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Portfolio Selection Based on Modified CoVaR in the Gaussian Framework

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Abstract: We study the Mean-Risk model, where risk is measured by the Modified CoVaR

$$\text{CoVaR}_{\alpha,\beta}^{\leq}(X|Y) = \text{VaR}_{\beta}(X | Y + \text{VaR}_{\alpha}(Y) \leq 0).$$

We prove that in Gaussian setting, for sufficiently small β , such model has a solution.

Keywords: Conditional Value at Risk (CoVaR); Portfolio selection; Gaussian copula

1. Introduction

The question of optimal asset allocation is naturally as old as the history of investing itself, yet the pioneering work of Markowitz is barely over seventy years long and it was as recently as 1990 that the Alfred Nobel Memorial Prize in Economic Sciences was awarded for the theory of portfolio choice (Harry Markowitz), Capital Asset Pricing Model (William Sharpe) and theory of corporate finance (Merton Miller). There are still ample opportunities to explore interesting questions inside that complex field of portfolio analysis. We will pursue the path of investigating conditional value-at-risk, selected instead of variance as the risk measure in the Markowitz model. The task before us is to find the portfolio with the lowest risk while simultaneously retaining the chosen expected return.

To fix the notation, in this paper we will base on the Profit/Loss (P/L) approach as for example in [1,2,7,11,12,20].

We will study random variables X, Y , which are modelling for example: welfare of the financial institutions, financial positions, gains of the investments, or rates of returns of stock prices and indices. So generally

"The higher value of X the better".

The above can be expressed in terms of quantiles. Namely Value-at-Risk at a significance level α is equal to minus upper α quantile of X or lower $1 - \alpha$ quantile of the loss $-X$

$$\text{VaR}_{\alpha}(X) = -Q_{\alpha}^{+}(X) = Q_{1-\alpha}^{-}(-X).$$

We recall that for a given random variable X and a given level $\kappa \in (0, 1)$ the set of κ quantiles is a closed interval $Q_{\kappa}(X)$, which might be reduced to a point. The end points of the interval $Q_{\kappa}(X)$ are referred to as upper and lower quantiles.

$$[Q_{\kappa}^{-}(X), Q_{\kappa}^{+}(X)] = Q_{\kappa}(X) = \{x : \mathbb{P}(X \leq x) \geq \kappa \text{ and } \mathbb{P}(X \geq x) \geq 1 - \kappa\}.$$

To switch to the alternative Loss/Profit (L/P) approach (applied for example in [3,9,15]) when random variables are modelling losses of the financial investments, actuarial risks or high water levels in hydrology, it is enough to change the sign of the variables

$$L = -X,$$

and remember that, by a convention, the subscript is changed. The significance level α is replaced by the confidence level $c = 1 - \alpha$

$$\text{VaR}_c(L) = \text{VaR}_{\alpha}(X).$$

Now assume, that we are measuring the risk given some stress event. For example, we want to determine how big bailout would be necessary to keep a financial institution X solvent with probability at least $1 - \beta$ when a financial institution Y would perform badly. Conditional Value-at-Risk (CoVaR) introduced in 2008 by Adrian and Brunnermeier ([1]) and its later modifications proved to be very useful tools for measuring (quantifying) such phenomena.

Let X and Y be random variables modelling positions. CoVaR is defined as VaR of X conditioned by Y . In more details:

$$\text{CoVaR}(X|Y) = \text{VaR}_\beta(X|Y \in E),$$

where a Borel subset of the real line E is modeling some adverse event concerning Y . Most often E consists of one point (a threshold) or is a half-line bounded by a threshold.

As we see, to deal with CoVaR, one has to model the dependence between X and Y . This can be achieved by means of copulas, as can be seen in [2,11,12].

Adrian and Brunnermeier ([1]) applied the construction with E consisting of one point. Such approach has a certain drawback, pointed out for example by Mainik and Schaanning in [15], which is due to the fact that the standard CoVaR is not compatible with the concordance ordering. Hence it is "breaking" the paradigm: *more dependence, more systemic risk* (see also [11]). To avoid this inconsistency a modified definition of CoVaR was introduced in 2013 and 2014 by Girardi and A.T. Ergün ([8]) and by Mainik and Schaanning ([15]), both in L/P setting.

The modified Conditional-VaR at a level (α, β) , which is a main objective of this survey, is defined as VaR at level β of X under the condition that $Y \leq -\text{VaR}_\alpha(Y)$.

In this survey we consider the portfolio optimization problem with risk measured by the modified CoVaR under the assumption of the normality of returns. It shows that the switch from "standard" CoVaR with one pointed E (see [20]) to modified CoVaR makes the problem more demanding. For example in the "modified" problem, it is not obvious that between portfolios with fixed value and expected value one can find a portfolio minimizing the risk.

The paper is organized in the following way:

In section 2, "Notation", we recall the basics about CoVaR and portfolio selection. In the next one, "The Mean-CoVaR model", we present our main results concerning the existence of optimal portfolios. The following one, "Proofs and auxiliary results", deals with Gaussian copulas and optimization problems. In the last section we provide numerical results concerning a given four-dimensional portfolio.

2. Notation

2.1. CoVaR in the Copula Setting

We present the definition after [2]

Definition 2.1.

$$\text{CoVaR}_{\alpha,\beta}^{\leq}(X|Y) = \text{VaR}_\beta(X | Y \leq -\text{VaR}_\alpha(Y)), \quad (1)$$

which can be expressed in terms of quantiles:

$$\text{CoVaR}_{\alpha,\beta}^{\leq}(X|Y) = -Q_\beta^+(X | Y \leq Q_\alpha^+(Y)).$$

We assume that the distribution functions of X and Y are continuous and their joint distribution is described by a copula C .¹

From the same article ([2]) we learn that, since

$$\mathbb{P}(X \leq x | Y \leq Q_{\alpha}^{+}(Y)) = \frac{\mathbb{P}(X \leq x, Y \leq Q_{\alpha}^{+}(Y))}{\alpha} = \frac{C(F_X(x), \alpha)}{\alpha},$$

the following is valid:

Theorem 2.1.

$$\text{CoVaR}(X|Y) = \text{VaR}_w(X), \quad (2)$$

where w is the maximal root of the equation:

$$C(\alpha, w) = \alpha\beta. \quad (3)$$

For more details the reader is referred to [11,12].

When furthermore we assume that the pair (X, Y) is normally distributed then

$$\text{CoVaR}(X|Y) = -\mathbb{E}(X) - \Phi^{-1}(w)\sigma(X), \quad (4)$$

and

$$C(\alpha, w, \rho(X, Y)) = \alpha\beta, \quad (5)$$

where $C(u, v, \rho)$ denotes a Gaussian copula with a parameter (correlation coefficient) ρ and Φ is a distribution function of a univariate standard normal probability law $\mathcal{N}_1(0, 1)$. Thus the impact of Y is encapsulated in the correlation coefficient. The implied significance level w is a function of α, β and ρ .

$$w = w(\alpha, \beta, \rho),$$

which is decreasing with ρ .

$$w(\alpha, \beta, -1) = 1 - \alpha(1 - \beta), \quad w(\alpha, \beta, 0) = \beta, \quad w(\alpha, \beta, 1) = \alpha\beta.$$

Furthermore ρ depends on X . For more details see Section 4.1.

2.2. Portfolio Selection

Portfolio (i.e. investment strategy) $x = (x_1, \dots, x_n)^T$ meets the natural condition of summing up to 1 and $\mu = (\mu_1, \dots, \mu_n)^T$ is defined as expected value of n -dimensional random variable of returns on **risky** assets, $R = (R_1, \dots, R_n)^T$. Now we add the assumption of **normality** of R , to be held throughout the rest of the article. To obtain a non-degenerate problem, two more assumptions are made. First, $\mu \not\parallel \mathbb{1}_n = (1, \dots, 1)^T$, i.e. not every asset has the same expected return. Second, the covariance matrix of R , $\Sigma = [\sigma_{ij}]_{i,j=1,\dots,n}$ is positive definite.

We also define $X = x^T R = \sum_{i=1}^n x_i R_i$, the univariate random variable of return on the portfolio. Obviously the expected value and the variance are given by

$$\mathbb{E}(X) = \mathbb{E}(x^T R) = x^T \mu,$$

¹ For more details about copulas the reader is referred to the monograph of Roger Nelsen [18] or other publications on the subject like [4–6,13,16].

$$\sigma^2(X) = \sigma^2(x^T R) = x^T \Sigma x = \|x\|_{\Sigma}^2.$$

The distribution of X is conditioned on one chosen variable $R_i, i \in \{1, \dots, n\}$. Without loss of generality let that be $Y = R_1$. The correlation coefficient $\rho = \rho(X, R_1)$ (provided $X \neq 0$) is equal to the cosine of the angle between vectors x and e_1 in the metric in $(R)^n$ induced by the scalar product given by the matrix Σ :

$$\rho = \sigma(X)^{-1} \sigma(R_1)^{-1} \text{cov}(X, R_1) = \frac{e_1^T \Sigma x}{\sqrt{e_1^T \Sigma e_1} \sqrt{x^T \Sigma x}} = \cos_{\Sigma}(x, e_1) \quad (6)$$

where $e_1 = (1, 0, \dots, 0)^T$.

We select portfolios with respect to two criteria. We maximize the expected value and minimize the risk measured by CoVaR.

