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On the Twistability of Partially Coherent, Schell-Model Sources

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Abstract: The problem of assessing the twistability of a given bonafide cross spectral density is here tackled for the class of Schell-model sources whose shift invariant degree of coherence is represented through a real and symmetric function, say $\mu(-\boldsymbol{r}) = \mu(\boldsymbol{r})$. On employing an abstract operatorial language, the problem of the determination of the, highly degenerate spectrum of a twisted operator $\hat{W}_u$ is addressed through a modal analysis based on the complete knowledge of the spectrum of the *sole* twist operator $\hat{T}_u$ found by Simon and Mukunda [J. Opt. Soc. Am. A **15,** 1361 (1998)]. To this end, the evaluation of the complete tensor of the matrix elements $\langle n', \ell' | \hat{W}_u | n, \ell \rangle$ is carried out within the framework of the so-called *extended Wigner distribution function*, a concept recently introduced by Van Valkenburgh [J. Mod. Opt. **55,** 3537 - 3549 (2008)]. As a nontrivial application of the algorithm here developed, the analytical determination of the spectrum of saturated twisted astigmatic Gaussian Schell-model sources is also presented.

Keywords: mathematical physics; classical optics; classical coherence theory; statistical optics

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1. Introduction

One of the most charming and still not completely explored topic of classical coherence theory is, without doubts, that dealing with the so-called *twisted sources* and the beams they generate. In a celebrated 1993 paper [1], Simon and Mukunda opened up "a new dimension" in coherence theory [2], through the introduction of a new, "genuinely two-dimensional" [1] axially symmetric twist phase to be imposed to "ordinary" Gaussian Schell-model (GSM) beams. In this way, the class of the celebrated *Twisted Gaussian Schell-model* (TGSM henceforth) beams was born. The multiplication of the cross-spectral density (CSD henceforth) of a standard GSM source by the new chiral phase term, produced a CSD whose striking properties continue to be explored, both theoretically as well as experimentally, after the original Simon/Mukunda paper [1] as well as a series of important follow-up works [2–4]. TGSM beams have received a considerable attention in the last thirty years, as could be witnessed by the, practically countless list of papers they inspired. Simon and Mukunda also posed the attention in Ref. [5] on a fundamental theoretical topic: to "stick" a twist phase to a typical bonafide untwisted CSD does not, in general, lead to a bonafide twisted CSD. Even when an "ordinary" GSM source is going to be twisted, it is known that the strength of the twist phase must satisfy severe numerical limitations, which are imposed by the coherence features of the *sole* spectral degree of the untwisted GSM source [1]. Important results were found in [6] for the class of all shape-invariant anisotropic GSM (AGSM henceforth) beams. In a sense, the above paper closed the fundamental question about the role played by, as well as the conditions under which the twist phase is viewed as an operator able to map *untwisted* bonafide CSD's onto *twisted* bonafide CSD within the general context of the so-called Gaussian optics [5,7].

In 2015, what it could be called the "twist mapping problem", was addressed for a considerably larger class, with respect to AGSM sources, of bonafide, still Schell-model CSD's [8]. In order to introduce our readers the main definitions and notations of the present paper, consider the mapping problem from a perspective as general as possible. Given a bonafide CSD associated to a planar partially

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coherent source, say $W(\boldsymbol{r}_1, \boldsymbol{r}_2)$, we shall ask under with conditions the new CSD, say $W_u(\boldsymbol{r}_1, \boldsymbol{r}_2)$, defined by

$$W_u(\boldsymbol{r}_1, \boldsymbol{r}_2) \;=\; W(\boldsymbol{r}_1, \boldsymbol{r}_2)\, \exp(-\mathrm{i} u \boldsymbol{r}_1 \times \boldsymbol{r}_2)\,, \qquad u > 0\,, \tag{1}$$

still represents a bonafide CSD. Here, u is the twist strength (which will be taken as positive), while the simbol $\times$ represents the z-component of the cross product between the transverse position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ across the source plane, which is orthogonal to the z-axis. In other words, the symbol $\boldsymbol{r}_1 \times \boldsymbol{r}_2$ is a "notational shortcut" for the mixed product $\boldsymbol{r}_1 \times \boldsymbol{r}_2 \cdot \hat{\boldsymbol{k}}$, with $\hat{\boldsymbol{k}}$ being the unit vector orthogonal to the source plane. Later, such formalism will be employed. In Ref. [8], the mapping problem was formally solved for the whole class of CSD's of the following, Schell-model type:

$$W(\boldsymbol{r}_1, \boldsymbol{r}_2) \;=\; \tau^*(\boldsymbol{r}_1)\, \tau(\boldsymbol{r}_2)\, \mu(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)\,, \tag{2}$$

where $\tau(\cdot)$ denotes the transmission function of an arbitrary (complex) amplitude filter and $\mu(\cdot)$ represents a, shift invariant, spectral degree of coherence endowed with *radial symmetry*. The necessary and sufficient condition for W to be bonafide, follows from Bochner's theorem, which implies the Fourier transform of μ to be identically nonnegative. The idea of Ref. [8] to solve the twist mapping problem was based on the fact that the twisted version of the CSD into Eq. (2) will be bonafide if and only if the corresponding *uniform* CSD, obtained by removing the complex filter $\tau(\boldsymbol{r})$, i.e.,

$$W_u(\boldsymbol{r}_1, \boldsymbol{r}_2) \;=\; \mu(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)\, \exp(-\mathrm{i} u\, \boldsymbol{r}_1 \times \boldsymbol{r}_2)\,, \qquad u > 0\,, \tag{3}$$

is bonafide too. Moreover, the solution of the twist problem was also critically based on the conjecture that the Wolf coherent modes [9] of the twisted uniform kernel $W_u(\boldsymbol{r}_1, \boldsymbol{r}_2)$ coincides with the so-called Laguerre-Gauss (LG henceforth) beams, say $\Phi_{j,m}(\boldsymbol{r})$, defined as follows [9]:

$$\Phi_{j,m}(\boldsymbol{r}) \;=\; \sqrt{\frac{u}{\pi}} \left[\frac{(j - |m|)!}{(j + |m|)!}\right]^{\frac{1}{2}} (r\sqrt{u})^{2|m|}\, \exp(\mathrm{i} 2 m \varphi)\, L_{j-|m|}^{2|m|}(u r^2)\, \exp\left(-\frac{u r^2}{2}\right). \tag{4}$$

Here, $j = 0, 1/2, 1, \ldots$ represents the infinite sequence of semi-integer and integer nonnegative numbers, while $m = -j, -j+1, \ldots, j-1, j$. In this way, it is sure that both radial $(j - |m|)$ and angular $(2|m|)$ indeces of Laguerre polynomials take on nonnegative *integer* values. Once the modal structure was established, in Ref. [8] the complete spectrum of W_u was found. Then, in order to assess the nonnegative definitiness of the twisted source, it was sufficient (but also necessary) to impose each eigenvalue to be nonnegative. That is all.

The results presented in Ref. [8] were based on a conjecture which was not fully justified from a rigorous mathematical viewpoint. One of the aims of the present paper is to provide such a justification. In doing so, we shall address the twist mapping problem for a class of Schell-model CSD's considerably more general than those defined in Eq. (2), namely those for which the spectral degree of coherence μ is not necessarily a radial function, but rather satisfies

$$\mu(-\boldsymbol{r}) \;=\; \mu(\boldsymbol{r})\,, \qquad \boldsymbol{r} \in \mathbb{R}^2\,, \tag{5}$$

which, due to the mandatory Hermiticity of W_u, implies that $\mu \in \mathbb{R}$. Our strategy is based on the approach outlined in [5], i.e., to interpret a typical CSD $W(\boldsymbol{r}_1, \boldsymbol{r}_2)$ as the position representation of a Hermitian operator, say $\hat{W}$, acting on elements, say $|\psi\rangle$, of the Hilbert space $\mathcal{H}$ of squared-integrable wavefunctions. To help our readers, it is worth giving a brief introduction of the formalism which will be used throughout the present paper. According to the notations of Ref. [5], let $|\psi\rangle$ the abstract state (ket) which corresponds, in the position representation, to the wavefunction $\psi(\boldsymbol{r})$, i.e.,

$$\psi(\boldsymbol{r}) \;=\; \langle \boldsymbol{r} | \psi \rangle\,, \tag{6}$$

where, as usual, the symbol $|\boldsymbol{r}\rangle$ denotes the eigenket of the position operator $\hat{\boldsymbol{r}}$, defined as

$$\hat{\boldsymbol{r}}|\boldsymbol{r}\rangle \;=\; \boldsymbol{r}|\boldsymbol{r}\rangle\,. \tag{7}$$

