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Gauge-Invariant Perturbation Theory on the Schwarzschild Background Spacetime Part II:—Even-Mode Perturbations—

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Abstract: This is the Part II paper of our series of papers on a gauge-invariant perturbation theory on the Schwarzschild background spacetime. After reviewing our general framework of the gauge-invariant perturbation theory and the proposal on the gauge-invariant treatments for $l = 0, 1$ mode perturbations on the Schwarzschild background spacetime in the Part I paper [K. Nakamura, arXiv:2110.13508 [gr-qc]], we examine the linearized Einstein equations for even-mode perturbations. We discuss the strategy to solve the linearized Einstein equations for these even-mode perturbations including $l = 0, 1$ modes. Furthermore, we explicitly derive the $l = 0, 1$ mode solutions to the linearized Einstein equations in both the vacuum and the non-vacuum cases. We show that the solutions for $l = 0$ mode perturbations includes the additional Schwarzschild mass parameter perturbation, which is physically reasonable. Then, we conclude that our proposal of the resolution of the $l = 0, 1$-mode problem is physically reasonable due to the realization of the additional Schwarzschild mass parameter perturbation and the Kerr parameter perturbation in the Part I paper.

Keywords: black hole; Schwarzschild spacetime; perturbation theory; gauge-invariance

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1. Introduction

Gravitational-wave observations are now carrying out through the ground-based detectors [1–4]. Furthermore, the projects of future ground-based gravitational-wave detectors [5,6] are also progressing to achieve more sensitive detectors. In addition to these ground-based detectors, some projects of space gravitational-wave antenna are also progressing [7–10]. Among them, the Extreme-Mass-Ratio-Inspiral (EMRI), which is a source of gravitational waves from the motion of a stellar mass object around a supermassive black hole, is a promising target of the Laser Interferometer Space Antenna [7]. To describe the gravitational wave from EMRIs, black hole perturbations are used [11]. Furthermore, the sophistication of higher-order black hole perturbation theories is required to support these gravitational-wave physics as a precise science. The motivation of this paper is in such theoretical sophistications of black hole perturbation theories toward higher-order perturbations for wide physical situations.

Although realistic black holes have their angular momentum and we have to consider the perturbation theory of a Kerr black hole for direct applications to the EMRI, we may say that further sophistications are possible even in perturbation theories on the Schwarzschild background spacetime. From the pioneering works by Regge and Wheeler [12] and Zerilli [13–15], there have been many studies on the perturbations in the Schwarzschild background spacetime [15–28]. In these works, perturbations on the Schwarzschild spacetime are decomposed through the spherical harmonics Y_{lm} because of the spherical symmetry of the background spacetime, and $l = 0$ and $l = 1$ modes should be separately treated. Furthermore, "*gauge-invariant*" treatments for $l = 0$ and $l = 1$ even-modes were unknown.

Owing to this situation, in the previous papers [29,30], we proposed the strategy of the gauge-invariant treatments of these $l = 0, 1$ mode perturbations, which is declared as Proposal 1 in Section 2 of this paper below. One of important premises of our gauge-invariant perturbations is the distinction of the first-kind gauge and the second-kind gauge. The first-kind gauge is the choice of the coordinate system on the single manifold and we often use this first-kind gauge when we predict or interpret the measurement results of experiments and observation. On the other hand, the second-kind gauge is the choice of the point-identifications between the points on the physical spacetime $\mathscr{M}_\epsilon$ and the background spacetime $\mathscr{M}$. This second-kind gauge have nothing to do with our physical spacetime $\mathscr{M}$. The proposal in the Part I paper [30] is a part of our developments of the general formulation of a higher-order gauge-invariant perturbation theory on a generic background spacetime toward unambiguous sophisticated nonlinear general-relativistic perturbation theories [31–36]. This general formulation of the higher-order gauge-invariant perturbation theory was applied to cosmological perturbations [37–44]. Even in cosmological perturbation theories, the same problem as the above $l = 0, 1$-mode problem exists as gauge-invariant treatments of homogeneous modes of perturbations. In this sense, we can expect that the proposal in the previous paper [30] will be a clue to the same problem in gauge-invariant perturbation theory on the generic background spacetime.

In addition to the proposal of the gauge-invariant treatments of $l = 0, 1$-mode perturbations on the Schwarzschild background spacetime, in the previous Part I paper, we also derived the linearized Einstein equations in a gauge-invariant manner following Proposal 1. From the parity of perturbations, we can classify the perturbations on the spherically symmetric background spacetime into even- and odd-mode perturbations. In the Part I paper [30], we also gave a strategy to solve the odd-mode perturbations including $l = 0, 1$ modes. Furthermore, we also derived the explicit solutions for the $l = 0, 1$ odd-mode perturbations to the linearized Einstein equations following Proposal 1.

This paper is the Part II paper of the series of papers on the application of our gauge-invariant perturbation theory to that on the Schwarzschild background spacetime. This series of papers is the full paper version of our short paper [29]. In this Part II paper, we discuss a strategy to solve the linearized Einstein equation for even-mode perturbations including $l = 0, 1$ mode perturbations. We also derive the explicit solutions to the $l = 0, 1$ mode perturbations with generic linear-order energy-momentum tensor. As the result, we show that the additional Schwarzschild mass parameter perturbation in the vacuum case. This is the realization of the Birkhoff theorem at the linear-perturbation level in a gauge-invariant manner. This result is physically reasonable, and it also implies that Proposal 1 is also physically reasonable. The other supports for Proposal 1 are also given by the realization of exact solutions with matter fields which will be discussed in the Part III paper [46]. Furthermore, brief discussions on the extension to the higher-order perturbations are given in the short paper [45].

