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Article

RHS and Quantum Mechanics: Some Extra Examples

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Abstract: The rigged Hilbert spaces (RHS) is the right mathematical context which includes many tools used in quantum physics, or even in some chaotic classical systems. It is particularly interesting that in RHS convive discrete and continuous basis, abstract basis along basis of special functions and representations of Lie algebras of symmetries by continuous operators. This is not possible in Hilbert spaces. In the present paper, we study a model showing all these features, based on the one dimensional Pöschl-Teller Hamiltonian. Also, RHS support representations of all kinds of ladder operators as continuous mappings. We give an interesting example based on one dimensional Hamiltonians with an infinite chain of SUSY partners, in which the factorization of Hamiltonians by continuous operators on RHS plays a crucial role.

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1. Introduction, Materials and Methods

The present article is the continuation of a series of papers intended to show the importance of Gelfand Triplets, also called Rigged Hilbert Spaces (RHS) in the description of a variety of situations in ordinary as well as relativistic Quantum Mechanics. Along the present article, we intend to give some examples of these applications. For the benefit of the reader, we review in this Section some introductory material with appropriate references.

Let us recall that a RHS is a triplet of spaces [1,2]:

$$\Phi \subset \mathcal{H} \subset \Phi^\times \,, \tag{1}$$

where: i.) the space $\mathcal{H}$ is an infinite dimensional complex separable Hilbert space. The separability is important as it implies that complete orthonormal sets, also called discrete orthonormal basis, are countably infinite; ii.) the space Φ is a dense[1] subspace of $\mathcal{H}$. It is endowed with a finer topology (has more open sets) than the Hilbert space topology; iii.) finally, the space $\Phi^\times$ is the space of all ant-linear continuous functionals[2] on Φ, endowed with a particular topology compatible with duality (we do not intend to explain this technicality, just give later an example).

Let ϕ be an arbitrary vector in the Hilbert space $\mathcal{H}$. Then, we may define a unique anti-linear continuous functional on Φ, $F_\phi \in \Phi^\times$, just by

$$F_\phi(\varphi) := \langle \varphi|\phi\rangle \,, \qquad \forall\, \varphi \in \Phi \,, \tag{2}$$

1 This means that in any neighbourhood of any vector $\phi \in \mathcal{H}$, there always exists a vector $\varphi \in \Phi$. The infinite dimensional character of the Hilbert space $\mathcal{H}$ assures the existence of dense subspaces in $\mathcal{H}$, different from $\mathcal{H}$.

2 These are mappings $F : \Phi \longmapsto \mathbb{C}$, $\mathbb{C}$ being the field of complex numbers, continuous with respect to the topologies on Φ and $\mathbb{C}$. Antilinearity means that if $F \in \Phi^\times$, $\psi, \varphi \in \Phi$ and $\alpha, \beta \in \mathbb{C}$ are complex numbers, then, $F(\alpha\psi + \beta\varphi) = \alpha^* F(\psi) + \beta^* F(\varphi)$, where the star denotes complex conjugation.

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where $\langle -|-\rangle$ denotes the scalar product on $\mathcal{H}$, where we assume linearity on the right and ant-linearity on the left. Continuity of F_ϕ on Φ, for each $\phi \in \mathcal{H}$ can be proven. This gives a one to one mapping (although never onto), $\phi \longmapsto F_\phi$, between $\mathcal{H}$ and $\Phi^\times$. Usually, we identify F_ϕ with ϕ, which justifies the inclusion $\mathcal{H} \subset \Phi^\times$.

RHS have been used in Physics to:

1.- Give a rigorous meaning of the Dirac formulation of Quantum Mechanics [3–9]. In particular, one gives meaning to the Dirac spectral representation for self-adjoint unbounded operators on Hilbert spaces. This result has a particular interest, which may justify that we give a brief account of it in here. Let A be a self adjoint operator on the Hilbert space $\mathcal{H}$, with domain $\mathcal{D}(A)$. Let $\Phi \subset \mathcal{H} \subset \Phi^\times$ a RHS such that: i.) $\Phi \subset \mathcal{D}(A)$; ii.) $A\Phi \subset \Phi$, which means that for any $\phi \in \Phi$, it results that $A\phi \in \Phi$. iii.) A is continuous on Φ (in general, not on $\mathcal{H}$). Then, it may be extended as a linear continuous operator on the anti-dual $\Phi^\times$ of Φ, by means of the duality formula:

$$\langle A\phi|F\rangle = \langle \phi|AF\rangle\,, \qquad \forall\, \phi \in \Phi\,, \qquad \forall\, F \in \Phi^\times\,, \tag{3}$$

where we are using the same notation to denote the original operator A and its extension to the anti-dual. Note that continuity is not necessary if we just want to extend the operator with linearity.

We say that $F_\lambda \in \Phi^\times$ is a generalized eigenvector of A with eigenvalue λ if $A\Phi \subset \Phi$, so that A can be extended to $\Phi^\times$, $AF_\lambda = \lambda\, F_\lambda$, where A denotes its extension to $\Phi^\times$. Equivalently, $\langle A\phi|F_\lambda\rangle = \lambda\langle\phi|F_\lambda\rangle, \forall\, \phi \in \Phi$.

Then, the Dirac spectral representation comes after the celebrated *Gelfand-Maurin Theorem*, which states the following:

Let A be an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(A)$. Then, thre exists a RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$ such that:

i.) $A\Phi \subset \Phi$ and A is continuous on Φ.

ii.) There exists a measure $d\mu(\lambda)$ defined on the spectrum, $\sigma(A)$ of A. Here, $\lambda \in \sigma(A)$. If $\sigma(A)$ is absolutely continuous and not degenerate, then, $d\mu(\lambda)$ is absolutely continuous with respect to the Lebesgue measure on $\sigma(A)$.

iii.) For almost all $\lambda \in \sigma(A)$, with respect to the measure $d\mu(\lambda)$, there is a $F_\lambda \in \Phi$, such that $AF_\lambda = \lambda\, F_\lambda$, so that F_λ is a generalized eigenvector of A with generalized eigenvalue on the spectrum of A, which includes the continuous spectrum of A, either absolutely or singular continuous. In order to fit with the Dirac notation, we may write $|\lambda\rangle \equiv F_\lambda$, so that $F_\lambda(\psi) = \langle\psi|\lambda\rangle^* =: \langle\lambda|\psi\rangle$ for all $\psi \in \Phi$.

iv.) Then, for any pair $\psi, \varphi \in \Phi$, we have the following spectral representation

$$\langle\psi|A^n\varphi\rangle = \int_{\sigma(A)} \lambda^n\, \langle\psi|\lambda\rangle\langle\lambda|\varphi\rangle\, d\mu(\lambda)\,, \qquad n = 0, 1, 2, \dots\,. \tag{4}$$

