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Robust Semi-Infinite Interval Equilibrium Problem Involving Data Uncertainty: Optimality Conditions and Duality

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Abstract: In this paper, we model uncertainty in both the objective function and the constraints for the robust semi-infinite interval equilibrium problem involving data uncertainty. We particularize these conditions for the robust semi-infinite mathematical programming problem with interval-valued functions by extending results from the literature. We introduce the dual robust version of the above problem, prove the Mond-Weir type weak and strong duality theorems, and illustrate with an example.

Keywords: robust optimization; equilibrium problem; semi-infinite programming; interval-valued functions; optimality conditions; duality

MSC: 90C46, 90C33, 90C34, 90C70

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1. Introduction

Uncertainty can be treated from fuzzy theory or interval analysis. We will use the latter method to capture uncertainty in the objective function and robust programming to capture uncertainty in the constraints.

On the other hand, in economics it is often interesting, in addition to looking for maxima and minima, to find the points where equilibrium is achieved. In the 1960s, Fan [1] studied the theory of equilibrium in Euclidean spaces. An *equilibrium problem (EP)* consists of finding $x \in S$ such that

$$F(x,y) \geq 0, \, \forall y \in S$$

where $S \subseteq \mathbb{R}^n$ is a nonempty closed set and $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bifunction function.

Far from being a particular problem, the equilibrium problem groups together other significant mathematical problems:

- The *optimality problem* where $F(x,y) = f(y) - f(x)$, $\forall x, y \in X$ where $f : X \to \mathbb{R}$.
- The *variational inequality problem* involving

$$F(x,y) =< T(x), y - x >, \, \forall x, y \in X$$

where $\mathcal{L}(X,Y)$ is the space of all continuous linear mappings from X to Y and $T : X \to \mathcal{L}(X,Y)$. The geometric interpretation is that the angle between the vectors $T(x)$ and $y - x$, is less than o equal 90^0.

A particular case of a variational problem is the *Signorini Problem*. This problem consists of finding the elastic equilibrium configuration of an anisotropic non-homogeneous elastic body, resting on a rigid frictionless surface and subject only to its mass forces. This problem can be formulated as follows:

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$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j})+\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}+cu-f=0 \text{ in } \Omega$$

$$u\geq 0 \text{ on } \partial\Omega$$

$$\frac{\partial u}{\partial v}=\sum_{i,j=1}^{n}a_{ij}\frac{\partial u}{\partial x_j}n_i\geq 0 \text{ on } \partial\Omega$$

$$u\frac{\partial u}{\partial v}=0 \quad \text{on } \partial\Omega$$

$$u\in H(\Gamma) \quad \text{on } \partial\Omega$$

where n_i are the components of the outer normal to $\partial\Omega$ and it represents the conceptual model of an elastic body Ω with boundary $\partial\Omega$ which is in contact with a rigid support body and is subject to volume forces f. We denote by u the displacement in $\partial\Omega$ produced by the deforming forces. This problem can be expressed by the variational inequality:

$$a(u,v-u)\geq(f,v-u),\ \forall v\in K$$

where

$$a(u,v)=\int_{\Omega}(\sum_{i,j}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j})+\sum_{i=1}^{n}b_iu\frac{\partial v}{\partial x_i}+cuv)dx$$

$$(f,v)=\int_{\Omega}fvdx$$

$$K=\{v\in H(\Omega):v(x)\geq 0 \text{ on } \partial\Omega\}$$

- *Fixed point problems*: given a closed set $C\subseteq\mathbb{R}^n$, a fixed point of a mapping $f:C\to C$ is any $\bar{x}\in C$ such that $\bar{x}=f(\bar{x})$. Finding a fixed point amounts to solving (EP) with

$$F(x,y)=<x-f(x),y-x>$$

- *Saddle point problems*: given two closed sets $C_1\subseteq\mathbb{R}^{n_1}$ and $C_2\subseteq\mathbb{R}^{n_2}$, a saddle point of a function $L:C_1\times C_2\to\mathbb{R}$ is any $\bar{x}=(\bar{x}_1,\bar{x}_2)\in C_1\times C_2$ such that

$$L(\bar{x}_1,y_2)\leq L(\bar{x}_1,\bar{x}_2)\leq L(y_1,\bar{x}_2)$$

holds for any $y=(y_1,y_2)\in C_1\times C_2$. Finding a saddle point of L amounts to solving EP with $C=C_1\times C_2$ and

$$F((x_1,x_2),(y_1,y_2))=L(y_1,x_2)-L(x_1,y_2)$$

- *Walras model of economic equilibrium*: which can be formulated as an Equilibrium problem. Let's assume we have a market structure with perfect competition. The model deals in n commodities. Then, given a price vector $p\in\mathbb{R}^n_+$, we can define the excess demand mapping as a function, $E:\mathbb{R}^n_+\to\Pi(\mathbb{R}^n)$, where $\Pi(\mathbb{R}^n)$ denotes the family of all subsets of $\mathbb{R}^n$.

We could define a price $p^*\in\mathbb{R}^n$ is said to be an equilibrium price vector if it solves

$$p^*\geq 0,\ \exists q^*\in E(p^*): \quad q^*\leq 0, \quad <p^*,q^*>=0$$

In 1990, Dafermos [2] proved that a price $p^*\in\mathbb{R}^n$ is said to be an equilibrium price vector if it solves the Equilibrium Problem EP consists of finding $p^*\geq 0$ such that $\exists q^*\in E(p^*)$ such that

$$F(p^*,p)=<-q^*,p-p^*>\geq 0, \quad \forall p\geq 0$$

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- The *Nash equilibrium problem*: when starting from n companies, each company i may possess I_i generating units. Let x denote the vector whose entry x_j stands for the power generation by unit j. Suppose that the price $p_i(s)$ is a decreasing affine function of s with $s = \sum_{i=1}^{N} x_i$, where N is the number of all generating units. We can formulate the benefit made by the company i as:

$$f_i(x) = -\sum_{j \in I_i} c_j(x_j) + p_i(s) \sum_{j \in I_i} x_j$$

where $c_j(x_j)$ is the cost for generating x_j by generating unit j. Let us may suppose that K_i be the strategy set of company i, which means that $\sum_{j \in I_i} x_i \in K_i$ must be fulfilled for every i. We denote the strategy set of the model as $K = K_1 \times K_2 \times \cdots \times K_n$.