The organization of this Part II paper is as follows. In Section 2, after briefly review the framework of the gauge-invariant perturbation theory, we summarize our proposal in Refs. [29,30]. Then, we also summarize the linearized even-mode Einstein equation on the Schwarzschild background spacetime which was derived in Ref. [30] following Proposal 1. In Section 3, following Proposal 1, we discuss a strategy to solve these even-mode Einstein equations including $l = 0, 1$ mode perturbations. In Section 4, we derive the explicit solutions to the linearized Einstein equation for the $l = 0$ mode perturbations in both the vacuum and the non-vacuum cases. In Section 5, we also derive the explicit solutions to the linearized Einstein equation for the $l = 1$ mode perturbations in both the vacuum and the non-vacuum cases. The final section (Section 6) is devoted to our summary and discussions.

We use the notation used in the previous papers [29,30,45] and the unit $G = c = 1$, where G is Newton's constant of gravitation and c is the velocity of light.

2. Brief Review of the General-Relativistic Gauge-Invariant Perturbation Theory

In this section, we review the premise of the series of our papers [29,30,46] and this paper. In Section 2.1, we briefly review the framework of the gauge-invariant perturbation theory [31,32]. This is an important premise of the series of our papers. In Section 2.2, we review the linear perturbation on spherically symmetric background spacetimes which includes our proposal in Ref. [29,30]. In Section 2.3, we review the linearized Einstein equations for even-mode perturbations on the Schwarzschild background spacetime which are to be solved in this paper.

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2.1. General Framework of Gauge-Invariant Perturbation Theory

In any perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime $(\mathscr{M}_{\rm ph}, \bar{g}_{ab})$, which is identified with our nature itself, and we want to describe this spacetime $(\mathscr{M}_{\rm ph}, \bar{g}_{ab})$ by perturbations. The other is the background spacetime $(\mathscr{M}, g_{ab})$, which is prepared as a reference by hand. Note that these two spacetimes are distinct. Furthermore, in any perturbation theory, we always write equations for the perturbation of the variable Q as follows:

$$Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p). \tag{1}$$

Equation (1) gives a relation between variables on different manifolds. Actually, $Q(\text{“}p\text{”})$ in Equation (1) is a variable on $\mathscr{M}_\epsilon = \mathscr{M}_{\rm ph}$, whereas $Q_0(p)$ and $\delta Q(p)$ are variables on $\mathscr{M}$. Because we regard Equation (1) as a field equation, Equation (1) includes an implicit assumption of the existence of a point identification map $\mathscr{X}_\epsilon : \mathscr{M} \to \mathscr{M}_\epsilon : p \in \mathscr{M} \mapsto \text{“}p\text{”} \in \mathscr{M}_\epsilon$. This identification map is a *gauge choice* in general-relativistic perturbation theories. This is the notion of the *second-kind gauge* pointed out by Sachs [47]. Note that this second-kind gauge is a different notion from the degree of freedom of the coordinate transformation on a single manifold, which is called the *first-kind gauge* [30,43,44].

To compare with the variable Q on $\mathscr{M}_\epsilon$ and its background value Q_0 on $\mathscr{M}$, we use the pull-back $\mathscr{X}_\epsilon^*$ of the identification map $\mathscr{X}_\epsilon : \mathscr{M} \to \mathscr{M}_\epsilon$ and we evaluate the pulled-back variable $\mathscr{M}_\epsilon^* Q$ on the background spacetime $\mathscr{M}$. Furthermore, in perturbation theories, we expand the pull-back operation $\mathscr{X}_\epsilon^*$ to the variable Q with respect to the infinitesimal parameter ϵ for the perturbation as

$$\mathscr{X}_\epsilon^* Q = Q_0 + \epsilon \,{}^{(1)}_{\mathscr{X}}Q + O(\epsilon^2). \tag{2}$$

Equation (2) are evaluated on the background spacetime $\mathscr{M}$. When we have two different gauge choices $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$, we can consider the *gauge-transformation*, which is the change of the point-identification $\mathscr{X}_\epsilon \to \mathscr{Y}_\epsilon$. This gauge-transformation is given by the diffeomorphism $\Phi_\epsilon := (\mathscr{X}_\epsilon)^{-1} \circ \mathscr{Y}_\epsilon : \mathscr{M} \to \mathscr{M}$. Actually, the diffeomorphism Φ_ϵ induces a pull-back from the representation $\mathscr{X}_\epsilon^* Q_\epsilon$ to the representation $\mathscr{Y}_\epsilon^* Q_\epsilon$ as $\mathscr{Y}_\epsilon^* Q_\epsilon = \Phi_\epsilon^* \mathscr{X}_\epsilon^* Q_\epsilon$. From general arguments of the Taylor expansion[48], the pull-back Φ_ϵ^* is expanded as

$$\mathscr{Y}_\epsilon^* Q_\epsilon = \mathscr{X}_\epsilon^* Q_\epsilon + \epsilon \pounds_{\xi_{(1)}} \mathscr{X}_\epsilon^* Q_\epsilon + O(\epsilon^2), \tag{3}$$

where $\xi^a_{(1)}$ is the generator of Φ_ϵ. From Equations (2) and (3), the gauge-transformation for the first-order perturbation ${}^{(1)}Q$ is given by

$${}^{(1)}_{\mathscr{Y}}Q - {}^{(1)}_{\mathscr{Y}}Q = \pounds_{\xi_{(1)}} Q_0. \tag{4}$$

We also employ the *order by order gauge invariance* as a concept of gauge invariance [41]. We call the kth-order perturbation ${}^{(k)}_{\mathscr{X}}Q$ as gauge invariant if and only if

$${}^{(k)}_{\mathscr{X}}Q = {}^{(k)}_{\mathscr{Y}}Q \tag{5}$$

for any gauge choice $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$.

Based on the above setup, we proposed a procedure to construct gauge-invariant variables of higher-order perturbations [31,32]. First, we expand the metric on the physical spacetime $\mathscr{M}_\epsilon$, which was pulled back to the background spacetime $\mathscr{M}$ through a gauge choice $\mathscr{X}_\epsilon$ as

$$\mathscr{X}_\epsilon^* \bar{g}_{ab} = g_{ab} + \epsilon \,{}_{\mathscr{X}}h_{ab} + O(\epsilon^2). \tag{6}$$

Although the expression (6) depends entirely on the gauge choice $\mathscr{X}_\epsilon$, henceforth, we do not explicitly express the index of the gauge choice $\mathscr{X}_\epsilon$ in the expression if there is no possibility of confusion.