If we omit the arbitrary $\psi, \varphi \in \Phi$, we obtain the following formal decomposition:

$$A = \int_{\sigma(A)} \lambda\, |\lambda\rangle\langle\lambda|\, d\mu(\lambda)\,. \tag{5}$$

Finally, the identity $I : \Phi \longmapsto \Phi^\times$ may be expressed as

$$I = \int_{\sigma(A)} |\lambda\rangle\langle\lambda|\, d\mu(\lambda)\,. \tag{6}$$

2.- Gamow vectors represent the purely exponential decaying part of a quantum resonance. Experiments have shown that, during almost all times of observation, the decay of unstable quantum states is exponential. There are deviations for very short and very large times, which are difficult to detect, particularly the latter ones. However, the Hamiltonian that describe the process is self adjoint and self adjoint operators produces unitary time evolutions that do not decay. The only possibility to describe an exponential time decay is by means of enlarging the Hamiltonian to the dual on a rigged

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Hilbert space. This has been well studied [10–13]. Hamiltonians admit spectral decompositions in terms of Gamow states and continuum states, as typically happens with the Friedrichs model [14].

3.- There have been multiple attempts to describe irreversibility in quantum processes. RHS have played an essential role in some of these attempts [15–20].

4.- Classical chaotic systems are studied through their power spectrum that often shows singularities called the Pollicot Ruelle resonances. As in the case of Hamiltonians with resonances, Koopman and Frobenius-Perron operators admit respective spectral decompositions in terms of these resonances. A complete formal study of this situation has been done in the RHS language [21–24].

5.- If on a RHS one replaces the Hilbert space by a Liouville space, which is also a Hilbert space, one gets a structure suitable to study certain problems of Quantum Statistical Mechanics including generalized states and singular structures [25]. However, this option has not been widely used.

6.- Axiomatic Theory of Quantum Fields includes interesting objects that fit into a formulation by RHS, such as Wightman functionals, Borchers algebras, generalized states and others [26,27]. Nevertheless, this line of construction of Quantum fields has been nearly abandoned some time ago.

7.- Distributions are usually functionals on a space $\Phi^\times$ of a RHS where the space Φ is a space of test functions. Distributions have many applications in Physics either Quantum or Classical. Analogously, differential equations of interest in Physics have often distributions as solutions.

8.- White noise and other stochastic processes may admit a description in terms of RHS [28,29].

9.- Last but not least. One of the ideas that lead to the notion of RHS was the analysis of Lie algebras of operators describing symmetries in Quantum Physics. Up to the knowledge of the authors, specific examples thereof have not been very much developed until recently. See [30] and quotations thereof. Near this context, stability properties of some RHS under the fractional Fourier transform have applied to signal theory [31].

The mentioned recent examples of applications of RHS in symmetries have shown that RHS are the suitable structures that include at the same time, discrete and continuous basis, Lie algebras of symmetries represented by continuous operators and special functions. Each particular example contains a different choice of these ingredients. For pedagogical and methodological reasons, our description is based on two RHS, one abstract and the other a representation of the former by functions. Both RHS are unitarily equivalent in the following sense: Let $\Phi \subset \mathcal{H} \subset \Phi^\times$ be the abstract RHS (the nature of vectors are not specified) and let $\mathcal{G}$ a Hilbert space of functions, say of the type $L^2(\Delta)$. Since the second is also a separable infinite dimensional Hilbert space, there is a unitary mapping connecting both: $U : \mathcal{H} \longmapsto \mathcal{G}$. Then, one may define $\Psi := U\Phi$, the image of Φ by U into $\mathcal{G}$. The topology on Φ may be *transported* by U into Ψ. Then, we have a new RHS: $\Psi \subset \mathcal{G} \subset \Psi^\times$. The operator U may be extended to a one to one mapping from $\Phi^\times$ onto $\Psi^\times$, also conserving the topologies, by means of a duality formula:

$$\langle U^\dagger \varphi | G \rangle = \langle \varphi | UG \rangle \,, \qquad \forall\, \varphi \in \Phi \,, \qquad \forall\, G \in \Psi^\times \tag{7}$$

where $U^\dagger$ is the adjoint of U. This extension of U has all good properties. The total construction can be seen in the following diagram:

$$\begin{array}{ccccc} \Phi & \subset & \mathcal{H} & \subset & \Phi^\times \\ U\downarrow & & U\downarrow & & U\downarrow \\ \Psi & \subset & \mathcal{G} & \subset & \Psi^\times \end{array} \,. \tag{8}$$

Obviously, $U^\dagger$ makes the inverse job.

Although in general, it studies systems without a continuous spectrum, the SUSY method to construct Hamiltonians with a discrete spectrum similar to the spectrum of a given Hamiltonian

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implies the use of ladder operators, which are in general unbounded. With the help of a suitable construction of RHS, one can make continuous these ladder operators.

2. Special Functions, Lie Algebras and Bases

Gelfand triplets are the suitable mathematical framework so as to include some important notions used in standard non-relativistic quantum mechanics for a particular system, such as: Special functions, discrete and continuous basis, generators of symmetry groups given as continuous essentially self adjoint operators and last but not least, the Hamiltonian of this system, also as a continuous operator. This is not possible on the standard formulation of quantum mechanics on Hilbert space. Thus, one may complete the basic formulation of quantum Theory under the basis proposed by Dirac [32].

The first and most celebrated example of this construction is provided by the one dimensional harmonic oscillator, where the main ingredients are:

- Let $\mathcal{S}$ be the one dimensional Schwartz space of all functions $f(x) : \mathbb{R} \longmapsto \mathbb{C}$, which are indefinitely differentiable at all point of the real line $\mathbb{R}$ and such that all these functions and their derivatives go to zero at the infinite faster than the inverse of a polynomial of arbitrary degree. With its usual topology [33], $\mathcal{S}$ is a locally convex Frèchet space (complete and metrizable), so that