We recall that $\bar{x} \in K$ is said to be an equilibrium point of the model if

$$f_i(\bar{x}) \geq f_i(\bar{x}[x_i]), \ \forall x_i \in K_i, \ \forall i = 1, 2, \ldots, n$$

where $\bar{x}[x_i]$ signifies the vector obtained from $\bar{x}$ by replacing $\bar{x}_i$ with x_i. Pickering

$$F(x,y) = \phi(x,y) - \phi(x,x)$$

with $\phi(x,y) = -\sum_{i=1}^{n} f_i(\bar{x}[x_i])$.

In recent years, computational studies have been carried out in parallel with theoretical studies. To find solutions to these equilibrium problems in a practical way we rely on auxiliary problems.

For any scalar $\lambda > 0$, we define a bifunction $F_\lambda : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$F_\lambda(x,y) = F(x,y) + \frac{\lambda}{2} d^2(x,y), \quad \forall x, y \in \mathbb{R}^n$$

Clearly, with the assumption of F, $F_\lambda(x,x) = 0, \quad x \in C$. It is considered the classical Auxiliary Equilibrium Problem (in short (AEP)) that is to find $\bar{x} \in C$ such that

$$F_\lambda(\bar{x}, y) \geq 0, \quad \forall y \in C$$

Li et al. [3] and Babu et al. [4] showed that the problems EP and AEP are equivalente under some assumptions. Other authors have proposed algorithms for the solution as an equilibrium problem such as Tran et al. [5], Yao et al. [6] or Nguyen et al. [7].

Robust optimization intends to find the solution by considering all the possible values of the parameters within their prescribed set of uncertainty including the worst case of all the existing scenarios. Goberna et al. [8] studied the radius of feasibility, optimality, and duality for a robust counterpart of a linear multiobjective problem affected by data uncertainty. Robust optimization has a wide spectrum of real-world applications, in particular, finance [9], energy [10], or internet routing [11].

In this paper, we study equilibrium problems with infinite constraints, which have been called *semi-infinite* equilibrium problems. The beginnings of this mathematical theory are from the beginning of the last century by Haar [12] and Charnes et al. [13]. Semi-infinite programming has countless applications in physics, portfolio problems, engineering design, etc. see ([14–16]) and the references cited therein or the work of Vaz et al. [17], where the authors describe robot trajectories as a semi-infinite programming problem. Recently, Upadhyay, Ghosh and Treanta [18] have studied the multiobjective semi-infinite programming problems in the novel field of Hadamard manifolds.

In many cases, knowledge about the parameters of a real-world system is imprecise or uncertain because, in general, these parameters cannot be observed or measured precisely as is the case with irrational numbers. Interval optimization problems serve as a surrogate option for dealing with uncertain parameters that cannot be precisely calculated. *Interval analysis* is based on the representation of an

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uncertain variable as an interval of real numbers. We owe to Ramon Moore [19] and Weldon Lodwick [20] the efforts to fix the arithmetic of working with intervals and to avoid that an accumulation of errors leads to a disastrous final result.

In 2001, Lodwick et al. [21] applied interval analysis to radiation therapy and Cecconelo et al. [22] applied the notion of constraint interval solutions to analyze the behavior of Sars-Covid. In 2007, Jiang et al. [23], treated the coefficients of friction as intervals and used a nonlinear interval number programming method. For a given molecular protein, in 2017, Costa et al. [24] assumed that the distances between its atoms are represented by intervals and proposed a new methodology to compute possible conformations using a constraint interval analysis approach. Also in 2017 Osuna et al. [25]) formulated the portfolio problem proposed by Markowitz using intervals.

Historical Background.

Some of the milestones in the study of equilibrium problems are the article by Ansari and Flores-Bazán [26] Euclidean spaces and 2010, the paper by Wei and Gong [27] where the authors studied optimality conditions for weakly constrained vector equilibrium solutions in real normed spaces. In 2013, Kim [28] studied Mond-Weir type duality for a robust multiobjective program associated with an uncertainty problem.

Recently, the most interesting works have been by Tung [30] and Ahmad et al [31]. In 2016, Jayswal, Ahmad, and Banerjee [29] studied the interval-valued optimization problem but not the equilibrium problems. Properties that are extended to the multiobjective case by Ahmad, Kaur, and Sharma [31] or to the case of constraints with uncertainty by Jaichander, Ahmad, and Kummari [32]. In 2020, Tung [30] dealt with semi-infinite convex with multiple interval-valued objective functions but no equilibrium problems. In 2022, Antczak and Farajzadeh [33] considered necessary and sufficient optimality conditions for a nondifferentiable semi-infinite vector optimization problem.

In 2019, the necessary and sufficient optimality conditions for vector equilibrium problems on Hadamard manifolds were provided by Ruiz-Garzón et al. [34]. Equilibrium problems with interval-valued functions have been studied by Ruiz-Garzón et al. [35] but not with uncertainty or by Tripathi and Arora [36] but without considering interval-valued functions or duality models.

To the best of our knowledge, there is no paper to study the optimality conditions for semi-infinite interval equilibrium problems involving data uncertainty. This paper is going in this direction. Our contributions are the following:

- Present equilibrium problems with infinite constraints and interval-valued objective functions and uncertainty in the constraints to handle imprecision.
- To achieve the necessary and sufficient conditions of optimality for the robust semi-infinite interval equilibrium problem involving data uncertainty.
- Particularize these conditions for the robust semi-infinite mathematical programming problem.
- To present and obtain duality theorems of the Mond-Weir type and illustrate with an example.

Organization. The paper is unfolded as follows: In Section 2, we discuss the notations and lemmas we will use intervals and semi-infinite programming. Section 3 is devoted to proving necessary and sufficient optimality conditions for robust semi-infinite interval equilibrium problems involving data uncertainty. We will illustrate the results with an example. In Section 4 we will see the optimality conditions of the semi-infinite interval programming problem involving data uncertainty and duality theorems for Mond-Weir-type dual problem. Results given by several authors can be considered as particular cases of those given here. We will end with some conclusions, further development, and references.

