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New Harmonic Number Series

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Abstract: Based on a recent representation of the psi function due to Guillera and Sondow and independently Boyadzhiev, new closed forms for various series involving harmonic numbers and inverse factorials are derived. A high point of the presentation is the rediscovery, by much simpler means, of a famous quadratic Euler sum originally discovered in 1995 by Borwein and Borwein..

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1. Introduction

Our main purpose in this note is to discover the harmonic number series associated with the following identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{n+z}{n}} = \zeta(2) - H_z^{(2)}, \quad z \in \mathbb{C} \setminus Z^-, \tag{1}$$

where $\zeta(s)$ is the Riemann zeta function and $H_z^{(2)}$ is a second order harmonic number (both definitions are given below). At $z = 0$ identity (1) subsumes the solution to the Basel problem and we will see that its derivative includes the well-known relation between the Apéry constant and a classical Euler sum, namely,

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\,\zeta(3),$$

as a special case, H_j being the jth harmonic number.

We will also evaluate the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\binom{n+z}{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)\binom{n+z}{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)\binom{n+z}{n}},$$

and derive the harmonic and odd harmonic number series associated with them.

Identity (1), stated without proof in Sofo and Srivastava [15, Equation (2.13)], is a consequence of the following representation of the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$:

$$\psi(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \ln(z+k), \tag{2}$$

which holds for all $z \in \mathbb{C}$ with $\Re(z) > 0$. This representation first appeared in [10, Theorem 5.1] and was rediscovered by Boyadzhiev [6, Identity (5.18)].

Harmonic numbers H_α and odd harmonic numbers O_α are defined for $0 \neq \alpha \in \mathbb{C} \setminus \mathbb{Z}^-$ by the recurrence relations

$$H_\alpha = H_{\alpha-1} + \frac{1}{\alpha} \qquad \text{and} \qquad O_\alpha = O_{\alpha-1} + \frac{1}{2\alpha-1},$$

with $H_0 = 0$ and $O_0 = 0$. Harmonic numbers are connected to the digamma function through the fundamental relation

$$H_\alpha = \psi(\alpha+1) + \gamma. \tag{3}$$

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Generalized harmonic numbers $H_\alpha^{(m)}$ and odd harmonic numbers $O_\alpha^{(m)}$ of order $m \in \mathbb{C}$ are defined by

$$H_\alpha^{(m)} = H_{\alpha-1}^{(m)} + \frac{1}{\alpha^m} \qquad \text{and} \qquad O_\alpha^{(m)} = O_{\alpha-1}^{(m)} + \frac{1}{(2\alpha-1)^m},$$

with $H_0^{(m)} = 0$ and $O_0^{(m)} = 0$ so that $H_\alpha = H_\alpha^{(1)}$ and $O_\alpha = O_\alpha^{(1)}$.

The recurrence relations imply that if $\alpha = n$ is a non-negative integer, then

$$H_n^{(m)} = \sum_{j=1}^{n} \frac{1}{j^m} \qquad \text{and} \qquad O_n^{(m)} = \sum_{j=1}^{n} \frac{1}{(2j-1)^m}.$$

Generalized harmonic numbers are linked to the polygamma functions $\psi^{(r)}(z)$ of order r defined by

$$\psi^{(r)}(z) = \frac{d^r}{dz^r}\psi(z) = (-1)^{r+1} r! \sum_{j=0}^{\infty} \frac{1}{(j+z)^{r+1}},$$

through

$$H_z^{(r)} = \zeta(r) + \frac{(-1)^{r-1}}{(r-1)!}\psi^{(r-1)}(z+1),$$

where $\zeta(s)$ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad s \in \mathbb{C}, \Re(s) > 1.$$

The analytical continuation to all $s \in \mathbb{C}$ with $\Re(s) > 0, s \neq 1$, is given by

$$\zeta(s) = (1 - 2^{1-s})^{-1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s}.$$

2. Proof of Identity (1)

For completeness and a better readability we first prove identity (1).