In this way, the connection between the CSD $W(\boldsymbol{r}_1,\boldsymbol{r}_2)$ and the associate abstract operator $\hat{W}$ will be given through its matrix elements, according to

$$W(\boldsymbol{r}_1,\boldsymbol{r}_2) \;=\; \langle \boldsymbol{r}_1|\hat{W}|\boldsymbol{r}_2\rangle\,. \tag{8}$$

The operator $\hat{W}$ acts on a ket $|\psi\rangle \in \mathcal{H}$ to generate another element of $\mathcal{H}$, which will denoted by $\hat{W}|\psi\rangle$, whose position representation can be obtained as follows:

$$\langle \boldsymbol{r}|\hat{W}|\psi\rangle \;=\; \int_{\mathbb{R}^2} \mathrm{d}^2\rho\,\langle \boldsymbol{r}|\hat{W}|\boldsymbol{\rho}\rangle\langle\boldsymbol{\rho}|\psi\rangle \;=\; \int_{\mathbb{R}^2} \mathrm{d}^2\rho\, W(\boldsymbol{r},\boldsymbol{\rho})\psi(\boldsymbol{\rho})\,, \tag{9}$$

where the formal completeness condition has been inserted:

$$\mathbb{I} \;=\; \int_{\mathbb{R}^2} \mathrm{d}^2\rho\,|\boldsymbol{\rho}\rangle\langle\boldsymbol{\rho}|\,, \tag{10}$$

with $\mathbb{I}$ being the identity operator. The other ingredient of our mathematical recipes is the commutation relationship between two operators. In particular, in Ref. [8] it was shown that, for radial spectral degree of coherence μ, the operator, say $\hat{W}_u$, associated with the twisted CSD defined into Eq. (3) and the operator, say $\hat{T}_u$, associated to the sole twist factor $T_u(\boldsymbol{r}_1,\boldsymbol{r}_2) = \exp(-\mathrm{i}u\boldsymbol{r}_1 \times \boldsymbol{r}_2)$, *do commute*, i.e.,

$$\hat{W}_u\,\hat{T}_u \;=\; \hat{T}_u\,\hat{W}_u\,. \tag{11}$$

It is not difficult to prove that Eq. (11) holds for any nonradial function $\mu(\boldsymbol{r})$ which satisfies Eq. (5). For the sake of simplicity, the proof has been confined into Appendix 6.

2. Commuting Operators and Spectral Degeneration

An important result proved in Ref. [5] has to do with the spectral theorem for the twist operator $\hat{T}_u$. In particular, within the abstract language of Hilbert space operators, Simon & Mukunda proved that

$$\hat{T}_u|j,m\rangle \;=\; \frac{2\pi}{u}\,(-1)^{j-m}\,|j,m\rangle\,, \qquad \left(\begin{array}{c} j = 0,1/2,1,\ldots \\ m = -j,-j+1,\ldots,j-1,j \end{array}\right), \tag{12}$$

where the eigenket $|j,m\rangle$ position representation coincides with the LG beam of Eq. (4), i.e.,

$$\langle \boldsymbol{r}|j,m\rangle \;=\; \Phi_{j,m}(\boldsymbol{r})\,, \qquad \left(\begin{array}{c} j = 0,1/2,1,\ldots \\ m = -j,-j+1,\ldots,j-1,j \end{array}\right). \tag{13}$$

From Eq. (12), it turns out that the spectrum of the twist operator $\hat{T}_u$ is highly degenerate. It is worth introducing two auxiliary integer parameters, say n and ℓ, defined as $n = j + m$, $\ell = j - m$, so that

$$|j,m\rangle \;=\; \left|\frac{n+\ell}{2},\frac{n-\ell}{2}\right\rangle, \qquad n,\ell = 0,1,2\ldots \tag{14}$$

In the following, given the one-to-one correspondance $(j,m) \iff (n,\ell)$, the eigenket $|j,m\rangle$ will be denoted by $|n,\ell\rangle$ in place of Eq. (14), without confusion. With these notations, Eq. (12) will be recast as

$$\hat{T}_u|n,\ell\rangle \;=\; \frac{2\pi}{u}\,(-1)^{\ell}\,|n,\ell\rangle\,, \qquad n,\ell = 0,1,2,\ldots \tag{15}$$

The high spectral degeneration of the twist operator represents one of the principal technical problem to be addressed, for the conditions under which a given bonafide Schell-model CSD can or cannot

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be twistable, to be established. In particular, the spectrum of the twist operator consists in only two eigenvalues, namely $2\pi/u$ and $-2\pi/u$, each of them infinitely degenerate. More precisely, the Hilbert space $\mathcal{H}$ can be thought of as the union of two subspaces, say $\mathbb{V}_\pm$, defined as follows:

$$\begin{cases} \mathbb{V}_+ &= \mathrm{span}\{|n,2k\rangle\}_{n,k=0}^{\infty}\,, \\ \mathbb{V}_- &= \mathrm{span}\{|n,2k+1\rangle\}_{n,k=0}^{\infty}\,, \end{cases} \tag{16}$$

i.e., each of them generated by all eigenkets corresponding to the same value of the eigenvalue. It is a well known fact that, since the operators $\hat{W}_u$ and $\hat{T}_u$ commute, the ket $\hat{W}_u|n,\ell\rangle$ must itself be an eigenket of $\hat{T}_u$ corresponding to the eigenvalue $\frac{2\pi}{u}(-1)^\ell$. In fact,

$$\hat{T}_u\hat{W}_u|n,\ell\rangle \;=\; \hat{W}_u\hat{T}_u|n,\ell\rangle \;=\; \frac{2\pi}{u}(-1)^\ell\,\hat{W}_u|n,\ell\rangle\,, \qquad n,\ell=0,1,2,\ldots \tag{17}$$

In Ref. [8], it was conjectured that, for any radial degree of coherence μ, the modes of the twisted CSD W_u of Eq. (3), coincide with the modes of T_u. One of the scopes of the present paper is also to provide a mathematical justification of it. In order to better clarify the terms of the problem, consider the following case, namely $\mu(x,y)=\exp(-x^2-\chi y^2)$, where χ represents a real parameter which will be let run within the interval $[1,\infty[$ (the choice $\chi=1$ corresponds to a Gaussian spectral degree of coherence, as for example in the classical GSM case). Consider then the fundamental state $|0,0\rangle$, such that

$$\langle \boldsymbol{r}|0,0\rangle = \Phi_{0,0}(\boldsymbol{r}) \;=\; \sqrt{\frac{u}{\pi}}\,\exp\left(-u\frac{x^2+y^2}{2}\right). \tag{18}$$

Then, it turns out [1]

$$\begin{aligned}\langle \boldsymbol{r}|\hat{W}_u|0,0\rangle \;&=\\ =\;&\frac{2\pi}{\sqrt{(u+2)(u+2\chi)}}\sqrt{\frac{u}{\pi}}\,\exp\left[-\frac{u(x-\mathrm{i}y)(u^2(x+iy)^2+4(x+\mathrm{i}y)+4u(x+\mathrm{i}\chi y))}{2(2+u)(u+2\chi)}\right],\end{aligned} \tag{19}$$

and it is not difficult to check that such wavefunction corresponds to an eigenket of the twist kernel $\hat{T}_u$, with eigenvalue equal to $2\pi/u$. Moreover, in the limit of $\chi\to 1^+$ (radial symmetry) the state $\hat{W}_u|0,0\rangle$ becomes proportional to $|0,0\rangle$, as conjectured in [8]. More precisely,

$$\lim_{\chi\to 1}\,\hat{W}_u|0,0\rangle \;=\; \frac{2\pi}{u+2}\,|0,0\rangle\,. \tag{20}$$

If $\chi>1$, the state $\hat{W}_u|0,0\rangle$ is expected to belong to the subspace $\mathbb{V}_+$ defined into Eq. (16). Accordingly, it is natural to write

$$\hat{W}_u|0,0\rangle \;=\; \sum_{n'=0}^{\infty}\sum_{\ell'=0}^{\infty}\langle n',\ell'|\hat{W}_u|0,0\rangle\,|n',\ell'\rangle\,, \tag{21}$$

where, since $|0,0\rangle\in\mathbb{V}_+$, only *even* values of the index ℓ' would be involved into the double series in Eq. (21). As it will be discussed in Sec. 4, the possibility of finding such a representation would allow, in principle, to solve the degenerate eigenvalue problem for the $\hat{W}_u$ operator and, consequently, to assess its (semi)positive definitiness.