We recall that a portfolio $x, x^T \mathbb{1}_n = 1$, is called efficient, if and only if it is maximal with respect to the generalized Markowitz ordering. Which means that there exists no portfolio y fulfilling the budget constraints $y^T \mathbb{1}_n = 1$, such that

$$\mathbb{E}(y^T R) \geq \mathbb{E}(x^T R) \quad \text{and} \quad \text{CoVaR}(y^T R | R_1) \leq \text{CoVaR}(x^T R | R_1) \quad (7)$$

and at least one inequality is strict.

3. The Mean-CoVaR Model

First, we want to find the portfolio x with fixed return which is minimizing CoVaR. The optimization problem presents itself as follows:

$$\begin{cases} \text{CoVaR}(x^T R | R_1) \rightarrow \min, \\ x^T \mu = E, \\ x^T \mathbb{1}_n = 1, \end{cases} \quad (8)$$

where $R \sim \mathcal{N}_n(\mu, \Sigma)$.

The solution of the above problem is closely related to the classical Markowitz problem

$$\begin{cases} x^T \Sigma x \rightarrow \min \\ x^T \mu = E \\ x^T \mathbb{1}_n = 1 \end{cases} \quad (9)$$

We recall that this problem has a unique solution $X_M \in \mathbb{R}^n$ when:

- the symmetric matrix Σ is positively defined i.e. is a matrix of a scalar product on $\mathbb{R}^n$;
- the vectors μ and $\mathbb{1}_n$ are linearly independent (i.e. not parallel);
- E is any real number.

The solution X_M is given by a formula

$$X_M^T(E) = (E, 1)G^{-1}(\mu, \mathbb{1}_n)^T \Sigma^{-1}, \quad (10)$$

where G is a Gram matrix of vectors μ and $\mathbb{1}_n$ with respect to the scalar product defined by the matrix Σ^{-1}

$$G = (\mu, \mathbb{1}_n)^T \Sigma^{-1} (\mu, \mathbb{1}_n), \quad (11)$$

where by $(\mu, \mathbb{1}_n)$ we denote a $n \times 2$ matrix with columns μ and $\mathbb{1}_n$. For details see [4.2.2](#).

The existence of the solution X_* of the optimization problem (8) depends on a portfolio X_{∞} , given by a formula:

$$X_{\infty}^T = e_1^T - X_M^T(\mu_1) = e_1^T - (\mu_1, 1)G^{-1}(\mu, \mathbb{1}_n)^T \Sigma^{-1}. \quad (12)$$

We recall that μ_1 , the first coefficient of the vector μ , is equal to $\mathbb{E}(Y)$. Note that X_∞ is orthogonal to vectors μ and $\mathbb{1}_n$ with respect to the standard scalar product and orthogonal to X_M with respect to the scalar product associated to the matrix Σ — for proofs see (52), (53). Besides, if we define q in the following way:

$$q^T = e_1^T \Sigma = X_\infty^T \Sigma + ((\mu_1, 1)G^{-1})(\mu, \mathbb{1}_n)^T, \quad (13)$$

then the vanishing of X_∞ implies that the vectors $q, \mu, \mathbb{1}_n$ are linearly dependent.

Theorem 3.1. For $q, \mu, \mathbb{1}_n$ linearly independent, the optimization problem (8)

1. has no solution when

$$\beta > \frac{1}{\alpha} C\left(\alpha, \frac{1}{2}, -\cos_\Sigma(e_1, X_\infty)\right);$$

2. has a solution when

$$\beta < \frac{1}{\alpha} C\left(\alpha, \frac{1}{2}, -\cos_\Sigma(e_1, X_\infty)\right);$$

where $C(u, v, \rho)$ is a Gaussian copula with correlation parameter ρ . Moreover, if X_* is a solution of the optimization problem (8), then the vector $X_* - X_M$ is parallel to X_∞ .

$$X_* = X_M(E) - \lambda(E)X_\infty, \quad (14)$$

where

$$\lambda(E) = \operatorname{Argmin}_{\lambda \geq 0} \left(-\Phi^{-1}(w(\alpha, \beta, \cos_\Sigma(e_1, X_M(E) - \lambda X_\infty))) \cdot \sqrt{\|X_M(E)\|_\Sigma^2 + \lambda^2 \|X_\infty\|_\Sigma^2} \right). \quad (15)$$

The proof is provided in Section 4.2.

Remark 3.1. For $\beta = \frac{1}{\alpha} C\left(\alpha, \frac{1}{2}, -\cos_\Sigma(e_1, X_\infty)\right)$ solution of the optimization problem (8) may not exist, see Example 5.1.

Next, we want to find the portfolio x which is minimizing CoVaR. The optimization problem presents itself as follows:

$$\begin{cases} \operatorname{CoVaR}(x^T R \mid R_1) \rightarrow \min, \\ x^T \mathbb{1}_n = 1, \end{cases} \quad (16)$$

where $R \sim \mathcal{N}_n(\mu, \Sigma)$.

Let the vector X_d denotes the derivative of $X_M(E)$ with respect to E . Then

$$X_d^T = \frac{dX_M(E)^T}{dE} = (1, 0)G^{-1}(\mu, \mathbb{1}_n)^T \Sigma^{-1} \quad (17)$$

and

$$\|X_d\|_\Sigma^2 = X_d^T \Sigma X_d = (1, 0)G^{-1}(1, 0)^T = \frac{\mathbb{1}_n^T \Sigma^{-1} \mathbb{1}_n}{\det(G)}. \quad (18)$$

Theorem 3.2. For $q, \mu, \mathbb{1}_n$ linearly independent, the optimization problem (16)

1. has no solution when

$$\beta > \min \left(\frac{1}{\alpha} C\left(\alpha, \Phi\left(\frac{-\xi}{\|X_d\|_\Sigma}\right), \xi \cos_\Sigma(e_1, X_d) - \sqrt{1 - \xi^2} \cos_\Sigma(e_1, X_\infty) \right) \mid \xi \in [0, 1] \right);$$

2. has a solutions when

$$\beta < \min \left(\frac{1}{\alpha} C \left(\alpha, \Phi \left(\frac{-\xi}{\|X_d\|_{\Sigma}} \right), \xi \cos_{\Sigma}(e_1, X_d) - \sqrt{1 - \xi^2} \cos_{\Sigma}(e_1, X_{\infty}) \right) \mid \xi \in [0, 1] \right);$$

where $C(u, v, \rho)$ is a Gaussian copula with the correlation parameter ρ .

The proof is provided in Section 4.2.

Remark 3.2. Note that when the set of the portfolios of the minimal risk is nonempty and bounded then the one with maximal expected value of return is an efficient portfolio. Moreover the existence of the nonempty, bounded set of portfolios of the minimal risk is a necessary and sufficient condition for the existence of the efficient portfolios.

4. Proofs and Auxiliary Results

4.1. Gaussian Copulas

Let us consider a Gaussian Copula:

$$C(u, v; \rho) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{e^{-\frac{\xi^2 - 2\rho\xi\eta + \eta^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} d\xi d\eta. \quad (19)$$

The correlation coefficient ρ has been added to the copula symbol for the convenience of notation as it is the only other parameter needed for the computation of a copula, due to all elliptical copulas being normalization invariant (c.f. [14], p. 174, Remark 4.3).

Meyer in [17] shows that the Gaussian copula can be extended continuously, approaching the lower and upper Fréchet-Hoeffding bounds, i.e. W and M , respectively; in dimension 2 both of those are copulas.

$$\lim_{\rho \rightarrow 1} C(u, v, \rho) = \min(u, v) = M(u, v), \quad (20)$$

$$\lim_{\rho \rightarrow -1} C(u, v, \rho) = \max(u + v - 1, 0) = W(u, v). \quad (21)$$

Following (c.f. [17]) we get an extension of the classical Fréchet-Hoeffding inequality:

Corollary 4.1.

$$\begin{aligned} W(u, v) = \max\{u + v - 1, 0\} &< C(u, v, \rho_1) < C(u, v, 0) = uv \\ &< C(u, v, \rho_2) < M(u, v) = \min\{u, v\} \end{aligned} \quad (22)$$

where $\rho_1 < 0 < \rho_2$.

Remark 4.1. The Double Gaussian distribution Φ_2 fulfills the following.