Theorem 1. *For all $z \in \mathbb{C} \setminus Z^-$ the following identity holds:*

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{n+z}{n}} = \zeta(2) - H_z^{(2)}.$$

Proof. Differentiating the representation (2) gives

$$\begin{aligned}
\psi^{(1)}(z) = \zeta(2) - H_{z-1}^{(2)} &= \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{z+k} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{(-1)^{k-1}}{z+k-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n} \binom{n}{k} \frac{(-1)^{k-1}}{z+k-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} k \binom{n}{k} \frac{(-1)^{k-1}}{z+k-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{\binom{n+z-1}{n}};
\end{aligned}$$

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and hence (1). Note that we used

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}k}{z+k}=\frac{1}{\binom{n+z}{n}},$$

which is a particular case of the Frisch identity ([1,2]). □

3. Required Identities

We will make frequent use of the following basic identity [12]:

$$\sum_{k=1}^{m}\frac{1}{\binom{k+n}{k}}=\frac{1}{n-1}-\frac{n}{n-1}\frac{1}{\binom{m+n}{m+1}},\quad 0,1\neq n\in\mathbb{C}. \tag{4}$$

and the identities stated in the following lemmata.

Lemma 1. *We have*

$$H_{k-1/2}=2\,O_k-2\ln 2, \tag{5}$$
$$H_{k-1/2}^{(2)}=-2\,\zeta(2)+4\,O_k^{(2)}, \tag{6}$$
$$H_{-1/2}^{(3)}=-6\,\zeta(3), \tag{7}$$
$$H_{-1/2}^{(4)}=-14\,\zeta(4), \tag{8}$$
$$H_{k-1/2}^{(m+1)}-H_{-1/2}^{(m+1)}=2^{m+1}O_k^{(m+1)}. \tag{9}$$

Lemma 2. *For integers u and v, we have*

$$\binom{u-1/2}{v}=\binom{2u}{u}\binom{u}{v}2^{-2v}\binom{2(u-v)}{u-v}^{-1}, \tag{10}$$
$$\binom{u}{1/2}=\frac{2^{2u+1}}{\pi}\binom{2u}{u}^{-1}, \tag{11}$$
$$\binom{u}{1/2-v}=\frac{(-1)^{v-1}}{v}\binom{u+v}{v}^{-1}\binom{2(u+v)}{u+v}^{-1}\binom{2(v-1)}{v-1}2^{2u+2}, \tag{12}$$
$$\binom{u+1/2}{v}=\binom{u}{v}^{-1}\binom{2u+1}{2v}2^{-2v}\binom{2v}{v}, \tag{13}$$
$$\binom{u+1/2}{v}=\frac{(-1)^{v-u-1}}{2^{2v-1}}\frac{2u+1}{v-u}\binom{v}{u}^{-1}\binom{2u}{u}\binom{2(v-u-1)}{v-u-1},\quad v>u, \tag{14}$$
$$\binom{-3/2}{u}=(-1)^u\,(2u+1)\,2^{-2u}\binom{2u}{u}. \tag{15}$$

Proof. These identities are readily derived using the following well-known Gamma function identities:

$$\Gamma\left(u+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2u}}\binom{2u}{u}\Gamma(u+1),\quad \Gamma\left(-u+\frac{1}{2}\right)=(-1)^u2^{2u}\binom{2u}{u}^{-1}\frac{\sqrt{\pi}}{\Gamma(u+1)},$$

together with the definition of the generalized binomial coefficients:

$$\binom{u}{v}=\frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}.$$

□

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4. Results

In this section we state new closed forms for infinite series involving inverse factorials. More results of this nature have been produced, among others, by Sofo [13,14], Boyadzhiev [7,8] and by the authors [3].

Theorem 2. *If m is a non-negative integer, then*

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n^2 \binom{2(n+m)}{n+m}\binom{n+m}{m}} = \frac{1}{\binom{2m}{m}}\left(3\,\zeta(2) - 4\,O_m^{(2)}\right). \tag{16}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n^2\binom{2n}{n}} = \frac{\pi^2}{2}. \tag{17}$$

