Technical Note

Weak Arbitrage Theorem Incorporating Loss Aversion

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Weak Arbitrage Theorem Incorporating Loss Aversion

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Abstract: We provide an extension of the "Arbitrage Theorem" with state-dependent linear utility functions for monetary gains and losses allowing for "loss aversion" in each possible state of nature.

Keywords: arbitrage theorem; loss aversion

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For some positive integer L ≥ 2, let {1, 2, …, L} denote the finite set of states of nature exactly one of which will be realized at a future date. We will refer to a **state of nature** as **SON** and its plural as **SONs**.

A row vector $x \in \mathbb{R}^L$ where for each j∈{1, …, L}, the jth coordinate of x denoted by x_j denotes the monetary return in SON j, from investment worth one unit of money, is said to be a **unit return vector**.

A vector $p \in \mathbb{R}^L_+$ satisfying $\sum_{j=1}^{L} p_j = 1$, such that for j∈{1, …, L}, p_j is the probability of occurrence of SON j, is a **probability vector**.

A pair (x, p) where x is a unit return vector and p is a probability vector is called a **unit risky investment pair**.

Given x, $y \in \mathbb{R}^L$, let $y^{T}x$ denote $\sum_{j=1}^{L} y_j x_j$.

The **expected value** of a unit risky investment pair denoted E(x, p) is $p^{T}x = \sum_{j=1}^{L} p_j\, x_j$.

The seminal contribution of Kahneman and Tversky (Kahneman and Tversky (1979)) noted the experimentally verified observation that agents tend to have a marginal utility of loss that is no less- if not higher-than the marginal utility of gain. Incorporating this idea in our analysis, we define the following.

A **linear utility profile** is an array in $(\mathbb{R}^2_{++})^L$, i.e., u = <$(u_j^-,\ u_j^+)$| j = 1, …, L>, satisfying $u_j^- \geq u_j^+$, for all j = 1, …, L.

The **expected utility** of a unit risky investment pair denoted by Eu(x, p) = $\sum_{j=1}^{L} p_j[u_j^- \min\{x_j, 0\} + u_j^+ \max\{x_j, 0\}]$.

For some positive integer K, let <$x^{(k)}$| k = 1, …, K> be an array of unit return vectors.

An **investment plan** is a (column) vector $\alpha \in \mathbb{R}^K$ where for each k∈{1, …, K}, the kth coordinate of α denoted α_k is the amount of money invested in the kth unit return vector.

Given an **array of unit return vectors** <$x^{(k)}$| k = 1, …, K> the **return** from an investment plan α in SON j is $\sum_{k=1}^{K} \alpha_k x_j^{(k)}$.

The well-known version of the Arbitrage Theorem that is applicable for a linear utility profile satisfying $u_j^- = u_j^+$, for all j = 1, …, L is available as Theorem 4.4.1 in Ross (2004). In the following theorem we extend the result- to the extent possible- when linear utility profiles may exhibit loss aversion.

Weak Arbitrage theorem incorporating loss aversion: Let <$x^{(k)}$| k = 1, …, K> be an array of unit return vector.

(i) Given a linear utility profile u = <(u_j^-, u_j^+) | j = 1, …, L> if there does not exist a probability vector p such that for all k = 1, …, K, Eu($x^{(k)}$, p) = 0, then there exists an investment plan α such that utility of return from α is strictly positive in SON j for all j∈{1, …, L}.

(ii) If there exists an investment portfolio α such that the return from α is strictly positive in SON j for all j∈{1, …, L}, then for each j∈{1, …, L} there exists a non-degenerate left-closed right open interval I_j with min{a | a∈I_j} = 1, such that for any linear utility profile u = <(u_j^-, u_j^+) | j = 1, …, L> satisfying $\frac{u_j^-}{u_j^+}$ ∈I_j for all j∈{1, …, L}, the following holds: there does not exist any probability vector p such that for all k = 1, …, K, Eu($x^{(k)}$, p) = 0.

Proof: (i) Let A be the (K+1)×L matrix whose (k, j)th term for k∈{1, …, K} and j∈{1, …, L} is $u_j^- \min\left\{x_j^{(k)},0\right\} + u_j^+ \min\left\{x_j^{(k)},0\right\}$ and for j∈{1, …, L} the (K+1, j)th term is 1.

Then by Farkas's lemma, the "non-existence" of p∈$\mathbb{R}_+^L$ satisfying Ap = $\binom{0}{1}$ where 0 is the K dimensional column vector all whose entries are 0, implies that there exists α∈$\mathbb{R}^K$ and a real number β such that $\sum_{k=1}^K \alpha_k [u_j^- \min\left\{x_j^{(k)},0\right\} + u_j^+ \min\left\{x_j^{(k)},0\right\}]$ + β ≥ 0 for all j∈{1 , …, L}, β < 0, but never both.

Thus, there exists α∈$\mathbb{R}^K$ such that $\sum_{k=1}^K \alpha_k [u_j^- \min\left\{x_j^{(k)},0\right\} + u_j^+ \max\left\{x_j^{(k)},0\right\}] > 0$ for all j ∈{1, …, L}.