For

$$\phi_2(x, y; \rho) = \frac{e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \quad \text{and} \quad \Phi_2(x, y; \rho) = \int_{-\infty}^y \int_{-\infty}^x \phi_2(\xi, v; \rho) d\xi dv \quad (23)$$

one has

$$\frac{\partial}{\partial \rho} \phi_2(x, y; \rho) = \frac{\partial^2}{\partial u \partial v} \phi_2(x, y, \rho),$$

so that:

$$\frac{\partial}{\partial \rho} \Phi_2(x, y; \rho) = \phi_2(x, y; \rho) = \frac{\partial^2}{\partial x \partial y} \Phi_2(x, y; \rho). \quad (24)$$

Thus

$$\Phi_2(x, y; \rho) = \phi(x)\phi(y) + \int_0^\rho \phi_2(x, y; r) dr. \quad (25)$$

In consequence,

$$C(u, v, \rho) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v); \rho)$$

is the strictly increasing function of not only its variables (which is obvious), but also of its parameter ρ . We recall that

$$\frac{\partial}{\partial \rho} C(u, v, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{\Phi^{-1}(u)^2 - 2\rho\Phi^{-1}(u)\Phi^{-1}(v) + \Phi^{-1}(v)^2}{2(1-\rho^2)}\right), \quad (26)$$

$$\frac{\partial}{\partial y} \Phi_2(x, y, \rho) = \varphi(y) \Phi\left(\frac{x - \rho y}{\sqrt{1-\rho^2}}\right), \quad (27)$$

$$\frac{\partial}{\partial v} C(u, v, \rho) = \Phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right). \quad (28)$$

Meyer in [17] provides us also with the following formula:

Corollary 4.2.

$$C(u, v, \rho) = uv + \int_0^\rho \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(-\frac{\Phi^{-1}(u)^2 - 2r\Phi^{-1}(u)\Phi^{-1}(v) + \Phi^{-1}(v)^2}{2(1-r^2)}\right) dr. \quad (29)$$

For $u = v = 1/2$ formula (29) simplifies.

$$\Phi(0, 0, \rho) = C(1/2, 1/2, \rho) = \frac{1}{4} + \int_0^\rho \frac{1}{2\pi\sqrt{1-r^2}} dr = \frac{1}{4} + \frac{1}{2\pi} \arcsin(\rho). \quad (30)$$

This result is usually attributed to Sheppard [19].

Corollary 4.2 implies:

Corollary 4.3. Given $C(\alpha, w, \rho) = \alpha\beta$ we have $\text{sign}(\beta - w) = \text{sign}(\rho)$, i.e. :

1. $w < \beta \Leftrightarrow \rho > 0$
2. $w = \beta \Leftrightarrow \rho = 0$
3. $w > \beta \Leftrightarrow \rho < 0$.

Remark 4.2. The unique root of the equation

$$C(\alpha, w; \rho) = \alpha\beta \quad (31)$$

solved for w , with a given $\rho \in (-1, 1)$, exists for any choice of significance levels α, β , as the horizontal sections of Gaussian copula are continuous, strictly increasing with respect to w :

$$\alpha\beta = C(\alpha, w; \rho) = \int_{-\infty}^{\Phi^{-1}(\alpha)} \int_{-\infty}^{\Phi^{-1}(w)} \frac{e^{-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} du dv. \quad (32)$$

Moreover, as $C(\alpha, w; \rho)$ is strictly increasing with respect to its parameter ρ and is a strictly increasing function of w , we have one-to-one correspondence in the implicit equation.

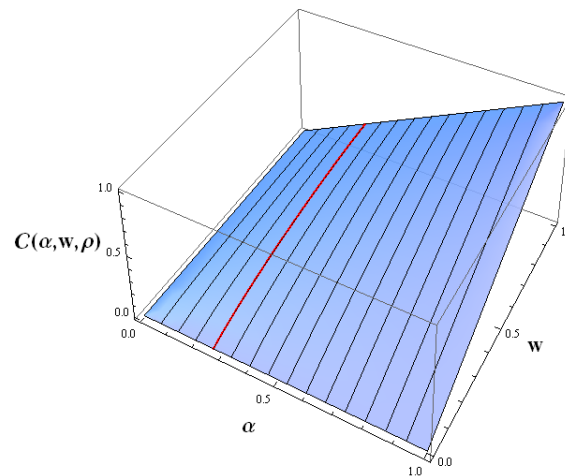


Figure 1. Horizontal sections of a chosen Gaussian copula

Remark 4.3. This holds true for $\rho = 1$ and the $M(\alpha, w)$ copula as well. We get $w = \alpha\beta$ (which by the Corollary 4.3 is the minimum value of w). Similarly for $\rho = -1$ and the $W(\alpha, w)$ copula we get $w = 1 - \alpha + \alpha\beta = 1 - \alpha(1 - \beta)$.

We denote the above root as $w(\alpha, \beta, \rho)$.

$$w : (0, 1)^2 \times (-1, 1) \longrightarrow (0, 1), \quad C(\alpha, w(\alpha, \beta, \rho), \rho) = \alpha\beta. \quad (33)$$

The following is true:

$$w(\alpha, \beta, -\rho) + w(\alpha, 1 - \beta, \rho) = 1. \quad (34)$$

Note that the equality (30) implies

$$w\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{\pi} \arcsin(\rho), \rho\right) = \frac{1}{2}. \quad (35)$$

Moreover $w(\alpha, \beta, \rho)$ is a strictly decreasing, implicit function of ρ and strictly increasing of β .

$$\frac{\partial w(\alpha, \beta, \rho)}{\partial \rho} = - \frac{\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{\Phi^{-1}(\alpha)^2 - 2\rho\Phi^{-1}(\alpha)\Phi^{-1}(w) + \Phi^{-1}(w)^2}{2(1-\rho^2)}\right)}{\Phi\left(\frac{\Phi^{-1}(\alpha) - \rho\Phi^{-1}(w)}{\sqrt{1-\rho^2}}\right)} \quad (36)$$

and

$$\frac{\partial \Phi^{-1}(w(\alpha, \beta, \rho))}{\partial \rho} = - \frac{\frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(\Phi^{-1}(\alpha) - \rho\Phi^{-1}(w))^2}{2(1-\rho^2)}\right)}{\Phi\left(\frac{\Phi^{-1}(\alpha) - \rho\Phi^{-1}(w)}{\sqrt{1-\rho^2}}\right)}. \quad (37)$$

By a direct calculation we get (compare [12]) for fixed β and ρ

$$\lim_{\alpha \rightarrow 0} w(\alpha, \beta, \rho) = \begin{cases} 0 & \rho > 0, \\ \beta & \rho = 0, \\ 1 & \rho < 0, \end{cases}$$

$$\lim_{\alpha \rightarrow 0} \frac{\partial w(\alpha, \beta, \rho)}{\partial \alpha} = \begin{cases} \infty & \rho > 0, \\ 0 & \rho = 0, \\ -\infty & \rho < 0. \end{cases}$$

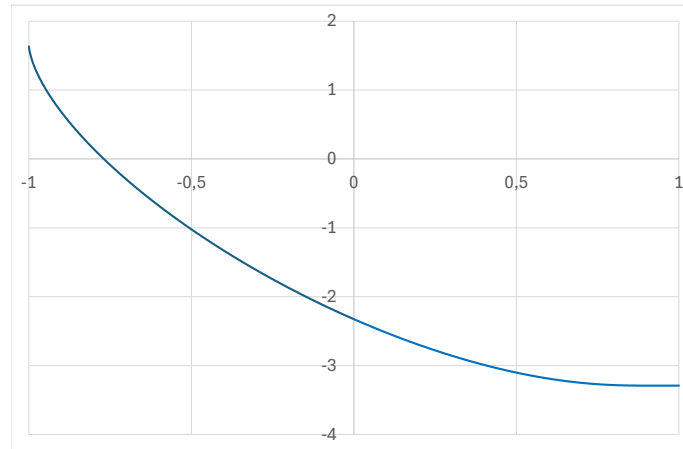


Figure 2. Graph of the function $\rho \rightarrow \Phi^{-1}(w(0.05, 0.01, \rho))$.

4.2. The Optimization Problems

4.2.1. The Basic Problem

We want to find the portfolio x minimizing CoVaR. The optimization problem (8) is equivalent to the following one:

$$\begin{cases} \text{CoVaR}(X | Y) = -x^T \mu - \Phi^{-1}(w) \sqrt{x^T \Sigma x} \rightarrow \min, \\ C(\alpha, w; \rho) = \alpha \beta, \\ \rho = \cos_{\Sigma}(e_1, x), \\ x^T \mu = E, \\ x^T \mathbb{1}_n = 1. \end{cases} \quad (38)$$

This poses an important question: what if the constraints cannot be satisfied? Now we note only that ρ (being a function of x , $\rho = \cos_{\Sigma}(e_1, x)$) plays an important role in answering that question. We recall some facts about the cosine when x is moving along a line.

Lemma 4.1. *Let*

$$x_{\lambda} = u + \lambda w,$$

where the nonzero vectors u and w are Σ -orthogonal. Then

$$(\cos_{\Sigma}(e_1, x_{\lambda}) - \cos_{\Sigma}(e_1, w)) \|x_{\lambda}\|_{\Sigma} = \|u\|_{\Sigma} \cos_{\Sigma}(e_1, u) + \cos_{\Sigma}(e_1, w) \frac{\|u\|_{\Sigma}^2}{\lambda \|w\|_{\Sigma} + \|x_{\lambda}\|_{\Sigma}}$$

and for λ tending to $+\infty$ we get

$$\cos_{\Sigma}(e_1, x_{\lambda}) = \cos_{\Sigma}(e_1, w) + \cos_{\Sigma}(e_1, u) \frac{\|u\|_{\Sigma}}{\|w\|_{\Sigma}} \lambda^{-1} + O(\lambda^{-2}).$$

Proof.

