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Article

Average case (s,t)-weak tractability of L_2-approximation with weighted covariance kernels

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Abstract: We study multivariate L_2-approximation problem APP_d defined over a Banach space in the average case setting. The space is equipped with a zero-mean Gaussian measure with weighted covariance kernel which depends on parameter sequences $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathbb{N}}$ with $1 < \alpha_1 \le \alpha_2 \le \cdots$ and $1 \ge \beta_1 \ge \beta_2 \ge \cdots > 0$. In this paper two interesting weighted covariance kernels are considered, which model the importance of the covariance kernels. Under the absolute error criterion or the normalized error criterion, we discuss (s,t)-weak tractability of the L_2-approximation problem $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ with the above two weighted covariance kernels for some positive numbers s and t in the average case setting, where (s,t)-weak tractability means that how the information complexity depends on d and ε^{-1} for large dimension d and small threshold ε. In particular, for all $s > 0$ and $t \in (0,1)$ we find the matching sufficient and necessary condition on the parameter sequences $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathbb{N}}$ to obtain average case (s,t)-weak tractability under the absolute error criterion or the normalized error criterion.

Keywords: L_2-approximation; weighted covariance kernels; information complexity; (s,t)-weak tractability

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1. Introduction

This paper is devoted to studying d-variate problems S_d with huge d. This is a hot topic in computational finance (see [1]) and computational chemistry (see [2]). We consider multivariate problem $S = \{S_d : F_d \to G_d\}_{d\in\mathbb{N}}$ in the average case setting, where F_d is a Banach space with a zero-mean Gaussian measure μ_d, and G_d is a Hilbert space. We approximate S_d by arbitrarily n continuous linear functionals. Let $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$. In the average case setting, for the absolute error criterion (ABS) or the normalized error criterion (NOR), information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ is defined as the minimal number of continuous linear functionals to approximate the multivariate problem S_d with the threshold less than $\varepsilon \mathrm{CRI}_d^{1/2}$, where $\mathrm{X} \in \{\mathrm{ABS}, \mathrm{NOR}\}$ and

$$\mathrm{CRI}_d := \begin{cases} 1, & \text{for } \ \mathrm{X} = \mathrm{ABS}, \\ \int_{F_d} \|S_d(f)\|_{G_d}^2 \mu_d(df), & \text{for } \ \mathrm{X} = \mathrm{NOR}. \end{cases}$$

In 1994, the notion of tractability was first introduced to describe the behavior of the information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ when d tends to infinity and ε tends to zero (see [3]). If the information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ is the function of d and ε^{-1} for large d and small ε, then the problem S is called algebraic tractability. In the average case setting, there are many papers discussing the algebraic tractability such as strongly polynomial tractability, polynomial tractability, quasi-polynomial tractability, unform weak tractability and (s,t)-weak tractability; see [4–6].

Recently, some authors are interested in algebraic tractability of multivariate approximation problems from Banach spaces equipped with zero-mean Gaussian measures with weighted covariance kernels in the average case setting. The weights can model how important the covariance kernels are. Some special weights were investigated such as analysis Korobov weights, Korobov weights, Euler weights, Wiener weights and Gaussian weights. In the average case setting, under ABS or NOR [7–9] discussed the L_2-approximation problem defined over a Banach space whose covariance kernel has analysis Korobov weight. [7] obtained the compete sufficient and necessary conditions for weak

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tractability, strongly polynomial tractability, polynomial tractability, quasi-polynomial tractability and uniform weak tractability; [8,9] got the compete sufficient and necessary conditions for (s,t)-weak tractability. For the L_2-approximation problem from a Banach space whose covariance kernel has Korobov weight, the matching sufficient and necessary conditions for weak tractability, strongly polynomial tractability and polynomial tractability under NOR in [10], quasi-polynomial tractability under NOR in [10,11], uniform weak tractability under ABS or NOR in [12], strongly polynomial tractability and polynomial tractability under ABS in [13], and (s,t)-weak tractability under ABS or NOR in [9] were studied in the average case setting. In the average case setting, for the L_2-approximation problem with Euler covariance kernel and Wiener covariance kernel under NOR, the sufficient and necessary conditions for weak tractability, strongly polynomial tractability, polynomial tractability and quasi-polynomial tractability in [14], uniform weak tractability in [15], and (s,t)-weak tractability in [9,16] were gotten. For the L_2-approximation problem with Gaussian covariance kernel under ABS or NOR, the matching necessary and sufficient conditions for strongly polynomial tractability and polynomial tractability in [17], quasi-polynomial tractability in [18], and weak tractability, uniform weak tractability and (s,t)-weak tractability in [19] were obtained in the average case setting.

It is interesting that different methods are used to solve (s,t)-weak tractability for different fixed s and t in the same article. Hence, in this paper we study (s,t)-weak tractability of multivariate L_2-approximation problems defined over Banach spaces with two different weighted covariance kernels in the average case setting. Those two weights come from the ideas of analysis Korobov weights (see [7–9]), Korobov weights (see [9–13]), and Gaussian ANOVA weights (see [20–23]). For ABS or NOR we obtain a compete sufficient and necessary condition on (s,t)-weak tractability of the above two L_2-approximation problems with all $s>0$ and $t\in(0,1)$.

We summarize the contents of this paper as follows. In Section 2 we present a general multivariate approximation problem equipped with a zero-mean Gaussian measure in the average case setting, and give some definitions about algebraic tractability. Section 3 discusses L_2-approximation problems with weighted covariance kernels in the average case setting. Section 3.1 and Section 3.2 introduce a variant of the korobov covariance kernel and a variant of the Gaussian ANOVA covariance kernel, respectively. In Section 4 we investigate sufficient and necessary conditions for (s,t)-weak tractability of the L_2-approximation problems with the above two weighted covariance kernels for all $s>0$ and $t\in(0,1)$ in the average case setting, and then give the proof. Section 5 provides a summary of this paper.

2. Average Case Algebraic Tractability of Multivariate Approximation Problems

First, some notions on the paper: we define $\mathbb{N}=\{1,2,\ldots\}$, $\mathbb{N}_0=\{0,1,\ldots\}$, $\ln^+ x=\max\{1,\ln x\}$, $\lceil a\rceil$ for the smallest integer not less than a.

We recall some concepts of multivariate approximation problems from functions defined over Banach spaces with zero-mean Gaussian measures in the average case setting; see [4].

We consider multivariate approximation problem $S_d: F_d\to G_d$ for each $d\in\mathbb{N}$, where F_d is a Banach space equipped with a zero-mean Gaussian measure μ_d, and G_d is a Hilbert space with an inner product $\langle\cdot,\cdot\rangle_{G_d}$. For every $f\in F_d$ we approximate $S_d(f)$ by algorithm of the form

$$A_{n,d}(f)=\Phi_{n,d}(L_1(f),\ldots,L_n(f)), \tag{1}$$

where $L_1,L_2,\ldots,L_n$ are continuous linear functionals on F_d, and $\Phi_{n,d}:\mathbb{R}^n\to G_d$ is an arbitrary mapping. We set $A_{0,d}=0$.

In this paper we approximate S_d in the average case setting. The average case error $e(A_{n,d})$ of the algorithm $A_{n,d}$ is defined as

$$e(A_{n,d}):=\left(\int_{F_d}\|S_d(f)-A_{n,d}(f)\|_{G_d}\mu_d(df)\right)^{\frac{1}{2}}.$$

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For any $n \in \mathbb{N}_0$ the nth minimal average case error is defined to be

$$e(n, S_d) := \inf_{A_{n,d}} e(A_{n,d}),$$

where the infimum is taken over all linear algorithms $A_{n,d}$ of the form (1). Then for $n = 0$ the error

$$e(0, S_d) = \left(\int_{F_d} \|S_d(f)\|_{G_d} \mu_d(df) \right)^{\frac{1}{2}}$$

is called the average case initial error. If there exists an algorithm $A^{\square}_{n,d}$ of the form (1) such that

$$e(A^{\square}_{n,d}) = e(n, S_d),$$

we call $A^{\square}_{n,d}$ the nth optimal algorithm of S_d.

