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Keywords: Linear multistep methods; Ordinary differential equations; Initial value problems; Phase-lag

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A Parametric Six-Step Method For Second Order IVPs With Oscillating Solutions

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Abstract: In this paper, we have developed an explicit symmetric six-step method. This new method is phase-fitted and incorporates a free coefficient as a parameter. Using a wide range of values for this parameter, we computationally define the periodicity interval. Numerical outcomes guided us toward identifying optimal values for the parameter, establishing the new method's high efficiency. The efficiency of the new method is demonstrated through its application to oscillatory problems, comparing it with well-known phase-fitted methods.

Keywords: Linear multistep methods; Ordinary differential equations; Initial value problems; Phase-lag

1. Introduction

This paper focuses on the numerical solution of the initial value problem (IVP) characterized by the following form

$$y''(t) = f(t, y), \qquad y(t_0) = y_0, \qquad y'(t_0) = y'_0 \tag{1}$$

whose solutions display a notable oscillatory nature. This kind of problem is commonly encountered across a wide spectrum of scientific disciplines, such as quantum mechanical scattering problems, celestial mechanics, and elsewhere.

Several authors have derived numerical methods of a single step, such as Runge-Kutta or Runge-Kutta-Nyström, which are based on properties such as phase-lag or trigonometric / exponential fitting (see [1–6]), while others use these properties for the development of multistep methods (see [7–20]). All of these methods incorporate frequency-dependent coefficients and demonstrate high efficiency in numerical solution of second-order IVPs with oscillating solutions.

In this research work, a new explicit parametric symmetric linear six-step method is produced. The new method is of sixth algebraic order with zero phase-lag and is used for the numerical solution of oscillatory problems. The approach of this study is based on the procedure of Anastassi and Simos [12] for the derivation of a linear four-step method, and should be considered as a technique which gives the opportunity to choose the appropriate value for the free parameter in order to numerically solve an IVP, depending on whether it needs a fast low accurate solution or a solution of high accuracy.

In particular, in Section 2, the basic theory of linear multistep methods is provided. In Section 3, the development of the new parametric method is presented, as well as the local truncation error and periodicity analysis for the new method. Furthermore, in Section 4, the results of the application to three well-known problems are demonstrated. Finally, the conclusions of this research article are presented in Section 5.

2. Theory of Linear Multistep Methods

The linear k-step method for the numerical integration of the initial value problem (1) is of the form

$$\sum_{i=0}^{k} a_i y_{n+i} = h^2 \sum_{i=0}^{k} b_i f(t_{n+i}, y_{n+i}) \tag{2}$$

The general method (2) it is known that is associated with the linear difference operator which is given from the expression

$$L(t) = \sum_{i=0}^{k} a_i y(t+ih) - h^2 \sum_{i=0}^{k} b_i y''(t+ih) \tag{3}$$

where $y \in C^2$

Definition 1. *The multistep method (2) is said to be of algebraic order p if the associated linear operator L in equation (3), vanishes for any linear combination of linearly independent functions* $1, t, t^2, ..., t^{p+1}$

At this point, we consider the scalar test equation

$$\frac{d^2 y(t)}{dt^2} = -\omega^2 y(t), \quad \omega \in \mathbb{R}^+ \tag{4}$$

By applying a 2k-step symmetric method to equation (4), the difference equation below is produced

$$A_k(v)y_{n+k} + ... + A_1(v)y_{n+1} + A_0(v)y_n + A_1(v)y_{n-1} + ... + A_k(v)y_{n-k} \tag{5}$$

where $A_0, A1, ..., A_k$ are polynomials in v of the form $A_i(v) = a_i + b_i v^2,\ i = 0(1)k$ and $v = \omega h$.

The characteristic equation which is associated with (5) is

$$A_k(v)s^k + ... + A_1(v)s + A_0(v)y_n + A_1(v)s^{-1} + ... + A_k(v)s^{-k} = 0 \tag{6}$$

The following definitions are derived from Lambert and Watson [21].