$$\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times \tag{9}$$

  is a Gelfand triplet or RHS.
- The normalized Hermite functions are all in $\mathcal{S}$ and form an orthonormal basis (complete orthonormal set) of $L^2(\mathbb{R})$. Thus, there is a *discrete basis* of *Special Functions* that span both $\mathcal{S}$ and $L^2(\mathbb{R})$.
- The Lie algebra associated to the Harmonic oscillator is the Heisenberg-Weyl Lie algebra. In the one dimensional case under consideration, the generators of this algebra are the identity operator I, the multiplication operator $Qf(x) = xf(x)$ and the derivation operator $Pf(x) = -if'(x)$, where the prime denotes derivation with respect to the variable x. Both Q and P are self-adjoint operators on suitable domains in $L^2(\mathbb{R})$ and both are essentially self adjoint on $\mathcal{S}$. Furthermore, both Q and P are continuous on $\mathcal{S}$ with the topology of the latter. Therefore, all the elements of the covering algebra spanned by I, Q and P are continuous on $\mathcal{S}$. This includes the Hamiltonian of the harmonic oscillator. In addition, all elements of the enveloping algebra may be extended to continuous operators on $\mathcal{S}^\times$, when we endow the latter with any topology compatible with the dual pair $\{\mathcal{S}, \mathcal{S}^\times\}$. It is important to remark that creation and annihilation operators are also continuous on $\mathcal{S}$ and $\mathcal{S}^\times$.
- The operator Q and P along to the RHS satisfy the Gelfand-Maurin Theorem [1], so that each one have generalized eigenvectors that satisfy equations (4-6). In particular for Q, we have that for $\psi(x), \varphi(x) \in \mathcal{S}$ and $n = 0, 1, 2, \ldots,$

$$\langle\psi|Q^n\varphi\rangle = \int_{-\infty}^{\infty} x^n \langle\psi|x\rangle\langle x|\varphi\rangle\, dx\,, \tag{10}$$

  with $Q|x\rangle = x\,|x\rangle \in \mathcal{S}^\times$, $x \in \mathbb{R}$. If we take $n = 0$ and omit the arbitrary $\psi \in \mathcal{S}$, we obtain the following formal expression for each $\varphi \in \mathcal{S}$:

$$\varphi = \int_{-\infty}^{\infty} |x\rangle\langle x|\varphi\rangle\, dx\,. \tag{11}$$

  This is a span of $\varphi \in \mathcal{S}$ in terms of the eigenfunctionals $\{|x\rangle\}_{x\in\mathbb{R}}$. Due to this span, we call *continuous basis* the set of functionals $\{|x\rangle\}_{x\in\mathbb{R}}$.

  In addition, there are some relations between discrete and continuous basis [31].

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There exists some other examples, see [30]. In the present contribution, we are illustrating some of the properties that have emerged for the harmonic oscillator and listed above for another solvable model such as the one dimensional Pöschl-Teller potential [34]. Its Hamiltonian has the form (where we have omitted irrelevant constants and $\ell > 0$ is fixed)

$$H_\ell := -\frac{d^2}{dx^2} + \frac{\ell(\ell-1)}{\cos^2 x}\,, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}\,, \qquad \ell > 0\,. \tag{12}$$

Normalized solutions of the Schrödinger equation, $H_\ell \psi_n = E_\ell^n \psi_\ell^n$ are of the form

$$\psi_\ell^n(x) = \sqrt{\frac{(\ell+n)\,\Gamma(2\ell+n)}{n!}}\,\sqrt{\cos x}\; P^{1/2-\ell}_{\ell+n-1/2}(sinx)\,, \qquad E_\ell^n = (\ell+n)^2\,, \tag{13}$$

where $P^\alpha_\beta(z)$ are the Legendre functions[3].

Functions $\psi_\ell^n(x)$, $n = 0,1,2,\dots$ with fixed $\ell > 0$, form an orthonormal basis (complete orthonormal set) of $L^2[-\pi/2,\pi/2]$. Each function $f(x) \in L^2[-\pi/2,\pi/2]$ admits a span of the form $f(x) = \sum_{n=0}^\infty a_n\,\psi_\ell^n(x)$, where the series converges on the L^2-norm and this norm is given for the function $f(x)$ as $||f||^2 = \sum_{n=0}^\infty |a_n|^2$.

Then, let us construct the space Φ_ℓ of all functions $f(x) \in L^2[-\pi/2,\pi/2]$, satisfying the following condition:

$$[p_\ell^s(f)]^2 = \sum_{n=0}^\infty |a_n|^2 (n+1)^{2s} < \infty\,. \tag{14}$$

Note that $p_\ell^s(f)$, $\ell > 0$ and $s = 0,1,2,\dots$, is a countably infinite set of norms, hence seminorms, on Φ_ℓ, with $p_\ell^0(f) = ||f||$. By construction, all Φ_ℓ, $\ell > 0$, are algebraically and topologically isomorphic to the one dimensional Schwartz space $\mathcal{S}$. For each $\ell > 0$, we have a Gelfand triplet:

$$\Phi_\ell \subset L^2[-\pi/2,\pi/2] \subset \Phi_\ell^\times\,. \tag{15}$$

Exactly as in the case of the Harmonic oscillator, the Pöschl-Teller Hamiltonian (12) defines ladder operators, which relate the functions in the orthonormal basis. They are constructed as follows: Let

$$B_n^+ := \left(\frac{E_\ell^n}{E_\ell^{n-1}}\right)^{1/4} \left(-\cos x\,\partial_x + \sqrt{E_\ell^{n-1}}\,\sin x\right); B_n^- := \left(\frac{E_\ell^{n-1}}{E_\ell^n}\right)^{1/4} \left(\cos x\,\partial_x + \sqrt{E_\ell^n}\,\sin x\right). \tag{16}$$

These transformations have a subindex showing on which eigenfunction they act. Thus, we have

$$B_n^-\,\psi_\ell^n = \sqrt{n(2\ell+n-1)}\,\psi_\ell^{n-1}\,, \qquad B_n^+\,\psi_\ell^{n-1} = \sqrt{n(2\ell+n-1)}\,\psi_\ell^n\,. \tag{17}$$

Then, we define the action on the eigenfunctions of the basis of the creation B^+ and the annihilation operator B^- as (we omit the subindex ℓ on the operators)

$$B^-\psi_\ell^n = B_n^-\psi_\ell^n\,, \qquad B^+\psi_\ell^n = B_n^+\psi_\ell^n\,, \quad n = 0,1,2,\dots\,. \tag{18}$$

[3] These functions have the form:

$$P^\alpha_\beta(z) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{1+z}{1-z}\right]^{\alpha/2} {}_2F_1\left(-\beta,\beta+1,1-\alpha,\frac{1-z}{2}\right),$$

where ${}_2F_1$ is the hypergeometric function.