Let $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$. Under the absolute error criterion (ABS) or the normalized error criterion (NOR) we define the information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ for $\mathrm{X} \in \{\mathrm{ABS}, \mathrm{NOR}\}$ as

$$n^{\mathrm{X}}(\varepsilon, S_d) := \min\left\{ n \in \mathbb{N}_0 : e(n, S_d) \le \varepsilon \mathrm{CRI}_d^{1/2} \right\},$$

where

$$\mathrm{CRI}_d := \begin{cases} 1, & \text{for } \mathrm{X} = \mathrm{ABS}, \\ e^2(0, S_d), & \text{for } \mathrm{X} = \mathrm{NOR}. \end{cases}$$

Recall some definitions of algebraic tractability (see [4–6]). Let $S = \{S_d\}_{d \in \mathbb{N}}$. For $\mathrm{X} \in \{\mathrm{ABS}, \mathrm{NOR}\}$, we say that

- S is strongly polynomially tractable iff there are non-negative numbers C and p such that

$$n^{\mathrm{X}}(\varepsilon, S_d) \le C\varepsilon^{-p} \ \text{ for all } \ \varepsilon \in (0, 1),\ d = 1, 2, \dots.$$

- S is polynomially tractable iff there are non-negative numbers C, p, and q such that

$$n^{\mathrm{X}}(\varepsilon, S_d) \le Cd^q\varepsilon^{-p} \ \text{ for all } \ \varepsilon \in (0, 1),\ d = 1, 2, \dots.$$

- S is quasi-polynomially tractable iff there are numbers $C > 0$ and $t > 0$ such that

$$n^{\mathrm{X}}(\varepsilon, S_d) \le C \exp\left(t(1 + \ln d)(1 + \ln \varepsilon^{-1}) \right) \ \text{ for all } \ \varepsilon \in (0, 1),\ d = 1, 2, \dots.$$

- S is uniform weakly tractable iff for all $s, t > 0$,

$$\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n^{\mathrm{X}}(\varepsilon, S_d)}{\varepsilon^{-s} + d^t} = 0.$$

- S is weakly tractable iff

$$\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n^{\mathrm{X}}(\varepsilon, S_d)}{\varepsilon^{-1} + d} = 0.$$

- S is (s, t)-weakly tractable for fixed $s > 0$ and $t > 0$ iff

$$\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n^{\mathrm{X}}(\varepsilon, S_d)}{\varepsilon^{-s} + d^t} = 0.$$

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Obviously, we have the relationships between the above algebraic tractability notions:

$$\begin{aligned}\text{strongly polynomial tractability} &\Rightarrow \text{polynomial tractability}\\ &\Rightarrow \text{quasi-polynomial tractability}\\ &\Rightarrow \text{uniform weak tractability}\\ &\Rightarrow (s,t)\text{-weak tractability for all } s,t>0,\end{aligned}$$

and

$$\text{weak tractability} \Leftrightarrow (1,1)\text{-weak tractability}.$$

We will discuss the nth minimal average case error $e(n, S_d)$ and the information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ more explicitly; see [4, Section 4.3].

Let $C_{\mu_d} : (F_d)^* \to F_d$ be the covariance operator of μ_d (see [4, p. 357-360]), and $\nu_d = \mu_d S_d^{-1}$ be the induced measure of μ_d. Then the induced measure ν_d is a zero-mean Gaussian measure on the Borel sets of G_d with covariance operator $C_{\nu_d} : G_d \to G_d$ given by

$$C_{\nu_d} = S_d C_{\mu_d} (S_d)^*,$$

where $(S_d)^*$ is the operator dual to S_d. The eigenpairs $\{(\lambda_{d,i}, \eta_{d,i})\}_{i\in\mathbb{N}}$ of C_{ν_d} satisfy

$$C_{\nu_d}\eta_{d,i} = \lambda_{d,i}\eta_{d,i} \ \text{ with } \ \lambda_{d,1} \ge \lambda_{d,2} \ge \cdots \ge 0,$$

where $\langle \eta_{d,i}, \eta_{d,j}\rangle_{G_d} = \delta_{i,j}$, and

$$\delta_{i,j} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

Then for $n \in \mathbb{N}_0$, we have that the nth optimal algorithm $A_{n,d}^{\square}$ of S_d satisfies

$$A_{n,d}^{\square} = \sum_{i=1}^{n} \langle S_d(f), \eta_{d,i}\rangle_{G_d} \eta_{d,i},$$

and the nth minimal average case error $e(n, S_d)$ has the form

$$e(n, S_d) = e(A_{n,d}^{\square}) = \left(\sum_{i=n+1}^{\infty} \lambda_{d,i} \right)^{\frac{1}{2}};$$

see [4, Section 6.1]. Hence for the absolute error criterion (ABS) or the normalized error criterion (NOR) the information complexity has the form

$$n^{\mathrm{X}}(\varepsilon, S_d) = \min\left\{ n \in \mathbb{N}_0 : \sum_{i=n+1}^{\infty} \lambda_{d,i} \le \varepsilon^2 \mathrm{CRI}_d \right\}, \tag{2}$$

where

$$\mathrm{CRI}_d = \begin{cases} 1, & \text{for X} = \text{ABS}, \\ \sum_{i=1}^{\infty} \lambda_{d,i}, & \text{for X} = \text{NOR}. \end{cases} \tag{3}$$

It is obvious that the algebraic tractability of $S = \{S_d\}$ depends on the behavior of the eigenvalues $\{\lambda_{d,i}\}_{i\in\mathbb{N}}$. Next, we present some relationships between the information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ and the eigenvalues $\{\lambda_{d,i}\}_{i\in\mathbb{N}}$ for ABS and NOR in the average setting.

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For any $\tau \in (0,1)$, since

$$\lambda_{d,j} \le \left(\frac{\sum_{i=1}^{j} \lambda_{d,i}^{\tau}}{j}\right)^{\frac{1}{\tau}} \le \left(\frac{\sum_{i=1}^{\infty} \lambda_{d,i}^{\tau}}{j}\right)^{\frac{1}{\tau}},$$

we have by (2) that

$$\begin{aligned} \varepsilon^2 \mathrm{CRI}_d &< \sum_{j=n^{\mathrm{X}}(\varepsilon,S_d)}^{\infty} \lambda_{d,j} \le \left(\sum_{i=1}^{\infty} \lambda_{d,i}^{\tau}\right)^{\frac{1}{\tau}} \sum_{j=n^{\mathrm{X}}(\varepsilon,S_d)}^{\infty} \frac{1}{j^{\frac{1}{\tau}}} \\ &\le \left(\sum_{i=1}^{\infty} \lambda_{d,i}^{\tau}\right)^{\frac{1}{\tau}} \int_{n^{\mathrm{X}}(\varepsilon,S_d)-1}^{\infty} \frac{1}{x^{\frac{1}{\tau}}} \\ &= \left(\sum_{i=1}^{\infty} \lambda_{d,i}^{\tau}\right)^{\frac{1}{\tau}} \frac{\tau}{(1-\tau)(n^{\mathrm{X}}(\varepsilon,S_d)-1)^{\frac{1-\tau}{\tau}}}. \end{aligned}$$

It means

$$n^{\mathrm{X}}(\varepsilon,S_d) \le \left\lceil \left(\frac{\tau}{1-\tau} \frac{\left(\sum_{i=1}^{\infty} \lambda_{d,i}^{\tau}\right)^{\frac{1}{\tau}}}{\mathrm{CRI}_d}\right)^{\frac{\tau}{1-\tau}} \varepsilon^{\frac{-2\tau}{1-\tau}} \right\rceil. \tag{4}$$