Definition 2. *A symmetric 2k-step method with characteristic equation of the form (6) has interval of periodicity* $(0, s_0^2)$*, if* $\forall s < s_0$ *the characteristic roots should satisfy the following conditions:*
$\lambda_0 = e^{I\phi(s)},\ \lambda_1 = e^{-I\phi(s)}$ *and* $|\lambda_i| \leq 1,\ i = 2(1)m-1,,$
where $s = \omega h$ *and* $\phi(s)$ *is a real function of s.*

Definition 3. *For any method corresponding to the characteristic equation (6), the phase-lag is defined as the term* $t = v - \phi(v)$.
If the quantity $t = O(v^{q+1})$ *as* $v \to 0$*, the order of phase-lag is q, where the characteristic roots are of the form* $\lambda_0 = e^{I\phi(v)},\ \lambda_1 = e^{-I\phi(v)},\ v = \omega h$ *and satisfy the conditions of Definition (2)*

Theorem 1. *The symmetric 2k-step method with characteristic equation given by (6) has phase-lag q and phase-lag constant c given by*

$$-cv^{q+2} + O(v^{q+4}) = \frac{2A_k(v)\cos(kv) + ... + 2A_j(v)\cos(jv) + ... + A_0(v)}{2k^2 A_k(v) + ... + 2j^2 A_j(v) + ... + 2A_1(v)} \tag{7}$$

3. Development of the New Symmetric Method

This section outlines the derivation approach for an explicit symmetric linear six-step method featuring frequency-dependent coefficients.

With this objective in mind, we examine the following scheme

$$y_3 = -y_{-3} - a_2(y_2 + y_{-2}) - a_1(y_1 + y_{-1}) + h^2(b_2(f_2 + f_{-2}) + b_1(f_1 + f_{-1}) + b_0 f_0) \tag{8}$$

The linear method we are about to formulate is of sixth algebraic order and incorporates a variable coefficient as a parameter. This coefficient will be utilized to explore both accuracy and the interval of periodicity. If we apply the six-step method (8) on the scalar equation (4), we generate the polynomials $A_k(v) \mid k = 0(1)2$, with $v = \omega h$, where ω represents the primary frequency of the problem and h

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denotes the integration step size. Subsequently, using Theorem (1), we derive the expression for the method's phase-lag (PL).

$$PL = \frac{2\,\cos(3\,v) + 2\left(b_2 v^2 + a_2\right)\cos(2\,v) + 2\left(b_1 v^2 + a_1\right)\cos(v) + b_0 v^2}{18 + 8\,b_2 v^2 + 8\,a_2 + 2\,b_1 v^2 + 2\,a_1} \tag{9}$$

To attain both zero phase-lag and the highest achievable algebraic order p for method (8), the following conditions must be satisfied

$$PL = 0 \tag{10}$$

$$L(t^i) = 0,\ i = 0, 1, ..., p+1 \tag{11}$$

Upon solving the aforementioned system, we obtain the expressions for the method's coefficients.

$$\begin{aligned}
a_1 &= -1 - a_2 \\
K &= v^2\Big((\cos(v))^2 + 1 - 2\,\cos(v)\Big) \\
b_0 &= \Big(-144\,(\cos(v))^3 + 144\,\cos(v) + 36\,\cos(v)a_2 - 112\,\cos(v)v^2 \\
&\quad -72\,a_2(\cos(v))^2 + 36\,a_2 + 32\,v^2 - 3\,v^2 a_2 - 57\,\cos(v)v^2 a_2 \\
&\quad -64\,v^2(\cos(v))^2 + 6\,v^2 a_2(\cos(v))^2\Big)\Big/12K \\
b_1 &= \Big(80\,v^2(\cos(v))^2 + 16\,v^2 + 96\,(\cos(v))^3 - 96\,\cos(v) - 24\,\cos(v)a_2 \\
&\quad +48\,a_2(\cos(v))^2 - 24\,a_2 + 21\,v^2 a_2 + 15\,v^2 a_2(\cos(v))^2\Big)\Big/12K \\
b_2 &= -\Big(48\,(\cos(v))^3 - 48\,\cos(v) - 12\,\cos(v)a_2 + 80\,\cos(v)v^2 \\
&\quad +24\,a_2(\cos(v))^2 - 12\,a_2 - 32\,v^2 + 3\,v^2 a_2 + 15\,\cos(v)v^2 a_2\Big)\Big/24K
\end{aligned} \tag{12}$$

As $v \to 0$, instead of using the above expressions, we use the Taylor series expansions.