[1] We did this evaluation with the help the laterst release (14.1) of *Mathematica*.

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The above example should be enough to convey our readers at least the complexity behind the "twistability problem," i.e., to find the necessary and sufficient conditions for the kernel $W_u(\boldsymbol{r}_1, \boldsymbol{r}_2)$ defined as

$$W_u(\boldsymbol{r}_1, \boldsymbol{r}_2) \;=\; \mu(\boldsymbol{r}_1 - \boldsymbol{r}_2)\, \exp(-\mathrm{i} u\, \boldsymbol{r}_1 \times \boldsymbol{r}_2)\,, \qquad u > 0\,, \tag{22}$$

to represent a bonafide CSD, under the sole hypothesis (5).

The strategy pursued in the present work is to extract the complete representation of the twisted operator $\hat{W}_u$ in terms of the orthonormal basis of the twist operator $\hat{T}_u$, similarly as done in Eq. (21), but for a typical state $|n, \ell\rangle$. To this end, the main technical problem is the evaluation of the typical matrix element $\langle n', \ell' | \hat{W}_u | n, \ell \rangle$. In the next section, it will be shown that the matrix element can be expressed in terms of the inner product between the degree of coherence $\mu(\boldsymbol{r})$ and a suitable LG mode of Eq. (4). To this end, the results of an important paper, published in 2008 by Van Valkenburgh [10], will be employed. In spite of the strong mathematical character of all steps, we preferred not to give them the aspect of an appendix, but rather, due to their key role, to arrange them into a section which indeed represents the technical core of the present paper.

3. Evaluation of $\hat{W}_u$'s Matrix Elements

The typical matrix element $\langle n', \ell' | \hat{W}_u | n, \ell \rangle$ is first explicited as a 4-dimensional integral, i.e.,

$$\begin{aligned} \langle n', \ell' | \hat{W}_u | n, \ell \rangle \;&=\; \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathrm{d}^2 r_1\, \mathrm{d}^2 r_2\, \langle n', \ell' | \boldsymbol{r}_1 \rangle\, \langle \boldsymbol{r}_1 | \hat{W}_u | \boldsymbol{r}_2 \rangle\, \langle \boldsymbol{r}_2 | n, \ell \rangle \;= \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathrm{d}^2 r_1\, \mathrm{d}^2 r_2\, W_u(\boldsymbol{r}_1, \boldsymbol{r}_2)\, \Phi^*_{j',m'}(\boldsymbol{r}_1)\, \Phi_{j,m}(\boldsymbol{r}_2)\,, \end{aligned} \tag{23}$$

where the pairs $(j, m) = \left(\frac{n+\ell}{2}, \frac{n-\ell}{2}\right)$ and $(j', m') = \left(\frac{n'+\ell'}{2}, \frac{n'-\ell'}{2}\right)$ have been re-introduced for the sake of comodity. In order to evaluate the integral into Eq. (23), the following properties of LG modes will also be employed:

$$\begin{cases} \Phi^*_{j,m}(\boldsymbol{r}) \;=\; \Phi_{j,-m}(\boldsymbol{r})\,, \\ \\ \Phi_{j,m}(\boldsymbol{r}) \;=\; (-1)^{2m}\, \Phi_{j,m}(-\boldsymbol{r})\,, \end{cases} \tag{24}$$

where $2m \in \mathbb{Z}$ and, similarly, for the pair (j', m'). On substituting from the second row of Eq. (24) into Eq. (23) and on letting $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, simple algebra gives

$$\langle n', \ell' | \hat{W}_u | n, \ell \rangle \;=\; (-1)^{2m} \int \mathrm{d}^2 r\, \mu(\boldsymbol{r}) \int \mathrm{d}^2 r_2\, \exp(-\mathrm{i} u \boldsymbol{r} \times \boldsymbol{r}_2)\, \Phi^*_{j',m'}(\boldsymbol{r} + \boldsymbol{r}_2)\, \Phi_{j,m}(-\boldsymbol{r}_2)\,. \tag{25}$$

The inner integral can be recast as a so-called *extended WDF* (EWDF henceforth), a concept introduced in Van Valkenburgh's paper [10]. To this end, a new integration variable, say $\boldsymbol{\rho} = \boldsymbol{r} + 2\boldsymbol{r}_2$, is introduced in place of $\boldsymbol{r}_2$. We have $-\boldsymbol{r}_2 = \dfrac{\boldsymbol{r}}{2} - \dfrac{\boldsymbol{\rho}}{2}$ and $\boldsymbol{r} + \boldsymbol{r}_2 = \dfrac{\boldsymbol{r}}{2} + \dfrac{\boldsymbol{\rho}}{2}$ which, after substituted into Eq. (25), give at once

$$\begin{aligned} &\int \mathrm{d}^2 r_2\, \exp(-\mathrm{i} u \boldsymbol{r} \times \boldsymbol{r}_2)\, \Phi^*_{j',m'}(\boldsymbol{r} + \boldsymbol{r}_2)\, \Phi_{j,m}(-\boldsymbol{r}_2) \;= \\ &= \frac{1}{4} \int \mathrm{d}^2 \rho\, \exp\!\left(-\mathrm{i} \frac{u}{2} \boldsymbol{r} \times \boldsymbol{\rho}\right) \Phi^*_{j',m'}\!\left(\frac{\boldsymbol{r}}{2} + \frac{\boldsymbol{\rho}}{2}\right) \Phi_{j,m}\!\left(\frac{\boldsymbol{r}}{2} - \frac{\boldsymbol{\rho}}{2}\right). \end{aligned} \tag{26}$$

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It should be realized how the $\boldsymbol{\rho}$-integral turns out to be *independent of* the twist parameter u. It is sufficient to change the integration variable $\boldsymbol{\rho}$ into the new variable $\boldsymbol{\xi} = \sqrt{u}\,\boldsymbol{\rho}$ and to taking again Eq. (4) into account. Accordingly, from Eq. (26) we have

$$\begin{aligned}\int \mathrm{d}^2 r_2\, \exp(-\mathrm{i}u\boldsymbol{r}\times\boldsymbol{r}_2)\,\Phi^*_{j',m'}(\boldsymbol{r}+\boldsymbol{r}_2)\,\Phi_{j,m}(\boldsymbol{r}_2) \;=\\ =\frac{(-1)^{2m}}{4}\int \mathrm{d}^2\xi\, \exp(-\mathrm{i}\boldsymbol{R}\times\boldsymbol{\xi})\,\bar{\Phi}^*_{j',m'}\left(\boldsymbol{R}+\frac{\boldsymbol{\xi}}{2}\right)\bar{\Phi}_{j,m}\left(\boldsymbol{R}-\frac{\boldsymbol{\xi}}{2}\right),\end{aligned} \tag{27}$$

where the dimensionless vector $\boldsymbol{R} = \dfrac{\boldsymbol{r}\sqrt{u}}{2}$ has been introduced and the symbol $\bar{\Phi}_{j,m}(\boldsymbol{r})$ stands for $\Phi_{j,m}(\boldsymbol{r})\big|_{u=1}$. The subsequent step consists in moving complex conjugation from the pair (j', m') to the pair (j, m), by using the first of Eq. (24), which gives

$$\begin{aligned}\int \mathrm{d}^2 r_2\, \exp(-\mathrm{i}u\boldsymbol{r}\times\boldsymbol{r}_2)\,\Phi^*_{j',m'}(\boldsymbol{r}+\boldsymbol{r}_2)\,\Phi_{j,m}(\boldsymbol{r}_2) \;=\\ =\frac{(-1)^{2m}}{4}\int \mathrm{d}^2\xi\, \exp(-\mathrm{i}\boldsymbol{R}\times\boldsymbol{\xi})\,\bar{\Phi}^*_{j,-m}\left(\boldsymbol{R}-\frac{\boldsymbol{\xi}}{2}\right),\bar{\Phi}_{j',-m'}\left(\boldsymbol{R}+\frac{\boldsymbol{\xi}}{2}\right).\end{aligned} \tag{28}$$