Since

$$\|x_{\lambda}\|_{\Sigma}^2 = \|u\|_{\Sigma}^2 + \lambda^2 \|w\|_{\Sigma}^2,$$

we get

$$\begin{aligned}
 & (\cos_{\Sigma}(e_1, x_{\lambda}) - \cos_{\Sigma}(e_1, w)) \|x_{\lambda}\|_{\Sigma} \\
 &= \frac{e_1^T \Sigma u}{\|e_1\|_{\Sigma}} + \frac{\lambda e_1^T \Sigma w}{\|e_1\|_{\Sigma}} - \cos_{\Sigma}(e_1, w) \|x_{\lambda}\|_{\Sigma} \\
 &= \|u\|_{\Sigma} \cos(e_1, u) + \lambda \|w\|_{\Sigma} \cos_{\Sigma}(e_1, w) - \cos_{\Sigma}(e_1, w) \|x_{\lambda}\|_{\Sigma} \\
 &= \|u\|_{\Sigma} \cos(e_1, u) + \cos_{\Sigma}(e_1, w) (\lambda \|w\|_{\Sigma} - \|x_{\lambda}\|_{\Sigma}) \\
 &= \|u\|_{\Sigma} \cos(e_1, u) - \cos_{\Sigma}(e_1, w) \frac{\|u\|_{\Sigma}^2}{\lambda \|w\|_{\Sigma} + \|x_{\lambda}\|_{\Sigma}}
 \end{aligned}$$

The tail expansion follows from the fact that for λ tending to $+\infty$

$$\|x_{\lambda}\|_{\Sigma}^{-1} = \lambda^{-1} \|w\|_{\Sigma}^{-1} (1 + O(\lambda^{-2})).$$

□

4.2.2. The Markowitz Problem

As was stated in Section 3 the classical Markowitz problem

$$\begin{cases} x^T \Sigma x \rightarrow \min \\ x^T \mu = E \\ x^T \mathbb{1}_n = 1 \end{cases} \quad (39)$$

has for every $E \in \mathbb{R}$, a unique solution $X_M(E) \in \mathbb{R}^n$ when:

- the symmetric matrix Σ is positively defined i.e. is a matrix of a scalar product on $\mathbb{R}^n$;
- the vectors μ and $\mathbb{1}_n$ are linearly independent (i.e. not parallel).

The solution $X_M(E)$ (X_M for short) is given by a formula

$$X_M^T = (E, 1) G^{-1} (\mu, \mathbb{1}_n)^T \Sigma^{-1}, \quad (40)$$

where G is a Gram matrix of vectors μ and $\mathbb{1}_n$ with respect to the scalar product defined by the matrix Σ^{-1}

$$G = (\mu, \mathbb{1}_n)^T \Sigma^{-1} (\mu, \mathbb{1}_n), \quad (41)$$

where by $(\mu, \mathbb{1}_n)$ we denote a $n \times 2$ matrix with columns μ and $\mathbb{1}_n$.

Indeed, X_M fulfills the constraints

$$X_M^T \circ (\mu, \mathbb{1}_n) = (E, 1) G^{-1} (\mu, \mathbb{1}_n)^T \Sigma^{-1} (\mu, \mathbb{1}_n) = (E, 1) G^{-1} G = (E, 1). \quad (42)$$

Furthermore for any nonzero Y , such that $Y^T \mu = 0 = Y^T \mathbb{1}_n$

$$(X_M + Y)^T \Sigma (X_M + Y) = X_M^T \Sigma X_M + Y^T \Sigma Y + 2 X_M^T \Sigma Y. \quad (43)$$

Since

$$Y^T \Sigma Y > 0$$

and

$$X_M^T \Sigma Y = (E, 1) G^{-1} (\mu, \mathbb{1}_n)^T \Sigma^{-1} \Sigma Y = (E, 1) G^{-1} (\mu, \mathbb{1}_n)^T Y = (E, 1) G^{-1} (\mu^T Y, \mathbb{1}_n^T Y) = 0, \quad (44)$$

the minimum is attained to X_M . Note that

$$X_M^T \Sigma X_M = (E, 1) G^{-1} (\mu, \mathbb{1}_n)^T \Sigma^{-1} \Sigma \Sigma^{-1} (\mu, \mathbb{1}_n) G^{-1} (E, 1)^T = (E, 1) G^{-1} (E, 1)^T, \quad (45)$$

$$X_M^T \Sigma e_1 = (E, 1) G^{-1} (\mu, \mathbb{1}_n)^T \Sigma^{-1} \Sigma e_1 = (E, 1) G^{-1} (\mu_1, 1)^T. \quad (46)$$

In the classical approach, where we put

$$\alpha_M = \mu^T \Sigma^{-1} \mu, \quad \beta_M = \mu^T \Sigma^{-1} \mathbb{1}_n, \quad \gamma_M = \mathbb{1}_n^T \Sigma^{-1} \mathbb{1}_n$$

and

$$G = \begin{pmatrix} \alpha_M & \beta_M \\ \beta_M & \gamma_M \end{pmatrix},$$

we get

$$X_M(E) = \left(\det(G) \Sigma \right)^{-1} \left(\begin{vmatrix} E & \beta_M \\ 1 & \gamma_M \end{vmatrix} \mu + \begin{vmatrix} \alpha_M & E \\ \beta_M & 1 \end{vmatrix} \mathbb{1}_n \right). \quad (47)$$

It is called the critical line. The variance of the portfolio and the correlation with R_1 equal respectively:

$$\|X_M(E)\|_{\Sigma}^2 = \frac{\gamma_M E^2 - 2\beta_M E + \alpha_M}{\det(G)} = \frac{\gamma_M \left(E - \frac{\beta_M}{\gamma_M}\right)^2}{\det(G)} + \frac{1}{\gamma_M}, \quad (48)$$

$$\cos_{\Sigma}(e_1, X_M(E)) = \frac{(\gamma_M \mu_1 - \beta_M)E + \alpha_M - \beta_M \mu_1}{\sqrt{\det(G)} \|e_1\|_{\Sigma} \sqrt{\gamma_M E^2 - 2\beta_M E + \alpha_M}}. \quad (49)$$

Note that $X_M(\beta_M/\gamma_M)$ is the portfolio of the minimal variance between portfolios fulfilling the constraint $\mathbb{1}_n^T x = 1$. Hence it is Σ -orthogonal to the critical line

$$X_M(\beta_M/\gamma_M) \perp_{\Sigma} X_M(E_1) - X_M(E_2).$$

Furthermore

$$\cos_{\Sigma}(e_1, X_M(\beta_M/\gamma_M)) = \frac{(\gamma_M \mu_1 - \beta_M)\beta_M + (\alpha_M - \beta_M \mu_1)\gamma_M}{\det(G) \|e_1\|_{\Sigma} \sqrt{\gamma_M}} = \frac{1}{\|e_1\|_{\Sigma} \sqrt{\gamma_M}} > 0. \quad (50)$$

For introduced earlier portfolio X_d given by

$$X_M(E) = E X_d + X_M(0)$$

we obtain

$$\|X_d\|_{\Sigma}^2 = \lim_{E \rightarrow +\infty} \frac{1}{E^2} \|X_M(E)\|_{\Sigma}^2 = \frac{\gamma_M}{\det(G)}. \quad (51)$$

We get also the estimates for the Sharpe ratio

$$S(x^T R) = \frac{E(x^T R)}{\sigma(x^T R)} = \frac{\mu^T x}{\|x\|_{\Sigma}}.$$

Lemma 4.2. For any portfolio fulfilling the constraint $\mathbb{1}_n^T x > 0$

$$\frac{\mu^T x}{\|x\|_{\Sigma}} \in \left(-\frac{\sqrt{\det(G)} + \beta_M^-}{\sqrt{\gamma}}, \frac{\sqrt{\det(G)} + \beta_M^+}{\sqrt{\gamma}} \right).$$

Proof.

Since both the expected value and standard deviation are homogeneous of degree 1 with respect to the positive multipliers, we may restrict ourself to the case $\mathbb{1}_n^T x = 1$. From the formula (48) we get for portfolios fulfilling the constraints $\mathbb{1}_n^T x = 1$ and $\mu^T x = E$

$$\frac{\det(G)}{\gamma_M} \|x\|_{\Sigma}^2 \geq \frac{\det(G)}{\gamma_M} \|x_M(E)\|_{\Sigma}^2 > \left(E - \frac{\beta_M}{\gamma_M}\right)^2.$$

Thus

$$\frac{\sqrt{\det(G)}}{\sqrt{\gamma_M}} \|x\|_{\Sigma} > \left|E - \frac{\beta_M}{\gamma_M}\right|,$$

and

$$-\frac{\sqrt{\det(G)}}{\sqrt{\gamma_M}} + \frac{\beta_M}{\gamma_M \|x\|_{\Sigma}} < \frac{E}{\|x\|_{\Sigma}} < \frac{\sqrt{\det(G)}}{\sqrt{\gamma_M}} + \frac{\beta_M}{\gamma_M \|x\|_{\Sigma}}.$$

Since for portfolios, fulfilling the constraint $\mathbb{1}_n^T x = 1$, formula (48) implies that $\|x\|_{\Sigma} \geq \frac{1}{\sqrt{\gamma_M}}$, we obtain

$$-\frac{\sqrt{\det(G)}}{\sqrt{\gamma_M}} - \frac{\beta_M^-}{\sqrt{\gamma_M}} < \frac{E}{\|x\|_{\Sigma}} < \frac{\sqrt{\det(G)}}{\sqrt{\gamma_M}} + \frac{\beta_M^+}{\sqrt{\gamma_M}}.$$