Proof. Write $m - 1/2$ for z in (1) to obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{n+m-1/2}{n}} = \zeta(2) - H_{m-1/2}^{(2)}, \tag{18}$$

from which (16) follows on account of Lemmata 1 and 2. □

Theorem 3. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\sum_{n=1}^{\infty} \frac{H_{n+z}}{n^2\binom{n+z}{n}} = H_z\left(\zeta(2) - H_z^{(2)}\right) + 2\left(\zeta(3) - H_z^{(3)}\right). \tag{19}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\,\zeta(3), \tag{20}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2(n+1)} = 2\,\zeta(3) - \zeta(2), \tag{21}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2(n+1)(n+2)} = \zeta(3) - \frac{3}{4}\,\zeta(2) + \frac{1}{4}. \tag{22}$$

Proof. Differentiate (1) with respect to z to obtain

$$\sum_{n=1}^{\infty} \frac{H_{n+z} - H_z}{n^2\binom{n+z}{n}} = 2\left(\zeta(3) - H_z^{(3)}\right), \tag{23}$$

and use (1) again to rewrite the second term on the left hand side of (23).

Identity (21) corresponds to an evaluation of (19) at $z = 1$ followed by the use of

$$H_{n+1} = H_n + \frac{1}{n+1}, \tag{24}$$

and the decomposition

$$\frac{1}{n^2(n+1)^2} = \frac{1}{n^2} + \frac{1}{(n+1)^2} - 2\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

□

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Corollary 1. *If m is a nonnegative integer, then*

$$\sum_{n=1}^{\infty} \frac{2^{2n}\, O_{n+m}}{n^2 \binom{2(n+m)}{n+m}\binom{n+m}{m}} = \frac{1}{\binom{2m}{m}} \left(O_m \left(3\,\zeta(2) - 4\, O_m^{(2)} \right) + \left(7\,\zeta(3) - 8\, O_m^{(3)} \right) \right). \tag{25}$$

Proof. Write $m - 1/2$ for z in (23) and use Lemmata 1 and 2. □

Theorem 4. *If $z \in \mathbb{C} \setminus Z^-$, then*

$$\sum_{n=1}^{\infty} \frac{(H_{n+z} - H_z)^2 + H_{n+z}^{(2)} - H_z^{(2)}}{n^2 \binom{n+z}{z}} = 6\,\zeta(4) - 6\, H_z^{(4)}. \tag{26}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n^2} = 6\,\zeta(4), \tag{27}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{2^{2n}}{\binom{2n}{n}} \left(O_n^2 + O_n^{(2)} \right) = \frac{\pi^2}{4}. \tag{28}$$

Proof. Differentiate (23) with respect to z. □

Remark 1. *The symmetry relation*

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} + \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{n^p} = \zeta(p)\zeta(q) + \zeta(p+q)$$

makes it easy to calculate

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^p} = \frac{1}{2}\left(\zeta(p)^2 + \zeta(2p) \right),$$

so that, in particular,

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} = \frac{7}{4}\zeta(4);$$

and using this in (27) therefore gives

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{17}{4}\zeta(4). \tag{29}$$

Thus (27) allows a much easier determination of the sum (29) which was originally evaluated by Borwein and Borwein [5] through Fourier series and contour integration.

Lemma 3. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\binom{n+z}{n}} = z\left(H_z^{(2)} - \zeta(2) \right) + 1, \tag{30}$$

Proof. Sum (1) over z from 1 to r, using (4) and the fact that [4, Equation (3.1)]

$$\sum_{z=1}^{r} H_z^{(2)} = (r+1)H_r^{(2)} - H_r,$$

to obtain

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\binom{n+r}{r}} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2 n} - 1 - (r-1)\zeta(2) + rH_r^{(2)}.$$

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But

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(n+1)^2n}&=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\\&=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\sum_{n=2}^{\infty}\frac{1}{n^2}\\&=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)+1-\sum_{n=1}^{\infty}\frac{1}{n^2}\\&=2-\zeta(2),\end{aligned}$$

which when substituted into the previous sum gives (30) after replacing r with z. □

Remark 2. *Identity* (30) *was also derived by Sofo and Srivastava [15, Equation (2.17)].*

Theorem 5. *If m is a non-negative integer, then*

$$\sum_{n=1}^{\infty}\frac{2^{2n}}{n(n+1)\binom{2(n+m)}{n+m}\binom{n+m}{m}}=\frac{1}{\binom{2m}{m}}\left(\left(m-\frac{1}{2}\right)\left(4\,O_m^{(2)}-3\,\zeta(2)\right)+1\right). \tag{31}$$