Now, the mathematical structure of the last $\boldsymbol{\xi}$-integral is the EWDF of the pair $\{\bar{\Phi}_{j,-m}, \bar{\Phi}_{j',-m'}\}$. In fact, the quantity $\boldsymbol{R}\times\boldsymbol{\xi}$ can be explicited as a mixed product as follows:

$$\boldsymbol{R}\times\boldsymbol{\xi}\cdot\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}\times\boldsymbol{R}\cdot\boldsymbol{\xi}, \tag{29}$$

which, once substituted into Eq. (28) gives

$$\begin{aligned}&\int \mathrm{d}^2 r_2\, \exp(-\mathrm{i}u\boldsymbol{r}\times\boldsymbol{r}_2)\,\Phi^*_{j',m'}(\boldsymbol{r}+\boldsymbol{r}_2)\,\Phi_{j,m}(\boldsymbol{r}_2) \;=\\ &=(-1)^{2m}\frac{\pi^2}{4\pi^2}\int \mathrm{d}^2\xi\, \exp\left(-\mathrm{i}\hat{\boldsymbol{k}}\times\boldsymbol{R}\cdot\boldsymbol{\xi}\right)\bar{\Phi}^*_{j,-m}\left(\boldsymbol{R}-\frac{\boldsymbol{\xi}}{2}\right)\bar{\Phi}_{j',-m'}\left(\boldsymbol{R}+\frac{\boldsymbol{\xi}}{2}\right) =\\ &=(-1)^{2m}\,\pi^2\,\mathcal{W}\{\bar{\Phi}_{j,-m}, \bar{\Phi}_{j',-m'}\}(\boldsymbol{R}, \hat{\boldsymbol{k}}\times\boldsymbol{R}).\end{aligned} \tag{30}$$

Here the symbol $\mathcal{W}\{\psi, \phi\}$ denotes the EWDF of the pair $\{\psi, \phi\}$, according to Ref. [10].

To help our readers, the main definitions and properties of the EWDF have briefly been resumed in Appendix 8, where it is also recalled the tight relationship between LG modes and factorized HG modes through the WDF and EWDF operators. In particular, from Eq. (67) we have

$$\begin{cases}\bar{\Phi}_{j,-m}(\boldsymbol{r}) = (-1)^{j-|m|}\,\tilde{W}\{h_{j-m,j+m}\}(\boldsymbol{r}) = (-1)^{\min\{n,\ell\}}\,\tilde{W}\{h_{\ell,n}\}(\boldsymbol{r}),\\ \bar{\Phi}_{j',-m'}(\boldsymbol{r}) = (-1)^{j'-|m'|}\,\tilde{W}\{h_{j'-m',j'+m'}\}(\boldsymbol{r}) = (-1)^{\min\{n',\ell'\}}\,\tilde{W}\{h_{\ell',n'}\}(\boldsymbol{r}),\end{cases} \tag{31}$$

where the function $\tilde{W}\{h_{h,k}\}(\boldsymbol{r})$ is defined into Appendix 8. In the same appendix, the Van Valkenburgh theorem has also been recalled (see Eq. (74)) which, once applied to the present case, gives

$$\mathcal{W}\{\bar{\Phi}_{j,-m}, \bar{\Phi}_{j',-m'}\}(\boldsymbol{R}, \hat{\boldsymbol{k}}\times\boldsymbol{R}) = \frac{(-1)^{\min\{n,\ell\}+\min\{n',\ell'\}}}{\pi}\,\tilde{W}\{h_{\ell,\ell'}\}(2X, -2Y)\tilde{W}\{h_{n,n'}\}(0, 0). \tag{32}$$

Here, for the sake of comodity, we came back again to the pairs (n, ℓ) and (n', ℓ'), in place of (j, m) and (j', m'), respectively, with (X, Y) being the Cartesian representation of the transverse vector$\boldsymbol{R}$. Now, as far as the factor $\tilde{W}\{h_{n,n'}\}(0, 0)$ is concerned, from Eq. (70) it is not difficult to obtain

$$\tilde{W}\{h_{n,n'}\}(0,0) \;=\; (-1)^{\min\{n,n'\}}\,\bar{\Phi}_{\frac{n+n'}{2},\frac{n-n'}{2}}(\boldsymbol{0}) \;=\; \frac{(-1)^n}{\sqrt{\pi}}\,\delta_{n,n'}\,, \tag{33}$$

where, in the last step, use has been made of Eq. (4) evaluated at $u = 1$. Equation (33) is an important result. It implies that the matrix elements of $\hat{W}_u$ are necessarily "diagonal" with respect to the index pair (n, n'), i.e.,

$$\boxed{n \neq n' \Longrightarrow \langle n', \ell' | \hat{W}_u | n, \ell \rangle \;=\; 0} \tag{34}$$

Equation (34) implies that the eigenket expansion of the state $\hat{W}_u|n, \ell\rangle$ will involve only kets of the form $|n, \ell'\rangle$. In other words, the double series expansion in Eq. (21) reduces to a single series, a nice mathematical surprise.

Generally speaking, we have

$$\hat{W}_u|n,\ell\rangle \;=\; \sum_{\ell'=0}^{\infty} \langle n,\ell'|\hat{W}_u|n,\ell\rangle\,|n,\ell'\rangle\,, \qquad\qquad \forall n,\ell \in \mathbb{N}. \tag{35}$$

Moreover, we would also expect that, if $\mu(-\boldsymbol{r}) = \mu(\boldsymbol{r})$, then ℓ' must have the same parity of ℓ. To prove this, it is sufficient to note that the factor $\tilde{W}\{h_{\ell,\ell'}\}(2X, -2Y)$ turns out to be

$$\tilde{W}\{h_{\ell,\ell'}\}(2X,-2Y) \;=\; (-1)^{\min\{\ell,\ell'\}}\,\bar{\Phi}_{\frac{\ell+\ell'}{2},\frac{\ell-\ell'}{2}}(2X,-2Y) \;=\; (-1)^{\min\{\ell,\ell'\}}\,\bar{\Phi}^*_{\frac{\ell+\ell'}{2},\frac{\ell-\ell'}{2}}(2\boldsymbol{R})\,, \tag{36}$$

where the second row of Eq. (24) has been used. Then, on substituting from Eqs. (33) and (36) into Eq. (32), after simple algebra we have

$$\begin{aligned}\mathcal{W}\{\bar{\Phi}_{j,-m},\,\bar{\Phi}_{j',-m'}\}(\boldsymbol{R},\,\hat{\boldsymbol{k}}\times\boldsymbol{R}) \;&=\; \frac{(-1)^{n+\min\{\ell,\ell'\}+\min\{n,\ell\}+\min\{n',\ell'\}}}{\pi\sqrt{\pi}}\,\bar{\Phi}^*_{\frac{\ell+\ell'}{2},\frac{\ell-\ell'}{2}}(2\boldsymbol{R})\,\delta_{n,n'} \;= \\ &=\; \frac{(-1)^{n+\min\{\ell,\ell'\}+\min\{n,\ell\}+\min\{n',\ell'\}}}{\pi\sqrt{u\pi}}\,\Phi^*_{\frac{\ell+\ell'}{2},\frac{\ell-\ell'}{2}}(\boldsymbol{r})\,\delta_{n,n'}\,,\end{aligned} \tag{37}$$

where in the last step use has been made of $2\boldsymbol{R} = \boldsymbol{r}\sqrt{u}$. Finally, on substituting from Eq. (37) into Eq. (30) and on taking into account that $(-1)^{2m} = (-1)^{n-\ell}$, we have

$$\boxed{\langle n',\ell'|\hat{W}_u|n,\ell\rangle \;=\; (-1)^{\ell+\min\{\ell,\ell'\}+\min\{n,\ell\}+\min\{n,\ell'\}}\,\delta_{n,n'}\,\sqrt{\frac{\pi}{u}}\,\int_{\mathbb{R}^2} \mathrm{d}^2 r\,\mu(\boldsymbol{r})\,\Phi^*_{\frac{\ell+\ell'}{2},\frac{\ell-\ell'}{2}}(\boldsymbol{r})} \tag{38}$$

which represents the most important technical result of the present paper. An immediate consequence of Eq. (38) is that, under the hypothesis $\mu(-\boldsymbol{r}) = \mu(\boldsymbol{r})$, it turns out to be

$$\langle n,\ell'|\hat{W}_u|n,\ell\rangle \;\equiv\; 0\,, \tag{39}$$

for *any* pair (ℓ, ℓ') such that $\ell - \ell'$ is an *odd integer*, as shown in Appendix. 7.

In other words, the orthonormal expansion of $\hat{W}_u|n, \ell\rangle$ in Eq. (35) must involve only kets belonging to the subspace $\mathbb{V}_\pm$ containing $|n, \ell\rangle$, as it was conjectured before.