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$$
\begin{aligned}
b_0 &= \frac{52}{15}+\frac{37}{40}a_2+\left(-\frac{118}{315}+\frac{53}{3360}a_2\right)v^2+\left(\frac{197}{9450}+\frac{11}{33600}a_2\right)v^4\\
&+\left(-\frac{13}{69300}-\frac{19}{8870400}a_2\right)v^6+\left(\frac{5267}{972972000}-\frac{6427}{10378368000}a_2\right)v^8\\
&+\left(-\frac{589}{40864824000}-\frac{11509}{290594304000}a_2\right)v^{10}\\
&+\left(-\frac{449}{308756448000}-\frac{56369}{29640619008000}a_2\right)v^{12}\\
&+\left(-\frac{1723}{15506914752000}-\frac{3011933}{37845142349414400}a_2\right)v^{14}\\
b_1 &= \frac{4}{5}+\frac{29}{30}a_2+\left(\frac{236}{945}-\frac{53}{5040}a_2\right)v^2+\left(-\frac{11}{50400}a_2-\frac{197}{14175}\right)v^4\\
&+\left(\frac{13}{103950}+\frac{19}{13305600}a_2\right)v^6+\left(-\frac{5267}{1459458000}+\frac{6427}{15567552000}a_2\right)v^8\\
&+\left(\frac{589}{61297236000}+\frac{11509}{435891456000}a_2\right)v^{10}\\
&+\left(\frac{449}{463134672000}+\frac{56369}{44460928512000}a_2\right)v^{12}\\
&+\left(\frac{1723}{23260372128000}+\frac{3011933}{56767713524121600}a_2\right)v^{14}\\
b_2 &= \frac{22}{15}+\frac{17}{240}a_2+\left(\frac{53}{20160}a_2-\frac{59}{945}\right)v^2+\left(\frac{197}{56700}+\frac{11}{201600}a_2\right)v^4\\
&+\left(-\frac{13}{415800}-\frac{19}{53222400}a_2\right)v^6+\left(-\frac{6427}{62270208000}a_2+\frac{5267}{5837832000}\right)v^8\\
&+\left(-\frac{589}{245188944000}-\frac{11509}{1743565824000}a_2\right)v^{10}\\
&+\left(-\frac{449}{1852538688000}-\frac{56369}{177843714048000}a_2\right)v^{12}\\
&+\left(-\frac{1723}{93041488512000}-\frac{3011933}{227070854096486400}a_2\right)v^{14}
\end{aligned}
\tag{13}
$$

3.1. Local Truncation Error and Periodicity Analysis

Using the expressions (13) into the method (8), we derive the principal error for the test equation (4). This error arises from the Taylor series expansion of the difference between both sides of Equation (8) and its expression is as follows.

$$PLTE=\frac{3776-159a_2}{60480}\left(y^{(8)}(t)+y^{(6)}(t)\omega^2\right)h^8 \tag{14}$$

At this point, we computationally determine the interval of periodicity. Following the application of method (8) using the coefficients outlined in (12) to address the test problem (4), we calculate the characteristic equation for the new parametric method as defined in Equation (6). The characteristic equation obtained is expressed as follows.

$$\lambda^6+B_2\lambda^5+B_1\lambda^4+B_0\lambda^3+B_1\lambda^2+B_2\lambda+1=0 \tag{15}$$

where:

$$
\begin{aligned}
B_0 &= -\frac{1}{12}\frac{\left(64\,(\cos(s))^2 - 6\,a_2(\cos(s))^2 + 57\,\cos(s)a_2 - 32 + 3\,a_2 + 112\,\cos(s)\right)s^2}{(\cos(s)-1)^2}\\
&\quad -\frac{1}{12}\frac{144\,(\cos(s))^3 - 144\,\cos(s) - 36\,a_2 + 72\,a_2(\cos(s))^2 - 36\,\cos(s)a_2}{(\cos(s)-1)^2}\\
B_1 &= \frac{1}{12}\frac{\left(15\,a_2(\cos(s))^2 + 80\,(\cos(s))^2 + 21\,a_2 + 16\right)s^2}{(\cos(s)-1)^2}\\
&\quad +\frac{1}{12}\frac{96\,(\cos(s))^3 + 36\,a_2(\cos(s))^2 - 36\,a_2 - 12\,(\cos(s))^2 - 72\,\cos(s) - 12}{(\cos(s)-1)^2}\\
B_2 &= -\frac{1}{24}\frac{(15\,\cos(s)a_2 + 3\,a_2 + 80\,\cos(s) - 32)s^2}{(\cos(s)-1)^2}\\
&\quad -\frac{1}{24}\frac{48\,(\cos(s))^3 + 36\,\cos(s)a_2 - 36\,a_2 - 48\,\cos(s)}{(\cos(s)-1)^2}\\
s &= \omega h
\end{aligned}
$$

We perform computational analysis on the characteristic equation (15) over a broad spectrum of coefficient values, a_2. In this process, we ascertain the periodicity interval by assessing the conditions stipulated in Definition 2 for the set $\{\hat{s}|\hat{s}^2 \in (0, s_0^2)\}$, where $\hat{s} = n\Delta s$, $n \in \mathbb{N}_0$ and $\Delta s = 10^{-2}$. The results of this investigation are illustrated in Figure 1.