We note that the fact $\ln\lceil x \rceil \le |\ln(2x)|$ holds for $x > 0$. Indeed, we have $\ln\lceil x \rceil < \ln(x+1) \le \ln(2x)$ for $x \ge 1$, and $\ln\lceil x \rceil = 0$ for $0 < x < 1$, which deduces $\ln\lceil x \rceil \le |\ln(2x)|$ for $x > 0$. Combing the inequality (4) and the above fact, we have

$$\begin{aligned} \ln n^{\mathrm{X}}(\varepsilon,S_d) &\le \ln \left\lceil \left(\frac{\tau}{1-\tau} \frac{\left(\sum_{i=1}^{\infty} \lambda_{d,i}^{\tau}\right)^{\frac{1}{\tau}}}{\mathrm{CRI}_d}\right)^{\frac{\tau}{1-\tau}} \varepsilon^{\frac{-2\tau}{1-\tau}} \right\rceil \\ &\le \left| \ln \left(2 \left(\frac{\tau}{1-\tau} \frac{\left(\sum_{i=1}^{\infty} \lambda_{d,i}^{\tau}\right)^{\frac{1}{\tau}}}{\mathrm{CRI}_d}\right)^{\frac{\tau}{1-\tau}} \varepsilon^{\frac{-2\tau}{1-\tau}} \right) \right| \\ &= \left| \frac{\tau}{1-\tau} \left(\ln \frac{\tau}{1-\tau} + \ln\left(\left(\sum_{i=1}^{\infty} \lambda_{d,i}^{\tau}\right)^{\frac{1}{\tau}} / \mathrm{CRI}_d \right) + 2\ln(\varepsilon^{-1}) \right) + \ln 2 \right|. \end{aligned} \tag{5}$$

3. L_2-Approximation with Weighted Covariance Kernels

Let $H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d)$ be a Banach space equipped with a zero-mean Gaussian measure μ_d with weighted covariance kernel

$$K_{W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}}(\boldsymbol{x},\boldsymbol{y}) = \int_{H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}} f(\boldsymbol{x}) f(\boldsymbol{y}) \mu_d(df) = \sum_{\boldsymbol{h} \in \mathbb{N}_0^d} W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{h}) \exp(2\pi i \boldsymbol{h} \cdot (\boldsymbol{x}-\boldsymbol{y})) \tag{6}$$

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for $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^d$, where $i = \sqrt{(-1)}$, $\boldsymbol{h} = (h_1, \ldots, h_d) \in \mathbb{N}_0^d$, $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{d} u_i v_i$, $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{R}^d$, $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d$, and $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ is the weight of the covariance kernel $K_{W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}}$. Here, $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathbb{N}}$ are parameter sequences satisfying

$$1 < \alpha_1 \le \alpha_2 \le \cdots \quad \text{and} \quad 1 \ge \beta_1 \ge \beta_2 \ge \cdots > 0. \tag{7}$$

Then the covariance operator of μ_d is given by $C_{\mu_d} : (H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d))^* \to H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d)$.

In this paper we discuss the L_2-approximation problem $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$

$$\mathrm{APP}_d : H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d) \to L_2([0,1]^d) \ \text{ with } \ \mathrm{APP}_d(f) = f, \tag{8}$$

for $f \in H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d)$. Then the covariance operator $C_{\nu_d} : L_2([0,1]^d) \to L_2([0,1]^d)$ of the induced measure $\nu_d = \mu_d \mathrm{APP}_d^{-1}$ has the form

$$(C_{\nu_d} f)(\boldsymbol{x}) = \int_{[0,1]^d} K_{W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}}(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) d\boldsymbol{y} \ \text{ for } \ \boldsymbol{x}, \boldsymbol{y} \in [0,1]^d. \tag{9}$$

By (6) and (9) we have that the eigenvalue sequence $\{\lambda_{d,i}\}_{i\in\mathbb{N}}$ with $\lambda_{d,1} \ge \lambda_{d,2} \ge \cdots \ge 0$ of C_{ν_d} is the sequence $\{W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{h})\}_{\boldsymbol{h}\in\mathbb{N}_0^d}$.

We will consider product weights

$$W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{h}) = \prod_{j=1}^{d} W_{\alpha_j,\beta_j}(h_j) \ \text{ for } \ \boldsymbol{h} = (h_1, \ldots, h_d) \in \mathbb{N}_0^d.$$

Then for any $\tau > 0$ we have

$$\sum_{i=1}^{\infty} \lambda_{d,i}^{\tau} = \sum_{\boldsymbol{h}\in\mathbb{N}_0^d} W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}^{\tau}(\boldsymbol{h}) = \sum_{\boldsymbol{h}\in\mathbb{N}_0^d} \prod_{j=1}^{d} W_{\alpha_j,\beta_j}^{\tau}(h_j) = \prod_{j=1}^{d} \sum_{h=0}^{\infty} W_{\alpha_j,\beta_j}^{\tau}(h). \tag{10}$$

We note that the weighted covariance kernels are restricted by their weights. So it is worth to investigate the weights. There are many papers discussing the Korobov weights (see [9–13]), the analysis Korobov weights (see [7–9]), and the Gaussian ANOVA weights (see [20–23]). According to ideas of the above weights we introduce two weights, which have faster decay rates than the Korobov weights and the Gaussian ANOVA weights, respectively.

3.1. A Variant of the Korobov Covariance Kernel

In this subsection we introduce a weighted covariance kernel $K_{W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}}$ with the weight $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ given by a variant of the korobov weight $\rho_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$, where $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathbb{N}}$ satisfy (7). The weight $\rho_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ is given as product form,

$$\rho_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{k}) := \prod_{j=1}^{d} \rho_{\alpha_j,\beta_j}(k_j), \quad \boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d,$$

where $\rho_{\alpha_j,\beta_j}(k_j)$ are univariate weights,

$$\rho_{\alpha,\beta}(k) := \begin{cases} 1, & \text{for } k = 0, \\ \beta A^{k^{\lceil\alpha\rceil}}, & \text{for } k \ge 1, \end{cases}$$

for fixed $A \in (0,1)$, $\alpha \in (1, +\infty)$ and $\beta \in (0,1]$. The idea of the weight $\rho_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ comes from the korobov weight (see [9–13]), and the analysis korobov weight (see [7–9]).

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The references [9–13] consider the L_2-approximation problem $\text{APP} = \{\text{APP}_d\}_{d\in\mathbb{N}}$ satisfying (8) from the Banach space $H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d)$ equipped with a zero-mean Gaussian measure μ_d whose weighted covariance kernel $K_{W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}}$ of the form (6) has the kovobov weight $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}} = r_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ of the form

$$r_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{k}) := \prod_{j=1}^{d} r_{\alpha_j,\beta_j}(k_j), \ \ \boldsymbol{k} = (k_1,\dots,k_d) \in \mathbb{N}_0^d,$$

with

$$r_{\alpha,\beta}(k) := \begin{cases} 1, & \text{for } k = 0, \\ \frac{\beta}{k^\alpha}, & \text{for } k \geq 1, \end{cases}$$

for $\alpha \in (1,+\infty)$ and $\beta \in (0,1]$, where the parameter sequences $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathbb{N}}$ satisfy (7). Using ABS or NOR the references [9–13] have solved the algebraic tractability of the above problem APP, and got the following results:

- For ABS or NOR, strongly polynomial tractability holds iff polynomial tractability holds iff

$$\liminf_{k\to\infty} \frac{\ln \frac{1}{\beta_k}}{\ln k} > 1.$$

- For NOR, quasi-polynomial tractability holds iff

$$\sup_{d\in\mathbb{N}} \frac{1}{\ln^+ d} \sum_{k=1}^{d} \beta_k \ln^+ \frac{1}{\beta_k} < \infty.$$