4. On the Twistability of Anisotropic Gaussian Schell-Model Sources

In a seminal 1998 paper, Simon and Mukunda explored the structure of the most general class of paraxial shape-invariant partially coherent Gaussian Schell-model beams [6]. Inspired by the

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achievements obtained by Simon and Mukunda, in the present section we shall deal with the spectral degree of coherence, already considered in Sec. 2, defined by

$$\mu(\boldsymbol{r}) = \exp\left(-x^2 - \chi\, y^2\right), \qquad \chi \geq 1. \tag{40}$$

On using group theory tools, Simon and Mukunda proved that, in order for a twisted *anisotropic* Gaussian source obtained starting from the degree of coherence of Eq. (40) to be bonafide, the following condition (translated in our notation) must be fulfilled:

$$u^2 \leq 4\chi. \tag{41}$$

The scope of the present section is to explore the role of the SM condition (41) inside the spectrum of the corresponding twisted operator $\hat{W}_u$. To this end, all matrix elements of the operator $\hat{W}_u$ obtained by twisting the degree of coherence in Eq. (40) will be evaluated in closed form. First of all, since the matrix elements $\langle n,\ell'|\hat{W}_u|n,\ell\rangle$ turn out to be real for any triplet (n,ℓ,ℓ'), in the following it will supposed $\ell' \geq \ell$. Moreover, all nonvanishing matrix elements correspond to indeces ℓ and ℓ' with the same parity. Accordingly, from Eq. (38) we have

$$\begin{aligned}\langle n,\ell+2M|\hat{W}_u|n,\ell\rangle &= \langle n,\ell|\hat{W}_u|n,\ell+2M\rangle = \\ &= (-1)^{\min\{n,\ell\}+\min\{n,\ell+2M\}}\sqrt{\frac{\pi}{u}}\int_{\mathbb{R}^2} \mathrm{d}^2 r\, \mu(\boldsymbol{r})\, \Phi^*_{\ell+M,M}(\boldsymbol{r}),\end{aligned} \tag{42}$$

where the integral can be explicited, as shown in Appendix 9, as follows:

$$\begin{aligned}\sqrt{\frac{\pi}{u}}\int_{\mathbb{R}^2} \mathrm{d}^2 r\, \mu(\boldsymbol{r})\, \Phi^*_{\ell+M,M} &= 2\pi\, u^M \sqrt{\frac{\ell!}{(\ell+2M)!}} \\ \times \int_0^\infty \mathrm{d}r\, r^{2M+1}\, L_\ell^{2M}(ur^2)\, I_M\left(\frac{r^2}{2}(\chi-1)\right)& \exp\left(-\frac{r^2}{2}(\chi+u+1)\right),\end{aligned} \tag{43}$$

where the symbol $I_n(\cdot)$ denotes the nth-order modified Bessel function of the first kind.

The integral into Eq. (43) can be expressed in closed form terms, starting from a notable expression recently found by Yuri Aleksandrovich Brychkov [11], which involves an important class of bivariate hypergeometric functions, the so-called Appell functions [12, Ch. 16]. First of all, the integration variable change $x = r^2/2$ can be done in Eq. (43), which becomes

$$\begin{aligned}\sqrt{\frac{\pi}{u}}\int_{\mathbb{R}^2} \mathrm{d}^2 r\, \mu(\boldsymbol{r})\, \Phi^*_{\ell+M,M} &= 2\pi\, (2u)^M \sqrt{\frac{\ell!}{(\ell+2M)!}} \\ \times \int_0^\infty \mathrm{d}x\, x^M\, L_\ell^{2M}(2ux)\, I_M((\chi-1)x)& \exp(-(\chi+u+1)x).\end{aligned} \tag{44}$$

Since all subsequent mathematical steps turn out to be not trivial, cumbersome, and somewhat boring, they have been confined into Appendix 10, where it is proved that, for $a > b \geq 0$, and for $n, m \in \mathbb{N}$,

$$\boxed{\begin{aligned}&\int_0^\infty \mathrm{d}x\, x^m\, \exp(-a\,x)\, I_m(b\,x)\, L_n^{2m}(c\,x) = \\ &= \left(\frac{b}{2}\right)^m \frac{(2m+n)!}{(m+n)!}\,\frac{(-bc/2)^n}{(a^2-b^2)^{n+m+1/2}}\, C_n^{-m-n}\left(\frac{a^2-b^2-ac}{bc}\right)\end{aligned}} \tag{45}$$

where the symbol $C_n^\lambda(\cdot)$ denotes the so-called ultraspherical, or Gegenbauer polynomials [12].

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Then, on substituting from Eq. (45) into Eq. (42), we have

$$\boxed{\begin{aligned}\langle n,\ell+2M|\hat{W}_u|n,\ell\rangle \;=\; \langle n,\ell|\hat{W}_u|n,\ell+2M\rangle \;=\; \frac{2\pi\,(-1)^{\min\{n,\ell\}+\min\{n,\ell+2M\}}}{\sqrt{(u+2)(u+2\chi)}}\\ \times\frac{\sqrt{\ell!\,(\ell+2M)!}}{(\ell+M)!}\left[\frac{u(\chi-1)}{(u+2)(u+2\chi)}\right]^{l+M} C_\ell^{-\ell-M}\left(\frac{u^2-4\chi}{2u(\chi-1)}\right)\end{aligned}} \tag{46}$$

Last equation constitutes another important result of the present paper, together with Eq. (38). Although with a more technical character, with respect to the latter, Eq. (46) unveils the whole information about the coherence features of the twisted, uniform source obtained from the anisotropic degree of coherence (40). In particular, the role played by the inequality (41) can be grasped from the matrix element arrangement. According to Simon and Mukunda, and to what has been exposed here so far, we assert that, under the condition $u^2 \leq 4\chi$, *any* twisted Schell-model source of the form

$$W(\boldsymbol{r}_1,\boldsymbol{r}_2) \;=\; \tau(\boldsymbol{r}_1)\,\tau^*(\boldsymbol{r}_2)\,\exp(-(x_1-x_2)^2-\chi\,(y_1-y_2)^2)\,\exp(-\mathrm{i}u\boldsymbol{r}_1\times\boldsymbol{r}_2)\,, \tag{47}$$

will be bonafide, irrespective the choice of the amplitude filter $\tau(\boldsymbol{r})$. That Eq. (41) represents a *necessary* condition can be proved by imposing that all “diagonal” matrix elements of the operator corresponding to the uniform CSD obtained from Eq. (47) by letting $\tau \equiv 1$, i.e., $\langle n,\ell|\hat{W}_u|n,\ell\rangle$, to be nonnegative, $\forall n,\ell \in \mathbb{N}^2$.

Proving such a statement is not difficult, thanks to the analytical form of the matrix elements given into Eq. (46) which, when it is written for $M=0$, gives at once

$$\langle n,\ell|\hat{W}_u|n,\ell\rangle \;=\; \frac{2\pi}{\sqrt{(u+2)(u+2\chi)}}\left[\frac{u(\chi-1)}{(u+2)(u+2\chi)}\right]^{\ell} C_\ell^{-\ell}\left(\frac{u^2-4\chi}{2u(\chi-1)}\right). \tag{48}$$

The first two factors are inherently nonnegative, due to the hypothesis $\chi \geq 1$. To find the conditions under which also the ultrapsherical polynomial is nonnegative ($\forall\ell \geq 0$), we shall employ a recent theorem proved by Driver and Duren [13,14], which states that all zeros of the polynomials $C_\nu^\lambda(x)$ are purely imaginary (zero included), provided that $\lambda < 1-\nu$. This, together with the elementary ultraspherical polynomials asymptotics [12, formula 18.5.10],

$$C_\ell^{-\ell}(x) \;\sim\; (-2x)^\ell \qquad\qquad |x| \to \infty\,, \tag{49}$$

confirms that, in order for the matrix elements into Eq. (48) to be identically nonnegative, the argument of the ultraspherical polynomial must be nonpositive $\forall\ell \geq 0$: the Simon Mukunda inequality in Eq. (41) is *necessary* for $\hat{W}_u$ to be semidefinite positive.