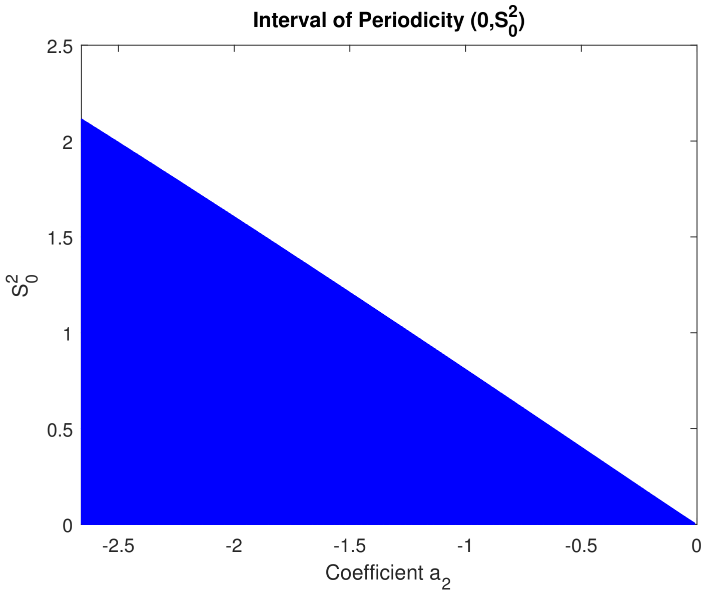

Figure 1. Method's periodicity interval with respect to the coefficient a_2

It is important to emphasize that the method exhibits a non-zero periodicity interval exclusively when a_2 falls within the range of $(-2.67, 0)$. Furthermore, the method displays convergence within this identical interval due to its inherent consistency. As depicted in Figure 1, the reduction in the value of a_2, corresponds to an expansion of the periodicity interval. In addition, a similar behavior is observed in the case of the local truncation error, evident from Equation (14), and notably from the expression $\frac{3776-159a_2}{60480}$. This expression reaches its maximum value as a_2 approaches -2.67^+ and its minimum value as a_2 tends towards 0^-. Therefore, the objective is to identify the optimal value of a_2 that simultaneously minimizes the error and ensures the stability of the method.

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4. Numerical Results

4.1. The Problems

The recently introduced method, described above, was applied to three widely recognized oscillatory problems.

- The Schrödinger equation - Resonance problem
- The "almost" periodic orbit problem by Stiefel and Bettis
- The Duffing equation

The description of these problems is given below.

4.1.1. Schrödinger Equation - Resonance Problem

The one-dimensional or radial Schrödinger equation has the form

$$y''(x) + \left(E - \frac{l(l+1)}{x^2} - V(x)\right)y(x) = 0. \tag{16}$$

where
$x \in \mathbb{R}^*$,
E is the *energy*,
$l(l+1)/x^2$ is the *centrifugal potential*,
$V(x)$ the *electric potential*

The function $W(x) = l(l+1)/x^2 + V(x)$ is the *effective potential*, where $\lim_{x\to\infty} V(x) = 0$ and so $\lim_{x\to\infty} W(x) = 0$. Furthermore, there is the boundary condition $y(0) = 0$ and a second boundary condition, which is determined by physical considerations for large values of x.

For our numerical illustration, we will consider the integration domain as $x \in [0, 15]$, applying the Woods-Saxon potential

$$V(x) = \frac{u_0}{1+q} + \frac{u_1 q}{(1+q)^2}, \qquad q = exp\left(\frac{x - x_0}{\alpha}\right), \quad where \tag{17}$$
$$u_0 = -50, \quad \alpha = 0.6, \quad x_0 = 7, \quad u_1 = -\frac{u_0}{\alpha}$$