2. Tools

In this section we will recall those semi-infinite programming lemmas (see [30]) and the definition of interval-valued function that we will use in this article.

Lemma 1. *If K is a nonempty compact subset of $\mathbb{R}^n$, then,*

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(i) The convex hull of K, $co(K)$, is a compact set;
(ii) If $0 \notin co(K)$, then the convex cone containing the origin generated by K, $pos(K)$ is a closed cone.

Lemma 2. *Suppose that S, P are arbitrary (possibly infinite) index sets, $a_s = (a_1(s), \ldots, a_n(s))$ maps S onto $\mathbb{R}^n$, and so does a_p. Suppose that the set $co(a_s, S \in S) + pos(a_p, p \in P)$ is closed. Then the following statements are equivalent:*

$$I: \begin{cases} < a_s, x >< 0, \ s \in S, \ S \neq \emptyset, \\ < a_p, x > \leq 0, \ p \in P \end{cases} \quad \text{has no solution } x \in \mathbb{R}^n;$$
$$II: \ 0 \in co(a_s, \ S \in S) + pos(a_p, p \in P).$$

Lemma 3. *Let $\{C_t \mid t \in v\}$ be an arbitrary collection of convex sets in $\mathbb{R}^n$ and $K = pos(\cup_{t \in v} C_t)$. Then, every nonzero vector of K can be expressed as a non-negative linear combination of n or fewer linear independent vectors, each belonging to a different C_t.*

We denote by $\mathcal{K}_C$ the family of all bounded closed intervals in $\mathbb{R}$. We will remember the LU-order to decide when an interval is smaller or larger than another one.

Definition 1. *Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in $\mathbb{R}$. We write,*

- $A \underline{\preceq} B \Leftrightarrow a^L \leq b^L$ *and* $a^U \leq b^U$.
- $A \preceq B \Leftrightarrow A \underline{\preceq} B$ *and* $A \neq B$, *i.e.,* $a^L \leq b^L$ *and* $a^U \leq b^U$*, with a strict inequality.*
- $A \prec B \Leftrightarrow a^L < b^L$ *and* $a^U < b^U$.

Let D be an open and non-empty subset of $M = \mathbb{R}$. The function $f : D \to \mathcal{K}_C$ is called an *interval-valued function*, i.e., $f(x)$ is a closed interval in $\mathbb{R}$ for each $x \in M$. We will denote $f(x) = [f^L(x), f^U(x)]$ where f^L and f^U are real-valued functions and satisfy $f^L(x) \leq f^U(x)$ for every $x \in M$.

3. Robust KKT Optimality Conditions

We then introduce the semi-infinite interval equilibrium problem with uncertainty in both the constraints and the objective function through interval-valued functions.

Consider function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}_C$ and $g_t : \mathbb{R}^n \times V_t \to \mathbb{R}$, where $V_t \subset M = \mathbb{R}^n$. In this case, $v_t \in \mathbb{R}^n$ is an uncertainty parameter that lies in some convex and compact set $V_t \subset \mathbb{R}^n$, $t \in T$. The set-valued mapping $V : T \rightrightarrows \mathbb{R}^n$ is defined as $V(t) := V_t$ for all $t \in T$. So the points in the graph of V_t are of the type (t, v_t).

A semi-infinite interval equilibrium problem involving data uncertainty is defined as finding $x \in E$ such that,

$$\text{(SIEPU)} \quad F(x,y) \underline{\succeq} [0,0], \ \forall y \in E$$
$$\text{subject to:}$$
$$g_t(x, v_t) \leq 0, \quad v_t \in V_t, \ \forall t \in T$$

where

$$E = \{x \in \mathbb{R}^n : \ g_t(x, v_t) \leq 0, \quad v_t \in V_t, \ \forall t \in T\}$$

and T is an arbitrary nonempty infinite index set.

The robust formulation for a semi-infinite interval equilibrium problem involving data uncertainty (called a robust semi-infinite interval equilibrium problem with uncertainty) is given as find

$$\text{(RSIEPU)} \quad F(x,y) \underline{\succeq} [0,0], \ \forall y \in S$$
$$\text{subject to:}$$
$$g_t(x, v_t) \leq 0, \quad \forall v_t \in V_t, \ \forall t \in T$$

Here, a v_t is an uncertain parameter for SIEPU. The notation S denotes the robust feasible region

$$S = \{x \in \mathbb{R}^n : g_t(x, v_t) \leq 0, \quad \forall v_t \in V_t, \forall t \in T\}$$

For a point $\bar{x} \in S$, the active constraint set is given as

$$A(\bar{x}) = \{(t, \bar{v}_t), t \in T : \exists \bar{v}_t \in V_t, g_t(\bar{x}, \bar{v}_t) = 0\}$$

The set of active constraint multipliers at $\bar{x} \in S$ is

$$\Lambda(\bar{x}) = \{\mu_t \in \mathbb{R}_+^{V_t} \mid (t, \bar{v}_t) \in A(\bar{x}), \mu_t(\bar{v}_t) g_t(\bar{x}, \bar{v}_t) = 0\}$$

where $\mathbb{R}_+^{V_t}$ is the collection of all the functions $\mu_t : V_t \to \mathbb{R}$, $t \in T$.

We assume that the condition $\mu_t(v_t) = 0$, $\forall v_t \in V_t$ satisfies for infinitely many values of $t \in T$ and there exist a finitely many $t \in T$ such that $\mu_t(v_t) \geq 0$, $\forall v_t \in V_t$.

If there exist a finite set $J(\bar{x}) \subset A(\bar{x})$ such that $\mu_t(\bar{v}_t) > 0$ for $t = 1, 2, \ldots, j$.