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Note that $B^-\psi_\ell^0 = \mathbf{0}$. Operators $B^\pm$ are unbounded on $L^2[-\pi/2,\pi/2]$ and, therefore, not continuous and not defined on the whole $L^2[-\pi/2,\pi/2]$. However, $B^\pm$ may be extended with continuity to Φ_ℓ. Let us define the action of $B^\pm$ on $f(x) \in \Phi_l$, $f(x) = \sum_{n=0}^\infty a_n\,\psi_\ell^n$, as:

$$B^+ f(x) := \sum_{n=0}^\infty a_n \sqrt{(n+1)(2\ell+n)}\,\psi_\ell^{n+1}\,, \quad B^- f(x) := \sum_{n=0}^\infty a_n \sqrt{n(2\ell+n-1)}\,\psi_\ell^{n-1}\,. \tag{19}$$

We need to show that the action of $B^\pm$ as in (19) is well defined and is in Φ_ℓ. First of all note that $2\ell+n = n+1+(2\ell-1) \le (n+1)+|2\ell-1|$. Thus,

$$\begin{aligned}[p_\ell^s(B^+f)]^2 &= \sum_{n=0}^\infty |a_n|^2(n+1)(2\ell+n)(n+1)^{2s} \\ &\le \sum_{n=0}^\infty |a_n|^2(n+1)^2(n+1)^{2s} + |2\ell-1|\sum_{n=0}^\infty |a_n|^2(n+1)(n+1)^{2s} \\ &\le (1+|2\ell-1|)\sum_{n=0}^\infty |a_n|^2(n+1)^{2(s+1)} = C_\ell^2[p_\ell^{s+1}(f)]^2\,, \qquad \forall f(x)\in\Phi_\ell\,,\end{aligned} \tag{20}$$

with $C_\ell = \sqrt{1+|2\ell-1|}$. Thus, $p_\ell^s(B^+f) \le C_\ell\, p_\ell^{s+1}(f)$ for all $f(x)\in\Phi_\ell$, which shows that $B^+f\in\Phi_\ell$. In addition, this inequality also shows the continuity of B^+ on Φ_ℓ. This comes from the following result [33]:

Let Φ a locally convex space for which the topology is given by the family of seminorms $\{p_i\}_{i\in I}$, where I is an index set. Let $A:\Phi \longmapsto \Phi$ a linear mapping. A is continuous on Φ if and only if for each seminorm p_i, there exists a positive constant C_i and $n(i)$ seminorms (the seminorms and the number $n(i)$ depends on the seminorm p_i) such that

$$p_i(Af) \le C_i\{p_{1(i)}(f)+p_{2(i)}(f)+\dots+p_{n(i)}(f)\}\,, \qquad \forall f\in\Phi\,. \tag{21}$$

The same result is true if the image space is another locally convex space, Ψ, different from Φ. Then, the seminorms p_i on the left hand side of the inequality (21) should be replaced by the seminorms defining the topology on Ψ. Same if the image space is the field of complex numbers $\mathbb{C}$ or any normed space. In this case, we have only one seminorm, which is the norm.

Thus, after (21), B^+ is continuous on Φ_ℓ. A similar proof goes for the consistency and continuity of B^- on Φ_ℓ.

Analogously, we may extend the Hamiltonian H_ℓ as in (12) to a continuous operator on Φ_ℓ using the following definition valid for any $f(x)\in\Phi_\ell$ (recall that $H_\ell\psi_\ell^n(x) = (\ell+n)^2\,\psi_\ell^n(x)$):

$$H_\ell f(x) := \sum_{n=0}^\infty a_n(\ell+n)^2\,\psi_\ell^n(x)\,. \tag{22}$$

The proof goes as in the previous cases. Furthermore, since H_ℓ is a symmetric operator and $(H_\ell \pm iI)\psi_\ell^n = [(\ell+n)^2+i]\psi_\ell^n$, so that the image of Φ_ℓ by $(H_\ell\pm iI)$ is dense in $L^2(-\pi/2,\pi/2)$, it results that H is essentially self adjoint on Φ_ℓ. Then, H_ℓ is self adjoint on its maximal domain and is positive. Therefore, it admits a unique positive self adjoint extension, $B := \sqrt{H_\ell}$. The operators $B^\pm, B$ satisfy the commutation relations of the generators of the Lie algebra $so(2,1)$, which are

$$[B,B^\pm] = \pm B^\pm\,, \qquad [B^-,B^+] = 2B\,. \tag{23}$$

This the elements of the algebra $so(2,1)$ as well as those of its enveloping algebra are continuously defined on Φ_ℓ and continuously extendable to $\Phi_\ell^\times$.

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Some Further Properties: Continuous Basis

Here, we are to investigate the existence of continuous basis for the model under discussion. For technical reasons, we are restricting ourselves to half integer values of ℓ and for simplicity, let us assume that $\ell = 1/2$. Then, the Legendre functions that appear in the right hand side of (13) are just the ordinary Legendre polynomials that admit the following upper bound:

$$|P_n(x)| \leq \sqrt{\frac{2}{\pi}}\,. \tag{24}$$

We know that the convergence of the series in the span $f(x) = \sum_{n=0}^{\infty} a_n\, \psi^n_{1/2}(x)$, $f(x) \in L^2[-\pi/2, \pi/2]$, makes sense in the norm topology. This convergence does not implies almost elsewhere pointwise convergence. However if $f(x) \in \Phi_{1/2}$ this convergence goes in the uniform and the absolute sense, and hence pontwise. The proof goes as follows: Take the series:

$$|f(x)| = \left|\sum_{n=0}^{\infty} a_n \psi^n_{1/2}(x)\right| \leq \sum_{n+0}^{\infty} |a_n|\, |\psi^n_{1/2}(x)| \leq \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} |a_n|\, \sqrt{n+1/2}$$

$$\leq \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} |a_n|\, \sqrt{n+1}\, \frac{(n+1)^2}{(n+1)^2} \leq \sqrt{\frac{2}{\pi}} \sqrt{\sum_{n=0}^{\infty} |a_n|^2 (n+1)^4}\, \sqrt{\sum_{n=0}^{\infty} \frac{1}{(n+1)^3}} = C\, p^2_{1/2}(f)\,, \tag{25}$$

where the meaning of C is obvious and the seminorm $p^2_{1/2}(f)$ has been defined in (14). The last inequality is the Cauchy-Schwartz inequality. The uniform and absolute convergence of the series $\sum_{n=0}^{\infty} a_n \psi^n_{1/2}(x)$ is then a consequence of the Weierstrass M-Theorem.