For positive energies ($E = k^2$), the potential $V(x)$ dies away faster than the centrifugal potential ($l(l+1)/x^2$), so for large values of x, Schrödinger equation effectively reduces to

$$y''(x) + (k^2 - \frac{l(l+1)}{x^2})y(x) = 0 \tag{18}$$

Equation (18) has two linearly independent solutions. These are $kxj_l(kx)$ and $kxn_l(kx)$, where j_l and n_l represent the spherical Bessel and Neumann functions, respectively. As $x \to \infty$, the solution to equation (16) takes the following asymptotic form.

$$\begin{aligned} y(x) &\simeq Akxj_l(kx) - Bkxn_l(kx) \\ &\simeq D\left[sin\left(kx - \frac{l\pi}{2}\right) + tan(\delta_l)cos\left(kx - \frac{l\pi}{2}\right)\right], \end{aligned} \tag{19}$$

where δ_l is the scattering phase shift, which may be approximated by the following expression.

$$tan(\delta_l) \simeq \frac{y(x_i)S(x_{i+1}) - y(x_{i+1})S(x_i)}{y(x_{i+1})C(x_i) - y(x_i)C(x_{i+1})}, \tag{20}$$

where $S(x) = kxj_l(kx)$, $C(x) = kxn_l(kx)$ and $x_i < x_{i+1}$ both exist in the asymptotic region.

In the case of positive energies ($E > 0$) and for $l = 0$, we approximate (δ_l) based on (20) and then compare its value to the exact value $\pi/2$. For this eigenvalue problem the boundary conditions are $y(0) = 0$ and $y(x) = cos(\sqrt{E}x)$ for large x.

We use the eigenenergy $E = 989.701916$ and for the frequency we use the suggestion of Ixaru and Rizea [22]

$$\omega = \begin{cases} \sqrt{E+50}, & x \in [0, 6.5] \\ \sqrt{E}, & x \in [6.5, 15] \end{cases}$$

4.1.2. Orbit problem by Stiefel and Bettis

Stiefel and Bettis (1969) [23] studied the "almost" periodic orbit problem, which is given below.

$$\frac{d^2y(t)}{dt^2} = -y(t) + \epsilon exp(it), \qquad y(t) \in C \tag{21}$$
$$y(0) = 1, \quad y'(0) = (1 - \frac{1}{2}\epsilon)i.$$

where $\epsilon = 0.001$.

The analytical solution of the problem (21) is

$$y(t) = cos(t) + \frac{1}{2}\epsilon t sin(t) + i[sin(t) - \frac{1}{2}\epsilon t cos(t)]$$

Equivalently, the problem (21) can be described from the system

$$\begin{aligned} \frac{d^2y_1(t)}{dt^2} &= -y_1 + 0.001\cos(t), \;\; y_1(0) = 1, \;\; y_1'(0) = 0, \\ \frac{d^2y_2(t)}{dt^2} &= -y_2 + 0.001\sin(t), \;\; y_2(0) = 0, \;\; y_2'(0) = 0.9995 \end{aligned} \tag{22}$$

The analytical solution of the system (22) is

$$\begin{aligned} y_1(t) &= \cos(t) + 0.0005t\sin(t), \\ y_2(t) &= \sin(t) - 0.0005t\cos(t) \end{aligned}$$

The estimated frequency for this problem is $\omega = 1$.

4.1.3. Duffing equation

The Duffing equation can be described by

$$\frac{d^2y(t)}{dt^2} = -y(t) - (y(t))^3 + Bcos(\omega t) \tag{23}$$

where $B = 0.002$

The analytical solution of (23) is

$$y(t) = A_1 cos(\omega t) + A_3 cos(3\omega t) + A_5 cos(5\omega t) + A_7 cos(7\omega t) + A_9 cos(9\omega t)$$

where
$A_1 = 0.200179477536$,

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$A_3 = 0.000246946143$,
$A_5 = 0.000000304014$,
$A_7 = 0.000000000374$ and
$A_9 = 0.000000000000$
The estimated frequency for this problem is $\omega = 1.01$.

4.2. The Compared Methods

To evaluate the efficiency of the new method, we compare it with five well-known sixth algebraic order phase-fitted explicit methods with variable coefficients. The methods used in the comparison are

- Method A: The new phase-fitted parametric method which was derived in Section 3 (for a wide range of parameter a_2).
- Method B: The 4-step phase-fitted method, developed by Simos [9].
- Method C: The Numerov-type hybrid method, developed by Konguetsof [14].
- Method D: The Runge-Kutta type hybrid method, developed by Alolyan and Simos [11].
- Method E: The zero-dissipative hybrid method, developed by Ahmad et. al [10].
- Method F: The high order method of the Runge-Kutta-Nyström pair, developed by Anastassi and Kosti [1].