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The important premise of our proposal was the following conjecture [31,32] for the linear metric perturbation h_{ab}:

Conjecture 1. *If the gauge-transformation rule for a perturbative pulled-back tensor field h_{ab} to the background spacetime $\mathscr{M}$ is given by ${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = \pounds_{\xi_{(1)}} g_{ab}$ with the background metric g_{ab}, there then exist a tensor field $\mathscr{F}_{ab}$ and a vector field Y^a such that h_{ab} is decomposed as $h_{ab} =: \mathscr{F}_{ab} + \pounds_Y g_{ab}$, where $\mathscr{F}_{ab}$ and Y^a are transformed as ${}_{\mathscr{Y}}\mathscr{F}_{ab} - {}_{\mathscr{X}}\mathscr{F}_{ab} = 0$ and ${}_{\mathscr{Y}}Y^a - {}_{\mathscr{X}}Y^a = \xi^a_{(1)}$ under the gauge transformation, respectively.*

We call $\mathscr{F}_{ab}$ and Y^a as the *gauge-invariant* and *gauge-variant* parts of h_{ab}, respectively.

The proof of Conjecture 1 is highly nontrivial [33,35], and it was found that the gauge-invariant variables are essentially non-local. Despite this non-triviality, once we accept Conjecture 1, we can decompose the linear perturbation of an arbitrary tensor field ${}^{(1)}_{\mathscr{X}}Q$, whose gauge-transformation is given by Equation (4), through the gauge-variant part Y_a of the metric perturbation in Conjecture 1 as

$$ {}^{(1)}_{\mathscr{X}}Q = {}^{(1)}\mathscr{Q} + \pounds_{{}_{\mathscr{X}}Y} Q_0. \tag{7}$$

As examples, the linearized Einstein tensor ${}^{(1)}_{\mathscr{X}}G_a{}^b$ and the linear perturbation of the energy-momentum tensor ${}^{(1)}_{\mathscr{X}}T_a{}^b$ are also decomposed as

$$ {}^{(1)}_{\mathscr{X}}G_a{}^b = {}^{(1)}\mathscr{G}_a{}^b[\mathscr{F}] + \pounds_{{}_{\mathscr{X}}Y} G_a{}^b, \quad {}^{(1)}_{\mathscr{X}}T_a{}^b = {}^{(1)}\mathscr{T}_a{}^b + \pounds_{{}_{\mathscr{X}}Y} T_a{}^b, \tag{8}$$

where G_{ab} and T_{ab} are the background values of the Einstein tensor and the energy-momentum tensor, respectively. The gauge-invariant part ${}^{(1)}\mathscr{G}_a{}^b$ of the linear-order perturbation of the Einstein tensor is given by

$$ {}^{(1)}\mathscr{G}_a{}^b[A] := {}^{(1)}\Sigma_a{}^b[A] - \frac{1}{2}\delta_a{}^b {}^{(1)}\Sigma_c{}^c[A], \tag{9}$$

$$ {}^{(1)}\Sigma_a{}^b[A] := -2\nabla_{[a}H_{d]}{}^{bd}[A] - A^{cb}R_{ac}, \quad H_{ba}{}^c[A] := \nabla_{(a}A_{b)}{}^c - \frac{1}{2}\nabla^c A_{ab}, \tag{10}$$

where A_{ab} is an arbitrary tensor field of the second rank. Then, using the background Einstein equation $G_a{}^b = 8\pi T_a{}^b$, the linearized Einstein equation ${}^{(1)}_{\mathscr{X}}G_{ab} = 8\pi {}^{(1)}_{\mathscr{X}}T_{ab}$ is automatically given in the gauge-invariant form

$$ {}^{(1)}\mathscr{G}_a{}^b[\mathscr{F}] = 8\pi {}^{(1)}\mathscr{T}_a{}^b \tag{11}$$

even if the background Einstein equation is nontrivial. We also note that, in the case of a vacuum background case, i.e., $G_a{}^b = 8\pi T_a{}^b = 0$, Equation (8) shows that the linear perturbations of the Einstein tensor and the energy-momentum tensor is automatically gauge-invariant of the second kind.

We can also derive the perturbation of the divergence of $\bar{\nabla}_a \bar{T}_b{}^a$ of the second-rank tensor $\bar{T}_b{}^a$ on $(\mathscr{M}_{\rm ph}, \bar{g}_{ab})$. Through the gauge choice $\mathscr{X}_\epsilon$, $\bar{T}_b{}^a$ is pulled back to $\mathscr{X}_\epsilon^* \bar{T}_b{}^a$ on the background spacetime $(\mathscr{M}, g_{ab})$, and the covariant derivative operator $\bar{\nabla}_a$ on $(\mathscr{M}_{\rm ph}, \bar{g}_{ab})$ is pulled back to a derivative operator $\bar{\nabla}_a (= \mathscr{X}_\epsilon^* \bar{\nabla}_a (\mathscr{X}_\epsilon^{-1})^*)$ on $(\mathscr{M}, g_{ab})$. Note that the derivative $\bar{\nabla}_a$ is the covariant derivative associated with the metric $\mathscr{X}_\epsilon \bar{g}_{ab}$, whereas the derivative ∇_a on the background spacetime $(\mathscr{M}, g_{ab})$ is the covariant derivative associated with the background metric g_{ab}. Bearing in mind the difference in these derivatives, the first-order perturbation of $\bar{\nabla}_a \bar{T}_b{}^a$ is given by

$$ {}^{(1)}\left(\bar{\nabla}_a \bar{T}_b{}^a\right) = \nabla_a {}^{(1)}\mathscr{T}_b{}^a + H_{ca}{}^a[\mathscr{F}]\, T_b{}^c - H_{ba}{}^c[\mathscr{F}]\, T_c{}^a + \pounds_Y \nabla_a T_b{}^a. \tag{12}$$