In particular,

$$\sum_{n=1}^{\infty}\frac{2^{2n}}{n(n+1)\binom{2n}{n}}=\frac{\pi^2}{4}+1, \tag{32}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n}}{n(n+1)^2\binom{2(n+1)}{n+1}}=\frac{3}{2}-\frac{\pi^2}{8}, \tag{33}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n}}{n(n+1)^2(n+2)\binom{2(n+2)}{n+2}}=\frac{23}{36}-\frac{\pi^2}{16}. \tag{34}$$

Theorem 6. *If $z\in\mathbb{C}\setminus\mathbb{Z}^-$, then*

$$\sum_{n=1}^{\infty}\frac{H_{n+z}}{n(n+1)\binom{n+z}{n}}=\left(\zeta(2)-H_z^{(2)}\right)(1-zH_z)+H_z-2z\left(\zeta(3)-H_z^{(3)}\right). \tag{35}$$

In particular,

$$\sum_{n=1}^{\infty}\frac{H_n}{n(n+1)}=\frac{\pi^2}{6}, \tag{36}$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n(n+1)^2}=\zeta(2)-\zeta(3). \tag{37}$$

Proof. Differentiate (30) with respect to z to obtain

$$\sum_{n=1}^{\infty}\frac{H_{n+z}-H_z}{n(n+1)\binom{n+z}{n}}=\zeta(2)-H_z^{(2)}-2z\left(\zeta(3)-H_z^{(3)}\right), \tag{38}$$

and hence (35) upon using (30) again to rewrite the second sum on the left hand side of (38). Identity (37) is an evaluation of (35) at $z=1$ where we used (24) and the fact that

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$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(n+1)^3}&=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}-\frac{1}{(n+1)^2}-\frac{1}{(n+1)^3}\right)\\&=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}-\sum_{n=1}^{\infty}\frac{1}{(n+1)^3}.\end{aligned}$$

□

Remark 3. *Identity (36) was also recorded by Chu in [9].*

Remark 4. *Subtraction of (37) from (21) gives the Euler sum*

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2(n+1)^2}=3\,\zeta(3)-2\,\zeta(2). \tag{39}$$

Corollary 2. *If m is a non-negative integer, then*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2^{2n}O_{n+m}}{n(n+1)\binom{2(n+m)}{n+m}\binom{n+m}{m}}&=\frac{1}{2\binom{2m}{m}}\left(3\zeta(2)-4O_m^{(2)}\right)(1-O_m(2m-1))\\&\quad+\frac{O_m}{\binom{2m}{m}}-\frac{(2m-1)}{2\binom{2m}{m}}\left(7\zeta(3)-8O_m^{(3)}\right).\end{aligned} \tag{40}$$

In particular,

$$\sum_{n=1}^{\infty}\frac{2^{2n}O_n}{n(n+1)\binom{2n}{n}}=\frac{\pi^2}{4}+\frac{7}{2}\zeta(3), \tag{41}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n}O_{n+1}}{n(n+1)^2\binom{2(n+1)}{n+1}}=\frac{5}{2}-\frac{7}{4}\zeta(3). \tag{42}$$

Proof. Write $m-1/2$ for z in (38) and use Lemmata 1 and 2. □

Theorem 7. *If $z\in\mathbb{C}\setminus\mathbb{Z}^-$, then*

$$\sum_{n=1}^{\infty}\frac{(H_{n+z}-H_z)^2+H_{n+z}^{(2)}-H_z^{(2)}}{n(n+1)\binom{n+z}{z}}=4\left(\zeta(3)-H_z^{(3)}\right)-6z\left(\zeta(4)-H_z^{(4)}\right). \tag{43}$$

In particular,

$$\sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{n(n+1)}=4\,\zeta(3), \tag{44}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n}\left(O_n^2+O_n^{(2)}\right)}{n(n+1)\binom{2n}{n}}=\frac{\pi^4}{8}+7\,\zeta(3). \tag{45}$$

Proof. Differentiate (38) with respect to z. □

Remark 5. *Since Xu showed that [17, Equation (2.30)]*

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n(n+1)}=\zeta(3),$$

identity (44) yields

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$$\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+1)}=3\,\zeta(3); \tag{46}$$

an identity that was also reported by Nimbran and Sofo in [11, Identity (2.24)].