4.3. Effectiveness and Optimal Values for the New Parametric Method

In Figures 2 - 4, we visualize the evaluation of the novel parametric approach, which is represented as $-log_{10}(error)$, relative to the coefficient a_2. The error in this context is determined differently for distinct scenarios: in the case of known accurate solutions such as the Stiefel-Bettis problem and the Duffing equation, the error corresponds to the maximum absolute error value of the integration interval. In the case of Schrödinger equation, the error is defined as $|\delta_l - \pi/2|$.

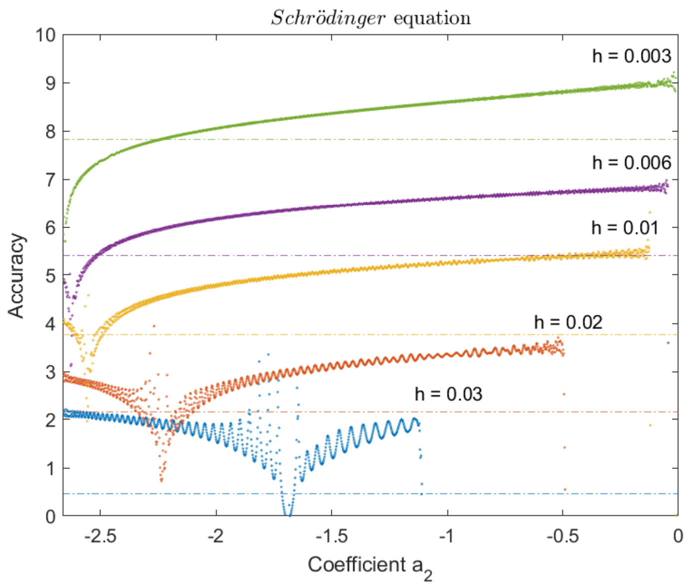

Figure 2. Efficiency of the new method for all values of a_2 - Schrödinger equation

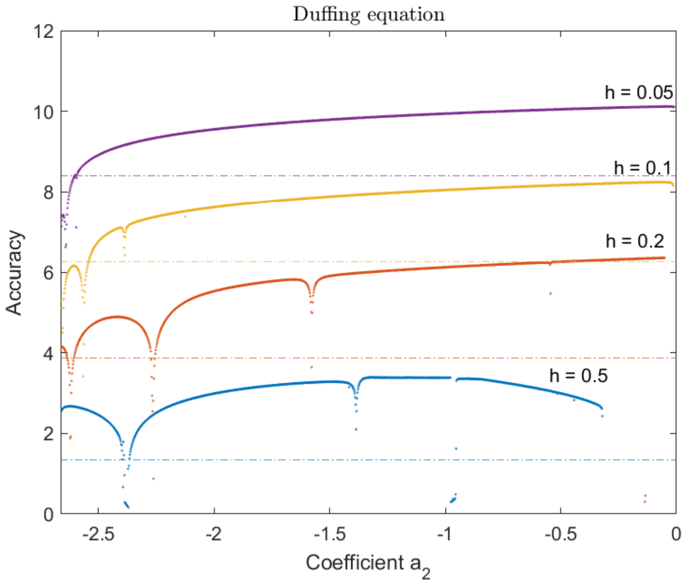

Figure 3. Efficiency of the new method for all values of a_2 - Duffing equation

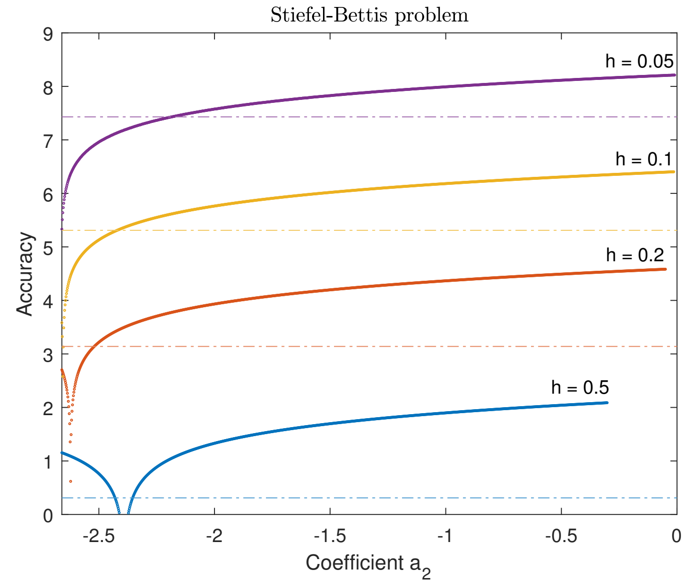

Figure 4. Efficiency of the new method for all values of a_2 - Stiefel-Bettis problem