The derivation of the formula (12) is given in Ref. [32]. If the tensor field $\bar{T}_b{}^a$ is the Einstein tensor $\bar{G}_a{}^b$, Equation (12) yields the linear-order perturbation of the Bianchi identity

$$\nabla_a{}^{(1)}\mathscr{G}_b{}^a[\mathscr{F}] + H_{ca}{}^a[\mathscr{F}]\, G_b{}^c - H_{ba}{}^c[\mathscr{F}]\, G_c{}^a = 0 \tag{13}$$

and if the background Einstein tensor vanishes $G_a{}^b = 0$, we obtain the identity

$$\nabla_a{}^{(1)}\mathscr{G}_b{}^a[\mathscr{F}] = 0. \tag{14}$$

By contrast, if the tensor field $\bar{T}_b{}^a$ is the energy-momentum tensor, Equation (12) yields the continuity equation of the energy-momentum tensor

$$\nabla_a{}^{(1)}\mathscr{T}_b{}^a + H_{ca}{}^a[\mathscr{F}]\, T_b{}^c - H_{ba}{}^c[\mathscr{F}]\, T_c{}^a = 0, \tag{15}$$

where we used the background continuity equation $\nabla_a T_b{}^a = 0$. If the background spacetime is vacuum $T_{ab} = 0$, Equation (15) yields a linear perturbation of the energy-momentum tensor given by

$$\nabla_a{}^{(1)}\mathscr{T}_b{}^a = 0. \tag{16}$$

We should note that the decomposition of the metric perturbation h_{ab} into its gauge-invariant part $\mathscr{F}_{ab}$ and into its gauge-variant part Y^a is not unique [41,43,44]. As explained in the Part I paper [30], for example, the gauge-invariant part $\mathscr{F}_{ab}$ has six components and we can create the gauge-invariant vector field Z^a through these components of the gauge-invariant metric perturbation $\mathscr{F}_{ab}$ such that the gauge-transformation of the vector field Z^a is given by ${}_{\mathscr{Y}}Z^a - {}_{\mathscr{X}}Z^a = 0$. Using this gauge-invariant vector field Z^a, the original metric perturbation can be expressed as follows:

$$h_{ab} = \mathscr{F}_{ab} - £_Z g_{ab} + £_{Z+Y} g_{ab} =: \mathscr{H}_{ab} + £_X g_{ab}. \tag{17}$$

The tensor field $\mathscr{H}_{ab} := \mathscr{F}_{ab} - £_Z g_{ab}$ is also regarded as the gauge-invariant part of the perturbation h_{ab} because ${}_{\mathscr{Y}}\mathscr{H}_{ab} - {}_{\mathscr{X}}\mathscr{H}_{ab} = 0$. Similarly, the vector field $X^a := Z^a + Y^a$ is also regarded as the gauge-variant part of the perturbation h_{ab} because ${}_{\mathscr{Y}}X^a - {}_{\mathscr{X}}X^a = \xi^a_{(1)}$. This non-uniqueness appears in the solutions derived in Sections 4 and 5, as in the case of the $l = 1$ odd-mode perturbative solutions in the Part I paper [30]. These non-uniqueness of gauge-invariant variable can be regarded as the first-kind gauge as explained in Part I paper [30], i.e., degree of freedom of the choice of the coordinate system on the physical spacetime $\mathscr{M}_\epsilon$. Since we often use the first-kind gauge when we predict and interpret the measurement results of observations and experiments, we should regard that this non-uniqueness of gauge-invariant variable of the second kind may have some physical meaning [30].

2.2. Linear Perturbations on Spherically Symmetric Background

Here, we consider the 2 + 2 formulation of the perturbation of a spherically symmetric background spacetime, which originally proposed by Gerlach and Sengupta [20–23]. Spherically symmetric spacetimes are characterized by the direct product $\mathscr{M} = \mathscr{M}_1 \times S^2$ and their metric is

$$g_{ab} = y_{ab} + r^2\gamma_{ab}, \tag{18}$$

$$y_{ab} = y_{AB}(dx^A)_a(dx^B)_b, \quad \gamma_{ab} = \gamma_{pq}(dx^p)_a(dx^q)_b, \tag{19}$$

where $x^A = (t, r)$, $x^p = (\theta, \phi)$, and γ_{pq} is the metric on the unit sphere. In the Schwarzschild spacetime, the metric (18) is given by

$$
\begin{aligned}
y_{ab} &= -f(dt)_a(dt)_b + f^{-1}(dr)_a(dr)_b, \quad f = 1 - \frac{2M}{r}, &(20)\\
\gamma_{ab} &= (d\theta)_a(d\theta)_b + \sin^2\theta(d\phi)_a(d\phi)_b = \theta_a\theta_b + \phi_a\phi_b, &(21)\\
\theta_a &= (d\theta)_a, \quad \phi_a = \sin\theta(d\phi)_a. &(22)
\end{aligned}
$$

On this background spacetime $(\mathscr{M}, g_{ab})$, the components of the metric perturbation are given by

$$h_{ab} = h_{AB}(dx^A)_a(dx^B)_b + 2h_{Ap}(dx^A)_{(a}(dx^p)_{b)} + h_{pq}(dx^p)_a(dx^q)_b. \quad (23)$$

Here, we note that the components h_{AB}, h_{Ap}, and h_{pq} are regarded as components of scalar, vector, and tensor on S^2, respectively. In many literatures, these components are decomposed through the decomposition [49–51] using the spherical harmonics $S = Y_{lm}$ as follows:

$$
\begin{aligned}
h_{AB} &= \sum_{l,m} \tilde{h}_{AB} S, &(24)\\
h_{Ap} &= r\sum_{l,m}\left[\tilde{h}_{(e1)A}\hat{D}_p S + \tilde{h}_{(o1)A}\epsilon_{pq}\hat{D}^q S\right], &(25)\\
h_{pq} &= r^2\sum_{l,m}\left[\frac{1}{2}\gamma_{pq}\tilde{h}_{(e0)}S + \tilde{h}_{(e2)}\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S + 2\tilde{h}_{(o2)}\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S\right], &(26)
\end{aligned}
$$

where $\hat{D}_p$ is the covariant derivative associated with the metric γ_{pq} on S^2, $\hat{D}^p = \gamma^{pq}\hat{D}_q$, $\epsilon_{pq} = \epsilon_{[pq]} = 2\theta_{[p}\phi_{q]}$ is the totally antisymmetric tensor on S^2.