Lemma 4. *If $z\in\mathbb{C}\setminus\mathbb{Z}^-$, then,*

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)\binom{n+z}{n}}=\frac{1}{2}z(z+1)\Big(H_z^{(2)}-\zeta(2)\Big)+\frac{z}{2}+\frac{1}{4}. \tag{47}$$

Proof. We first derive the following identity:

$$\sum_{n=1}^{\infty}\frac{1}{n(n+2)\binom{n+z}{n}}=\frac{1}{2}z(z-1)\Big(\zeta(2)-H_z^{(2)}\Big)-\frac{z}{2}+\frac{3}{4}, \tag{48}$$

by summing both sides of (30) over z from 1 to r and replacing r with z in the final identity. Note that

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}=\frac{1}{4},$$

since

$$\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+1}+\frac{1}{n+2}\right).$$

Note also that [4, Equation (3.2)]:

$$2\sum_{z=1}^{r}zH_z^{(2)}=r(r+1)H_r^{(2)}+H_r-r. \tag{49}$$

Identity (47) is obtained by subtracting (48) from (30). □

Theorem 8. *If m is a non-negative integer, then*

$$\begin{aligned}\sum_{n=1}^{\infty}&\frac{2^{2n}}{n(n+1)(n+2)\binom{2(n+m)}{n+m}\binom{n+m}{m}}\\&=\frac{1}{2\binom{2m}{m}}\left(\left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\left(4O_m^{(2)}-3\zeta(2)\right)+m\right).\end{aligned} \tag{50}$$

In particular,

$$\sum_{n=1}^{\infty}\frac{2^{2n}}{n(n+1)(n+2)\binom{2n}{n}}=\frac{\pi^2}{16}, \tag{51}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n}}{n(n+1)^2(n+2)\binom{2(n+1)}{n+1}}=1-\frac{3\pi^2}{32}, \tag{52}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n+1}}{n(n+1)^2(n+2)^2\binom{2(n+2)}{n+2}}=\frac{14}{9}-\frac{5\pi^2}{32}. \tag{53}$$

Proof. Set $z=m-1/2$ in (47). □

Theorem 9. *If $z\in\mathbb{C}\setminus\mathbb{Z}^-$, then*

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$$\sum_{n=1}^{\infty} \frac{H_{n+z}}{n(n+1)(n+2)\binom{n+z}{n}} = \left(z + \frac{1}{2} - \frac{1}{2}z(z+1)H_z\right)\left(\zeta(2) - H_z^{(2)}\right) + \left(\frac{z}{2} + \frac{1}{4}\right)H_z - z(z+1)\left(\zeta(3) - H_z^{(3)}\right) - \frac{1}{2}. \tag{54}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)(n+2)} = \frac{\pi^2}{12} - \frac{1}{2}, \tag{55}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)^2(n+2)} = \frac{\pi^2}{12} + \frac{1}{2} - \zeta(3). \tag{56}$$

Proof. Differentiate (47) to obtain

$$\sum_{n=1}^{\infty} \frac{H_{n+z} - H_z}{n(n+1)(n+2)\binom{n+z}{n}} = \left(z + \frac{1}{2}\right)\left(\zeta(2) - H_z^{(2)}\right) - z(z+1)\left(\zeta(3) - H_z^{(3)}\right) - \frac{1}{2}. \tag{57}$$

Identity (56) is an evaluation of (54) at $z = 1$, where we also used (24) and the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^3(n+2)} = \frac{5}{4} - \zeta(3),$$

since

$$\frac{1}{n(n+1)^3(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \frac{1}{(n+1)^3}.$$

