t^{s-1} f(t)\Gamma(\delta - s)\eta(\delta - s)dt = \int_0^\infty t^{\delta-1} f(t) \cdot t^{s-\delta}\Gamma(\delta - s)\eta(\delta - s)dt$$

$$= \int_0^\infty t^{\delta-1} f(t) \int_0^\infty x^{-s} \frac{x^{\delta-1}}{e^{xt}+1} dxdt = \int_0^\infty x^{-s} \int_0^\infty t^{\delta-1} f(t) \frac{x^{\delta-1}}{e^{xt}+1} dtdx$$

$$= \int_0^\infty x^{-s+\delta-1} \int_0^\infty f(t) \frac{t^{\delta-1}}{e^{xt}+1} dtdx = (\mathcal{M}[\mathcal{L}_\delta[f]])(\delta - s),$$

where $\mathcal{M}$ denotes the Mellin transform (1.2).

Since $\Gamma(\delta - s)$ and $\eta(\delta - s)$ have no zeros in $\Re s < \delta - 1$, one obtains

$$(\mathcal{M}[f])(s) = \frac{(\mathcal{M}[\mathcal{L}_\delta[f]])(\delta - s)}{\Gamma(\delta - s)\eta(\delta - s)}, \quad \text{for } \Re s < \delta - 1. \tag{2.2}$$

The previous results are summarized in the next result

Theorem 1. *Set $\delta > 0$. Assume that f is a measurable function on $\mathbb{R}_+$ such that the integral in (1.3) converges and let $f \in L_{\Re s,1}(\mathbb{R}_+)$, then*

$$(\mathcal{M}[f])(s) = \frac{(\mathcal{M}[\mathcal{L}_\delta[f]])(\delta - s)}{\Gamma(\delta - s)\eta(\delta - s)}, \text{ for } \Re s < \delta - 1,$$

where $\mathcal{M}$ denotes the Mellin transform (1.2) and $\mathcal{L}_\delta$ denotes the Widder-Lambert type integral transform (1.3).

3. Plancherel's Theorem and the Salem Equivalence to the Riemann Hypothesis

As it has been exposed in [9] (p. 694), the Mellin transform is defined for $f \in L_{a,2}(\mathbb{R}_+)$ by the integral

$$(\mathcal{M}[f])(s) = \int_0^\infty t^{s-1} f(t)dt, \quad (\mathrm{Re}\, s = a) \tag{3.1}$$

being convergent in mean with respect to the norm in $L_2((a - i\infty, a + i\infty))$.

Also, for the case when $f \in L_{a,1}(\mathbb{R}_+) \cap L_{a,2}(\mathbb{R}_+)$, the Mellin transform (3.1) agrees almost everywhere with the usual Mellin transform (1.2) (see [10] for details).

Thus, according to the Plancherel theorem for the Mellin transform given by Titchmarsh [10] (Theorem 71, p. 94-95) and using Theorem 1 above one has

Theorem 2. *Set $\delta > 0$. Assume that f is a measurable function on $\mathbb{R}_+$ such that the integral in (1.3) converges and let $f \in L_{a,1}(\mathbb{R}_+) \cap L_{a,2}(\mathbb{R}_+)$, for some arbitrary a with $a < \delta - 1$, then*

$$\int_0^\infty |f(x)|^2 x^{2a-1} dx = \frac{1}{2\pi} \int_{-\infty}^\infty \left| \frac{(\mathcal{M}[\mathcal{L}_\delta[f]])(\delta - a - it)}{\Gamma(\delta - a - it)\eta(\delta - a - it)} \right|^2 dt, \tag{3.2}$$

where $\mathcal{M}$ denotes the Mellin transform (1.2) and $\mathcal{L}_\delta$ denotes the Widder-Lambert type integral transform (1.3).

Proof. In fact, since $f \in L_{a,2}(\mathbb{R}_+)$ one obtains from [10] (Theorem 71, pp. 94-95)

$$\int_0^\infty |f(x)|^2 x^{2a-1} dx = \frac{1}{2\pi} \int_{-\infty}^\infty |(\mathcal{M}[f])(a + it)|^2 dt,$$

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where $\mathcal{M}$ denotes the $L_{a,2}$ Mellin transform (3.1).
Now, since $f \in L_{a,1}(\mathbb{R}_+)$ then the Mellin transform (3.1) agrees almost everywhere on $\mathbb{R}_+$ with the usual Mellin transform (1.2). Thus, using Theorem 1 one obtains the relation (3.2). □

Corollary 1. *Assuming the hypothesis of Theorem 2 one has*

$$If \quad \mathcal{L}_\delta[f] = 0 \ almost \ everywhere \quad on \ \mathbb{R}_+ \quad then \quad f = 0 \quad almost \ everywhere \ on \ \mathbb{R}_+.$$

Proof. Since $\mathcal{L}_\delta[f] = 0$ almost everywhere on $\mathbb{R}_+$ then the right-hand side of (3.2) is zero. Thus

$$\int_0^\infty |f(x)|^2 x^{2a-1} dx = 0,$$

and so $f = 0$ almost everywhere on $\mathbb{R}_+$. □

Now, concerning the Salem equivalence of the Riemann hypothesis and having into account that for any bounded measurable function the integral in (1.3) converges, one obtains

Corollary 2. *Let f be a bounded measurable function such that $f(t) = O(t^{1/2})$ as $t \to 0^+$. Set $\frac{1}{2} < \delta < 1$, and assume that f is a solution of the homogeneous equation (1.1), then $f = 0$ almost everywhere on $\mathbb{R}_+$.*

Proof. Observe that for $\frac{1}{2} < \delta < 1$ one obtains $\frac{1}{2} = 1 - \frac{1}{2} > 1 - \delta$. Taking a such that $\frac{1}{2} > -a > 1 - \delta$, one has that the class of functions of this Corollary satisfies the hypothesis of Theorem above. Then the result holds. □

4. Final Observations

The primary contribution of this paper lies in the application of the Plancherel theorem for the Mellin transform, in conjunction with the relationship between the Mellin and the Widder-Lambert type transforms presented in Section 2. Specifically, we employ Theorem 71 from Titchmarsh's book [10] (Theorem 71, pp. 94-95) which focuses on Plancherel's theorem for the Mellin transform, rather than relying on any inversion formulas. Importantly, in the present paper with respect to references [5] and [6] we do not utilize any inversion formulas, such as the Post-Widder or L_2 inversion formulas. Instead we make use of a Plancherel theorem. This approach represents the novelty of our work.

In summary, the validity of the Riemann hypothesis could be supported by proving the Corollary 2 for bounded measurable functions with not other restrictions. However, this paper shows that bounded measurable functions that satisfy $f(t) = O(t^{1/2})$ as $t \to 0^+$ meet Salem's criterion. Therefore, any counterexample to Salem's equivalence (and thus to Riemann hypothesis) must not have the property $f(t) = O(t^{1/2})$ as $t \to 0^+$.

These findings open up potential avenues for further exploration of the Riemann hypothesis in the field of integral transforms.

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