If we employ the decomposition (24)–(26) with $S = Y_{lm}$ to the metric perturbation h_{ab}, special treatments for $l = 0, 1$ modes are required [12–28]. This is due to the fact that the set of harmonic functions

$$\left\{S, \hat{D}_p S, \epsilon_{pq}\hat{D}^q S, \frac{1}{2}\gamma_{pq}S, \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\right)S, 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S\right\} \quad (27)$$

loses its linear-independence for $l = 0, 1$ modes. Actually, the inverse-relation of the decomposition formulae (24)–(26) requires the Green functions of the derivative operators $\hat{\Delta} := \hat{D}^r\hat{D}_r$ and $\hat{\Delta} + 2 := \hat{D}^r\hat{D}_r + 2$. Since the eigen modes of these operators are $l = 0$ and $l = 1$, respectively, this is the reason why the special treatments for these modes are required. However, these special treatments become an obstacle when we develop higher-order perturbation theory [52].

To resolve this $l = 0, 1$ mode problem, in Part I paper [29,30], we chose the scalar function S as

$$S = S_\delta = \begin{cases} Y_{lm} & \text{for} \quad l \geq 2; \\ k_{(\hat{\Delta}+2)m} & \text{for} \quad l = 1; \\ k_{(\hat{\Delta})} & \text{for} \quad l = 0. \end{cases} \quad (28)$$

and use the decomposition formulae (24)–(26), where the functions $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)}$ satisfy the equation

$$\hat{\Delta}k_{(\hat{\Delta})} = 0, \quad \left(\hat{\Delta} + 2\right)k_{(\hat{\Delta}+2)} = 0, \quad (29)$$

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respectively. As shown in Part I paper [30], the set of harmonic functions (27) becomes the linear-independent set including $l = 0, 1$ modes if we employ

$$k_{(\hat{\Delta})} = 1 + \delta \ln \left(\frac{1 - \cos\theta}{1 + \cos\theta} \right)^{1/2}, \quad \delta \in \mathbb{R}, \tag{30}$$

$$k_{(\hat{\Delta}+2,m=0)} = \cos\theta + \delta \left(\frac{1}{2} \cos\theta \ln \frac{1 + \cos\theta}{1 - \cos\theta} - 1 \right), \quad \delta \in \mathbb{R}, \tag{31}$$

$$k_{(\hat{\Delta}+2,m=\pm1)} = \left[\sin\theta + \delta \left(+\frac{1}{2} \sin\theta \ln \frac{1 + \cos\theta}{1 - \cos\theta} + \cot\theta \right) \right] e^{\pm i\phi}. \tag{32}$$

These choices guarantee the one-to-one correspondence between the components $\{h_{AB}, h_{Ap}, h_{pq}\}$ and the mode coefficients $\{\tilde{h}_{AB}, \tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e0)}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}$ with the decomposition formulae (24)-(26) owing to the linear-independence of the set of the harmonic functions (27) when $\delta \neq 0$. Then, the mode-by-mode analysis including $l = 0, 1$ is possible when $\delta \neq 0$. However, the mode functions (30)–(32) are singular if $\delta \neq 0$. When $\delta = 0$, we have $k_{(\hat{\Delta})} \propto Y_{00}$ and $k_{(\hat{\Delta}+2)m} \propto Y_{1m}$. Using the above harmonics functions S_δ in Equation (28), we propose the following strategy:

Proposal 1. *We decompose the metric perturbation h_{ab} on the background spacetime with the metric (18)–(21) through Equations (24)–(26) with the harmonic function S_δ given by Equation (28). Then, Equations (24)–(26) become invertible including $l = 0, 1$ modes. After deriving the mode-by-mode field equations such as linearized Einstein equations by using the harmonic functions S_δ, we choose $\delta = 0$ as regular boundary condition for solutions when we solve these field equations.*

As shown in the Part I paper [30], once we accept Proposal 1, the Conjecture 1 becomes the following statement:

Theorem 1. *If the gauge-transformation rule for a perturbative pulled-back tensor field h_{ab} to the background spacetime $\mathscr{M}$ is given by ${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = \pounds_{\xi_{(1)}} g_{ab}$ with the background metric g_{ab} with spherical symmetry, there then exist a tensor field $\mathscr{F}_{ab}$ and a vector field Y^a such that h_{ab} is decomposed as $h_{ab} =: \mathscr{F}_{ab} + \pounds_Y g_{ab}$, where $\mathscr{F}_{ab}$ and Y^a are transformed as ${}_{\mathscr{Y}}\mathscr{F}_{ab} - {}_{\mathscr{X}}\mathscr{F}_{ab} = 0$ and ${}_{\mathscr{Y}}Y^a - {}_{\mathscr{X}}Y^a = \xi^a_{(1)}$ under the gauge transformation, respectively.*

Furthermore, including $l = 0, 1$ modes, the components of the gauge-invariant part $\mathscr{F}_{ab}$ of the metric perturbation h_{ab} is given by

$$\mathscr{F}_{AB} = \sum_{l,m} \tilde{F}_{AB} S_\delta, \tag{33}$$

$$\mathscr{F}_{Ap} = r \sum_{l,m} \tilde{F}_A \epsilon_{pq} \hat{D}^q S_\delta, \quad \hat{D}^p \mathscr{F}_{Ap} = 0, \tag{34}$$

$$\mathscr{F}_{pq} = = \frac{1}{2} \gamma_{pq} r^2 \sum_{l,m} \tilde{F} S_\delta. \tag{35}$$

Thus, we have resolved the zero-mode problem in the perturbations on the spherically symmetric background spacetimes. Through the gauge-invariant variables (33)–(35), we derived the linearized Einstein equations in Part I paper [30].