- For ABS or NOR, weak tractability holds iff

$$\lim_{k\to\infty} \beta_k = 0.$$

- For ABS or NOR, uniform weak tractability holds iff

$$\lim_{k\to\infty} \beta_k k^p = 0 \ \text{ for all } \ p \in (0,1).$$

- For ABS or NOR, (s,t)-weak tractability with $s > 0$ and $t > 1$ always holds.
- For ABS or NOR, (s,t)-weak tractability with $s > 0$ and $t \in (0,1)$ holds iff

$$\lim_{k\to\infty} k^{1-t} \beta_k \ln^+ \frac{1}{\beta_k} = 0.$$

Another covariance kernel is the analysis korobov covariance kernel, which is famous for its fast exponentially decaying weight. The analysis korobov weight is given as

$$\omega_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{k}) := \prod_{j=1}^{d} \omega_{\alpha_j,\beta_j}(k_j), \ \ \boldsymbol{k} = (k_1,\dots,k_d) \in \mathbb{N}_0^d,$$

with

$$\omega_{\alpha,\beta}(k) := \begin{cases} 1, & \text{for } k = 0, \\ \omega^{\beta k^\alpha}, & \text{for } k \geq 1, \end{cases}$$

for fixed $\omega \in (0,1)$, $\alpha > 0$ and $\beta > 0$, where the parameter sequences $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathbb{N}}$ satisfy

$$\inf_{k\in\mathbb{N}} \alpha_k > 0 \ \text{ and } \ 0 < \beta_1 \leq \beta_2 \leq \cdots.$$

In the average case setting, the references [7–9] investigate the algebraic tractability of the L_2-approximation problem $\text{APP} = \{\text{APP}_d\}_{d\in\mathbb{N}}$ satisfying (8) from the Banach space $H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d)$ equipped with a zero-

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mean Gaussian measure μ_d whose weighted covariance kernel $K_{W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}}$ of the form (6) has the analysis kovobov weight $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}} = \omega_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$. They obtained that (see [7–9]):

- For ABS or NOR, strongly polynomial tractability holds iff polynomial tractability holds iff

$$\liminf_{k\to\infty} \frac{\beta_k}{\ln k} > \frac{1}{\ln \omega^{-1}}.$$

- For ABS or NOR, weak tractability holds iff

$$\lim_{k\to\infty} \beta_k = \infty.$$

- For NOR, quasi-polynomial tractability holds iff

$$\sup_{d\in\mathbb{N}} \frac{1}{\ln^+ d} \sum_{k=1}^{d} \beta_k \omega^{\beta_k} < \infty.$$

- For ABS or NOR, uniform weak tractability holds iff

$$\lim_{k\to\infty} \omega^{\beta_k} k^p = 0 \quad \text{for all} \quad p \in (0,1).$$

- For ABS or NOR, (s,t)-weak tractability with $s > 0$ and $t > 1$ always holds.
- For ABS or NOR, (s,t)-weak tractability with $s > 0$ and $t \in (0,1)$ holds iff

$$\lim_{k\to\infty} k^{1-t} \beta_k \omega^{\beta_k} = 0.$$

Remark 1. *We note that the variant of the korobov weight $\rho_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ descends faster than the korobov weight $r_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$, but slower than the analysis korobov weight $\omega_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$.*

3.2. A Variant of the Gaussian ANOVA Covariance Kernel

In this subsection, we present a weighted covariance kernel $K_{W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}}$ with the weight $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ given as a variant of the Gaussian ANOVA weight $\sigma_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$, where $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathbb{N}}$ satisfy (7). The weight $\sigma_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ is of product form, and determined by

$$\sigma_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{k}) := \prod_{j=1}^{d} \sigma_{\alpha_j,\beta_j}(k_j),$$

where $\sigma_{\alpha_j,\beta_j}(k_j)$ are univariate weights,

$$\sigma_{\alpha,\beta}(k) := \begin{cases} 1, & \text{for } k = 0, \\ \beta A^{k!}, & \text{for } 1 \le k < \lceil \alpha \rceil, \\ \beta A^{k!/(k-\lceil \alpha \rceil)!}, & \text{for } k \ge \lceil \alpha \rceil, \end{cases}$$

for fixed $A \in (0,1)$, $\alpha \in (1,+\infty)$ and $\beta \in (0,1]$.

The weight $\sigma_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ is similar but different with the Gaussian ANOVA weight $\psi_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ given by

$$\psi_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{k}) := \prod_{j=1}^{d} \psi_{\alpha_j,\beta_j}(k_j)$$

with

$$\psi_{\alpha,\beta}(k) := \begin{cases} 1, & \text{for } k = 0, \\ \frac{\beta}{k!}, & \text{for } 1 \le k < \lceil \alpha \rceil, \\ \frac{\beta (k-\lceil \alpha \rceil)!}{k!}, & \text{for } k \ge \lceil \alpha \rceil, \end{cases}$$

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for $\alpha \in (1, +\infty)$ and $\beta \in (0,1]$, where $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathbb{N}}$ satisfy (7) (see [22,23]). In the worst case setting, the reference [21] and the reference [22] investigate the algebraic tractability of $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ satisfying (8) defined over the reproducing kernel Hilbert space $H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d)$, where the reproducing kernel function has the Gaussian ANOVA weight $\psi_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$. But in the average case setting, there are no results about the algebraic tractability of the problem $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ satisfying (8) from the Banach space $H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d)$ equipped with a zero-mean Gaussian measure μ_d with the Gaussian ANOVA covariance kernel or the variant of the Gaussian ANOVA covariance kernel.

Remark 2. *Noting that the variant of the Gaussian ANOVA weight $\sigma_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ has faster decay rate than the Gaussian ANOVA weight $\psi_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$.*

Remark 3. *Set $W_{\alpha_j,\beta_j}(k) \in \{\rho_{\alpha_j,\beta_j}(k), \sigma_{\alpha_j,\beta_j}(k)\}$ for all $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$W_{\alpha_j,\beta_j}(k) \le \beta_j A^{\frac{k^{\lceil\alpha_1\rceil}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}}} \quad \textit{for all} \quad j,k \in \mathbb{N}.$$

Especially, we have $W_{\alpha_j,\beta_j}(0) = 1$ for all $j \in \mathbb{N}$.

Proof. (1) Set $W_{\alpha_j,\beta_j}(k) = \rho_{\alpha_j,\beta_j}(k)$ for all $j,k \in \mathbb{N}$. We have

$$\rho_{\alpha_j,\beta_j}(k) \le \rho_{\alpha_1,\beta_j}(k) = \beta_j A^{k^{\lceil\alpha_1\rceil}} \le \beta_j A^{\frac{k^{\lceil\alpha_1\rceil}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}}}$$

for all $j,k \in \mathbb{N}$.

(2) Set $W_{\alpha_j,\beta_j}(k) = \sigma_{\alpha_j,\beta_j}(k)$ for all $j,k \in \mathbb{N}$. For $1 \le k < \lceil\alpha_j\rceil$ and $j \in \mathbb{N}$ we have

$$\sigma_{\alpha_j,\beta_j}(k) = \beta_j A^{k!} \le \beta_j A \le \beta_j A^{\left(\frac{k}{\lceil\alpha_j\rceil}\right)^{\lceil\alpha_j\rceil}} = \beta_j A^{\frac{k^{\lceil\alpha_j\rceil}}{\lceil\alpha_j\rceil^{\lceil\alpha_j\rceil}}}.$$