What could be surprisingly, at least for the present case, is that the same condition turns out to be also sufficient for this, as proved in [6] for the particular case of the ATGSM beams. Several numerical simulations, again carried out via *Mathematica*, have been performed to check such an intriguing result within the new framework of the matrix element representation of $\hat{W}_u$. Rather than annoying our readers with a dull description of such simulations, our attention will be focused on a single case, in which analytical results can be found. Before doing this, it is worth resuming the terms of the problem, in consideration of what shown in the preceding sections: due to the high spectral degeneration of the $\hat{T}_u$ operator, the action of $\hat{W}_u$ on a typical $\hat{T}_u$ eigenstate $|n,\ell\rangle$ produces the state $\hat{W}_u|n,\ell\rangle$, which belongs to $\mathbb{V}_+$ ($\mathbb{V}_-$) if ℓ is an even (odd) integer. Thanks to the “diagonal” character of the matrix elements (expressed by Eq. (38), Eq. (21) reduces to Eq. (35), which dramatically restricts the (infinite) set of the eigenstates involved in the representation of $\hat{W}_u|n,\ell\rangle$. The complete knowledge of the symmetric matrix $\{\langle n,\ell'|\hat{W}_u|n,\ell\rangle\}_{\ell,\ell'=0}^{\infty}$ obtained through Eq. (21), would allow, in principle, to solve such a degenerate eigenvalue problem.

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As anticipated, we shall limit ourselves to the so-called *saturated case*, namely $u^2 = 4\chi$, which corresponds to the maximum twist strength admissible for a given χ. Then, on letting $u = 2\sqrt{\chi}$ the matrix element (46) can be rearranged as follows:

$$\begin{aligned}\langle n, \ell + 2M|\hat{W}_u|n, \ell\rangle &= \langle n, \ell|\hat{W}_u|n, \ell + 2M\rangle = \frac{\pi}{\chi^{1/4}\,(1+\sqrt{\chi})} \\ &\times (-1)^{\min\{n,\ell\}+\min\{n,\ell+2M\}} \frac{\sqrt{\ell!\,(\ell+2M)!}}{(\ell+M)!}\,\xi^{l+M}\,C_\ell^{-\ell-M}(0)\,,\end{aligned} \tag{50}$$

where the real parameter ξ, defined as

$$\xi = \frac{1}{2}\,\frac{\sqrt{\chi}-1}{\sqrt{\chi}+1}\,, \tag{51}$$

is such that $\xi \in [0, \frac{1}{2}[$. The structure of Eq. (50) is particularly interesting. First of all, due to the fact that the Gegenbauer polynomial $C_\ell^\lambda(x)$ has the same parity of the index ℓ, all elements corresponding to odd values of ℓ are necessarily null. More importantly, on again re-introducing the index $\ell' = \ell + 2M$, the matrix element into (50) can be *factorized* as follows:

$$\langle n, \ell|\hat{W}_u|n, \ell'\rangle = \phi_{n,\ell}\,\phi_{n,\ell'}\,, \tag{52}$$

where

$$\phi_{n,\ell} = \sqrt{\frac{\pi}{\chi^{1/4}\,(1+\sqrt{\chi})}} \times \begin{cases} (-1)^{\min\{n,\ell\}}\,\dfrac{\sqrt{\ell!}}{(\ell/2)!}\,\xi^{\ell/2}\,, & \ell \text{ even}\,, \\ \\ 0\,, & \ell \text{ odd}\,, \end{cases} \tag{53}$$

and use has been made of the fact that [12, Table 18.6.1],

$$C_{2n}^{(\lambda)}(0) = (-1)^n\,\frac{(\lambda)_n}{n!}\,. \tag{54}$$

From Eq. (52), it follows that the spectrum of the matrix $\{\langle n, \ell'|\hat{W}_u|n, \ell\rangle\}_{\ell,\ell'=0}^{\infty}$ turns out to be of the form $\{\Lambda, 0, 0, 0, \ldots\}$, where Λ coincides with its trace. On using [15, formula 5.2.13.1], we have

$$\Lambda = \sum_{\ell=0}^{\infty} \langle n, \ell|\hat{W}_u|n, \ell\rangle = \frac{\pi}{\chi^{1/4}\,(1+\sqrt{\chi})} \sum_{k=0}^{\infty} \frac{(2k)!}{k!^2}\,\xi^{2k} = \frac{\pi}{2\sqrt{\chi}}\,, \tag{55}$$

where in the last step use has been made of Eq. (51). Since the value of Λ does not depend on the index n, we conclude that the complete spectrum of the twisted operator $\hat{W}_u$ consists in an infinite list of infinite lists, i.e.,

$$\left\{\left\{\frac{\pi}{2\sqrt{\chi}}, 0, 0, 0, \ldots\right\}, \left\{\frac{\pi}{2\sqrt{\chi}}, 0, 0, 0, \ldots\right\}, \left\{\frac{\pi}{2\sqrt{\chi}}, 0, 0, 0, \ldots\right\}, \ldots\right\}, \tag{56}$$

which confirms the positive semidefiniteness of $\hat{W}_u$.

5. Conclusions

More than thirty years have now passed since the birth of TGSM beams, which marked the introduction of *twist* into the scenario of classical coherence theory. Despite hundreds of works published on the subject, there remain still unexplored areas which would deserve to be studied. On of them concerns with the problem of identifying the conditions under which Schell-model CSDs can made twistable simply by imposing the twist term $\exp(-iu\boldsymbol{r}_1 \times \boldsymbol{r}_2)$, seems to be far from being closed. In the present paper, a possible strategy has been outlined to solve, at least in principle, such a

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nontrivial technical problem for the largest class of Schell-model sources. In particular, on employing the operatorial approach introduced in [5], a complete characterization of a twisted, uniform operator $\hat{W}_u$ in terms of the eigenstates of the sole twist operator $\hat{T}_u$, has analytically been performed, also thank to the important results of Ref. [10]. Such a choice has been driven by the fact that, for the class of *real* symmetric functions $\mu(\boldsymbol{r}) = \mu(-\boldsymbol{r})$ functions considered here, the operators $\hat{W}_u$ and $\hat{T}_u$ must commute. The obtained scenario, allowed the results of Ref. [8], where the subclass of *radial* functions $\mu(|\boldsymbol{r}|)$ was studied, to be given a solid mathematical justification. The class of twisted sources obtained by nonradial Gaussian degrees of coherence has also been analyzed, in consideration of the numerical limitation of the twist strength, found in [5].

6. Proof of Eq. (11)

In order to find the conditions under which Eq. (11) holds, we have to impose that, for any $|\psi\rangle \in \mathcal{H}$, the following equation must hold:

$$\hat{W}_u \hat{T}_u |\psi\rangle = \hat{T}_u \hat{W}_u |\psi\rangle, \qquad \forall |\psi\rangle \in \mathcal{H}. \tag{57}$$

The matrix elements of the operator $\hat{W}_u \hat{T}_u$ are given, in the position representation, by

$$\begin{aligned}
\langle \boldsymbol{r}_1 | \hat{W}_u \hat{T}_u | \boldsymbol{r}_2 \rangle &= \int_{\mathbb{R}^2} \mathrm{d}^2\rho \, \langle \boldsymbol{r}_1 | \hat{W}_u | \boldsymbol{\rho} \rangle \langle \boldsymbol{\rho} \hat{T}_u | \boldsymbol{r}_2 \rangle = \int_{\mathbb{R}^2} \mathrm{d}^2\rho \, W_u(\boldsymbol{r}_1, \boldsymbol{\rho}) \, T_u(\boldsymbol{\rho}, \boldsymbol{r}_2) = \\
&= \int_{\mathbb{R}^2} \mathrm{d}^2\rho \, \mu(\boldsymbol{r}_1 - \boldsymbol{\rho}) \, \exp(-\mathrm{i}u(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{\rho}) = \int_{\mathbb{R}^2} \mathrm{d}^2\rho \, \mu(\boldsymbol{\xi}) \, \exp(-\mathrm{i}u\boldsymbol{\xi} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2)) = \\
&= \exp(-\mathrm{i}u\boldsymbol{r}_1 \times \boldsymbol{r}_2) \int_{\mathbb{R}^2} \mathrm{d}^2\xi \, \mu(\boldsymbol{\xi}) \, \exp(-\mathrm{i}u\boldsymbol{\xi} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2)),
\end{aligned} \tag{58}$$

where the new integration variable $\boldsymbol{\xi} = \boldsymbol{r}_1 - \boldsymbol{\rho}$ has been introduced.