Definition 2. *A point $\bar{x} \in S$ is said to be an optimal solution to RSIEPU if there exists no $\bar{x} \in S$ such that $F(\bar{x}, y) \prec [0, 0]$, $\forall y \in S$.*

Remark 1. *The definition 2 is equivalent to there exists no $\bar{x} \in S$ such that $F_{\bar{x}}^L(y) < 0$, $\forall y \in S$. In this case, the RSIEPU is a problem of real functions, not interval-valued functions.*

Let us remember the classic concepts:

Definition 3. *Let S be a given nonempty subset of M and $\bar{x} \in clS$.*

(a) *The contingent cone of S at $\bar{x}$ is:*

$$\mathcal{T}(S, \bar{x}) = \{v \in \mathbb{R}^n \mid \exists t_k \to 0, \exists v_k \in \mathbb{R}^n, v_k \to v, \forall k \in \mathbb{N}, \bar{x} + t_k v_k \in S\}$$

(b) *The negative polar cone of S in M is:*

$$S^- = \{x^* \in M \mid < x^*, x > \leq 0, \quad \forall x \in S\}$$

(c) *The strictly negative polar cone of S in M is:*

$$S^s = \{x^* \in M \mid < x^*, x > < 0, \quad \forall x \in S \backslash \{0\}\}$$

The properties that constraints must satisfy to ensure that the Karush-Kuhn-Tucker conditions are necessary conditions of local optimality are called constraint qualification and constitute a fundamental hypothesis for establishing the validity of the Karush-Kuhn-Tucker theorem. We will use one of them, the Abadie constraint qualification.

Definition 4. *The Abadie constraint qualification (ACQ) holds at $\bar{x} \in S$ if the negative polar cone $\left(\bigcup_{(t,\bar{v}_t) \in A(\bar{x})} \nabla g_t(\bar{x}, \bar{v}_t)\right)^- \subseteq \mathcal{T}(S, \bar{x})$ and the set*

$$pos\left(\bigcup_{(t,\bar{v}_t) \in A(\bar{x})} \nabla g_t(\bar{x}, \bar{v}_t)\right)$$

is closed.

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We will start by proving the necessary condition of optimality where we will make use of the lemmas of Section 2.

Theorem 1 (Necessary optimality condition)**.** *Let S be a nonempty convex subset of M and let $F^L, F^U : S \to \mathbb{R}$, $g_t : S \times V_t \to \mathbb{R}$, $t \in T$ be differential mappings at $\bar{x} \in S$ a feasible point. Let $F(\bar{x}, \bar{x}) = F_{\bar{x}}(\bar{x}) = [F^L_{\bar{x}}(\bar{x}), F^U_{\bar{x}}(\bar{x})] = [0, 0]$.*

Suppose that $\bar{x}$ is an optimal solution of RSIEPU and ACQ holds at $\bar{x}$. Then, there exist $\lambda^L \in \mathbb{R}_+$, $\mu_t \in \Lambda(\bar{x})$ and $v_t \in V_T$, $t \in T$, such that

$$0 = \lambda^L \nabla F^L_{\bar{x}}(y) + \sum_{(t,\bar{v}_t) \in T \times V_t} \mu_t(\bar{v}_t) \nabla g_t(\bar{x}, \bar{v}_t), \ \forall y \in S \tag{1}$$

$$\mu_t(\bar{v}_t) g_t(\bar{x}, \bar{v}_t) = 0 \tag{2}$$

Proof. Our first objective is to prove

$$\left(\nabla F^L_{\bar{x}}(y)\right)^s \cap \mathcal{T}(S, \bar{x}) = \emptyset \tag{3}$$

Case 1. If $[0, 0] = \nabla F_{\bar{x}}(y))$ then $\left(\nabla F^L_{\bar{x}}(y)\right)^s = \emptyset$.

Then, expression (3) is satisfied.

Case 2. If $[0, 0] \neq \nabla F_{\bar{x}}(y)$. By reductio ad absurdum, suppose to the contrary, that there exists $v \in \left(\nabla F^L_{\bar{x}}(y)\right)^s \cap \mathcal{T}(S, \bar{x})$ then $\nabla F^L_{\bar{x}}(y) v < 0$.

Since $v \in \mathcal{T}(S, \bar{x})$, there exists $t_k \to 0$ and $v_k \to v$ such that $\bar{x} + t_k v_k \in S$ for all k. It follows that

$$\lim_{t_k \to 0} \frac{1}{t_k} [F^L_{\bar{x}}(\bar{x} + t_k v_k) - F^L_{\bar{x}}(\bar{x})] < 0$$

Therefore

$$F^L_{\bar{x}}(\bar{x} + t_k v_k) - F^L_{\bar{x}}(\bar{x}) < 0$$

By the assumption that $F(\bar{x}, \bar{x}) = F_{\bar{x}}(\bar{x}) = [0, 0]$ then

$$F^L_{\bar{x}}(\bar{x} + t_k v_k) < 0$$

which contradicts the fact that $\bar{x}$ is an optimal solution of RSIEPU and therefore the expression (3) holds.

From (3) and ACQ we have that

$$\left(\nabla F^L_{\bar{x}}(y)\right)^s \cap \left(\bigcup_{(t,\bar{v}_t) \in A(\bar{x})} \nabla g_t(\bar{x}, \bar{v}_t) \right)^- \subseteq$$

$$\left(\nabla F^L_{\bar{x}}(y)\right)^s \cap \mathcal{T}(S, \bar{x}) = \emptyset$$

And therefore, there is no $\bar{v}_t \in V_t$ fulfilling

$$\begin{cases} \nabla F^L_{\bar{x}}(y) \bar{v}_t < 0, \\ \nabla g_t(\bar{x}, \bar{v}_t) < 0, \quad (t, \bar{v}_t) \in A(\bar{x}) \end{cases}$$

Furthermore, from Lemma 1 we are assured that $co\left(\nabla F^L_{\bar{x}}(y)\right)$ is a compact set, and therefore,

$$co\left(\nabla F^L_{\bar{x}}(y)\right) + pos\left(\bigcup_{(t,\bar{v}_t) \in A(\bar{x})} \nabla g_t(\bar{x}, \bar{v}_t) \right)$$

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is closed.