On the other hand, for $k \ge \lceil\alpha_j\rceil$ and $j \in \mathbb{N}$ we get

$$\begin{aligned}\sigma_{\alpha_j,\beta_j}(k) &= \beta_j A^{k!/(k-\lceil\alpha_j\rceil)!} = \beta_j A^{k(k-1)\cdots(k-\lceil\alpha_j\rceil+1)} \\ &\le \beta_j A^{(k-\lceil\alpha_j\rceil+1)^{\lceil\alpha_j\rceil}} = \beta_j A^{k^{\lceil\alpha_j\rceil}\left(1-\frac{\lceil\alpha_j\rceil-1}{k}\right)^{\lceil\alpha_j\rceil}} \\ &\le \beta_j A^{k^{\lceil\alpha_j\rceil}\left(1-\frac{\lceil\alpha_j\rceil-1}{\lceil\alpha_j\rceil}\right)^{\lceil\alpha_j\rceil}} = \beta_j A^{\frac{k^{\lceil\alpha_j\rceil}}{\lceil\alpha_j\rceil^{\lceil\alpha_j\rceil}}}.\end{aligned}$$

It follows that for all $j,k \in \mathbb{N}$

$$\sigma_{\alpha_j,\beta_j}(k) \le \beta_j A^{\frac{k^{\lceil\alpha_j\rceil}}{\lceil\alpha_j\rceil^{\lceil\alpha_j\rceil}}}.$$

Since $\sigma_{\alpha_j,\beta_j}(k) \le \sigma_{\alpha_1,\beta_j}(k)$ for all $j,k \in \mathbb{N}$, we further obtain

$$\sigma_{\alpha_j,\beta_j}(k) \le \beta_j A^{\frac{k^{\lceil\alpha_1\rceil}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}}}.$$

Therefore, by (1) and (2) we have

$$W_{\alpha_j,\beta_j}(k) \le \beta_j A^{\frac{k^{\lceil\alpha_1\rceil}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}}}$$

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for $W_{\alpha_j,\beta_j}(k) \in \{\rho_{\alpha_j,\beta_j}(k), \sigma_{\alpha_j,\beta_j}(k)\}$ and all $j,k \in \mathbb{N}$.

(3) For all $j \in \mathbb{N}$ it is obvious from

$$\rho_{\alpha_j,\beta_j}(0) = \sigma_{\alpha_j,\beta_j}(0) = 1$$

that $W_{\alpha_j,\beta_j}(0) = 1$. □

Remark 4. *Set $W_{\alpha_j,\beta_j}(k) \in \{\rho_{\alpha_j,\beta_j}(k), \sigma_{\alpha_j,\beta_j}(k)\}$ for all $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then for all $j \in \mathbb{N}$ due to $\rho_{\alpha_j,\beta_j}(1) = \sigma_{\alpha_j,\beta_j}(1) = \beta_j A$, we have $W_{\alpha_j,\beta_j}(1) = \beta_j A$.*

4. Average Case (s,t)-Weak Tractability of L_2-Approximation with the Two Weighted Covariance Kernels and the Main Result

In this section, we consider (s,t)-weak tractability of the L_2-approximation $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ satisfying (8) defined over the Banach space $H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d)$ with a zero-mean Gaussian measure μ_d in the average case setting. Here, the covariance kernel $K_{W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}}$ with weight $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ of the measure μ_d is given by (6), and the parameter sequences $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathbb{N}}$ satisfy (7). In this paper, we consider two product weights: the variant of the korobov weight $\rho_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ and the variant of the Gaussian ANOVA weight $\sigma_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$.

Let $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}} = \prod_{j=1}^{d} W_{\alpha_j,\beta_j}(h_j)$ with $W_{\alpha_j,\beta_j}(k) \in \{\rho_{\alpha_j,\beta_j}(k), \sigma_{\alpha_j,\beta_j}(k)\}$ for all $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then from Lemma 3 we have

$$W_{\alpha_j,\beta_j}(0) = 1 \geq W_{\alpha_j,\beta_j}(k) \quad \text{for all} \quad j,k \in \mathbb{N}, \tag{11}$$

which yields

$$\lambda_{d,1} = W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{0}) = \prod_{j=1}^{d} W_{\alpha_j,\beta_j}(0) = 1. \tag{12}$$

We conclude from (3), (10) with $\tau = 1$ and (11) that

$$\mathrm{CRI}_d = \begin{cases} 1, & \text{for X} = \text{ABS}, \\ \prod_{j=1}^{d}\left(1 + \sum_{k=1}^{\infty} W_{\alpha_j,\beta_j}(k)\right), & \text{for X} = \text{NOR}. \end{cases} \tag{13}$$

By (2) and (13) we obtain

$$n^{\mathrm{NOR}}(\varepsilon, \mathrm{APP}_d) \leq n^{\mathrm{ABS}}(\varepsilon, \mathrm{APP}_d). \tag{14}$$

Theorem 1. *Let the parameter sequences $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathbb{N}}$ satisfy (7). Consider the L_2-approximation APP from the space $H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d)$ with the covariance weight $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}} \in \{\rho_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}, \sigma_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ in the average case setting. For any $t \in (0,1)$ and $s > 0$, (s,t)-weak tractability holds under ABS or NOR iff*

$$\lim_{j\to\infty} \beta_j j^{1-t} = 0.$$

Proof. Necessity. Let $s > 0$ and $t \in (0,1)$. Assume that (s,t)-weak tractability holds for ABS or NOR.

By the inequality (14), we only need to assume that (s,t)-weak tractability holds for NOR. Due to the definition of the information complexity (2) for NOR, we have

$$\mathrm{CRI}_d - \sum_{i=1}^{n^{\mathrm{NOR}}(\varepsilon,\mathrm{APP}_d)} \lambda_{d,i} = \sum_{i=n^{\mathrm{NOR}}(\varepsilon,\mathrm{APP}_d)+1}^{\infty} \lambda_{d,i} \leq \varepsilon^2 \mathrm{CRI}_d,$$

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which means

$$(1-\varepsilon^2)\mathrm{CRI}_d \le \sum_{i=1}^{n^{\mathrm{NOR}}(\varepsilon,\mathrm{APP}_d)} \lambda_{d,i}.$$

We further deduce from (13) and (12) that

$$\begin{aligned}(1-\varepsilon^2)\prod_{j=1}^{d}(1+\sum_{k=1}^{\infty} W_{\alpha_j,\beta_j}(k)) = (1-\varepsilon^2)\mathrm{CRI}_d &\le \sum_{i=1}^{n^{\mathrm{NOR}}(\varepsilon,\mathrm{APP}_d)} \lambda_{d,i} \\ &\le n^{\mathrm{NOR}}(\varepsilon,\mathrm{APP}_d)\lambda_{d,1} = n^{\mathrm{NOR}}(\varepsilon,\mathrm{APP}_d). \end{aligned} \tag{15}$$

From (15) we get

$$\ln n^{\mathrm{NOR}}(\varepsilon,\mathrm{APP}_d) \ge \ln(1-\varepsilon^2) + \sum_{j=1}^{d} \ln(1+\sum_{k=1}^{\infty} W_{\alpha_j,\beta_j}(k)). \tag{16}$$

Set $\varepsilon = \frac{1}{2}$. It follows from the assumption, inequality (16) and Remark 4 that

$$\begin{aligned} 0 = \lim_{d\to\infty} \frac{\ln n^{\mathrm{NOR}}(\frac{1}{2},\mathrm{APP}_d)}{2^s+d^t} &\ge \lim_{d\to\infty} \frac{\ln\frac{3}{4} + \sum_{j=1}^{d}\ln(1+\sum_{k=1}^{\infty} W_{\alpha_j,\beta_j}(k))}{2^s+d^t} \\ &\ge \lim_{d\to\infty} \frac{\ln\frac{3}{4} + \sum_{j=1}^{d}\ln(1+W_{\alpha_j,\beta_j}(1))}{2^s+d^t} \\ &= \lim_{d\to\infty} \frac{\ln\frac{3}{4} + \sum_{j=1}^{d}\ln(1+\beta_j A)}{2^s+d^t} \\ &= \lim_{d\to\infty} \frac{\sum_{j=1}^{d}\ln(1+\beta_j A)}{d^t}. \end{aligned}$$

Due to the fact $\ln(1+x) > \frac{x}{2}$ for all $x \in (0,1)$, $\beta_j A \in (0,1)$ for all $j \in \mathbb{N}$, and Stolz theorem, we further have

$$\begin{aligned} 0 = \lim_{d\to\infty} \frac{\ln n^{\mathrm{NOR}}(\frac{1}{2},\mathrm{APP}_d)}{2^s+d^t} &\ge \lim_{d\to\infty} \frac{\sum_{j=1}^{d}\frac{\beta_j A}{2}}{d^t} \\ &= \frac{A}{2}\lim_{d\to\infty}\frac{\beta_d}{d^t-(d-1)^t} \\ &= \frac{A}{2t}\lim_{d\to\infty}\beta_d d^{1-t} \ge 0. \end{aligned}$$

It yields $\lim_{d\to\infty}\beta_d d^{1-t} = 0$ for any $t \in (0,1)$.