□

4.2.3. Critical Plane

First we show that the solution X_* of the optimization problem (38) is belongs to the linear plane spanned by two Σ -orthogonal vectors $X_M(E)$ and X_{∞} , which was defined in (12) as:

$$X_{\infty}^T = e_1^T - X_M^T(\mu_1) = e_1^T - (\mu_1, 1)G^{-1}(\mu, \mathbb{1}_n)^T \Sigma^{-1}.$$

Note that X_{∞} is orthogonal to vectors μ and $\mathbb{1}_n$ with respect to the standard scalar product and orthogonal to X_M with respect to the scalar product associated to the matrix Σ . Indeed

$$\begin{aligned} X_{\infty}^T(\mu, \mathbb{1}_n) &= e_1^T(\mu, \mathbb{1}_n) - (\mu_1, 1)G^{-1}(\mu, \mathbb{1}_n)^T \Sigma^{-1}(\mu, \mathbb{1}_n) \\ &= (\mu_1, 1) - (\mu_1, 1)G^{-1}G = 0, \end{aligned} \quad (52)$$

$$\begin{aligned} X_{\infty}^T \Sigma X_M(E) &= \left(e_1^T - (\mu_1, 1)G^{-1}(\mu, \mathbb{1}_n)^T \Sigma^{-1}\right) \Sigma \Sigma^{-1}(\mu, \mathbb{1}_n)G^{-1}(E, 1)^T \\ &= \left(e_1^T(\mu, \mathbb{1}_n) - (\mu_1, 1)G^{-1}(\mu, \mathbb{1}_n)^T \Sigma^{-1}(\mu, \mathbb{1}_n)\right) G^{-1}(E, 1)^T \\ &= \left((\mu_1, 1) - (\mu_1, 1)G^{-1}G\right) G^{-1}(E, 1)^T = 0. \end{aligned} \quad (53)$$

Furthermore the Σ -scalar product of X_{∞} and e_1 is positive, unless $X_{\infty} = 0$. Indeed

$$X_{\infty}^T \Sigma X_{\infty} = X_{\infty}^T \Sigma e_1 - X_{\infty}^T \Sigma X_M(\mu_1) = X_{\infty}^T \Sigma e_1 + 0, \quad (54)$$

$$\cos_{\Sigma}(e_1, X_{\infty}) = \frac{\|X_{\infty}\|_{\Sigma}}{\|e_1\|_{\Sigma}}. \quad (55)$$

Besides, since by (13) we have:

$$q^T = e_1^T \Sigma = X_{\infty}^T \Sigma + (e_1^T(\mu, \mathbb{1}_n)G^{-1})(\mu, \mathbb{1}_n)^T,$$

we noted that the vanishing of X_{∞} implies that the vectors $q, \mu, \mathbb{1}_n$ are linearly dependent.

4.2.4. Auxiliary Optimization Problem

Now let X_* be a solution of the optimization problem (8) then it solves the following auxiliary problem.

Let t^2 be a scalar square of X_* ,

$$t^2 = X_*^T \Sigma X_*. \quad (56)$$

Since $(-\Phi^{-1}(w(\alpha, \beta, \rho)))$ is strictly increasing with ρ , X_* is a solution of the optimization problem

$$\begin{cases} e_1^T \Sigma x \rightarrow \min, \\ x^T \mu = E, \\ x^T \mathbb{1}_n = 1, \\ x^T \Sigma x = t^2. \end{cases} \quad (57)$$

Lemma 4.3. *The vector*

$$X_{test} = X_M(E) - \lambda X_\infty, \quad \lambda = \frac{\sqrt{t^2 - X_M(E)^T \Sigma X_M(E)}}{\sqrt{X_\infty^T \Sigma X_\infty}} \quad (58)$$

is a unique solution of the optimization problem (57) with $t^2 \geq X_M(E)^T \Sigma X_M(E)$.

Proof.

Step 1. X_{test} fulfills the constraints. Indeed, due to equalities (42), (52) and (53) we have

$$X_{test}^T(\mu, \mathbb{1}_n) = X_M(\mu, \mathbb{1}_n) - \lambda X_\infty(\mu, \mathbb{1}_n) = (E, 1) - (0, 0); \quad (59)$$

$$\begin{aligned} X_{test}^T \Sigma X_{test} &= X_M^T \Sigma X_M - 2\lambda X_M^T \Sigma X_\infty + \lambda^2 X_\infty^T \Sigma X_\infty \\ &= X_M^T \Sigma X_M + \frac{t^2 - X_M^T \Sigma X_M}{X_\infty^T \Sigma X_\infty} X_\infty^T \Sigma X_\infty = t^2. \end{aligned} \quad (60)$$

Step 2. Minimum by perturbations.

Let Y be nonzero vector such that

$$(\mu, \mathbb{1}_n)^T Y = (0, 0)^T \quad \text{and} \quad (X_{test} + Y)^T \Sigma (X_{test} + Y) = t^2. \quad (61)$$

Since $\lambda > 0$ and $X_M \Sigma Y = 0$ (compare equality (44)),

$$\begin{aligned} (X_{test} + Y)^T \Sigma e_1 &= (X_{test} + Y)^T \Sigma e_1 + \frac{1}{2\lambda} ((X_{test} + Y)^T \Sigma (X_{test} + Y) - t^2) \\ &= X_{test}^T \Sigma e_1 + \frac{1}{2\lambda} (X_{test}^T \Sigma X_{test} - t^2) + e_1^T \Sigma Y + \frac{1}{\lambda} X_{test}^T \Sigma Y + \frac{1}{2\lambda} Y^T \Sigma Y \\ &= X_{test}^T \Sigma e_1 + 0 + e_1^T \Sigma Y + \frac{1}{\lambda} (X_M^T - \lambda X_\infty^T) \Sigma Y + \frac{1}{2\lambda} Y^T \Sigma Y \\ &= X_{test}^T \Sigma e_1 + e_1^T \Sigma Y - X_\infty^T \Sigma Y + \frac{1}{2\lambda} Y^T \Sigma Y \\ &= X_{test}^T \Sigma e_1 + e_1^T \Sigma Y - (e_1^T - e_1^T(\mu, \mathbb{1}_n)G^{-1}(\mu, \mathbb{1}_n)^T \Sigma^{-1}) \Sigma Y + \frac{1}{2\lambda} Y^T \Sigma Y \\ &= X_{test}^T \Sigma e_1 + e_1^T(\mu, \mathbb{1}_n)G^{-1}(\mu, \mathbb{1}_n)^T Y + \frac{1}{2\lambda} Y^T \Sigma Y \\ &= X_{test}^T \Sigma e_1 + \frac{1}{2\lambda} Y^T \Sigma Y > X_{test}^T \Sigma e_1. \end{aligned} \quad (62)$$

□

Since X_{test} is the unique solution of the optimization problem (57) it must be equal to X_* .

4.2.5. Proof of Theorem 3.1

We fix $E \in \mathbb{R}$ and consider the limiting properties of

$$X_\lambda = X_M(E) - \lambda X_\infty,$$

when λ tends to $+\infty$.

Lemma 4.4.

$$\lim_{\lambda \rightarrow \infty} \text{CoVaR}(X_\lambda^T R | R_1) = \begin{cases} -\infty & \text{if } \beta > \frac{1}{\alpha} C\left(\alpha, \frac{1}{2}, \rho_\infty\right), \\ +\infty & \text{if } \beta < \frac{1}{\alpha} C\left(\alpha, \frac{1}{2}, \rho_\infty\right), \\ -E + \frac{\frac{1}{2\pi\sqrt{1-\rho_\infty^2}} \exp\left(-\frac{\Phi^{-1}(\alpha)^2}{2(1-\rho_\infty^2)}\right)}{\Phi\left(\frac{\Phi^{-1}(\alpha)}{\sqrt{1-\rho_\infty^2}}\right)} \cos_\Sigma(e_1, X_M(E)) \|X_M(E)\|_\Sigma & \text{if } \beta = \frac{1}{\alpha} C\left(\alpha, \frac{1}{2}, \rho_\infty\right), \end{cases}$$

where $\rho_\infty = -\cos_\Sigma(e_1, X_\infty)$.

Proof.