Each of these figures illustrates the accuracy of the new method, represented by small dots (appearing as solid lines) across all values of the parameter a_2 and for various integration step sizes h. As expected, smaller integration steps result in higher accuracy. The variety of colors in the graphs corresponds to different integration step lengths, as indicated in the figures' legends. The dashed line, shown in a specific color, signifies the highest level of accuracy achieved by alternative methods in the comparative analysis (see Section 4.2) while requiring the same computational effort — measured as log_{10} of the number of function evaluations — as the new parametric method.

For instance, in the case of the Stiefel and Bettis problem (Figure 4), when the new method is employed with a step size of $h = 0.05$, its initial accuracy is relatively low (< 5.5) for values of $a_2 \to -2.66^+$. However, it progressively improves, surpassing an accuracy level of 8 as $a_2 \to 0^-$. Among the other methods subjected to the same number of function evaluations required by the new method (for $h = 0.05$), the most efficient demonstrates an accuracy of approximately 7.4, denoted by the same color dashed horizontal line (as accuracy for the other methods does not depend on the coefficient a_2). The intersection points reveal that the new parametric method consistently outperforms

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the other methods under consideration for all values of a_2 exceeding approximately -1.6. Points where the graph lacks small dots indicate that the parametric method exhibits instability. This typically occurs when employing large step-size values (h) and for high values of the coefficient (a_2).

Furthermore, in terms of effectiveness, there exists a substantial range of values for the coefficient a_2, where the new method consistently delivers the highest level of accuracy. These specific intervals commence at the point of intersection between the small dots and the dashed line. For the Schrödinger equation, this interval begins at approximately -1.6, for the Duffing equation at -2.2, and for the Stiefel-Bettis problem it starts at -2.1.

Regarding the upper limit of this interval, it varies depending on whether our priority is to obtain a highly accurate solution or a faster, albeit less accurate one. In pursuit of enhanced accuracy, our objective is to choose a value for a_2 that closely approaches the upper limit of this interval. When our aim is to achieve a quicker solution with reduced accuracy, for example, an accuracy level of 1 or 2, it becomes necessary to adjust the upper limit of the interval within which a_2 operates in order to maintain method's stability. Specifically, these adjusted upper limits are as follows: -1.2 for the Schrödinger equation and -0.3 for the rest of the problems.

The new method's accuracy exhibits occasional spikes across different combinations of h and a_2 values. These spikes are a result of resonances triggered by the presence of spurious roots within the characteristic equation of the method. It's essential to recognize that these resonances vary for different values of a_2, and this phenomenon is inherent in all linear multistep methods comprising more than two steps. Quinlan's extensive investigation of this behavior, with further details available in [24], sheds light on the matter. Fortunately, when considering the recommended optimal a_2 values outlined below, the new method consistently outperforms the compared methods.

Considering the examination presented in the preceding paragraphs, the recommended values for the coefficient a_2 are presented in Table 1.

Table 1. Optimal values of the parameter a_2.

Value of a_2	**Accuracy**
$a_2 = -\frac{1}{20}$	for high accuracy (> 8)
$a_2 = -1.5$	for fast solution with low accuracy (≈ 2)
$a_2 \in (-1.5, -\frac{1}{20})$	for medium accuracy

4.4. Comparison of the New Parametric Method with Other Phase-Fitted Methods Based on Optimal Values and Upper Limits for a_2

At this point, we demonstrate the accuracy of the new method against the phase-fitted methods described in Section 4.2 by choosing a value for the coefficient a_2 based on the upper limits discussed in Section 4.3. We choose to measure the efficiency of the method by computing the accuracy in the decimal digits, which is $-log_{10}$ (*maximum error through the integration intervals*) versus the computational effort measured by the log_{10} (*number of function evaluations required*).

Following the analysis preceded in Section 4.3, we choose for the parameter a_2 the value -1.2, which is the upper limit of the interval for the Schrödinger equation. In Figures 5 - 7, we display the results. According to the results, the proposed method demonstrates superior effectiveness in all three problems, achieving higher accuracy than the other methods in the comparison.