Sufficiency. Assume that $\lim_{d\to\infty}\beta_d d^{1-t} = 0$ for any $t \in (0,1)$. We will prove that (s,t)-weak tractability holds for ABS or NOR.

By the inequality (14), we only need to prove that (s,t)-weak tractability holds for ABS. We set

$$\tau_k = \max\left\{\frac{1}{\ln^+(\beta_k^{-1})}, \frac{1}{\ln(2k+1)}\right\} \quad \text{for} \quad k \in \mathbb{N}. \tag{17}$$

Obviously, $\lim_{k\to\infty}\beta_k = 0$ and thus $\lim_{k\to\infty}\tau_k = 0$, i.e., $\tau_k \in (0,1)$ for sufficiently large k. Set $\tau = 1-\tau_d$. It follows from inequality (5) for ABS that

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln n^{\mathrm{ABS}}(\varepsilon,\mathrm{APP}_d)}{\varepsilon^{-s}+d^t} \le \lim_{\varepsilon^{-1}+d\to\infty}\frac{\left|\frac{1-\tau_d}{\tau_d}\left(\ln\frac{1-\tau_d}{\tau_d}+2\ln(\varepsilon^{-1})+\frac{\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d}\right)}{1-\tau_d}\right)+\ln 2\right|}{\varepsilon^{-s}+d^t}. \tag{18}$$

Note that

$$\begin{aligned}
0 &\le \lim_{\varepsilon^{-1}+d\to\infty}\frac{\left|\frac{1-\tau_d}{\tau_d}\left(\ln\frac{1-\tau_d}{\tau_d}+2\ln(\varepsilon^{-1})\right)+\ln 2\right|}{\varepsilon^{-s}+d^t}\\
&\le \lim_{\varepsilon^{-1}+d\to\infty}\frac{\left|(\ln(2d+1)-1)\left(\ln(\ln(2d+1)-1)+2\ln(\varepsilon^{-1})\right)+\ln 2\right|}{\varepsilon^{-s}+d^t}\\
&\le \lim_{d\to\infty}\frac{(\ln(2d+1)-1)\ln(\ln(2d+1)-1)}{d^t}+\lim_{\varepsilon^{-1}+d\to\infty}\frac{2(\ln(2d+1)-1)\ln(\varepsilon^{-1})}{\varepsilon^{-s}+d^t}\\
&= \lim_{\varepsilon^{-1}+d\to\infty}\frac{2(\ln(2d+1)-1)\ln(\varepsilon^{-1})}{\varepsilon^{-s}+d^t}\\
&\le \lim_{\varepsilon^{-1}+d\to\infty}\frac{(\ln(2d+1)-1)^2+\left(\ln(\varepsilon^{-1})\right)^2}{\varepsilon^{-s}+d^t}\\
&\le \lim_{d\to\infty}\frac{(\ln(2d+1)-1)^2}{d^t}+\lim_{\varepsilon^{-1}\to\infty}\frac{\left(\ln(\varepsilon^{-1})\right)^2}{\varepsilon^{-s}}=0,
\end{aligned}$$

and thus in the inequality (18) we only need to prove

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\left|\frac{1}{\tau_d}\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d}\right)\right|}{\varepsilon^{-s}+d^t}=\lim_{\varepsilon^{-1}+d\to\infty}\frac{\frac{1}{\tau_d}\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d}\right)}{\varepsilon^{-s}+d^t}=0, \tag{19}$$

where we used

$$\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d}\right)\ge\ln\lambda_{d,1}^{1-\tau_d}=0$$

by (12).

From (10) with $\tau = 1-\tau_d$ and Lemma 3 we have

$$\begin{aligned}
\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d}\right) &= \ln\left(\prod_{j=1}^{d}\left(1+\sum_{k=1}^{\infty}\left(W_{\alpha_j,\beta_j}(k)\right)^{1-\tau_d}\right)\right)\\
&= \sum_{j=1}^{d}\ln\left(1+\sum_{k=1}^{\infty}\left(W_{\alpha_j,\beta_j}(k)\right)^{1-\tau_d}\right)\\
&\le \sum_{j=1}^{d}\ln\left(1+\sum_{k=1}^{\infty}\beta_j^{1-\tau_d}A^{\frac{(1-\tau_d)k^{\lceil\alpha_1\rceil}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}}}\right).
\end{aligned} \tag{20}$$

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Since $0 < A^{\frac{(1-\tau_d)k^{\lceil\alpha_1\rceil}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}}} \le A^{\frac{(1-\tau_1)k^{\lceil\alpha_1\rceil}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}}}$ and $\sum\limits_{k=1}^{\infty} A^{\frac{(1-\tau_1)k^{\lceil\alpha_1\rceil}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}}}$ is convergent. It means that there exists a constant $C > 0$ such that

$$\sum_{k=1}^{\infty} A^{\frac{(1-\tau_d)k^{\lceil\alpha_1\rceil}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}}} < C \tag{21}$$

for all $d \in \mathbb{N}$. By (20) and (21) we have

$$\ln\left(\sum_{i=1}^{\infty} \lambda_{d,i}^{1-\tau_d}\right) \le \sum_{j=1}^{d} \ln\left(1 + C\beta_j^{1-\tau_d}\right) \le C\sum_{j=1}^{d} \beta_j^{1-\tau_d}, \tag{22}$$

where in the last inequality we used $\ln(1+x) \le x$ for $x \ge 0$. Since (17), we have $\tau_j^{-1} \le \ln^+ \beta_j^{-1} \le 1 + \ln \beta_j^{-1}$ and thus $\beta_j \le e^{1-\tau_j^{-1}}$ for all $j \in \mathbb{N}$. We further get for all $1 \le j \le d$ that

$$\beta_j^{1-\tau_d} \le \beta_j^{1-\tau_j} \le e^{(1-\tau_j^{-1})(1-\tau_j)} = e^{1-\tau_j^{-1}} e^{-\tau_j(1-\tau_j^{-1})}. \tag{23}$$

We note that

$$\begin{aligned} e^{1-\tau_j^{-1}} = e e^{-\tau_j^{-1}} &= \frac{e}{\min\{e^{\ln^+ \beta_j^{-1}}, e^{\ln(2j+1)}\}} \\ &\le \frac{e}{\min\{e^{\ln \beta_j^{-1}}, e^{\ln(2j+1)}\}} = e\max\{\beta_j, \frac{1}{2j+1}\}, \end{aligned} \tag{24}$$

and

$$e^{-\tau_j(1-\tau_j^{-1})} = e^{-\tau_j+1} \le e. \tag{25}$$

Combing (23), (24) and (25), we have

$$\sum_{j=1}^{d} \beta_j^{1-\tau_d} \le e^2 \sum_{j=1}^{d} \max\{\beta_j, \frac{1}{2j+1}\}. \tag{26}$$