□

Theorem 10. *If m is a non-negative integer, then*

$$\begin{aligned}\sum_{n=1}^{\infty} &\frac{2^{2n} O_{n+m}}{n(n+1)(n+2)\binom{2(n+m)}{n+m}\binom{n+m}{m}} \\ &= \frac{1}{2\binom{2m}{m}}\left(m - O_m\left(m - \frac{1}{2}\right)\left(m + \frac{1}{2}\right)\right)\left(3\zeta(2) - 4O_m^{(2)}\right) \\ &\quad - \frac{1}{2\binom{2m}{m}}\left(\left(m - \frac{1}{2}\right)\left(m + \frac{1}{2}\right)\left(7\zeta(3) - 8O_m^{(3)}\right) - mO_m + \frac{1}{2}\right).\end{aligned} \tag{58}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{2^{2n} O_n}{n(n+1)(n+2)\binom{2n}{n}} = \frac{7}{8}\zeta(3) - \frac{1}{4}, \tag{59}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n} O_{n+1}}{n(n+1)^2(n+2)\binom{2(n+1)}{n+1}} = \frac{\pi^2}{32} - \frac{21}{16}\zeta(3) + \frac{11}{8}. \tag{60}$$

Proof. Set $z = m - 1/2$ in (57) and use Lemmata 1 and 2. □

Theorem 11. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

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$$\sum_{n=1}^{\infty}\frac{(H_{n+z}-H_z)^2+H_{n+z}^{(2)}-H_z^{(2)}}{n(n+1)(n+2)\binom{n+z}{z}}=H_z^{(2)}-\zeta(2)-2(2z+1)\Big(H_z^{(3)}-\zeta(3)\Big)+3z(z+1)\Big(H_z^{(4)}-\zeta(4)\Big). \tag{61}$$

In particular,

$$\sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{n(n+1)(n+2)}=2\zeta(3)-\zeta(2), \tag{62}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n}\Big(O_n^2+O_n^{(2)}\Big)}{n(n+1)(n+2)\binom{2n}{n}}=\frac{\pi^4}{32}-\frac{\pi^2}{8}. \tag{63}$$

Proof. Differentiate (57) with respect to z. □

Lemma 5. *If $z\in\mathbb{C}\setminus\mathbb{Z}^-$, then*

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)(n+3)\binom{n+z}{n}}=\frac{1}{12}z(z+1)(z+2)\Big(H_z^{(2)}-\zeta(2)\Big)+\frac{z^2}{12}+\frac{5z}{24}+\frac{1}{18}. \tag{64}$$

Proof. Sum (48) over z using (4), (49) and [4, Equation (3.6)]

$$\sum_{z=1}^{r}z^2H_z^{(2)}=\frac{r(r+1)(2r+1)}{6}H_r^{(2)}-\frac{1}{6}H_r+\frac{r}{3}-\frac{r^2}{6},$$

to obtain

$$\sum_{n=1}^{\infty}\frac{1}{n(n+3)\binom{n+z}{n}}=\frac{1}{6}z(z-1)(z-2)\Big(H_z^{(2)}-\zeta(2)\Big)+\frac{1}{6}z^2-\frac{7}{12}z+\frac{11}{18}. \tag{65}$$

Identity (64) is obtained by adding (65) and (47) and subtracting (48). □

Theorem 12. *If m is a non-negative integer, then*

$$\begin{aligned}\sum_{n=1}^{\infty}&\frac{2^{2n}}{n(n+1)(n+2)(n+3)\binom{2(n+m)}{n+m}\binom{n+m}{m}}\\&=\frac{1}{12\binom{2m}{m}}\left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)\Big(4O_m^{(2)}-3\zeta(2)\Big)\\&\quad+\frac{1}{4\binom{2m}{m}}\left(\frac{m^2}{3}+\frac{m}{2}-\frac{1}{9}\right).\end{aligned} \tag{66}$$

In particular,

$$\sum_{n=1}^{\infty}\frac{2^{2n}}{n(n+1)(n+2)(n+3)\binom{2n}{n}}=\frac{\pi^2}{64}-\frac{1}{36}, \tag{67}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n}}{n(n+1)^2(n+2)(n+3)\binom{2(n+1)}{n+1}}=\frac{29}{72}-\frac{5\pi^2}{128}, \tag{68}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n+1}}{n(n+1)^2(n+2)^2(n+3)\binom{2(n+2)}{n+2}}=\frac{65}{72}-\frac{35\pi^2}{384}. \tag{69}$$