2.3. Even-Mode Linearized Einstein Equations

Since the odd-mode perturbations are discussed in Part I paper [30], we consider the linearized even-mode Einstein equations on the Schwarzschild background spacetime in this paper. The Schwarzschild spacetime is vacuum solution to the Einstein equation $G_a{}^b = 0 = T_a{}^b$. Since we proved Theorem 1 on the spherically symmetric background spacetime, the linearized Einstein equation is

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given in the following gauge-invariant form as Equation (11). To evaluate the Einstein equation (11) through the mode-by-mode analysis including $l = 0, 1$, we also consider the mode-decomposition of the gauge-invariant part ${}^{(1)}\mathscr{T}_{ab} := g_{bc}{}^{(1)}\mathscr{T}_{a}{}^{c}$ of the linear-perturbation of the energy momentum tensor through the set (27) of the harmonics as follows:

$$\begin{aligned}{}^{(1)}\mathscr{T}_{ab} \;=\; & \sum_{l,m}\tilde{T}_{AB}S_\delta(dx^A)_a(dx^B)_b + r\sum_{l,m}\left\{\tilde{T}_{(e1)A}\hat{D}_pS_\delta + \tilde{T}_{(o1)A}\epsilon_{pr}\hat{D}^rS_\delta\right\}2(dx^A)_{(a}(dx^p)_{b)} \\ & +r^2\sum_{l,m}\left\{\tilde{T}_{(e0)}\frac{1}{2}\gamma_{pq}S_\delta + \tilde{T}_{(e2)}\left(\hat{D}_p\hat{D}_qS_\delta - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^rS_\delta\right)\right. \\ & \left.\qquad +\tilde{T}_{(o2)}2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^rS_\delta\right\}(dx^p)_a(dx^q)_b. \end{aligned} \tag{36}$$

Since the background spacetime is vacuum, the pull-backed divergence of the energy-momentum tensor is given by Equation (16) and the even-mode components of Equation (16) in terms of the mode coefficients defined by Equation (37) are given by

$$\bar{D}^C\tilde{T}_C{}^B + \frac{2}{r}(\bar{D}^Dr)\tilde{T}_D{}^B - \frac{1}{r}l(l+1)\tilde{T}^B_{(e1)} - \frac{1}{r}(\bar{D}^Br)\tilde{T}_{(e0)} = 0, \tag{37}$$

$$\bar{D}^C\tilde{T}_{(e1)C} + \frac{3}{r}(\bar{D}^Cr)\tilde{T}_{(e1)C} + \frac{1}{2r}\tilde{T}_{(e0)} - \frac{1}{2r}(l-1)(l+2)\tilde{T}_{(e2)} = 0. \tag{38}$$

Owing to the linear-independence of the set (27) of the harmonics, we can evaluate the gauge-invariant linearized Einstein equation (11) through the mode-by-mode analyses including $l = 0, 1$ modes. As summarized in the Part I paper [30], the traceless even part of the (p, q)-component of the linearized Einstein equation (11) is given by

$$\tilde{F}_D{}^D = -16\pi r^2\tilde{T}_{(e2)}. \tag{39}$$

Using this equation, the even part of (A, q)-component, equivalently (p, B)-component, of the linearized Einstein equation (11) yields

$$\bar{D}^D\tilde{\mathbb{F}}_{AD} - \frac{1}{2}\bar{D}_A\tilde{F} = 16\pi\left[r\tilde{T}_{(e1)A} - \frac{1}{2}r^2\bar{D}_A\tilde{T}_{(e2)}\right] =: 16\pi S_{(ec)A} \tag{40}$$

through the definition of the traceless part $\tilde{\mathbb{F}}_{AB}$ of the variable $\tilde{F}_{AB}$:

$$\tilde{\mathbb{F}}_{AB} := \tilde{F}_{AB} - \frac{1}{2}y_{AB}\tilde{F}_C{}^C. \tag{41}$$

Using Equations (39) and (40), and the background Einstein, the trace part of (p, q)-component of the linearized Einstein equation (11) yields Equation (38).

Finally, through Equations (39) and (40) and the background Einstein equations, the trace part of the (A, B)-component of the linearized Einstein equation (11) is given by

$$\left(\bar{D}_D\bar{D}^D + \frac{2}{r}(\bar{D}^Dr)\bar{D}_D - \frac{(l-1)(l+2)}{r^2}\right)\tilde{F} - \frac{4}{r^2}(\bar{D}_Cr)(\bar{D}_Dr)\tilde{\mathbb{F}}^{CD} = 16\pi S_{(F)}, \tag{42}$$

$$S_{(F)} := \tilde{T}_C{}^C + 4(\bar{D}_Dr)\tilde{T}^D_{(e1)} - 2r(\bar{D}_Dr)\bar{D}^D\tilde{T}_{(e2)} - (l(l+1)+2)\tilde{T}_{(e2)}. \tag{43}$$

On the other hand, the traceless part of the (A, B)-component of the linearized Einstein equation (11) is given by

$$\left[-\bar{D}_D\bar{D}^D - \frac{2}{r}(\bar{D}_D r)\bar{D}^D + \frac{4}{r}(\bar{D}^D\bar{D}_D r) + \frac{l(l+1)}{r^2}\right]\tilde{\mathbb{F}}_{AB}$$
$$+\frac{4}{r}(\bar{D}^D r)\bar{D}_{(A}\tilde{\mathbb{F}}_{B)D} - \frac{2}{r}(\bar{D}_{(A}r)\bar{D}_{B)}\tilde{F}$$
$$= 16\pi S_{(\mathbb{F})AB}, \tag{44}$$
$$S_{(\mathbb{F})AB} := T_{AB} - \frac{1}{2}y_{AB}T_C{}^C - 2\left(\bar{D}_{(A}(r\tilde{T}_{(e1)B)}) - \frac{1}{2}y_{AB}\bar{D}^D(r\tilde{T}_{(e1)D})\right)$$
$$+2\left((\bar{D}_{(A}r)\bar{D}_{B)} - \frac{1}{2}y_{AB}(\bar{D}^D r)\bar{D}_D\right)(r\tilde{T}_{(e2)})$$
$$+r\left(\bar{D}_A\bar{D}_B - \frac{1}{2}y_{AB}\bar{D}^D\bar{D}_D\right)(r\tilde{T}_{(e2)})$$
$$+2\left((\bar{D}_A r)(\bar{D}_B r) - \frac{1}{2}y_{AB}(\bar{D}^C r)(\bar{D}_C r)\right)\tilde{T}_{(e2)}$$
$$+2y_{AB}(\bar{D}^C r)\tilde{T}_{(e1)C} - r y_{AB}(\bar{D}^C r)\bar{D}_C\tilde{T}_{(e2)}. \tag{45}$$