From (22) and (26) and we have

$$\begin{aligned} 0 \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{\frac{1}{\tau_d}\ln\left(\sum\limits_{i=1}^{\infty} \lambda_{d,i}^{1-\tau_d}\right)}{\varepsilon^{-s} + d^t} &\le \lim_{d\to\infty} \frac{\ln\left(\sum\limits_{i=1}^{\infty} \lambda_{d,i}^{1-\tau_d}\right)}{\tau_d d^t} \\ \le \lim_{d\to\infty} \frac{C\sum\limits_{j=1}^{d} \beta_j^{1-\tau_d}}{\tau_d d^t} &\le \lim_{d\to\infty} \frac{Ce^2 \sum\limits_{j=1}^{d} \max\{\beta_j, \frac{1}{2j+1}\}}{\tau_d d^t}. \end{aligned} \tag{27}$$

Next, we will prove

$$\lim_{d\to\infty} \frac{\sum\limits_{j=1}^{d} \max\{\beta_j, \frac{1}{2j+1}\}}{\tau_d d^t} = 0.$$

It follows from (17) that

$$0 \le \lim_{d\to\infty} \frac{\sum\limits_{j=1}^{d} \frac{1}{2j+1}}{\tau_d d^t} \le \lim_{d\to\infty} \frac{\int_{j=0}^{d} \frac{1}{2x+1}dx}{\tau_d d^t} \le \lim_{d\to\infty} \frac{\frac{1}{2}\ln^2(2d+1)}{d^t} = 0,$$

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i.e.,

$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \frac{1}{2j+1}}{\tau_d d^t} = 0. \tag{28}$$

Due to $\lim_{d\to\infty} d^{1-t}\beta_d = 0$ for any $t \in (0,1)$, we have $\lim_{d\to\infty} d^{1-t/2}\beta_d = 0$. This means that there exists a positive number $N \in \mathbb{N}$ such that $\beta_j \le j^{t/2-1}$ for all $j > N$. It follows that

$$\begin{aligned}\sum_{j=1}^{d} \beta_j &= \sum_{j=1}^{N} \beta_j + \sum_{j=N+1}^{d} \beta_j \le N + \sum_{j=N+1}^{d} j^{t/2-1} \\ &\le N + \sum_{j=2}^{d} j^{t/2-1} \le N + \int_1^d x^{t/2-1} dx \\ &= N + \frac{2d^{t/2}-2}{t} \le N + \frac{2d^{t/2}}{t}\end{aligned}$$

for sufficiently large d, which yields by (17) that

$$\begin{aligned}0 \le \lim_{d\to\infty} \frac{\sum_{j=1}^{d} \beta_j}{\tau_d d^t} &\le \lim_{d\to\infty} \frac{\ln(2d+1) \sum_{j=1}^{d} \beta_j}{d^t} \\ &\le \lim_{d\to\infty} \frac{\ln(2d+1)(N + \frac{2d^{t/2}}{t})}{d^t} \\ &= \lim_{d\to\infty} \frac{N \ln(2d+1)}{d^t} + \lim_{d\to\infty} \frac{2d^{t/2} \ln(2d+1)}{td^t} \\ &= \lim_{d\to\infty} \frac{2 \ln(2d+1)}{td^{t/2}} = 0,\end{aligned}$$

i.e.,

$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \beta_j}{\tau_d d^t} = 0. \tag{29}$$

We conclude from (28) and (29) that

$$\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \max\{\beta_j, \frac{1}{2j+1}\}}{\tau_d d^t} = 0.$$

Then we have by (27) that

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\frac{1}{\tau_d} \ln\left(\sum_{i=1}^{\infty} \lambda_{d,i}^{1-\tau_d}\right)}{\varepsilon^{-s} + d^t} = 0,$$

and thus (19) holds. Hence (s,t)-weak tractability holds for any $t \in (0,1)$ and $s > 0$ for ABS or NOR. Therefore, we finish the proof. □

Example 1. *An example for (s,t)-weak tractability with $s > 0$ and $t \in (0,1)$.*

Assume that $\alpha_j = 2j$ and $\beta_j = \frac{1}{j+2}$ satisfy (7) for all $j \in \mathbb{N}$. Obviously, we have

$$\lim_{j\to\infty} \beta_j j^{1-t} = \lim_{j\to\infty} \frac{j^{1-t}}{j+2} = 0.$$

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Next, we will prove that the problem APP defined over the space $H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d)$ with the covariance weight $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}} \in \{\rho_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}, \sigma_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ is (s,t)-weakly tractable for $s > 0$ and $t \in (0,1)$ under ABS or NOR. By the inequality (14), we only need to prove that (s,t)-weak tractability holds for $s > 0$ and $t \in (0,1)$ under ABS.

Let $s > 0$ and $t \in (0,1)$. Choose

$$\tau_d' = \frac{1}{\ln(d+2)}, \quad \text{for} \quad d \in \mathbb{N}.$$

Set $\tau = 1 - \tau_d'$ in the inequality (5) for ABS. Then we have

$$0 \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{ABS}(\varepsilon, APP_d)}{\varepsilon^{-s}+d^t} \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{\left|\frac{1-\tau_d'}{\tau_d'}\left(\ln\frac{1-\tau_d'}{\tau_d'} + 2\ln(\varepsilon^{-1}) + \frac{\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d'}\right)}{1-\tau_d'}\right) + \ln 2\right|}{\varepsilon^{-s}+d^t}$$

$$\le \lim_{\varepsilon^{-1}+d\to\infty} \frac{\left|\frac{1-\tau_d'}{\tau_d'}\left(\ln\frac{1-\tau_d'}{\tau_d'} + 2\ln(\varepsilon^{-1})\right) + \ln 2\right|}{\varepsilon^{-s}+d^t} + \lim_{\varepsilon^{-1}+d\to\infty} \frac{\left|\frac{1}{\tau_d'}\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d'}\right)\right|}{\varepsilon^{-s}+d^t}.$$

Since

$$\begin{aligned}
0 &\le \lim_{\varepsilon^{-1}+d\to\infty} \frac{\left|\frac{1-\tau_d'}{\tau_d'}\left(\ln\frac{1-\tau_d'}{\tau_d'} + 2\ln(\varepsilon^{-1})\right) + \ln 2\right|}{\varepsilon^{-s}+d^t} \\
&= \lim_{\varepsilon^{-1}+d\to\infty} \frac{\left|(\ln(d+2)-1)\left(\ln(\ln(d+2)-1) + 2\ln(\varepsilon^{-1})\right) + \ln 2\right|}{\varepsilon^{-s}+d^t} \\
&\le \lim_{d\to\infty} \frac{(\ln(d+2)-1)\ln(\ln(d+2)-1) + \ln 2}{d^t} + \lim_{\varepsilon^{-1}+d\to\infty} \frac{2(\ln(d+2)-1)\ln(\varepsilon^{-1})}{\varepsilon^{-s}+d^t} \\
&= \lim_{\varepsilon^{-1}+d\to\infty} \frac{2(\ln(d+2)-1)\ln(\varepsilon^{-1})}{\varepsilon^{-s}+d^t} \\
&\le \lim_{\varepsilon^{-1}+d\to\infty} \frac{(\ln(d+2)-1)^2 + \left(\ln(\varepsilon^{-1})\right)^2}{\varepsilon^{-s}+d^t} \\
&\le \lim_{d\to\infty} \frac{(\ln(d+2)-1)^2}{d^t} + \lim_{\varepsilon^{-1}\to\infty} \frac{\left(\ln(\varepsilon^{-1})\right)^2}{\varepsilon^{-s}} = 0,
\end{aligned}$$

i.e.,

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\left|\frac{1-\tau_d'}{\tau_d'}\left(\ln\frac{1-\tau_d'}{\tau_d'} + 2\ln(\varepsilon^{-1})\right) + \ln 2\right|}{\varepsilon^{-s}+d^t} = 0,$$

next, we only need to prove

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\left|\frac{1}{\tau_d'}\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d'}\right)\right|}{\varepsilon^{-s}+d^t} = 0.$$