By the lemma 2, we obtain

$$0 \in co\Big(\nabla F^L_{\bar{x}}(y)\Big) + pos\left(\bigcup_{(t,\bar{v}_t)\in A(\bar{x})} \nabla g_t(\bar{x},\bar{v}_t)\right)$$

According to Lemma 3, there exist $\lambda^L \in \mathbb{R}_+$ and $\mu_t \in \Lambda(\bar{x})$ such that

$$0 = \lambda^L \nabla F^L_{\bar{x}}(y) + \sum_{(t,\bar{v}_t)\in T\times V_t} \mu_t(\bar{v}_t)\nabla g_t(\bar{x},\bar{v}_t), \ \forall y \in S$$

Thus, the first condition of KKT (1) is satisfied.

To prove second KKT condition, let W be the set that $(y,z) \in S \times S$, $\exists\, x \in S$ such that

$$y - \nabla F^L_{\bar{x}}(x)(\,x - \bar{x}) > 0$$

$$z - [g(\bar{x},\bar{v}_t) + \nabla g_t(\bar{x},\bar{v}_t)(x-\bar{x})] > 0$$

and we can see that W is a nonempty open convex set. It is clear

$$([\nabla F^L_{\bar{x}}(\bar{x})(\bar{x}-\bar{x})) + t'c], \nabla g_t(\bar{x},\bar{v}_t)(\bar{x}-\bar{x})) + t'k) \in W$$

for all c, k and $t' > 0$. By separation theorem, there exist $\lambda^L \in \mathbb{R}_+$ and $\mu \in \Lambda(\bar{x})$ such that

$$\lambda^L[\nabla F^L_{\bar{x}}(\bar{x})(\bar{x}-\bar{x}) + t'c] + \mu_t(\bar{v}_t)[g_t(\bar{x},\bar{v}_t) + \nabla g_t(\bar{x},\bar{v}_t)(\bar{x}-\bar{x}) + t'k] =$$

$$t'\lambda^L c + \mu_t(\bar{v}_t)g_t(\bar{x},\bar{v}_t) + t'\mu(\bar{v}_t)k > 0$$

Letting $t' \to 0$, we obtain $\mu_t(\bar{v}_t)g(\bar{x},\bar{v}_t) \geq 0$. As $g_t(\bar{x},\bar{v}_t) \leq 0$ and $\mu_t \geq 0$, then we have that

$$\mu_t(\bar{v}_t)g_t(\bar{x},\bar{v}_t) \leq 0$$

thus us

$$\mu_t(\bar{v}_t)g(\bar{x},\bar{v}_t) = 0$$

And, therefore the KKT conditions were verified. □

We will now tackle the proof of sufficient optimality conditions, for which we will need convexity assumptions.

Theorem 2 (Sufficient optimality condition). *Let S be a nonempty convex subset of M and let F^L, F^U : $S \to \mathbb{R}$, $g_t : S \times V_t \to \mathbb{R}$, $t \in T$ be differential mappings at $\bar{x} \in S$ a feasible point. Let $F(\bar{x},\bar{x}) = F_{\bar{x}}(\bar{x}) = [F^L_{\bar{x}}(\bar{x}), F^U_{\bar{x}}(\bar{x})] = [0,0]$.*

Assume that convexity of $F^L_{\bar{x}}(x)$ and $g_t(x,v_t)$ at $\bar{x}$ and there exist $\lambda^L \in \mathbb{R}_+$, $\mu_t \in \Lambda(\bar{x})$ and $v_t \in V_T$, $t \in T$, such that KKT optimality conditions (1) and (2) hold then $\bar{x}$ is an optimal solution for RSIEPU.

Proof. As $\bar{x} \in S$ satisfying (1), there exist $\nabla F^L_{\bar{x}}(y)$ and $\nabla g_t(\bar{x},\bar{v}_t)$ where $J(\bar{x})$ is a finite subset of $A(\bar{x})$, such that

$$\sum_{(t,\bar{v}_t)\in J(\bar{x})} \mu_t(\bar{v}_t)\nabla g_t(\bar{x},\bar{v}_t) = -\lambda^L \nabla F^L_{\bar{x}}(\bar{x}) \tag{4}$$

Since $x \in S$ and $g_t(x,\bar{v}_t) \leq 0$, $\forall (t,\bar{v}_t) \in J(\bar{x})$, we get that

$$g_t(x,\bar{v}_t) \leq 0 = g_t(\bar{x},\bar{v}_t), \ \forall (t,\bar{v}_t) \in J(\bar{x})$$

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Due to convexity g_t at $\bar{x}$, we have that

$$\sum_{(t,\bar{v})\in J(\bar{x})} \mu_t(\bar{v}_t)\nabla g_t(\bar{x},\bar{v}_t)(x-\bar{x}) \leq \sum_{(t,\bar{v}_t)\in J(\bar{x})} \mu_t(g_t(x,\bar{v}_t) - g_t(\bar{x},\bar{v}_t)) \leq 0 \tag{5}$$

By (4) and (5)

$$\lambda^L \nabla F^L_{\bar{x}}(\bar{x})(x-\bar{x}) \geq 0 \tag{6}$$

Let us now assume by reductio ad absurdum that $\bar{x}$ is not an optimal solution for RSIEPU. Then there exists $x \in S$ satisfying

$$F(\bar{x},x) = F_{\bar{x}}(x) \prec [0,0]$$

The above inequalities, together with $\lambda^L \in \mathbb{R}_+$ imply that

$$\lambda^L(F^L_{\bar{x}}(x) - F^L_{\bar{x}}(\bar{x})) < 0$$

Using the convexity of $F^L_{\bar{x}}(x)$ at $\bar{x}$,

$$0 > F^L_{\bar{x}}(x) - F^L_{\bar{x}}(\bar{x}) \geq \nabla F^L_{\bar{x}}(\bar{x})(x-\bar{x}) \tag{7}$$

And, from inequality (7) we obtain that

$$\lambda^L \nabla F^L_{\bar{x}}(\bar{x})(x-\bar{x}) < 0$$

which contradicts (6). □

Through generalized convexity conditions, we can also obtain sufficient optimality conditions.