Next and as already commented in the Introduction, let $\mathcal{H}$ be an abstract infinite dimensional separable Hilbert space and $U : L^2[-\pi/2, \pi/2] \longmapsto \mathcal{H}$ unitary. As outlined in the Introduction, there exists an abstract Gelfand triple, $\Psi \subset \mathcal{H} \subset \Phi^{\times}$, such that

$$\begin{array}{ccccc} \Phi_{1/2} & \subset & L^2[-\pi/2, \pi/2] & \subset & \Phi^{\times}_{1/2} \\ U\downarrow & & U\downarrow & & U\downarrow \\ \Psi & \subset & \mathcal{H} & \subset & \Psi^{\times} \end{array}\,. \tag{26}$$

Take $f \in \Psi$ and define a functional $|x\rangle$ such that $\langle f|x\rangle := f^*(x) = U^{-1}f$. This functional is obviously antilinear. Its continuity comes after (25), since

$$|\langle f|x\rangle| = |f^*(x)| \leq C\, p^2_{1/2}(f)\,, \qquad \forall\, f \in \Psi\,, \tag{27}$$

and the comment on the paragraph after (21). Note that U and its inverse U^{-1} are bijective and bicontinuous among the spaces as marked in (26). In the sequel, we are taken the complex conjugate $\langle x|f\rangle := \langle f|x\rangle^* = f(x)$. Now, take two arbitrary vectors $f, g \in \Psi$ and consider $f(x) = U^{-1}f$ and $g(x) = U^{-1}g$, which are both in $\Phi_{1/2}$. Since a unitary operator preserves the scalar product, we have that

$$\langle g|f\rangle = \int_{-\pi/2}^{\pi/2} g^*(x)\, f(x)\, dx = \int_{-\pi/2}^{\pi/2} \langle g|x\rangle\langle x|f\rangle\, dx\,. \tag{28}$$

Then omitting the arbitrary $g \in \Psi$, we have for all $f \in \Psi$ the following formal expression:

$$f = \int_{-\pi/2}^{\pi/2} |x\rangle\langle x|f\rangle\, dx = \int_{-\pi/2}^{\pi/2} f(x)\, |x\rangle\, dx\,, \tag{29}$$

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so that every $f \in \Psi$ can be formally written in terms of the continuous functionals $\{|x\rangle\}_{x\in[-\pi/2,\pi/2]}$. This construction preserves linearity on Ψ. Hence, the set of functionals is often called a *continuous basis* on Ψ.

Next, define $|n\rangle := U[\psi^n_{1/2}(x)]$, $n = 0, 1, 2, \ldots$. Obviously, $\{|n\rangle\}_{n\in\{0\}\cup\mathbb{N}}$ is an orthonormal basis in $\mathcal{H}$, which is also in Ψ. As mentioned before, an orthonormal basis is often denoted as *discrete basis*. Let us see that there exists a formal relation between discrete and continuous basis. We know that $\sum_{n=0}^{\infty} |n\rangle\langle n| = I$, where I is the identity on $\mathcal{H}$ and this series converges on the strong operator sense.

Then, if $f \in \Psi$, we have formally that $\langle f|x\rangle = \sum_{n=0}^{\infty}\langle f|n\rangle\langle n|x\rangle$, so that, omitting the arbitrary $f \in \Psi$, we have the following formal relation (note that $\langle n|x\rangle = \psi^n_{1/2}(x)$ after the above definition and taking into account that the functions $\psi^n_{1/2}(x)$ are real):

$$|x\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|x\rangle = \sum_{n=0}^{\infty} \psi^n_{1/2}(x)\, |n\rangle\,, \tag{30}$$

which gives the *continuous basis* in terms of the *discrete basis*. Then, from (29), we have

$$|n\rangle = \int_{-\pi/2}^{\pi/2} \psi^n_{1/2}(x)\, |x\rangle\, dx\,, \qquad n = 0, 1, 2, \ldots\,, \tag{31}$$

which is the inversion formula of (30) .

In the case of the harmonic oscillator, the generalized basis $\{|x\rangle\}_{x\in[-\pi/2,\pi/2]}$ is given by a complete set of eigenfunctionals of the multiplication operator $Qf(x) = xf(x)$. In the present case, we cannot guarantee that $xf(x) \in \Phi_{1/2}$ if $f(x) \in \Phi_{1/2}$. Nevertheless, there is a way out. Taking into account that

$$\sin x\, P_n(\sin x) = \frac{n+1}{2n+1}\, P_{n+1}(\sin x) + \frac{n}{2n+1}\, P_{n-1}(\sin x)\,, \tag{32}$$

we have that the same relation is fulfilled by the eigenfunctions $\psi^n_{1/2}(x)$. Then, define the multiplication operator $\sin Q$ as $(\sin Q)f(x) = \sin x\, f(x)$. The operator $\sin Q$ is obviously bounded on $L^2[-\pi/2, \pi/2]$. It is also well defined and continuous on $\Phi_{1/2}$. Its action on any function $f(x) \in \Phi_{1/2}$ is given by

$$(\sin Q)f(x) = \sum_{n=0}^{\infty} a_n\, \sin x\, \psi^n_{1/2}(x)$$
$$= \sum_{n=0}^{\infty} a_n\, \frac{n+1}{2n+1}\, \sin x\, \psi^{n+1}_{1/2}(x) + \sum_{n=0}^{\infty} a_n\, \frac{n}{2n+1}\, \sin x\, \psi^{n-1}_{1/2}(x)\,. \tag{33}$$

Then, the stability of $\Phi_{1/2}$ by $\sin Q$ as well as the continuity of the latter is a simple exercise using (33).

Now, define $T := U^{-1}[\sin Q]U$ on Ψ and take $f, g \in \Psi$. We have

$$\langle g|Tf\rangle = \int_{-\pi/2}^{\pi/2} g^*(x)(\sin x)\, f(x)\, dx = \int_{-\pi/2}^{\pi/2} \sin x\, \langle g|x\rangle\langle x|f\rangle\, dx\,, \tag{34}$$

so that omitting the arbitrary $g, f \in \Phi_{1/2}$, we have

$$T = \int_{-\pi/2}^{\pi/2} \sin x\, |x\rangle\langle x|\, dx\,, \tag{35}$$

which may be looked as a sort of spectral decomposition of $\sin Q$.

As a final property on the trigonometric Pöschl-Teller, we would like to remark that the Pöschl-Teller coherent states given by ($z \in \mathbb{C}$ fixed)

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$$\psi_z(x,t) = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\, e^{-i(\ell+n)^2 t}\, \psi_\ell^n(x) \in \Phi_\ell\,, \tag{36}$$

which is trivial.

3. Gelfand Triplets Associated to SUSY

In the standard non-relativistic quantum mechanics, super-symmetry (SUSY) is a procedure that serves to find from a given Hamiltonian with discrete spectrum to another one with a similar or equal spectrum. In some cases, the new Hamiltonian gives rise to a second one, then the second one to a third one and so on, making an infinite sequence of SUSY transformations. If we depart from a Hamiltonian with an infinite number of bound states, we may produce an infinite sequence of Hamiltonians with the same property of having an infinite number of bound states. This is not always the case, although we have examples thereof. Let us explain first how SUSY works under the mentioned circumstances and, then, we construct a Gelfand triplet suitable for the whole scheme.