Since $C(\alpha, w, \rho) = \alpha\beta$ and for a fixed α copula C is strictly increasing with w (see Remark 4.2) we have

$$\begin{aligned} \text{sgn}(\Phi^{-1}(w)) &= \text{sgn}(w - \frac{1}{2}) \\ &= \text{sgn}(C(\alpha, w, \rho) - C(\alpha, \frac{1}{2}, \rho)) = \text{sgn}(\alpha\beta - C(\alpha, \frac{1}{2}, \rho)) \end{aligned}$$

Note that since we have Lemma 4.1 and $w(\alpha, \beta, \rho)$ is a continuous function,

$$\begin{aligned} &\lim_{\lambda \rightarrow \infty} -\Phi^{-1}(w(\alpha, \beta, \cos_\Sigma(X_\lambda, e_1))) \\ &= -\Phi^{-1}(w(\alpha, \beta, \cos_\Sigma(-X_\infty, e_1))) \begin{cases} < 0 & \text{when } \beta > \frac{1}{\alpha} C\left(\alpha, \frac{1}{2}, -\cos_\Sigma(e_1, X_\infty)\right), \\ > 0 & \text{when } \beta < \frac{1}{\alpha} C\left(\alpha, \frac{1}{2}, -\cos_\Sigma(e_1, X_\infty)\right), \\ = 0 & \text{when } \beta = \frac{1}{\alpha} C\left(\alpha, \frac{1}{2}, -\cos_\Sigma(e_1, X_\infty)\right). \end{cases} \end{aligned}$$

Therefore for $\alpha\beta \neq C\left(\alpha, \frac{1}{2}, \cos_\Sigma(e_1, X_\infty)\right)$

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \text{CoVaR}(X_\lambda | Y) &= -E - \lim_{\lambda \rightarrow \infty} \Phi^{-1}(w(\alpha, \beta, \cos_\Sigma(X_\lambda, e_1))) \sqrt{X_\lambda^T \Sigma X_\lambda} \\ &= -\text{sgn}(\alpha\beta - C(\alpha, \frac{1}{2}, \rho)) \cdot \infty. \end{aligned} \quad (63)$$

The last case $\alpha\beta = C(\alpha, \frac{1}{2}, \rho_\infty)$, $\rho_\infty = -\cos_\Sigma(e_1, X_\infty)$, is a bit more complicated. Note that the above assumption implies that

$$w(\alpha, \beta, \rho_\infty) = \frac{1}{2}.$$

We apply Lemma 4.1 and basing on the formula (37), we obtain

$$\begin{aligned}
 & \lim_{\lambda \rightarrow \infty} \Phi^{-1}(w(\alpha, \beta, \cos_{\Sigma}(X_{\lambda}, e_1))) \sqrt{X_{\lambda}^T \Sigma X_{\lambda}} \\
 &= \lim_{\lambda \rightarrow \infty} \frac{\Phi^{-1}(w(\alpha, \beta, \cos_{\Sigma}(X_{\lambda}, e_1)))}{\cos_{\Sigma}(X_{\lambda}, e_1) - \cos_{\Sigma}(-X_{\infty}, e_1)} \cdot (\cos_{\Sigma}(X_{\lambda}, e_1) - \cos_{\Sigma}(-X_{\infty}, e_1)) \|X_{\lambda}\|_{\Sigma} \\
 &= \lim_{\rho \rightarrow \rho_{\infty}} \frac{\Phi^{-1}(w(\alpha, \beta, \rho)) - \Phi^{-1}(w(\alpha, \beta, \rho_{\infty}))}{\rho - \rho_{\infty}} \\
 &\quad \cdot \lim_{\lambda \rightarrow \infty} (\cos_{\Sigma}(X_{\lambda}, e_1) - \cos_{\Sigma}(-X_{\infty}, e_1)) \|X_{\lambda}\|_{\Sigma} \\
 &= \frac{\partial \Phi^{-1}(w(\alpha, \beta, \rho_{\infty}))}{\partial \rho} \cos_{\Sigma}(e_1, X_M) \|X_M\|_{\Sigma} \\
 &= -\frac{\frac{1}{2\pi\sqrt{1-\rho_{\infty}^2}} \exp\left(-\frac{\Phi^{-1}(\alpha)^2}{2(1-\rho_{\infty}^2)}\right)}{\Phi\left(\frac{\Phi^{-1}(\alpha)}{\sqrt{1-\rho_{\infty}^2}}\right)} \cos_{\Sigma}(e_1, X_M) \|X_M\|_{\Sigma}.
 \end{aligned}$$

□

Theorem 3.1 is a direct consequence of Lemma 4.4. Indeed:

Point 1. follows from the fact that each member of the family X_{λ} , $\lambda \in [0, \infty)$, is fulfilling the constraints of the optimization problem (38). Since due to lemma 4.4 our target function $\text{CoVaR}(X_{\lambda}|Y)$ tends to $-\infty$ when $\lambda \rightarrow \infty$, the optimization problem has no solution.

Point 2: Since due to lemma 4.4 our target function $\text{CoVaR}(X_{\lambda}|Y)$ tends to $+\infty$ when $\lambda \rightarrow \infty$, there exists $\lambda_1 > 0$ such that

$$\forall \lambda \geq \lambda_1 \quad \text{CoVaR}(X_{\lambda}|Y) \geq 2 \text{CoVaR}(X_M|Y).$$

Hence the minimum is attained to some X_{λ} with $\lambda \in [0, \lambda_1]$.

Furthermore for a solution X_* of the optimization problem (38) the vector $X_* - X_M$ is parallel to X_{∞} . □

4.2.6. Proof of Theorem 3.2

We start with the following basic case:

Lemma 4.5. Let $(x_n)_{n=1}^{\infty}$ be a sequence of portfolios fulfilling the constraint $\mathbb{1}_n^T x = 1$, such that:

(i). The variance tends to infinity

$$\lim_{n \rightarrow \infty} \|x_n\|_{\Sigma}^2 = \infty;$$

(ii). The Sharpe ratio has a limit s_*

$$\lim_{n \rightarrow \infty} \frac{\mu^T x_n}{\|x_n\|_{\Sigma}} = s_*;$$

(iii). The correlation coefficient with R_1 has a limit ρ_*

$$\lim_{n \rightarrow \infty} \cos_{\Sigma}(e_1, x_n) = \rho_*.$$

Then

$$\lim_{n \rightarrow +\infty} \text{CoVaR}(x_n^T R | R_1) = \begin{cases} -\infty & \text{if } \beta > \frac{1}{\alpha} C(\alpha, \Phi(-s_*), \rho_*), \\ +\infty & \text{if } \beta < \frac{1}{\alpha} C(\alpha, \Phi(-s_*), \rho_*). \end{cases}$$

Proof.

We observe that

$$\frac{\text{CoVaR}(x_n^T R | R_1)}{\|x_n\|_\Sigma} = -\frac{\mu^T x_n}{\|x_n\|_\Sigma} - \Phi^{-1}(w(\alpha, \beta, \cos_\Sigma(e_1, x_n))) \xrightarrow{n \rightarrow \infty} -s_* - \Phi^{-1}(w(\alpha, \beta, \rho_*)). \quad (64)$$

Since

$$\begin{aligned} \text{sgn}(-s_* - \Phi^{-1}(w(\alpha, \beta, \rho_*))) &= \text{sgn}(\Phi(-s_*) - w(\alpha, \beta, \rho_*)) \\ &= \text{sgn}(C(\alpha, \Phi(-s_*), \rho_*) - C(\alpha, w(\alpha, \beta, \rho_*), \rho_*)) \\ &= \text{sgn}(C(\alpha, \Phi(-s_*), \rho_*) - \alpha\beta), \end{aligned} \quad (65)$$

we get for $\beta \neq \frac{1}{\alpha}C(\alpha, \Phi(-s_*), \rho_*)$

$$\lim_{n \rightarrow +\infty} \text{CoVaR}(x_n^T R | R_1) = \text{sgn}(C(\alpha, \Phi(-s_*), \rho_*) - \alpha\beta) \cdot \infty. \quad (66)$$

□

If we add to assumptions from the above lemma, the additional assumption that the sequence x_n belongs to the critical half-plane, than we get the dependence between the limiting values s_* and ρ_* .

Lemma 4.6. *Let*

$$x_n = X_M(E_n) - \lambda_n X_\infty, \quad E_n, \lambda_n > 0.$$

Then the assumptions from Lemma 4.5 imply that:

$$s_*^2 \leq \frac{1}{\|X_d\|_\Sigma^2};$$

and

$$\rho_* = s_* \|X_d\|_\Sigma \cos_\Sigma(e_1, X_d) - \sqrt{1 - s_*^2 \|X_d\|_\Sigma^2} \cos_\Sigma(e_1, X_\infty).$$

Proof.

From the definition of the Sharpe ratio we get

$$s_n = \frac{E_n}{\sqrt{\|X_M(E_n)\|_\Sigma^2 + \lambda_n^2 \|X_\infty\|^2}}.$$

Thus

$$\frac{\lambda_n^2}{E_n^2} \|X_\infty\|^2 = \frac{1}{s_n^2} - \frac{\|X_M(E)\|_\Sigma^2}{E^2}.$$

This implies that the ratio $\frac{\lambda_n}{E_n}$ has a limit. Basing on the formula (48) we get

$$\|X_\infty\|^2 \lim_{n \rightarrow \infty} \frac{\lambda_n^2}{E_n^2} = \frac{1}{s_*^2} - \frac{\gamma_M}{\det(G)}. \quad (67)$$

Obviously the above implies that

$$s_*^2 \leq \frac{\det(G)}{\gamma_M} = \frac{1}{\|X_d\|_\Sigma^2}.$$

Next

$$\rho_n = \cos_{\Sigma}(e_1, x_n) = \frac{e_1^T \Sigma X_M(E_n) - \lambda_n e_1^T \Sigma X_{\infty}}{\|e_1\|_{\Sigma} \|X_n\|_{\Sigma}} \quad (68)$$

$$= \frac{e_1^T \Sigma X_M(E_n) - \lambda_n e_1^T \Sigma X_{\infty}}{\|e_1\|_{\Sigma} \frac{E_n}{s_n}} \quad (69)$$

$$= \frac{s_n}{\|e_1\|_{\Sigma}} \left(\frac{e_1^T \Sigma X_M(E_n)}{E_n} - \frac{\lambda_n}{E_n} e_1^T \Sigma X_{\infty} \right) \quad (70)$$

$$\xrightarrow{n \rightarrow \infty} \frac{s_*}{\|e_1\|_{\Sigma}} \left(e_1^T \Sigma X_d - \sqrt{\frac{1}{s_*^2} - \frac{\gamma_M}{\det(G)}} \cos_{\Sigma}(e_1, X_{\infty}) \|e_1\|_{\Sigma} \right) \quad (71)$$

$$= s_* \cos_{\Sigma}(e_1, X_d) \|X_d\|_{\Sigma} - \sqrt{1 - s_*^2 \frac{\gamma_M}{\det(G)}} \cos_{\Sigma}(e_1, X_{\infty}) \quad (72)$$

$$= s_* \|X_d\|_{\Sigma} \cos_{\Sigma}(e_1, X_d) - \sqrt{1 - s_*^2 \|X_d\|_{\Sigma}^2} \cos_{\Sigma}(e_1, X_{\infty}). \quad (73)$$