Theorem 3 (Sufficient optimality condition). *Let S be a nonempty convex subset of M and let $F^L, F^U : S \to \mathbb{R}$, $g_t : S \times V_t \to \mathbb{R}$, $t \in T$ be differential mappings at $\bar{x} \in S$ a feasible point. Let $F(\bar{x},\bar{x}) = F_{\bar{x}}(\bar{x}) = [F^L_{\bar{x}}(\bar{x}), F^U_{\bar{x}}(\bar{x})] = [0,0]$.*

Assume that pseudoconvexity of $F^L_{\bar{x}}(x)$ and quasiconvexity of $g_t(x,v_t)$ at $\bar{x}$ and there exist $\lambda^L \in \mathbb{R}_+$, $\mu_t \in \Lambda(\bar{x})$ and $v_t \in V_T$, $t \in T$, such that KKT optimality conditions (1) and (2) hold then $\bar{x}$ is an optimal solution for RSIEPU.

Proof. Assume to the contrary that, $\bar{x}$ is not an optimal solution for RSIEPU then there is an $x \in S$ such that $F^L_{\bar{x}}(x) \prec F^L_{\bar{x}}(\bar{x})$ and the pseudoconvexity of F^L at $\bar{x}$ we obtain $\nabla F^L_{\bar{x}}(\bar{x})(x-\bar{x}) < 0$. Since $\lambda \geq 0$, therefore

$$\lambda^L \nabla F^L_{\bar{x}}(\bar{x})(x-\bar{x}) < 0 \tag{8}$$

As $x \in S$ and for $\mu_t \in \Lambda(\bar{x})$, we get $\mu_t(\bar{v}_t)g_t(x,\bar{v}_t) \leq 0$, $t \in T$. This along with $\mu_t(\bar{v}_t)g_t(\bar{x},\bar{v}_t) = 0$ yields

$$\mu_t(\bar{v}_t)g_t(x,\bar{v}_t) - \mu_t(\bar{v}_t)g_t(\bar{x},\bar{v}_t) \leq 0,\ t \in T$$

The quasiconvexity of $g_t(x,v_t)$ at $\bar{x}$ implies

$$\sum_{(t,\bar{v}_t)\in T\times V_t} \mu_t(\bar{v}_t)\nabla g_t(\bar{x},\bar{v}_t)(x-\bar{x}) \leq 0\ \forall y \in S$$

then

$$-\sum_{(t,\bar{v}_t)\in T\times V_t} \mu_t(\bar{v}_t)\nabla g_t(\bar{x},\bar{v}_t)(x-\bar{x}) = \lambda^L \nabla F^L_{\bar{x}}(y)(x-\bar{x}) \geq 0 \tag{9}$$

But (9) is a contradiction to (8). Hence $\bar{x}$ is an optimal solution of RSIEPU. □

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Remark 2. *The results obtained in this paper extend the theorems given by Wei and Gong [27] in normed spaces and the optimality conditions given in Ruiz-Garzón et al. [35] from semi-infinite interval equilibrium problems to uncertainty constraints. As well as the results achieved by Tripathi and Arora [36] involving data uncertainty to interval-valued functions.*

Remark 3. *It should be noted that the first KKT condition (1) does not involve the upper bound of the interval-valued function.*

To sum up, we have obtained the necessary and sufficient KKT optimality conditions for the solutions of the semi-infinite interval equilibrium problems with constraints. We will illustrate the above optimality conditions with the following example:

Example 1. *Consider the problem RSIEPU: find $\bar{x}$ such that*

$$F(x,y) = [x(y-x),\ 4x(y-x)] \succeq 0,\ \forall y \in S$$

$$\begin{aligned} &\textit{subject to:} \\ &\quad g_t(x,v_t) = tv_t x - t - 1 \leq 0, \\ &\quad \forall v_t \in V_t = [-t, t+3],\ \forall t \in T = [-1,1] \end{aligned}$$

Where $g_t(x,v_t) \leq 0$, $\forall v_t \in V_t$, $\forall t \in T \Leftrightarrow x \in [0,1/2]$, $S = [0,1/2]$ and $F^L, F^U : S \to \mathbb{R}$, with $g_t : S \times V_t \to \mathbb{R}$ are differentiable functions. For $\bar{x} = 0 \in S$,

$$\nabla F^L_{\bar{x}}(y) = y, \quad \nabla g_t(\bar{x},\bar{v}_t) = \{tv_t\},\ \forall t \in T,\ A(\bar{x}) = \{-1\}$$

$$\left(\bigcup_{(t,\bar{v}_t)\in J(\bar{x})} \nabla g_t(\bar{x},\bar{v}_t) \right)^{-} = \mathbb{R}_+$$

$$pos\left(\bigcup_{(t,\bar{v}_t)\in J(\bar{x})} \nabla g_t(\bar{x},\bar{v}_t) \right) = -\mathbb{R}_+$$

is closed, i.e., ACQ holds at $\bar{x}$. Now, there exist $\lambda^L = 1$ and

$$\mu_t = \begin{cases} y, & \textit{if} \quad t = -1;\ v_{-1} = 1 \\ 0, & \textit{for all} \quad t \in (-1,1],\ v_t \in [-t, t+3] \end{cases}$$

such that

$$0 = \lambda^L \nabla F^L_{\bar{x}}(y) + \sum_{(t,\bar{v}_t)\in J(\bar{x})} \mu_t(\bar{v}_t) \nabla \mu_t g_t(\bar{x},\bar{v}_t)$$

$$0 = y + y(-1)$$

that F^L and $g_t(x,v_t)$ are convex and therefore F^L is pseudoconvex and $g_t(x,v_t)$ is quasiconvex at $\bar{x} = 0$. Hence, all assumptions in Theorem 3 hold that $\bar{x}$ is a solution of RSIEPU.

4. Particular Case

4.1. Robust Dual Model

We can consider the semi-infinite interval programming problem involving uncertainty in the data as a particular case of equilibrium problems.

Thus we will be able to study the semi-infinite interval programming problem with constraints (SIPU) defined as:

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$$\text{(SIPU)} \quad \begin{array}{l} \min f(x) = [f^L(x), f^U(x)] \\ \text{subject to:} \\ \quad g_t(x, v_t) \le 0, \quad v_t \in V_t, \forall t \in T \end{array}$$

where

$$E = \{x \in \mathbb{R}^n : g_t(x, v_t) \le 0, \quad v_t \in V_t, \forall t \in T\}$$

and T is an arbitrary nonempty infinite index set where $f : S \to \mathcal{K}_C$, $g_t : S \times V_t \to \mathbb{R}$, $t \in T$ be differential mappings at $\bar{x} \in S$.