Similarly, for the product $\hat{T}_u \hat{W}_u$ we have

$$\begin{aligned}
\langle \boldsymbol{r}_1 | \hat{T}_u \hat{W}_u | \boldsymbol{r}_2 \rangle &= \int_{\mathbb{R}^2} \mathrm{d}^2\rho \, T_u(\boldsymbol{r}_1, \boldsymbol{\rho}) \, W_u(\boldsymbol{\rho}, \boldsymbol{r}_2) = \\
&= \int_{\mathbb{R}^2} \mathrm{d}^2\rho \, \exp(-\mathrm{i}u\boldsymbol{r}_1 \times \boldsymbol{\rho}) \, \mu(\boldsymbol{\rho} - \boldsymbol{r}_2) \, \exp(-\mathrm{i}u\boldsymbol{\rho} \times \boldsymbol{r}_2) = \\
&= \exp(-\mathrm{i}u\boldsymbol{r}_1 \times \boldsymbol{r}_2) \int_{\mathbb{R}^2} \mathrm{d}^2\xi \, \mu(-\boldsymbol{\xi}) \, \exp(-\mathrm{i}u\boldsymbol{\xi} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2)),
\end{aligned} \tag{59}$$

where now we set $-\boldsymbol{\xi} = \boldsymbol{\rho} - \boldsymbol{r}_2$. On comparing Eq. (58) with Eq. (59), we conclude that the commutator $[\hat{W}_u, \hat{T}_u]$ vanishes iff $\mu(\boldsymbol{\xi}) \equiv \mu(-\boldsymbol{\xi})$, $\square$.

7. Proof of Eq. (39)

Consider again the quantity $\langle n, \ell' | \hat{W}_u | n, \ell \rangle$ given by Eq. (38),

$$\langle n, \ell' | \hat{W}_u | n, \ell \rangle = \sqrt{\frac{\pi}{u}} \, (-1)^{\ell + \min\{\ell, \ell'\} + \min\{n, \ell'\} + \min\{n, \ell\}} \int_{\mathbb{R}^2} \mathrm{d}^2 r \, \mu(\boldsymbol{r}) \, \Phi^*_{\frac{\ell'+\ell}{2}, \frac{\ell-\ell'}{2}}(\boldsymbol{r}), \tag{60}$$

and introduce two auxiliary variables, say $J = \frac{\ell+\ell'}{2} \geq 0$ and $M = \frac{\ell-\ell'}{2}$.

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Suppose now that $2M$ be an *odd* integer number. Since by hypothesis $\mu(-\boldsymbol{r}) = \mu(\boldsymbol{r}) \in \mathbb{R}$, on using the second of Eq. (24), the integral into Eq. (60) becomes

$$\begin{aligned}&\int_{\mathbb{R}^2} \mathrm{d}^2 r\, \mu(\boldsymbol{r})\, \Phi^*_{J,M}(\boldsymbol{r}) \,=\, \int_{\mathbb{R}^2} \mathrm{d}^2 r\, \mu(-\boldsymbol{r})\, \Phi^*_{J,M}(-\boldsymbol{r}) \,=\, \int_{\mathbb{R}^2} \mathrm{d}^2 r\, \mu(\boldsymbol{r})\, \Phi^*_{J,M}(-\boldsymbol{r}) \,= \\ &= (-1)^{2M} \int_{\mathbb{R}^2} \mathrm{d}^2 r\, \mu(\boldsymbol{r})\, \Phi^*_{J,M}(\boldsymbol{r}) \,=\, -\int_{\mathbb{R}^2} \mathrm{d}^2 r\, \mu(\boldsymbol{r})\, \Phi^*_{J,M}(\boldsymbol{r})\,,\end{aligned} \tag{61}$$

which implies that the integral is null and, consequently, also the matrix element. Accordingly, $\langle n, \ell' | \hat{W}_u | n, \ell \rangle = 0$ for any odd $\ell - \ell'$, $\square$.

8. On the Van Valkenburgh Extended Wigner Distribution Function

We start from the definition of Wigner distribution function (WDF henceforth) of a given coherent wavefield $\psi(\boldsymbol{r})$, namely [16]

$$\mathcal{W}\{\psi\}(\boldsymbol{\rho}, \boldsymbol{p}) \,=\, \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \mathrm{d}^2 \xi\, \exp(-\mathrm{i}\boldsymbol{\xi} \cdot \boldsymbol{p})\, \psi^*\left(\boldsymbol{\rho} - \frac{\boldsymbol{\xi}}{2}\right) \psi\left(\boldsymbol{\rho} + \frac{\boldsymbol{\xi}}{2}\right). \tag{62}$$

In 2008, Van Valkenburgh proposed the following natural generalization of the WDF [10]:

$$\mathcal{W}\{\psi, \phi\}(\boldsymbol{\rho}, \boldsymbol{p}) \,=\, \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \mathrm{d}^2 \xi\, \exp(-\mathrm{i}\boldsymbol{\xi} \cdot \boldsymbol{p})\, \psi^*\left(\boldsymbol{\rho} - \frac{\boldsymbol{\xi}}{2}\right) \phi\left(\boldsymbol{\rho} + \frac{\boldsymbol{\xi}}{2}\right), \tag{63}$$

where now $\psi(\boldsymbol{r})$ e $\phi(\boldsymbol{r})$ denote *arbitrary* complex wavefields. In particular, if $\psi \equiv \phi$, then the EWDF defined into Eq. (63) must coincide with the ordinary WDF defined into Eq. (62), i.e.,

$$\mathcal{W}\{\psi, \psi\}(\boldsymbol{\rho}, \boldsymbol{p}) \,\equiv\, \mathcal{W}\{\psi\}(\boldsymbol{\rho}, \boldsymbol{p})\,. \tag{64}$$

A quarter of century ago, Simon and Agarval first evaluated the WDF of LG modes (4), namely [16]

$$\mathcal{W}\{\Phi_{j,m}\}(\boldsymbol{\rho}, \boldsymbol{p}) \,=\, \frac{(-1)^{2j}}{\pi^2}\, \mathcal{L}_{j+m}(Q_0 + Q_2)\, \mathcal{L}_{j-m}(Q_0 - Q_2)\,, \tag{65}$$

where $\mathcal{L}_n(x) = L_n(x) \exp(-x/2)$ and the quantities Q_0 and Q_2 were defined by

$$\begin{cases} Q_0 \,=\, u\rho^2 + \dfrac{p^2}{u}\,, \\[2ex] Q_2 \,=\, 2\,\boldsymbol{\rho} \times \boldsymbol{p}\,. \end{cases} \tag{66}$$

Eigth years later, such a fundamental result was analyzed from a different point of view by Van Valkenburgh [10], who proved that any LG mode $\Phi_{j,m}$ can be interpreted as an EWDF and, more importantly, conceived a beautiful and simple analytical algorithm to express the EWDF of the product of two *arbitrarily chosen* LG modes, thus extending the achievement obtained in [16].

In order to "tune" the notations of the Van Valkenburgh paper with those, slightly different, of the Simon/Agarwal paper, it will assumed $u = 1$. In this way, it can be proved that [10]

$$\bar{\Phi}_{j,m}(\boldsymbol{r}) \,=\, (-1)^{j-|m|}\, \tilde{W}\{h_{j+m,j-m}\}(x, y)\,, \tag{67}$$

where $h_{n,\ell}(x, y) = h_n(x)\, h_\ell(y)$, with the so-called HG functions $h_n(x)$ being defined as

$$h_n(x) \,=\, \pi^{-1/4} n!^{-1/2} 2^{-n/2} \exp\left(-\frac{1}{2}x^2\right) H_n(x), \qquad n = 0, 1, 2, \ldots \tag{68}$$

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As far as the symbol $\tilde{W}\{h_{n,\ell}\}(x,y)$ is concerned, again from Ref. [10] we have

$$\tilde{W}\{h_{n,\ell}\}(x,y) \,=\, \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\xi \, \exp(-\mathrm{i}\xi y) \, h_{n,\ell}\left(\frac{x-\xi}{\sqrt{2}}, \frac{x+\xi}{\sqrt{2}}\right). \tag{69}$$

In particular, Eq. (67) can be inverted as follows:

$$\tilde{W}\{h_{n,\ell}\}(x,y) \,=\, (-1)^{\min\{n,\ell\}} \bar{\Phi}_{\frac{n+\ell}{2},\frac{n-\ell}{2}}(\boldsymbol{r}), \tag{70}$$

which plays a role of pivotal importance for the scope of the present paper.