□

In order to prove point 1 of Theorem 3.2 we provide for any $\xi \in [0, 1]$ such that

$$\beta > \frac{1}{\alpha} C \left(\alpha, \frac{\xi}{\|X_d\|_{\Sigma}}, \xi \cos_{\Sigma}(e_1, X_d) - \sqrt{1 - \xi^2} \cos_{\Sigma}(e_1, X_{\infty}) \right),$$

a sequence x_n such that

$$\lim_{n \rightarrow \infty} \text{CoVaR}(x_n^T R, R_1) = -\infty. \quad (74)$$

We put

$$x_n = X_M(n) - \sqrt{\frac{1}{\xi^2} - 1} \frac{\|X_d\|_{\Sigma}}{\|X_{\infty}\|_{\Sigma}} n X_{\infty}.$$

We get

$$\|x_n\|_{\Sigma}^2 = \|X_M(n)\|_{\Sigma}^2 + n^2 \left(\frac{1}{\xi^2} - 1 \right) \|X_d\|_{\Sigma}^2,$$

and

$$s_*^2 = \lim_{n \rightarrow \infty} \frac{n^2}{\|x_n\|_{\Sigma}^2} = \lim_{n \rightarrow \infty} \frac{1}{\frac{\|X_M(n)\|_{\Sigma}^2}{n^2} + \left(\frac{1}{\xi^2} - 1 \right) \|X_d\|_{\Sigma}^2} \quad (75)$$

$$= \frac{1}{\|X_d\|_{\Sigma} + \left(\frac{1}{\xi^2} - 1 \right) \|X_d\|_{\Sigma}^2} = \frac{\xi^2}{\|X_d\|_{\Sigma}^2}. \quad (76)$$

Hence, due to Lemma 4.6

$$\begin{aligned} \rho_* &= s_* \sqrt{\frac{\gamma_M}{\det(G)}} \cos_{\Sigma}(e_1, X_d) - \sqrt{1 - s_*^2 \frac{\gamma_M}{\det(G)}} \cos_{\Sigma}(e_1, X_{\infty}) \\ &= \xi \cos_{\Sigma}(e_1, X_d) - \sqrt{1 - \xi^2} \cos_{\Sigma}(e_1, X_{\infty}) \end{aligned} \quad (77)$$

Finally Lemma 4.5 implies that the limit from (74) equals $-\infty$.

Next we prove point 2 of Theorem 3.2. We assume that

$$\beta < \min \left(\frac{1}{\alpha} C \left(\alpha, \Phi \left(\frac{-\xi}{\|X_d\|_{\Sigma}} \right), \xi \cos_{\Sigma}(e_1, X_d) - \sqrt{1 - \xi^2} \cos_{\Sigma}(e_1, X_{\infty}) \right) \mid \xi \in [0, 1] \right). \quad (78)$$

We denote by $\inf_*$ the infimum of the CoVaR,

$$\begin{aligned} \inf_* &= \inf \{ \text{CoVaR}_{\alpha, \beta}(x^T R, R_1) \mid x^T \mathbb{1}_n = 1 \} \\ &= \inf(-x^T \mu - \Phi^{-1}(w(\alpha, \beta, \cos_{\Sigma}(e_1, x)) \|x\|_{\Sigma}) \mid x^T \mathbb{1}_n = 1). \end{aligned} \quad (79)$$

Let the sequence $(x_n)_1^{\infty}$ be a sequence of portfolios approaching $\inf_*$,

$$x_n^T \mathbb{1}_n = 1, \quad \lim_{n \rightarrow \infty} (-x_n^T \mu - \Phi^{-1}(w(\alpha, \beta, \cos_{\Sigma}(e_1, x_n)) \|x_n\|_{\Sigma})) = \inf_*.$$

We show that the assumption (78) implies that the sequence x_n must be bounded. Indeed, otherwise there would exist a subsequence x_{n_k} such that

$$\|x_{n_k}\|_{\Sigma} \rightarrow +\infty.$$

Since the Sharpe ratios

$$s_{n_k} = \frac{x_{n_k}^T \mu}{\|x_{n_k}\|_{\Sigma}}$$

and correlation coefficients

$$\cos_{\Sigma}(e_1, x_n) = \frac{e_1^T \Sigma x_n}{\|e_1\|_{\Sigma} \|x_n\|_{\Sigma}}$$

belong to a bounded sets (see Lemma 4.2) we might select the subsequence x_{n_k} in such a way that these ratios and coefficients have limits. We denote these limits by s_* and ρ_*

$$s_* = \lim_{k \rightarrow \infty} \frac{x_{n_k}^T \mu}{\|x_{n_k}\|_{\Sigma}}, \quad (80)$$

$$\rho_* = \lim_{k \rightarrow \infty} \cos_{\Sigma}(e_1, x_{n_k}) \quad (81)$$

Due to Lemma 4.3 ($t = \|x_{n_k}\|_{\Sigma}$, $E = x_{n_k}^T \mu$) and Lemma 4.6

$$\begin{aligned} \rho_* &= \lim_{k \rightarrow \infty} \cos_{\Sigma}(e_1, x_{n_k}) \geq \lim_{k \rightarrow \infty} \cos_{\Sigma}(e_1, X_M(x_{n_k}^T \mu) - \lambda_k X_{\infty}) \\ &= s_* \|X_d\|_{\Sigma} \cos_{\Sigma}(e_1, X_d) - \sqrt{1 - s_*^2 \|X_d\|_{\Sigma}^2} \cos_{\Sigma}(e_1, X_{\infty}). \end{aligned} \quad (82)$$

Thus the assumption (78) and Lemma 4.5 imply that

$$\lim_{k \rightarrow \infty} \text{CoVaR}(x_{n_k}^T R, R_1) = +\infty. \quad (83)$$

which contradicts the assumption that the sequence x_n is approaching the infimum of CoVaR.

5. Examples

We consider the following data:

$$\Sigma = \begin{pmatrix} 1 & \frac{1}{5} & 1 & -1 \\ \frac{1}{5} & 1 & 0 & -1 \\ 1 & 0 & 9 & 0 \\ -1 & -1 & 0 & 4 \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 3 \end{pmatrix}$$

We get

$$\det \Sigma = 17.16,$$

$$G = \frac{1}{17.16} \begin{pmatrix} 587.6 & 217.36 \\ 217.36 & 82.28 \end{pmatrix}.$$

$$X_M(E)^T = \frac{1}{64.24} ((142, -98, 25.36, -5.12) + E(-44, 46.2, -10.12, 7.92)),$$

$$X_\infty^T = \frac{1}{64.24} (10.24, 5.6, -5.12, -10.72).$$

Finally

$$\cos_\Sigma(e_1, X_\infty) = 2\sqrt{\frac{53}{803}}.$$

We fix E and study the halflife of portfolios

$$X_\lambda = X_M(E) - \lambda X_\infty, \quad \lambda \geq 0.$$

Example 5.1. We have $\rho_\infty = -\cos_\Sigma(e_1, X_\infty) = -2\sqrt{\frac{53}{803}}$. We choose

$$\alpha = \Phi\left(-\frac{3}{5}\right), \beta = \frac{\Phi_2(-\frac{3}{5}, 0; \rho_\infty)}{\Phi(-\frac{3}{5})}, E = \frac{637}{220}.$$

This way we get

$$\beta = \frac{1}{\alpha} C\left(\alpha, \frac{1}{2}, -\cos_\Sigma(e_1, X_\infty)\right), \quad \Phi^{-1}(w(\rho_\infty)) = 0, \text{ and } \cos_\Sigma(X_M, e_1) = 0.$$

Therefore

$$\text{CoVaR}(X_\lambda | Y) = -\frac{637}{220} - \Phi^{-1}(w(\rho(\lambda))) \sqrt{\frac{212\lambda^2}{803} + \frac{2561}{8800}}$$

where

$$\rho(\lambda) = \cos_\Sigma(X_\lambda, e_1) = \frac{-212\lambda}{803\sqrt{\frac{212\lambda^2}{803} + \frac{2561}{8800}}} > \rho_\infty,$$

for any $\lambda > 0$ and in consequence $-\Phi^{-1}(w(\rho(\lambda))) > 0$. From Lemma 4.4, we have that:

$$\lim_{\lambda \rightarrow \infty} \text{CoVaR}(X_\lambda | Y) = -\frac{637}{220} < \text{CoVaR}(X_\lambda | Y)$$

Thus the global minimum cannot exist. We illustrate this at Figure 3.