We introduce the robust formulation for the previous problem:

$$\text{(RSIPU)} \quad \begin{array}{l} \min f(x) = [f^L(x), f^U(x)] \\ \text{subject to:} \\ \quad g_t(x, v_t) \le 0, \quad \forall v_t \in V_t, \forall t \in T \end{array}$$

The notation S denotes the robust feasible region:

$$S = \{x \in \mathbb{R}^n : g_t(x, v_t) \le 0, \quad \forall v_t \in V_t, \forall t \in T\}$$

Definition 5. *The point $\bar{x}$ is an optimal solution for RSIPU if there is no $x \in S$ satisfying $f(x) \prec f(\bar{x})$.*

We can therefore obtain the following result for the robust semi-infinite interval mathematical programming problem (RSIPU).

Corollary 1. *Let S be a nonempty convex subset of M and let $f^L, f^U : S \to \mathbb{R}$, $g_t : S \times V_t \to \mathbb{R}$, $t \in T$ be differential mappings at $\bar{x} \in S$ a feasible point.*

a) Suppose that $\bar{x}$ is an optimal solution of RSIPU and ACQ holds at $\bar{x}$. Then, there exist $\lambda^L \in \mathbb{R}_+$, $\mu_t \in \Lambda(\bar{x})$ and $v_t \in V_T$, $t \in T$, such that

$$0 = \lambda^L \nabla f^L(\bar{x}) + \sum_{(t,\bar{v}_t) \in T \times V_t} \mu_t(\bar{v}_t) \nabla g_t(\bar{x}, \bar{v}_t) \tag{10}$$

$$\mu_t(\bar{v}_t) g_t(\bar{x}, \bar{v}_t) = 0 \tag{11}$$

(b) Assume that pseudoconvexity of $f^L(\bar{x})$ and quasiconvexity of $g_t(x, v_t)$ at $\bar{x}$ and there exist $\lambda^L \in \mathbb{R}_+$, $\mu_t \in \Lambda(\bar{x})$ and $v_t \in V_T$, $t \in T$, such that KKT optimality conditions (10) and (11) hold then $\bar{x}$ is an optimal solution for RSIPU.

Proof. The proof is similar to that of previous theorems with no more than considering RSIPU as a particular case of RSIEPU, simply takes $F(x, y) = f(y) - f(x), \forall x, y \in M$. □

Remark 4. *The results obtained by Tung [30] can be considered as particular cases of those obtained here involving data uncertainty.*

In the development of mathematical optimization the dual model is important since the solutions of the dual and primal models are related, in addition to the advantages of using one or the other model depending on the occasion.

We present the following Mond-Weir type dual robust semi-infinite program DRSIPU:

$$\text{(DRSIPU)} \; \begin{array}{l} \max f(u) = [f^L(u), f^U(u)] \\ \text{subject to:} \\ 0 = \lambda^L \nabla f^L(u) + \sum_{(t,\bar{v}_t) \in T \times V_t} \mu_t(\bar{v}_t) \nabla g_t(u, \bar{v}_t) \end{array}$$

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$$\mu_t(\bar{v}_t)g_t(u,\bar{v}_t) \geq 0,\ t \in T$$
$$\lambda^L \in \mathbb{R}_+,\ \mu_t \in \Lambda(\bar{x})\ \text{ and } v_t \in V_T,\ t \in T$$

Theorem 4 (Weak Duality)**.** *Let x be a feasible solution to RSIPU and (u, λ, μ_t, v_t) be a feasible solution to DRSIPU. Suppose that $f^L(u)$ is pseudoconvex at u and $g_t(u, v_t)$ is quasiconvex at u then the following cannot hold $f(x) \prec f(u)$.*

Proof. From the feasibility hypothesis of x for RSIPU we have $\mu_t(\bar{v}_t)g_t(x,\bar{v}_t) \leq 0$ and the dual feasibility of (u, λ, μ_t, v_t) gives $\mu_t(\bar{v}_t)g_t(u,\bar{v}_t) \geq 0$ for $\mu_t \in \Lambda(\bar{x})$. Combining, we get

$$\mu_t(\bar{v}_t)g_t(x,\bar{v}_t) - \mu_t(\bar{v}_t)g_t(u,\bar{v}_t) \leq 0$$

By quasiconvexity of $g_t(u, v_t)$ at u, we get

$$\sum_{(t,\bar{v}_t)\in T\times V_t} \mu_t(\bar{v}_t)\nabla g_t(u,\bar{v}_t) \leq 0$$

Now first dual feasibility condition, we have

$$\lambda^L \nabla f^L(u) = -\sum_{(t,\bar{v}_t)\in T\times V_t} \mu_t(\bar{v}_t)\nabla g_t(u,\bar{v}_t)$$

Hence

$$\lambda^L \nabla f^L(u)(x-u) = -\sum_{(t,\bar{v}_t)\in T\times V_t} \mu_t(\bar{v}_t)\nabla g_t(u,\bar{v}_t)(x-u) \geq 0 \tag{12}$$

Let us now assume by reductio ad absurdum that the hypothesis does not hold, i.e. $f(x) \prec f(u)$ then $f^L(x) < f^L(u)$. The pseudoconvexity of f^L at u implies

$$\lambda^L \nabla f^L(u)(x-u) < 0 \tag{13}$$

But (12) and (13) contradict each other. So, we conclude that $f(x) \prec f(u)$ is not true. □

Let's illustrate this theorem with an example:

Example 2. *Let us consider the robust semi-infinite interval programming problem involving data uncertainty (RSIPU) defined as:*

$$\begin{array}{ll} (RSIPU) & \min f(x) = [f^L(x), f^U(x)] = [x^2+x, x^2+x+1] \\ & \textit{subject to:} \\ & \quad g_t(x,v_t) = tv_t x - t - 1 \leq 0, \\ & \quad \forall v_t \in V_t = [-t, t+3],\ \forall t \in T = [-1,1] \end{array}$$