Since, $u_j^- \geq u_j^+$, for all j = 1, …, L, $\sum_{k=1}^K \alpha_k [u_j^- \min\left\{x_j^{(k)},0\right\} + u_j^+ \max\left\{x_j^{(k)},0\right\}] \leq \sum_{k=1}^K \alpha_k [u_j^+ \min\left\{x_j^{(k)},0\right\} + u_j^+ \max\left\{x_j^{(k)},0\right\}] = u_j^+ \left(\sum_{k=1}^K \alpha_k [\min\left\{x_j^{(k)},0\right\} + \max\left\{x_j^{(k)},0\right\}] \right) = u_j^+ (\sum_{k=1}^K \alpha_k x_j^{(k)})$ for all j∈{1 , …, L}.

Thus, $\sum_{k=1}^K \alpha_k [u_j^- \min\left\{x_j^{(k)},0\right\} + u_j^+ \max\left\{x_j^{(k)},0\right\}] > 0$ for all j∈{1 , …, L} implies $u_j^+ (\sum_{k=1}^K \alpha_k x_j^{(k)}) > 0$ for all j∈{1 , …, L}.

Since, $u_j^+ > 0$, for all j∈{1 , …, L}, it must be the case that $\sum_{k=1}^K \alpha_k x_j^{(k)} > 0$ for all j∈{1 , …, L}.

(ii) Now suppose, there exists an investment portfolio α such that $\sum_{k=1}^K \alpha_k x_j^{(k)} > 0$ for all j∈{1, …, L}.

Thus, $\sum_{k=1}^K \alpha_k [\min\left\{x_j^{(k)},0\right\} + \max\left\{x_j^{(k)},0\right\}] = \sum_{k=1}^K \alpha_k \min\left\{x_j^{(k)},0\right\} + \sum_{k=1}^K \alpha_k \max\left\{x_j^{(k)},0\right\} > 0$ for all j∈{1 , …, L}.

Hence, $\sum_{k=1}^K \alpha_k \max\left\{x_j^{(k)},0\right\} > - \sum_{k=1}^K \alpha_k \min\left\{x_j^{(k)},0\right\} \geq 0$ for all j∈{1 , …, L}.

For j∈{1 , …, L}, let I_j = [1, + ∞) if $- \sum_{k=1}^K \alpha_k \min\left\{x_j^{(k)},0\right\} = 0$ and I_j = $\left[1, \frac{\sum_{k=1}^K \alpha_k \max\left\{x_j^{(k)},0\right\}}{- \sum_{k=1}^K \alpha_k \min\left\{x_j^{(k)},0\right\}}\right)$ if $- \sum_{k=1}^K \alpha_k \min\left\{x_j^{(k)},0\right\} > 0$.

Clearly I_j is a non-degenerate left closed and right open interval in $\mathbb{R}_{++}$ satisfying min{a | a∈I_j} = 1, for all j∈{1 , …, L}.

For j∈{1 , …, L}, let u_j^-, $u_j^+ > 0$ be such that $\frac{u_j^-}{u_j^+}$∈ I_j. Thus $u_j^- \geq u_j^+$, for all j∈{1 , …, L}.

Let u = <(u_j^-, u_j^+) | j = 1, …, L> be a linear utility profile.

Thus, $u_j^+ \left(\sum_{k=1}^K \alpha_k \max\left\{x_j^{(k)},0\right\} \right) > u_j^- \left(\sum_{k=1}^K \alpha_k \min\left\{x_j^{(k)},0\right\} \right)$ for all j∈{1 , …, L}, i.e., $u_j^+ (\sum_{k=1}^K \alpha_k \max\left\{x_j^{(k)},0\right\}) + u_j^- (\sum_{k=1}^K \alpha_k \min\left\{x_j^{(k)},0\right\}) > 0$, for j∈{1 , …, L}.

Let β < 0 be such that - β = min{$u_j^+ (\sum_{k=1}^K \alpha_k \max\left\{x_j^{(k)},0\right\}) + u_j^- (\sum_{k=1}^K \alpha_k \min\left\{x_j^{(k)},0\right\})$ | j∈{1 , …, L}}.

Thus, β < 0 and $u_j^+ (\sum_{k=1}^K \alpha_k \max\left\{x_j^{(k)},0\right\}) + u_j^- (\sum_{k=1}^K \alpha_k \min\left\{x_j^{(k)},0\right\}) + \beta \geq 0$ for all j∈{1 , …, L}, i.e., β < 0 and $\sum_{k=1}^K \alpha_k [u_j^- \min\left\{x_j^{(k)},0\right\} + u_j^+ \min\left\{x_j^{(k)},0\right\}]$ + β ≥ 0 for all j∈{1 , …, L}.

Let A be the (K+1)×L matrix whose (k, j)th term for k∈{1, …, K} and j∈{1, …, L} is $u_j^- \min\left\{x_j^{(k)},0\right\} + u_j^+ \min\left\{x_j^{(k)},0\right\}$ and for j∈{1, …, L} the (K+1, j)th term is 1.

Then, by Farkas's lemma, there does not exist a column vector p∈$\mathbb{R}_+^L$ such that Ap = $\binom{0}{1}$ where 0 is the K dimensional column vector all whose entries are 0, i.e., there does not exist any probability vector p such that for all k = 1, …, K, Eu($x^{(k)}$, p) = 0. Q.E.D.

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