Equations (39), (40), (42), and (44) are all independent equations of the linearized Einstein equation for even-mode perturbations. These equations are coupled equations for the variables $\tilde{F}_C{}^C$, F, and $\tilde{\mathbb{F}}_{AB}$ and the energy-momentum tensor for the matter field. When we solve these equations, we have to take into account of the continuity Equations (37) and (38) for the matter fields. We note that these equations are valid not only for $l \geq 2$ modes but also $l = 0, 1$ modes in our formulation. For $l \geq 2$ modes, we can derive the Zerilli equation, while we can derive formal solutions for $l = 0, 1$ modes. The derivations of these formal solutions for $l = 0, 1$ modes are the main ingredients of this paper.

3. Component Treatment of the Even-Mode Linearized Einstein Equations

To summarize the even-mode Einstein equations, we consider the static chart of y_{AB} as Equation (20). On this chart, the components of the Christoffel symbol $\bar{\Gamma}_{AB}{}^C$ associated with the covariant derivative $\bar{D}_A$ is summarized as

$$\bar{\Gamma}_{tt}{}^t = 0, \quad \bar{\Gamma}_{tr}{}^t = \frac{f'}{2f}, \quad \bar{\Gamma}_{rr}{}^t = 0, \quad \bar{\Gamma}_{tt}{}^r = \frac{ff'}{2}, \quad \bar{\Gamma}_{tr}{}^r = 0, \quad \bar{\Gamma}_{rr}{}^r = -\frac{f'}{2f}, \tag{46}$$

where $f' := \partial_r f$.

First, Equation (39) is a direct consequence of the even-mode Einstein equation. Here, we introduce the components $X_{(e)}$ and $Y_{(e)}$ of the traceless variable $\tilde{\mathbb{F}}_{AB}$ by

$$\tilde{\mathbb{F}}_{AB} =: X_{(e)}\left\{-f(dt)_A(dt)_B - f^{-1}(dr)_A(dr)_B\right\} + 2Y_{(e)}(dt)_{(A}(dr)_{B)}. \tag{47}$$

Through these components $X_{(e)}$ and $Y_{(e)}$, t- and r- components of Equation (40) are given by

$$\partial_t X_{(e)} + f\partial_r Y_{(e)} + f'Y_{(e)} - \frac{1}{2}\partial_t\tilde{F} = 16\pi S_{(ec)t}, \tag{48}$$
$$\frac{1}{f}\partial_t Y_{(e)} + \partial_r X_{(e)} + \frac{f'}{f}X_{(e)} + \frac{1}{2}\partial_r\tilde{F} = -16\pi S_{(ec)r}. \tag{49}$$

The source term $S_{(ec)A}$ is defined by

$$S_{(ec)A} := r\tilde{T}_{(e1)A} - \frac{1}{2}r^2\bar{D}_A\tilde{T}_{(e2)}. \tag{50}$$

Furthermore, the evolution Equation (42) for the variable $\tilde{F}$ is given by

$$-\partial_t^2\tilde{F} + f\partial_r(f\partial_r\tilde{F}) + \frac{2}{r}f^2\partial_r\tilde{F} - \frac{(l-1)(l+2)}{r^2}f\tilde{F} + \frac{4}{r^2}f^2X_{(e)} = 16\pi G f S_{(F)}. \tag{51}$$

The source term $S_{(F)}$ is defined by

$$\begin{aligned} S_{(F)} &:= \tilde{T}_E{}^E + 4(\bar{D}_D r)\tilde{T}_{(e1)}{}^D - 2r(\bar{D}_D r)\bar{D}^D\tilde{T}_{(e2)} - (l(l+1)+2)\tilde{T}_{(e2)} \qquad (52)\\ &= -\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr} + 4f\tilde{T}_{(e1)r} - 2rf\partial_r\tilde{T}_{(e2)} - (l(l+1)+2)\tilde{T}_{(e2)}. \qquad (53)\end{aligned}$$

The component expression of Equation (44) with the constraints (48) and (49) are given by

$$\begin{aligned} &\partial_t^2 X_{(e)} - f\partial_r(f\partial_r X_{(e)}) - \frac{2(1-2f)f}{r}\partial_r X_{(e)} - \frac{(1-f)(1-5f)-l(l+1)f}{r^2}X_e \\ &-\frac{(1-3f)f}{r}\partial_r\tilde{F} \\ =\;& -16\pi\left(S_{(\mathbb{F})tt} + \frac{2f(3f-1)}{r}S_{(ec)r}\right), \qquad (54)\\ &\partial_t^2 Y_{(e)} - f\partial_r\left(f\partial_r Y_{(e)}\right) - \frac{2(1-2f)f}{r}\partial_r Y_{(e)} - \frac{(1-f)(1-5f)-l(l+1)f}{r^2}Y_{(e)} \\ &+\frac{1-3f}{r}\partial_t\tilde{F} \\ =\;& 16\pi\left(fS_{(\mathbb{F})tr} - \frac{2(1-2f)}{r}S_{(ec)t}\right). \qquad (55)\end{aligned}$$