Proof. Set $z=m-1/2$ in (64). □

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Theorem 13. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_{n+z}}{n(n+1)(n+2)(n+3)\binom{n+z}{n}} &= \left(\frac{1}{12}z(z+1)(z+2)H_z - \frac{z(z+2)}{4} - \frac{1}{6}\right)\left(H_z^{(2)} - \zeta(2)\right) \\ &\quad + \frac{1}{6}z(z+1)(z+2)\left(H_z^{(3)} - \zeta(3)\right) + \left(\frac{z^2}{12} + \frac{5z}{24} + \frac{1}{18}\right)H_z - \frac{z}{6} - \frac{5}{24}.\end{aligned} \tag{70}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)(n+2)(n+3)} = \frac{\pi^2}{36} - \frac{5}{24}. \tag{71}$$

Proof. Differentiate (64) to obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_{n+z} - H_z}{n(n+1)(n+2)(n+3)\binom{n+z}{n}} &= -\left(\frac{z(z+2)}{4} + \frac{1}{6}\right)\left(H_z^{(2)} - \zeta(2)\right) \\ &\quad + \frac{1}{6}z(z+1)(z+2)\left(H_z^{(3)} - \zeta(3)\right) - \frac{z}{6} - \frac{5}{24},\end{aligned} \tag{72}$$

and hence (70) by a repeated use of (64). □

Theorem 14. *If m is a non-negative integer, then*

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{2^{2n} O_{n+m}}{n(n+1)(n+2)(n+3)\binom{2(n+m)}{n+m}\binom{n+m}{m}} \\ &= \frac{1}{2\binom{2m}{m}}\left(\frac{1}{6}\left(m - \frac{1}{2}\right)\left(m + \frac{1}{2}\right)\left(m + \frac{3}{2}\right)O_m - \frac{m^2}{4} - \frac{m}{4} + \frac{1}{48}\right)\left(4O_m^{(2)} - 3\zeta(2)\right) \\ &\quad + \frac{1}{12\binom{2m}{m}}\left(m - \frac{1}{2}\right)\left(m + \frac{1}{2}\right)\left(m + \frac{3}{2}\right)\left(8O_m^{(3)} - 7\zeta(3)\right) \\ &\quad + \frac{1}{2\binom{2m}{m}}\left(\frac{1}{36}\left(6m^2 + 9m - 2\right)O_m - \frac{m}{6} - \frac{1}{8}\right).\end{aligned} \tag{73}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{2^{2n} O_n}{n(n+1)(n+2)(n+3)\binom{2n}{n}} = \frac{7}{32}\zeta(3) - \frac{\pi^2}{192} - \frac{1}{16}, \tag{74}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n} O_{n+1}}{n(n+1)^2(n+2)(n+3)\binom{2(n+1)}{n+1}} = \frac{137}{288} + \frac{\pi^2}{48} - \frac{35}{64}\zeta(3). \tag{75}$$

Proof. Write $m - 1/2$ for z in (72) and use Lemmata 1 and 2. □

Theorem 15. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(H_{n+z} - H_z)^2 + H_{n+z}^{(2)} - H_z^{(2)}}{n(n+1)(n+2)(n+3)\binom{n+z}{z}} &= \frac{1}{2}(z+1)\left(H_z^{(2)} - \zeta(2)\right) \\ &\quad - \left(z(z+2) + \frac{2}{3}\right)\left(H_z^{(3)} - \zeta(3)\right) \\ &\quad + \frac{1}{2}z(z+1)(z+2)\left(H_z^{(4)} - \zeta(4)\right) + \frac{1}{6}.\end{aligned} \tag{76}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n(n+1)(n+2)(n+3)} = \frac{2}{3}\zeta(3) - \frac{\pi^2}{12} + \frac{1}{6}, \tag{77}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n}\left(O_n^2 + O_n^{(2)}\right)}{n(n+1)(n+2)(n+3)\binom{2n}{n}} = \frac{\pi^4}{128} - \frac{\pi^2}{32} - \frac{7}{48}\zeta(3) + \frac{1}{24}. \tag{78}$$

Proof. Differentiate (72) with respect to z. □

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