Where $g_t(x,v_t) \leq 0,\ \forall v_t \in V_t,\ \forall t \in T \Leftrightarrow x \in [0,1/2]$, $S = [0,1/2]$ and $f^L, f^U : S \to \mathbb{R}$, with $g_t : S \times V_t \to \mathbb{R}$ are differentiable functions. For $\bar{x} = 0 \in S$,

$$\nabla f^L(y) = 1, \quad \nabla g_t(\bar{x},\bar{v}_t) = \{tv_t\}, \forall t \in T,\ A(\bar{x}) = \{-1\}$$

$$\left(\bigcup_{(t,\bar{v}_t)\in J(\bar{x})} \nabla g_t(\bar{x},\bar{v}_t)\right)^- = \mathbb{R}_+$$

$$pos\left(\bigcup_{(t,\bar{v}_t)\in J(\bar{x})} \nabla g_t(\bar{x},\bar{v}_t)\right) = -\mathbb{R}_+$$

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is closed, i.e., ACQ holds at $\bar{x}$. Now, there exist $\lambda^L = 1$ and

$$\mu_t = \begin{cases} 1, & \text{if} \quad t = -1;\ v_{-1} = 1 \\ 0, & \text{for all} \quad t \in (-1,1],\ v_t \in [-t, t+3] \end{cases}$$

such that

$$0 = \lambda^L \nabla f^L(y) + \sum_{(t,\bar{v}_t) \in J(\bar{x})} \mu_t(\bar{v}_t) \nabla \mu_t g_t(\bar{x}, \bar{v}_t)$$

$$0 = 1(1) + 1(-1)$$

that f^L is pseudoconvex and $g_t(x)$ is quasiconvex function at $\bar{x} = 0$. Hence, all assumptions in Theorem 1 hold that $\bar{x}$ is a solution of RSIPU.

Let us formulate the dual model:

$$\begin{aligned} &(DRSIPU) \max f(u) = [u^2 + u, u^2 + u + 1] \\ &\quad \text{subject to:} \\ &\quad 0 = 1 + \textstyle\sum_{(t,\bar{v}_t) \in T \times V_t} \mu_t(\bar{v}_t) t v_t \\ &\quad \mu_t(\bar{v}_t)(t v_t u - t - 1) \geq 0,\ t \in T \\ &\quad \lambda^L \in \mathbb{R}_+,\ \mu_t \in \Lambda(\bar{x}) \text{ and } v_t \in V_T,\ t \in T \end{aligned}$$

The point $u = 0$ is a feasible solution to RSIPU and (0, 1, 1, 1) satisfies the feasibility conditions of the dual model DRSIPU. The assumptions of the weak duality theorem hold at these points. Hence the weak duality relation holds between RSIPU and DRSIPU.

Remark 5. *This weak duality theorem (4) extends to uncertainty constraints proposition 4 by Tung [30], proposition 5.1 by Jayswal et al. [29], theorem 4.1 by Ahmad et al. [31] to interval-valued functions or the Mond-Weir dual problem given by Jaichander et al. [32].*

Theorem 5 (Strong duality)**.** *Let $\bar{x}$ be an optimal solution to RSIPU and ACQ constraint qualification is satisfied at $\bar{x}$. Then there exist $\bar{\lambda}^L \in \mathbb{R}_+$, $\bar{\mu}_t \in \Lambda(\bar{x})$ and $\bar{v}_t \in V_T$, $t \in T$, such that $(\bar{x}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ is a feasible solution to DRSIPU and the two objective values are equal. Further, if the hypothesis of weak duality holds for all feasible solutions $(\bar{y}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$, then $(\bar{x}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ is an optimal solution to DRSIPU.*

Proof. Since $\bar{x}$ is an optimal solution to RSIPU and ACQ constraint qualification is satisfied at $\bar{x}$, then by Theorem 1 there exist $\bar{\lambda}^L \in \mathbb{R}_+$, $\bar{\mu}_t \in \Lambda(\bar{x})$ and $\bar{v}_t \in V_T$, $t \in T$, such that

$$0 = \bar{\lambda}^L \nabla F^L_{\bar{x}}(y) + \sum_{(t,\bar{v}_t) \in T \times V_t} \bar{\mu}_t(\bar{v}_t) \nabla g_t(\bar{x}, \bar{v}_t), \ \forall y \in S \tag{14}$$

$$\bar{\mu}_t(\bar{v}_t) g_t(\bar{x}, \bar{v}_t) = 0 \tag{15}$$

which yields that $(\bar{x}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ is a feasible solution to DRSIPU and the corresponding objective values are equal. Suppose that $(\bar{x}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ is not an optimal solution to DRSIPU, then there exists a feasible solution $(\bar{y}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ to DRSIPU such that $f(x) \prec f(y)$ which contradicts the weak duality. Hence $(\bar{x}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ is an optimal solution to DRSIPU. □

Remark 6. *This strong duality theorem (5) extends uncertainty constraints proposition 5 by Tung [30], proposition 5.2 by Jayswal et al. [29], theorem 4.2 by Ahmad et al. [31] to interval-valued functions or the Mond-Weir dual problem given by Jaichander et al. [32].*

5. Conclusions

What has been achieved with this article is summarized in the following advances concerning the existing literature:

- Introduce robust semi-infinite interval equilibrium problem involving data uncertainty by addressing the treatment of uncertainty in the objective function and the constraints.
- To achieve the necessary and sufficient conditions of optimality for the robust semi-infinite interval equilibrium problem involving data uncertainty. The results obtained in this paper extend the theorems given by Wei and Gong [27] given in normed spaces and the optimality conditions given in Ruiz-Garzón et al. [35] from semi-infinite interval equilibrium problems to uncertainty constraints. As well as the results achieved by Tripathi and Arora [36] involving data uncertainty to interval-valued functions.
- We introduce the Mond-Weir type dual robust semi-infinite program and we have proved the weak and strong theorem of duality. We have generalized results by Tung [30], Jayswal et al. [29], Ahmad et al. [31] and Jaichander et al. [32].

Finally, I believe it is appropriate to continue to persevere in obtaining applications of these results in the economic field.

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