The point of departure is the factorization method [35,36] that we briefly sketch here. Let H be a Hamiltonian with an infinite number of bound states, which, in addition, it may be factorized by an operator B and its formal adjoint B^+ as

$$H_0 = -\frac{d^2}{dx^2} + V_0(x) = BB^+ - \lambda = B^+B + \lambda'\,. \tag{37}$$

A typical example is the Hamiltonian of the harmonic oscillator, where B and B^+ are the annihilation and creation operators, respectively, and $\lambda = \lambda' = 1/2$ (writing the harmonic oscillator Hamiltonian with a factor $1/2$ in front of the derivative). In general, this decomposition is not unique and there exists A_0 and its formal adjoint A_0^+ such that

$$H_0 = A_0 A_0^+ - \lambda\,, \quad A_0 := \frac{d}{dx} + \beta(x)\,, \quad A_0^+ = -\frac{d}{dx} + \beta(x)\,, \tag{38}$$

where $\beta(x)$ fulfils a solvable Riccati equation. However, the Hamiltonian

$$H_1 := A_0^+ A_0 - \lambda = -\frac{d^2}{dx^2} + V_1(x)\,, \tag{39}$$

is different from H_0. Many examples appear in the literature. We cite a few of them only [35,36,39–41,47,48]. The Hamiltonians H_0 and H_1 satisfy the following intertwining relation:

$$H_1 A_0^+ = A_0^+ H_0\,. \tag{40}$$

In addition, if $H_0\psi_0^n = E_n\psi_0^n$, the sequence of vectors

$$\psi_n^1 := \frac{A_0^+\,\psi_0^{n-1}}{\sqrt{E_{n-1} + 2\lambda_n}}\,, \qquad n = 1,2,3,\ldots \tag{41}$$

are orthonormal and satisfy the relations $H_1\,\psi_n^1 = E_n\,\psi_n^1$, $n = 1,2,\ldots$. Under some conditions that we are supposing to be fulfilled here, the process may go to the infinity as we may depict in the following diagram:

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$$
\begin{array}{ccccccccc}
\psi_0^0 & \xrightarrow{A_0^+} & \psi_1^1 & \xrightarrow{A_1^+} & \psi_2^2 & \xrightarrow{A_2^+} & \psi_3^3 & \cdots \\
B_{0,0}^+ \downarrow & & B_{1,1}^+ \downarrow & & B_{2,2}^+ \downarrow & & B_{3,3}^+ \downarrow & \cdots \\
\psi_0^1 & \xrightarrow{A_0^+} & \psi_1^2 & \xrightarrow{A_1^+} & \psi_2^3 & \xrightarrow{A_2^+} & \psi_3^4 & \cdots \\
B_{0,1}^+ \downarrow & & B_{1,2}^+ \downarrow & & B_{2,3}^+ \downarrow & & A_{3.4}^+ \downarrow & \cdots \\
\psi_0^2 & \xrightarrow{A_0^+} & \psi_1^3 & \xrightarrow{A_1^+} & \psi_2^4 & \xrightarrow{A_2^+} & \psi_3^5 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & &
\end{array} . \tag{42}
$$

The above diagram requires an explanation. Let us start with the vertical lines. The sequence of vectors $\{\psi_0^n\}$, $n = 0, 1, 2, \ldots$, are the normalized eigenvectors of H_0. They are an orthonormal basis that span a Hilbert space, $\mathcal{H}_0$. $B_{0,n}^+$ transform ψ_0^n into ψ_0^{n+1}, etc. The sequence of vectors $\{\psi_1^n\}$, $n = 1, 2, \ldots$ are the normalized eigenvectors of H_1. They are an orthonormal basis that span a Hilbert space, $\mathcal{H}_1$. $B_{1,n}^+$ transform ψ_1^n into ψ_1^{n+1}, etc, and so on. Thus, we have a sequence of separable infinite dimensional Hilbert spaces $\{\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots\}$ with their respective discrete basis (complete orthonormal sets).

Horizontal lines. Although the generalization of (41) is $\psi_\ell^n = (A_{\ell-1}^+ \psi_{n-1}^{\ell-1})/\sqrt{E_{n-1} + 2\lambda}$, we have omitted the square roots in the diagram for simplicity. Otherwise this diagram would be excessively burdened with notation. We assume that all vectors (indeed eigenfunctions) ψ_ℓ^n are normalized.

Next, let us consider all Hilbert spaces $\mathcal{H}_\ell$, $\ell = 0, 1, 2, \ldots$ as independent[4]. This allows for the construction of the orthogonal direct sum $\oplus_{\ell=0}^\infty \mathcal{H}_\ell$. This sum is well defined as a Hilbert space [33] with orthonormal basis $\{\psi_\ell^n\}$ with $n, \ell = 0, 1, 2, \ldots$.

The sequence of Hamiltonians H_ℓ are defined as $H_\ell := A_{\ell-1}^+ A_{\ell-1} - \lambda$. These Hamiltonians are iso-spectral in the sense that $H_\ell \psi_\ell^n = E_n \psi_\ell^n$, $n = \ell, \ell+1, \ldots$.

For any $f_\ell \in \mathcal{H}_\ell$, we have the span $f_\ell = \sum_{n=\ell}^\infty a_n^\ell \psi_n^\ell$. Let us consider the space $\mathcal{S}_\ell$ of all functions f_ℓ such that

$$
[p_\ell^s(f_\ell)]^2 := \sum_{n=\ell}^\infty |a_\ell^n|^2 (n+1)^{2s} < \infty , \qquad s = 0, 1, 2 \ldots . \tag{43}
$$

We have discussed before in the present article the properties of $\mathcal{S}_\ell$ endowed with the seminorms $p_\ell^s(f_\ell)$ as defined in (43). In particular $\mathcal{S}_\ell \subset \mathcal{H}_\ell \subset \mathcal{S}_\ell^\times$ is a Gelfand triplet for each $\ell = 0, 1, 2, \ldots$. Then, define

$$
\Phi_\ell := \bigoplus_{k=0}^{\ell} \mathcal{S}_k . \tag{44}
$$

On Φ_ℓ, we define the following set of seminorms: If $f_k \in \mathcal{S}_k$, $k = 0, 1, 2 \ldots, \ell$,

$$
[p_\ell^{s_0, s_1, \ldots, s_\ell}(f_0 + f_1 + \cdots + f_\ell)]^2 := [p_0^{s_0}(f_0)]^2 + [p_1^{s_1}(f_1)]^2 + \cdots + [p_\ell^{s_\ell}(f_\ell)]^2 . \tag{45}
$$