It follows from (10) with $\tau = 1 - \tau_d'$ and Lemma 3 that

$$\begin{aligned}
\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d'}\right) &= \ln\left(\prod_{j=1}^{d}\left(1 + \sum_{k=1}^{\infty}\left(W_{\alpha_j,\beta_j}(k)\right)^{1-\tau_d'}\right)\right) \\
&= \sum_{j=1}^{d}\ln\left(1 + \sum_{k=1}^{\infty}\left(W_{\alpha_j,\beta_j}(k)\right)^{1-\tau_d'}\right)
\end{aligned}$$

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$$
\begin{aligned}
&\le \sum_{j=1}^{d} \ln\left(1+\sum_{k=1}^{\infty}\beta_j^{1-\tau_d'} A^{\frac{(1-\tau_d')k^{\lceil\alpha_1\rceil}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}}}\right) \\
&= \sum_{j=1}^{d} \ln\left(1+\frac{1}{(j+2)^{1-\tau_d'}}\sum_{k=1}^{\infty} A^{\frac{(1-\tau_d')k^2}{4}}\right). \qquad (30)
\end{aligned}
$$

We note that

$$
A^{\frac{(1-\tau_d')k^2}{4}} = A^{\frac{\left(1-\frac{1}{\ln(d+2)}\right)k^2}{4}} \le A^{\frac{\left(1-\frac{1}{\ln 3}\right)k^2}{4}}
$$

and $\sum_{k=1}^{\infty} A^{\frac{(1-\frac{1}{\ln 3})k^2}{4}}$ *is convergent. Then there exists a constant* $C>0$ *such that*

$$
\sum_{k=1}^{\infty} A^{\frac{(1-\tau_d')k^2}{4}} \le \sum_{k=1}^{\infty} A^{\frac{(1-\frac{1}{\ln 3})k^2}{4}} \le C.
$$

We further get from (30) *that*

$$
\begin{aligned}
\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d'}\right) &\le \sum_{j=1}^{d}\ln\left(1+\frac{C}{(j+2)^{1-\tau_d'}}\right) \le C\sum_{j=1}^{d}\frac{1}{(j+2)^{1-\tau_d'}} \\
&= C\sum_{j=1}^{d}\frac{1}{(j+2)^{1-\frac{1}{\ln(d+2)}}} = C\sum_{j=1}^{d}\frac{(j+2)^{\frac{1}{\ln(d+2)}}}{j+2} \\
&\le C\sum_{j=1}^{d}\frac{(d+2)^{\frac{1}{\ln(d+2)}}}{j+2} = Ce\sum_{j=1}^{d}\frac{1}{j+2} \\
&\le Ce\left(\int_0^d \frac{1}{x+2}dx\right) = Ce(\ln(d+2)-\ln 2),
\end{aligned}
$$

which conclude that

$$
\begin{aligned}
0 &\le \lim_{\varepsilon^{-1}+d\to\infty}\frac{\left|\frac{1}{\tau_d'}\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d'}\right)\right|}{\varepsilon^{-s}+d^t} \\
&= \lim_{\varepsilon^{-1}+d\to\infty}\frac{|Ce\ln(d+2)(\ln(d+2)-\ln 2)|}{\varepsilon^{-s}+d^t} \\
&\le \lim_{d\to\infty}\frac{|Ce\ln(d+2)(\ln(d+2)-\ln 2)|}{d^t} \\
&\le \lim_{d\to\infty}\frac{|Ce\ln(d+2)(\ln(d+2))|}{d^t} = 0,
\end{aligned}
$$

i.e.,

$$
\lim_{\varepsilon^{-1}+d\to\infty}\frac{\left|\frac{1}{\tau_d'}\ln\left(\sum_{i=1}^{\infty}\lambda_{d,i}^{1-\tau_d'}\right)\right|}{\varepsilon^{-s}+d^t} = 0.
$$

Hence we have

$$
\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln n^{ABS}(\varepsilon, APP_d)}{\varepsilon^{-s}+d^t} = 0,
$$

which yields that (s,t)*-weak tractability holds for ABS. Therefore, APP is* (s,t)*-weakly tractable for any* $s>0$ *and* $t\in(0,1)$ *under ABS or NOR.*

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Example 2. *An example not for (s,t)-weak tractability with any $s>0$ and $t\in(0,1)$.*

Assume that $\alpha_j = 2^j$ and $\beta_j = \frac{1}{\ln(3j)}$ for all $j\in\mathbb{N}$. Obviously, we have

$$\lim_{j\to\infty}\beta_j j^{1-t} = \lim_{j\to\infty}\frac{j^{1-t}}{\ln(3j)} = \infty.$$

Next, we will prove that the problem APP defined over the space $H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}([0,1]^d)$ with the covariance weight $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}\in\{\rho_{d,\boldsymbol{\alpha},\boldsymbol{\beta}},\sigma_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ is not (s,t)-weakly tractable for any $s>0$ and $t\in(0,1)$ under ABS or NOR. Due to the inequality (14), we only need to prove that (s,t)-weak tractability does not hold for any $s>0$ and $t\in(0,1)$ under NOR.

Let $s>0$ and $t\in(0,1)$. We conclude from inequality (16) with $\varepsilon=\frac{1}{2}$ and Remark 4 that

$$\begin{aligned}
\lim_{d\to\infty}\frac{\ln n^{NOR}(\frac{1}{2},APP_d)}{2^s+d^t} &\geq \lim_{d\to\infty}\frac{\ln\frac{3}{4}+\sum_{j=1}^{d}\ln(1+\sum_{k=1}^{\infty}W_{\alpha_j,\beta_j}(k))}{2^s+d^t}\\
&\geq \lim_{d\to\infty}\frac{\ln\frac{3}{4}+\sum_{j=1}^{d}\ln(1+W_{\alpha_j,\beta_j}(1))}{2^s+d^t}\\
&= \lim_{d\to\infty}\frac{\ln\frac{3}{4}+\sum_{j=1}^{d}\ln(1+\beta_j A)}{2^s+d^t}\\
&= \lim_{d\to\infty}\frac{\sum_{j=1}^{d}\ln(1+\beta_j A)}{2^s+d^t}\\
&= \lim_{d\to\infty}\frac{\sum_{j=1}^{d}\ln(1+\frac{A}{\ln(3j)})}{2^s+d^t}\\
&= \lim_{d\to\infty}\frac{\ln(1+\frac{A}{\ln(3d)})}{d^t-(d-1)^t}\\
&= \lim_{d\to\infty}\frac{1}{t}d^{1-t}\ln(1+\frac{A}{\ln(3d)})\\
&= \lim_{d\to\infty}\frac{Ad^{1-t}}{t\ln(3d)} = +\infty,
\end{aligned}$$

where in the fourth equality we used Stolz theorem. Hence APP is not (s,t)-weak tractable for any $s>0$ and $t\in(0,1)$ under ABS or NOR.

5. Conclusions

In this paper we study average case (s,t)-weak tractability with any $s>0$ and $t\in(0,1)$ for the L_2-approximation problem $\mathrm{APP}=\{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ from the Banach space $H_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$ equipped with a zero-mean Gaussian measure μ_d with the covariance kernel $K_{W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}}$ with weight $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}$, where $1<\alpha_1\leq\alpha_2\leq\cdots$ and $1\geq\beta_1\geq\beta_2\geq\cdots>0$ are parameters. We obtain a compete result for $W_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}\in\{\rho_{d,\boldsymbol{\alpha},\boldsymbol{\beta}},\sigma_{d,\boldsymbol{\alpha},\boldsymbol{\beta}}\}$ that APP is (s,t)-weakly tractable under ABS or NOR for any $s>0$ and $t\in(0,1)$ iff

$$\lim_{j\to\infty}\beta_j j^{1-t} = 0.$$

We will further investigate other algebraic tractability notions about multivariate approximation problems from Banach spaces equipped with zero-mean Gaussian measures with different weighted covariance kernels and hope to get more good results.

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