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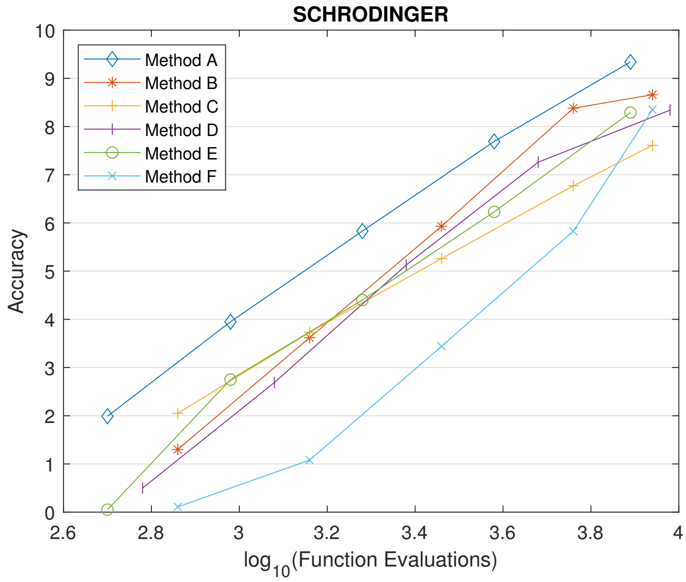

Figure 5. Comparison for the Schrödinger equation

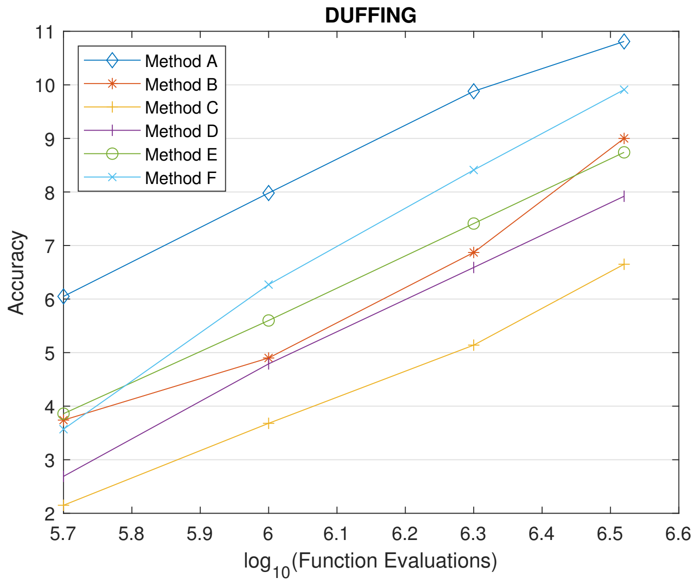

Figure 6. Comparison for the Duffing equation

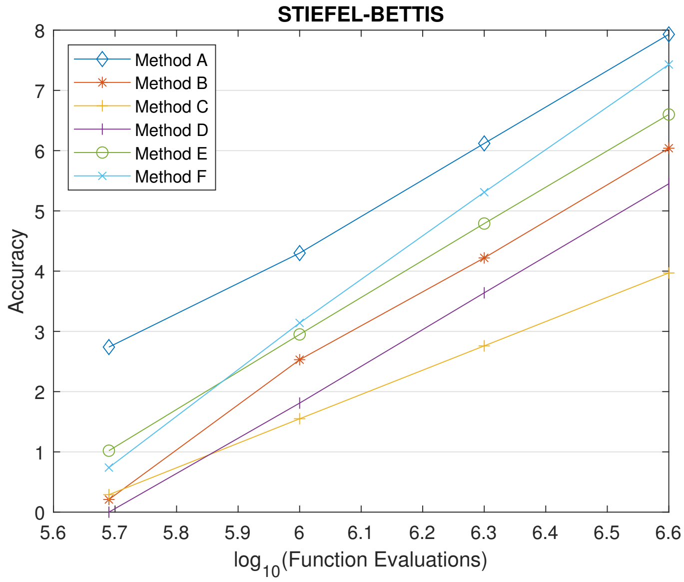

Figure 7. Comparison for the Stiefel-Bettis problem

5. Conclusions

This study introduces a six-step parametric phase-fitting method with frequency-dependent coefficients for the numerical solution of oscillatory problems. We evaluated the efficiency of our new method across a wide spectrum of parameter's value and conducted a comparative assessment against established methods in the existing literature. The numerical results guided us toward identifying optimal values for the parameter, establishing the high efficiency of the new method.

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