To illustrate the Van Valkenburgh theorem, consider *his own* definition of EWDF, precisely [10]

$$W_2\{\psi,\phi\}(\boldsymbol{\rho},\boldsymbol{p}) \,=\, \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}^2\xi \, \exp(-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{p}) \, \psi^*\left(\frac{\boldsymbol{\rho}-\boldsymbol{\xi}}{\sqrt{2}}\right)\phi\left(\frac{\boldsymbol{\rho}+\boldsymbol{\xi}}{\sqrt{2}}\right), \tag{71}$$

which, as said above, slightly differs from the definition given into Eq. (63). Nevertheless, it is trivial to prove that

$$\mathcal{W}\{\psi,\phi\}(\boldsymbol{\rho},\,\boldsymbol{p}) \,=\, \frac{1}{\pi} W_2\{\psi,\phi\}\left(\boldsymbol{\rho}\sqrt{2},\boldsymbol{p}\sqrt{2}\right). \tag{72}$$

Now, in Ref. [10] it has been proved that:

$$W_2\{\tilde{W}\{h_{n,\ell}\},\tilde{W}\{h_{n',\ell'}\}\}(\boldsymbol{\rho},\boldsymbol{p}) \,=\, \tilde{W}\{h_{n,n'}\}\left(\frac{x+p_y}{\sqrt{2}}, \frac{p_x-y}{\sqrt{2}}\right)\tilde{W}\{h_{\ell,\ell'}\}\left(\frac{x-p_y}{\sqrt{2}}, \frac{p_x+y}{\sqrt{2}}\right), \tag{73}$$

which, on taking Eq. (72) into account, leads to

$$\mathcal{W}\{\tilde{W}\{h_{n,\ell}\},\tilde{W}\{h_{n',\ell'}\}\}(\boldsymbol{\rho},\,\boldsymbol{p}) \,=\, \frac{1}{\pi}\tilde{W}\{h_{n,n'}\}(x+p_y,\,p_x-y)\tilde{W}\{h_{\ell,\ell'}\}(x-p_y,\,p_x+y). \tag{74}$$

9. Proof of Eq. (43)

We start from Eq. (4) written in polar coordinates (r,φ),

$$\begin{aligned}\sqrt{\frac{\pi}{u}} \int_{\mathbb{R}^2} \mathrm{d}^2 r \, \mu(\boldsymbol{r}) \, \Phi^*_{\ell,\ell+2M}(\boldsymbol{r}) \,&=\, u^M \sqrt{\frac{\ell!}{(\ell+2M)!}} \int_0^\infty \mathrm{d}r \, r^{2M+1} \, L^{2M}_{J-M}(ur^2) \, \exp\left(\frac{ur^2}{2}\right) \\ &\times \int_0^{2\pi} \mathrm{d}\varphi \, \exp\left[-r^2\left(\cos^2\varphi + \chi\sin^2\varphi\right)\right] \exp(-2\mathrm{i}M\varphi).\end{aligned} \tag{75}$$

The φ-integral can be evaluated elementarily,

$$\begin{aligned}&\int_0^{2\pi} \mathrm{d}\varphi \, \exp\left[-r^2\left(\cos^2\varphi+\chi\sin^2\varphi\right)\right] \exp(-2\mathrm{i}M\varphi) \,= \\ &= \exp\left[-\frac{r^2}{2}(\chi+1)\right] \int_0^{2\pi} \mathrm{d}\varphi \, \exp\left[-\frac{r^2}{2}(\chi-1)\cos 2\varphi\right] \exp(-2\mathrm{i}M\varphi) \,= \\ &= 2\pi \, \exp\left[-\frac{r^2}{2}(\chi+1)\right] I_M\left(\frac{r^2}{2}(\chi-1)\right),\end{aligned} \tag{76}$$

where $I_n(\cdot)$ denotes the nth-order modified Bessel function of the first kind. On substituting from Eq. (76) into Eq. (75), after rearranging Eq. (43) follows, □.

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10. Proof of Eq. (45)

The starting point is the following result by Brichkov [11, formula 14.2.18]:

$$\int_0^\infty \mathrm{d}x\, x^m \exp(-a\,x)\, I_m(b\,x)\, L_n^{2m}(c\,x) \;=\; \frac{b^m \Gamma(2m+1)(2m+1)_n}{2^m n!(a+b)^{2m+1}\Gamma(m+1)}$$
$$= F_2\left(2m+1; m+\frac{1}{2}, -n; 2m+1, 2m+1; \frac{2b}{a+b}, \frac{c}{a+b}\right), \tag{77}$$

where the symbol F_2 denotes one of the so-called Appell functions [12, Ch. 16]. In this particular case, we shall limit ourselves to real values of (a, b, c) and to integer values of (n, m). Moreover, in the following it will be assumed that $a > b \geq 0$.

The Appell function F_2 into Eq. (77) can be recast in terms of the Appell function F_1 by using the following transformation formula [12, formula 16.16.3]:

$$F_2(\alpha; \beta, -n; \alpha, \alpha; x, y) \;=\; (1-y)^n\, F_1\left(\beta; \alpha+n, -n; \alpha; x, \frac{x}{1-y}\right), \tag{78}$$

where $F_1(\cdot)$ denotes another member of the Appell function set. Simple algebra then gives

$$F_2\left(2m+1; m+\frac{1}{2}, -n; 2m+1, 2m+1; \frac{2b}{a+b}, \frac{c}{a+b}\right) =$$
$$= \left(\frac{a+b-c}{a+b}\right)^n F_1\left(m+\frac{1}{2}; 2m+n+1, -n; 2m+1; \frac{2b}{a+b}, \frac{2b}{a+b-c}\right). \tag{79}$$

Equation (81) can further be simplified, by expressing the Appell function F_1 in terms of the, better known Gauss hypergeometric function ${}_2F_1$ through the following connection formulas [12]:

$$F_1(\beta; \alpha+n, -n; \alpha; \xi, \eta) \;=\; (1-\eta)^{-\beta}\, {}_2F_1\left(\beta, \alpha+n; \alpha; \frac{\xi-\eta}{1-\eta}\right) =$$
$$= (1-\eta)^{-\beta} \left(\frac{1-\xi}{1-\eta}\right)^{-\beta-n} {}_2F_1\left(\alpha-\beta, -n; \alpha; \frac{\xi-\eta}{1-\eta}\right) = \tag{80}$$
$$= \left(\frac{1-\eta}{1-\xi}\right)^n (1-\xi)^{-\beta}\, {}_2F_1\left(\alpha-\beta, -n; \alpha; \frac{\xi-\eta}{1-\eta}\right),$$

which, when substituted into Eq. (81) gives at once

$$F_2\left(2m+1; m+\frac{1}{2}, -n; 2m+1, 2m+1; \frac{2b}{a+b}, \frac{c}{a+b}\right) =$$
$$= \left(\frac{a+b-c}{a+b}\right)^n F_1\left(m+\frac{1}{2}; 2m+n+1, -n; 2m+1; \frac{2b}{a+b}, \frac{2b}{a+b-c}\right) = \tag{81}$$
$$= \left(\frac{a-b-c}{a-b}\right)^n \left(\frac{a+b}{a-b}\right)^{m+1/2} {}_2F_1\left(m+\frac{1}{2}, -n; 2m+1; \frac{2bc}{(a+b)(b+c-a)}\right).$$

Finally, on substituting from Eq. (81) into Eq. (77), after simplifying and rearranging, it is obtained

$$\int_0^\infty \mathrm{d}x\, x^m \exp(-a\,x)\, I_m(b\,x)\, L_n^{2m}(c\,x) = \left(\frac{b}{2}\right)^m \frac{(2m+n)!}{n!m!} \frac{1}{(a^2-b^2)^{m+1/2}} \times \left(\frac{a-b-c}{a-b}\right)^n {}_2F_1\left(m+\frac{1}{2}, -n; 2m+1; \frac{2bc}{(a+b)(b+c-a)}\right). \tag{82}$$

Last equation can further be simplified on taking into account that

$${}_2F_1(-n,\gamma;2\gamma;z) = \frac{n!}{\left(\gamma+\frac{1}{2}\right)_n} \frac{1}{4^n} z^n\, C_n^{\frac{1}{2}-\gamma-n}\left(1-\frac{2}{z}\right), \tag{83}$$

where the symbol $C_n^\lambda(\cdot)$ denotes the ultraspherical, or Gegenbauer polynomial. Then, on substituting from Eq. (45) into Eq. (84), eventually we arrive at

$$\int_0^\infty \mathrm{d}x\, x^m \exp(-a\,x)\, I_m(b\,x)\, L_n^{2m}(c\,x) = \left(\frac{b}{2}\right)^m \frac{(2m+n)!}{(m+n)!} \frac{(-bc/2)^n}{(a^2-b^2)^{n+m+1/2}} C_n^{-m-n}\left(\frac{a^2-b^2-ac}{bc}\right), \tag{84}$$

which coincides with Eq. (45), □.

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