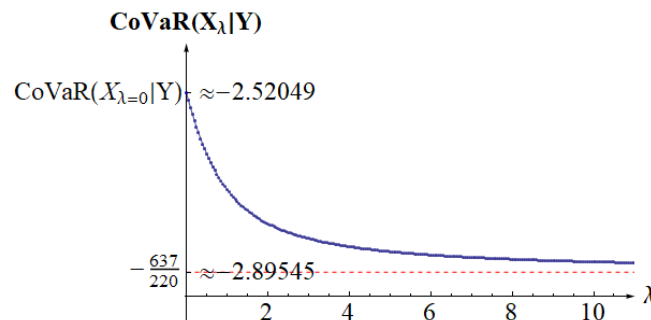


Figure 3. Numerically calculated CoVaR as the function of λ

Example 5.2. We consider the same data. However, now we choose

$$\alpha = \Phi\left(-\frac{3}{5}\right), \beta = \frac{\Phi_2\left(-\frac{3}{5}, 0; \rho_\infty - \frac{1}{100}\right)}{\Phi\left(-\frac{3}{5}\right)}.$$

This way we get $\beta < \frac{1}{\alpha} C\left(\alpha, \frac{1}{2}, -\cos_\Sigma(e_1, X_\infty)\right)$ and the optimization problem has a solution. We have

$$\text{CoVaR}(X | Y) = -E - \Phi^{-1}(w(\rho(\lambda))) \sqrt{\frac{2057E^2 - 10868E + 424\lambda^2 + 14690}{1606}}$$

where $\rho(\lambda) = \frac{\sqrt{2}(-660E - 212\lambda + 1911)}{\sqrt{803(2057E^2 - 10868E + 424\lambda^2 + 14690)}}$.

For $E = 2$ we have $\rho(x_M, e_1) > 0$. We get numerically calculated minimum $-0.815187 \dots$ for $\lambda = 4.211162 \dots$ as illustrated at Figure 4.

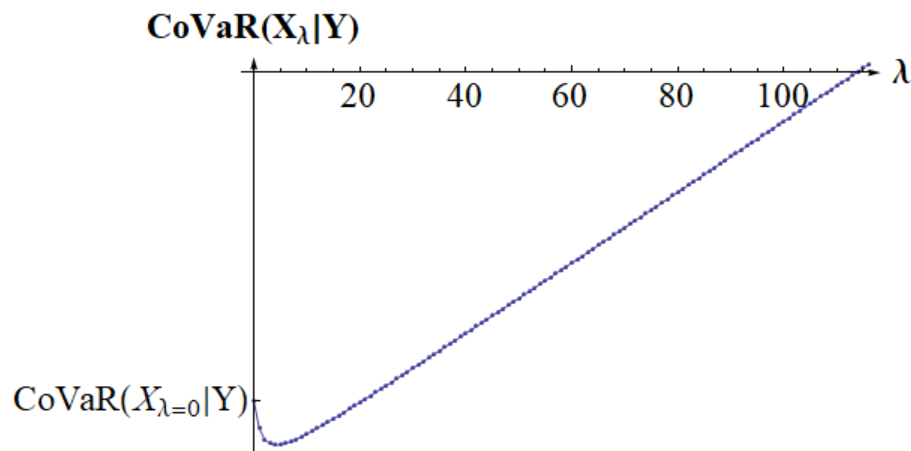


Figure 4. Numerically calculated CoVaR for $E = 2$

For $E = -1$ we have $\text{CoVaR}|_{\lambda=0} > 0$. We get numerically calculated minimum $6.254844 \dots$ for $\lambda = 24.788285 \dots$ as illustrated at Figure 5. As the reader can guess, it has been easier to subtract a certain constant from CoVaR to show how the new function is positive for $\lambda > 210$.

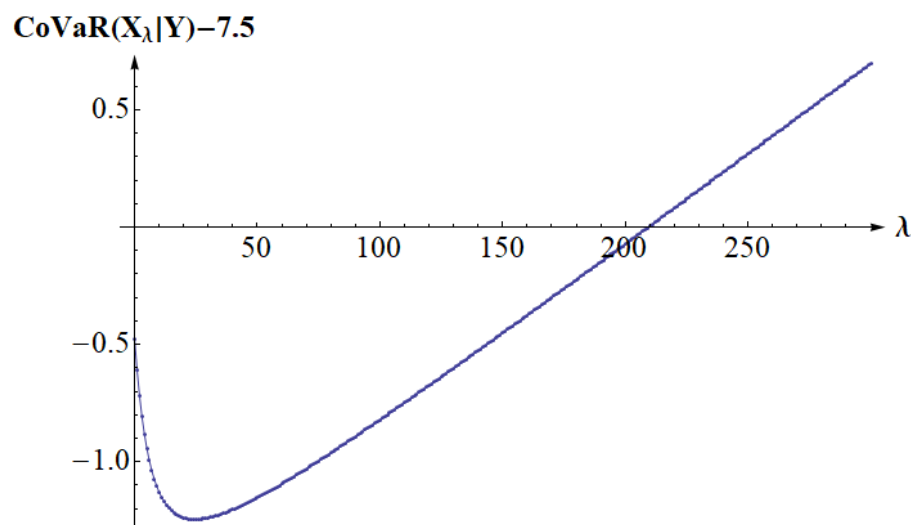


Figure 5. Numerically calculated CoVaR -7.5 for $E = -1$

For $E = \frac{637}{220}$ we have $\rho(x_M, e_1) = 0$. We get numerically calculated minimum $-2.812375 \dots$ for $\lambda = 5.271369 \dots$ as illustrated at Figure 6.

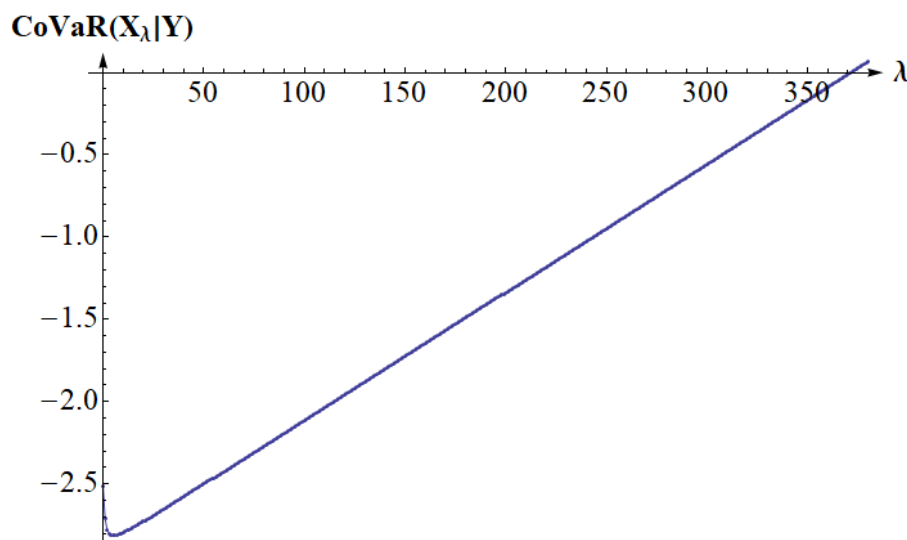


Figure 6. Numerically calculated CoVaR for $E = \frac{637}{220}$

For $E = 3$ we get $\rho(x_M, e_1) < 0$. We get numerically calculated minimum $-3.036088 \dots$ for $\lambda = 5.991312 \dots$ as illustrated at Figure 7.

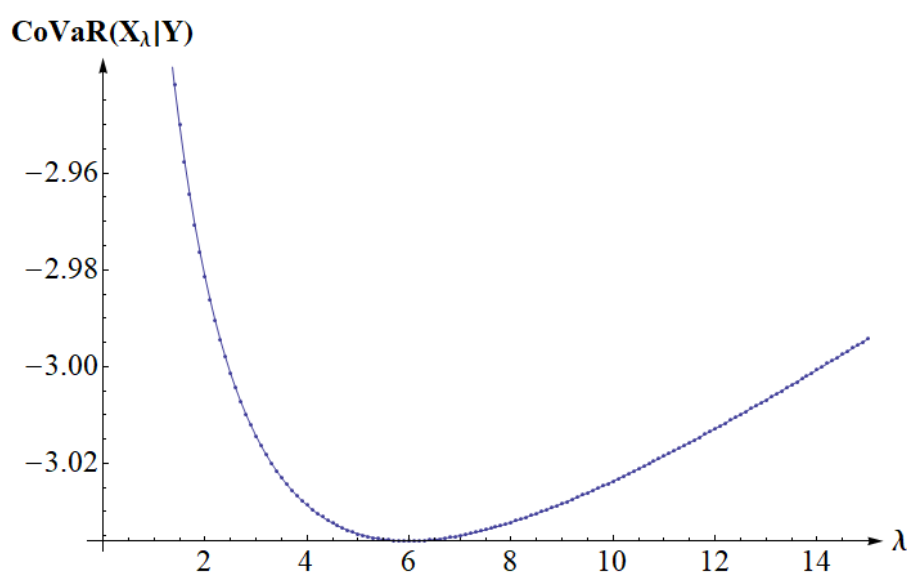


Figure 7. Numerically calculated CoVaR for $E = 3$

All the numerical calculations seem to point to the existence of unique solution under condition from Theorem 3.2.

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