With these set of seminorms, Φ_ℓ, $\ell = 0, 1, 2, \ldots$, is a Frèchet space and the triplets

$$
\Phi_\ell \subset \bigoplus_{k=0}^{\ell} \mathcal{H}_k \subset \Phi_\ell^\times , \tag{46}
$$

are Gelfand triplets for all $\ell = 0, 1, 2, \ldots$. Note that

[4] We make this Ansatz even being aware that the operators A_n^+ are transformations between functions and that the spaces $\mathcal{H}_\ell$ could be even identical or one a subspace of one other. We shall further comment this point.

$$\Phi_0 \subset \Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_\ell \subset \cdots \subset \bigcup_{\ell=0}^{\infty} \Phi_\ell =: \Phi\,. \tag{47}$$

Now, take the identity $I_\ell : \Phi_\ell \longmapsto \Phi_{\ell+1}$, such that $I_\ell(f_0 + f_1 + \ldots f_\ell) = f_0 + f_1 + \cdots + f_\ell + \mathbf{0}$, with $f_k \in \Phi_k$. Each of the identities I_ℓ, $\ell = 0, 1, 2, \ldots$, are continuous mappings, since

$$[p_{\ell+1}^{s_0,s_1,\ldots,s_{\ell+1}}[I_\ell(f_0 + f_1 + \cdots + f_\ell)]]^2 = [p_\ell^{s_0,s_1,\ldots,s_\ell}(f_0 + f_1 + \cdots + f_\ell)]^2\,. \tag{48}$$

Needless to say that $\Phi = \cup_{\ell=0}^{\infty}\Phi_\ell$ is a linear space. Then, let us endow it with the *strict inductive limit topology*[5] produced on Φ by the family $\{\Phi_\ell\}_{\ell\in\{0\}\cup\mathbb{N}}$. One usually calls LF the spaces which are strict inductive limits of Frèchet spaces (from "Limit Frèchet"). Thus Φ is a Frèchet space.

Now, take $f(x) \in \Phi$. After the definition of Φ, there exists Φ_k such that $f(x) \in \Phi_k$. Then,

$$f(x) = f_0 + f_1 + \cdots + f_k = \sum_{n=0}^{\infty} a_0^n\,\psi_0^n + \sum_{n=1}^{\infty} a_1^n\,\psi_1^n + \cdots + \sum_{n=k}^{\infty} a_k^n\,\psi_k^n\,. \tag{49}$$

Recalling diagram (42), let us define the linear operator A^+ on Φ as, for any $f(x) \in \Phi$, we have

$$A^+ f(x) = A_0^+ f_0 + A_1^+ f_1 + \cdots + A_k^+ f_k = \sum_{n=1}^{\infty} a_0^{n-1}\sqrt{E_{n-1} - \lambda}\,\psi_1^n$$
$$+ \sum_{n=2}^{\infty} a_1^{n-1}\sqrt{E_{n-1} - \lambda}\,\psi_2^n + \cdots + \sum_{n=k+1}^{\infty} a_k^{n-1}\sqrt{E_{n-1} - \lambda}\,\psi_{k+1}^n\,. \tag{50}$$

Obviously, $A^+\Phi_\ell \longmapsto \Phi_{\ell+1}$, $\ell = 0, 1, 2, \ldots$, and is a linear mapping.

In order to show the topological properties of A, as well as those of the operators included in the diagram (42) and also of Hamiltonians H_ℓ, $\ell = 0, 1, 2, \ldots$, we need to specify which model are using. In this presentation, we consider two of them, those for which H_0 is: i.) the standard one dimensional Harmonic oscillator and ii.) the standard Pöscl-Teller one dimensional Hamiltonian.

3.1. H_0 is the One Dimensional Harmonic Oscillator

SUSY partners of the one dimensional Harmonic Oscillator have been extensively studied by several authors. Let us cite here [34,39,40,42,43,45]. In this case, we have the following equations ($n, \ell = 0, 1, 2, \ldots$):

$$H_\ell \psi_\ell^n = \left(n + \frac{1}{2}\right)\psi_\ell^n\,, \qquad \psi_\ell^n = \frac{A_0^+\psi_{\ell-1}^{n-1}}{\sqrt{n}}\,, \qquad A_{\ell-1}\,\psi_\ell^n = \sqrt{n}\,\psi_{\ell-1}^{n-1}\,. \tag{51}$$

Analogously as A^+, we may define A on each of the Φ_ℓ as ($f_i \in \mathcal{S}_i$)

$$A(f_1 + f_2 + \cdots + f_\ell) := A_1 f_1 + A_2 f_2 + \cdots + A_\ell f_\ell$$
$$= \sum_{n=1}^{\infty} a_1^n\,\sqrt{n}\,\psi_0^{n-1} + \sum_{n=1}^{\infty} a_2^n\,\sqrt{n}\,\psi_1^{n-1} + \cdots + \sum_{n=1}^{\infty} a_\ell^n\,\sqrt{n}\,\psi_{\ell-1}^{n-1}\,. \tag{52}$$

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[5] This topology is the finest topology that makes all the identity mappings $J_\ell : \Phi_\ell \longmapsto \Phi$, i.e., the so called *final topology* in the language by Bourbaki [37].

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Thus, $A : \Phi_\ell \longmapsto \Phi_\ell$, for all $\ell = 1, 2, \dots$ is a linear mapping. As for the case of A^+, A can be extended to a linear mapping on Φ.

Theorem.- The mappings A^+ and A are continuous linear mappings on Φ.

Proof.- Obviously, there are linear. Consider A^+. Let us show that it is continuous as a mapping $A^+ : \Phi_\ell \longmapsto \Phi_{\ell+1}$ for all $\ell = 0, 1, 2, \dots$.

$$[p_{k+1}^{s_0, s_1, \dots, s_{k+1}}(A^+ f)]^2 = \sum_{n=1}^{\infty} |a_0^{n-1}|^2 (n+1)^{2s_0} (n+1)$$

$$+ \sum_{n=2}^{\infty} |a_1^{n-1}|^2 (n+1)^{2s_1} (n+1) + \dots + \sum_{n=k+1}^{\infty} |a_k^{n-1}|^2 (n+1)^{2s_k} (n+1)$$

$$\leq \sum_{n=1}^{\infty} |a_0^{n-1}|^2 (n+1)^{2(s_0+1)} + \sum_{n=2}^{\infty} |a_1^{n-1}|^2 (n+1)^{2(s_1+1)} + \dots + \sum_{n=k+1}^{\infty} |a_k^{n-1}|^2 (n+1)^{2(s_k+1)}$$

$$= [p_k^{s_0+1, s_1+1, \dots, s_k+1}(f)]^2 . \tag{53}$$

with $k = 0, 1, 2, \dots$. This proves the continuity from Φ_ℓ to $\Phi_{\ell+1}$ because of (21). Then, note that the canonical injection $J_\ell : \Phi_\ell \longmapsto \Phi$ is continuous due to the definition of the strict inductive limit topology on Φ. Then for all $\ell = 0, 1, 2, \dots$, we have that the mappings

$$J_{\ell+1} \circ A^+ : \Phi_\ell \longmapsto \Phi \tag{54}$$

are continuous. Then after a well known result [38] page 58, B^+ is continuous as an operator on Φ[6]. The proof of the continuity of B on Φ is similar. ∎

Take now $\mathcal{S}_\ell$ and $f(x) \in \mathcal{S}_\ell$, $f_\ell(x) = \sum_{k=\ell}^{\infty} a_\ell^k \psi_\ell^k(x)$. Define B_ℓ^+ on $\mathcal{S}_\ell$ as

$$B_\ell^+ f_\ell := \sum_{k=\ell}^{\infty} a_\ell^k B_{\ell,k}^+ \psi_\ell^k(x) = \sum_{k=\ell}^{\infty} a_\ell^k \sqrt{n+1}\, \psi_\ell^{k+1} . \tag{55}$$

Obviously, $B_\ell^+ \psi_\ell^k(x) = \sqrt{n+1}\, \psi_\ell^{k+1}$. Since $[p_\ell^s(B_\ell^+ f_\ell)]^2 \leq [p_\ell^{s+1}(f_\ell)]^2$, $s = 0, 1, 2, \dots$, B_ℓ^+ is continuous on $\mathcal{S}_\ell$, $\ell = 0, 1, 2, \dots$. Analogously,

$$B_\ell f_\ell := \sum_{k=\ell} a_\ell^k B_{\ell,k} \psi_\ell^k = \sum_{k=\ell+1}^{\infty} \sqrt{n}\, \psi_\ell^{k-1} . \tag{56}$$

Note that B_0^+ and B_0 are the creation and annihilation operators of the harmonic oscillator, respectively, B_1^+ and B_1 of the first SUSY partner of the oscillator, etc. The Hamiltonian H_ℓ has the same eigenvalues of the harmonic oscillator except for the $\ell - 1$ first eigenvalues. Thus, for $f_\ell \in \mathcal{S}_\ell$

$$H_\ell f_\ell := \sum_{k=\ell}^{\infty} a_\ell^k (n + 1/2)\, \psi_\ell^k . \tag{57}$$

Considering the above discussion, it is quite straightforward to show that B_ℓ^+, B_ℓ and H_ℓ are continuous operators on $\mathcal{S}_\ell$ and, therefore, continuously extendable to the dual $\mathcal{S}_\ell^\times$, $\ell = 0, 1, 2, \dots$. Their extension to all Φ_ℓ is immediate as it is the proof of the continuity of these extensions to Φ.

---

6 This result establishes that if Y is a locally convex space, X is a strict inductive limit of the spaces X_n and $f : X \longmapsto Y$ a linear mapping, the f is continuous if and only if all mappings $f : X_n \longmapsto Y$ are continuous, where each of the f_n is the restriction of f to X_n.

3.2. H_0 is the One Dimensional Trigonometric Pöschl-Teller Hamiltonian

Although we have not written the operators $B_{\ell,k}$ on the diagram (42), it is clear how they go. This construction goes for many cases. When E_n is polynomially bounded by $P_k(n+1)$ all considerations of continuity go exactly the same as in the above example. In particular if H_0 is either the one dimensional Pöschl-Teller trigonometric potential or the one dimensional infinite square well [49], they obey to the above scheme.

Let us take the one dimensional Pöschl-Teller Hamiltonian as in (12). Then, a SUSY transform takes (12) and transforms it into a similar Hamiltonian where ℓ has been replaces by $\ell+1$ and so on. Thus, vectors in the first column in (42) should be denoted by $\{\psi_\ell^n\}$, just replacing 0 by ℓ, $n = 0, 1, 2\dots$. Vectors in the second column are denoted as $\{\psi_{\ell+1}^n\}$, $n = 1, 2, \dots$ and so on. At the same time, we replace A_k^+ by $A_{\ell+k}^+$, $k = 0, 1, 2, \dots$. Now [48],

$$A^+\psi_\ell^n = \sqrt{(n+1)[(2(\ell+1)+n+1]}\,\psi_{\ell+1}^{n+1}\,, \qquad \ell, n = 0, 1, 2\dots\,, \tag{58}$$

and

$$A\psi_\ell^n = \sqrt{n(2\ell+n)}\,\psi_{\ell-1}^{n-1}\,, \qquad \ell, n = 1, 2\dots\,. \tag{59}$$

Then, all the construction goes essentially as in the above example. In particular, A^+ and A are linear continuous operators on Φ.

4. Concluding Remarks

Gelfand triplets, also called Rigged Hilbert Spaces (RHS) are the suitable framework for a rigorous mathematical formulation of the Dirac formalism of Quantum Mechanics, quantum systems (relativistic and non-relativistic and sometimes even classical) showing resonances or different types of singularities. In addition, RHS are the suitable arena to describe with the due mathematical rigor objects of common use in standard quantum mechanics such us continuous and discrete basis, basis of special functions and Lie algebras of continuous operators of the symmetries of a given quantum model. All them inside a common framework, a property that does not have the standard formalism on Hilbert spaces.

In the present paper, we give an interesting example of the latter based on the one dimensional Pöschl-Teller potential. This is a very interesting example showing all features as described in the end of the last paragraph. We believe that this example is very illustrative and at the same time non-trivial.

The factorization method and the SUSY quantum mechanics provides an efficient method to obtain Hamiltonians with similar spectrum than one given. They use of ladder operators in this factorization, as well as in the process of relating eigenvectors of the different Hamiltonians resulting after the SUSY transformations. All these ladder operators are not bounded operators on Hilbert space, so that its proper mathematical manipulation would require of a cumbersome and non-trivial analysis, contrary to the formal analysis which is performed by physicists. The context of RHS solves this problem for a strict mathematical point of view. In the present article, we describe a particularly standard model in which the seed Hamiltonian, H_0, from where all other come after reiterative SUSY transformations, is the Harmonic Oscillator. We have chosen this seed Hamiltonian because its factorization and partners are well know. In addition, it has an infinite number of partners, a fact that adds some further mathematical interest to the problem. Same analysis can be made when the seed Hamiltonian is the one dimensional trigonometric Pöschl-Teller.

There are many examples and studies of this factorization method and SUSY transformations [40–43], which may be the point of departures of other similar studies in the future. This may be the case with respect to the spectrum generating algebras [34,46–48].

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