Here, we note that (rr)-component of Equation (44) with the constraint (49) is equivalent to Equation (54). The source terms $S_{(\mathbb{F})tt}$ and $S_{(\mathbb{F})tr}$ in Equations (54) and (55) are given by

$$\begin{aligned} S_{(\mathbb{F})tt} &= \frac{1}{2}\left(\tilde{T}_{tt} + f^2\tilde{T}_{rr}\right) - r\partial_t\tilde{T}_{(e1)t} - 3f^2\tilde{T}_{(e1)r} - rf^2\partial_r\tilde{T}_{(e1)r} \\ &\quad + \frac{r^2}{2}\partial_t^2\tilde{T}_{(e2)} + \frac{r^2}{2}f^2\partial_r^2\tilde{T}_{(e2)} + 3rf^2\partial_r\tilde{T}_{(e2)} + 2f^2\tilde{T}_{(e2)}, \qquad (56)\\ S_{(\mathbb{F})tr} &= \tilde{T}_{tr} - r\partial_t\tilde{T}_{(e1)r} - r\partial_r\tilde{T}_{(e1)t} - \tilde{T}_{(e1)t} + \frac{1-f}{f}\tilde{T}_{(e1)t} \\ &\quad + r^2\partial_t\partial_r\tilde{T}_{(e2)} + 2r\partial_t\tilde{T}_{(e2)} - \frac{r(1-f)}{2f}\partial_t\tilde{T}_{(e2)}. \qquad (57)\end{aligned}$$

The components of Equation (37) is given by

$$-\partial_t\tilde{T}_{tt} + f^2\partial_r\tilde{T}_{rt} + \frac{(1+f)f}{r}\tilde{T}_{rt} - \frac{f}{r}l(l+1)\tilde{T}_{(e1)t} = 0, \tag{58}$$

$$-\partial_t\tilde{T}_{tr} + \frac{1-f}{2rf}\tilde{T}_{tt} + f^2\partial_r\tilde{T}_{rr} + \frac{(3+f)f}{2r}\tilde{T}_{rr} - \frac{f}{r}l(l+1)\tilde{T}_{(e1)r} - \frac{f}{r}\tilde{T}_{(e0)} = 0, \tag{59}$$

where Equation (58) is the t-component and Equation (59) is the r-component, respectively. Furthermore, Equation (38) is given by

$$-\partial_t\tilde{T}_{(e1)t} + f^2\partial_r\tilde{T}_{(e1)r} + \frac{(1+2f)f}{r}\tilde{T}_{(e1)r} + \frac{f}{2r}\tilde{T}_{(e0)} - \frac{f}{2r}(l-1)(l+2)\tilde{T}_{(e2)} = 0. \tag{60}$$

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From the time derivative of Equations (48) and (49), we obtain

$$\begin{aligned} &\partial_t^2 X_{(e)} - f\partial_r(f\partial_r X_{(e)}) - 2\frac{1-f}{r}f\partial_r X_{(e)} - \frac{(1-3f)(1-f)}{r^2}X_{(e)} \\ &-\frac{1}{2}\partial_t^2\tilde{F} - \frac{1}{2}f\partial_r(f\partial_r\tilde{F}) - \frac{1-f}{2r}f\partial_r\tilde{F} \\ &-16\pi\partial_t S_{(ec)t} - 16\pi\partial_r(f^2 S_{(ec)r}) = 0. \end{aligned} \quad (61)$$

$$\begin{aligned} &-\partial_t^2 Y_{(e)} + f\partial_r(f\partial_r Y_{(e)}) + 2f\frac{1-f}{r}\partial_r Y_{(e)} + \frac{(1-3f)(1-f)}{r^2}Y_{(e)} \\ &-\frac{1-f}{2r}\partial_t\tilde{F} - f\partial_r\partial_t\tilde{F} - 16\pi\partial_r(fS_{(ec)t}) - 16\pi f\partial_t S_{(ec)r} = 0. \end{aligned} \quad (62)$$

From Equations (54) and (61), we obtain

$$\begin{aligned} &4f\partial_r(fX_{(e)}) + \frac{2}{r}l(l+1)(fX_{(e)}) + r\partial_t^2\tilde{F} + rf\partial_r(f\partial_r\tilde{F}) + (5f-1)f\partial_r\tilde{F} \\ = \; &-32\pi r\left[S_{(\mathbb{F})tt} + \partial_t S_{(ec)t} + f^2\partial_r S_{(ec)r} + \frac{4}{r}f^2 S_{(ec)r}\right]. \end{aligned} \quad (63)$$

Furthermore, from Equations (55) and (62), we obtain

$$\begin{aligned} &4f\partial_r(fY_{(e)}) + \frac{2}{r}l(l+1)(fY_{(e)}) - 2rf\partial_r\partial_t\tilde{F} - (5f-1)\partial_t\tilde{F} \\ = \; &32\pi r\left[fS_{(\mathbb{F})tr} + f\partial_t S_{(ec)r} - \frac{1-3f}{r}S_{(ec)t} + f\partial_r S_{(ec)t}\right]. \end{aligned} \quad (64)$$

Equations (48) and (64) yields

$$\begin{aligned} l(l+1)Y_{(e)} = \; &r\partial_t\left(2X_{(e)} + r\partial_r\tilde{F}\right) + \frac{3f-1}{2f}r\partial_t\tilde{F} \\ &+16\pi r^2\left[S_{(\mathbb{F})tr} + \partial_t S_{(ec)r} - \frac{1-f}{rf}S_{(ec)t} + \partial_r S_{(ec)t}\right]. \end{aligned} \quad (65)$$

Similarly, Equations (63) and (51) yield

$$\begin{aligned} &4f\partial_r(fX_{(e)}) + \frac{2}{r}[l(l+1)+2f](fX_{(e)}) \\ &+2f\partial_r(rf\partial_r\tilde{F}) + (5f-1)f\partial_r\tilde{F} - \frac{(l-1)(l+2)}{r}f\tilde{F} \\ = \; &-32\pi r\left[S_{(\mathbb{F})tt} + \partial_t S_{(ec)t} + f^2\partial_r S_{(ec)r} + \frac{4}{r}f^2 S_{(ec)r} - \frac{f}{2}S_{(F)}\right]. \end{aligned} \quad (66)$$

Thus, we may regard that the independent components of the Einstein equations for the even-mode perturbations are summarized as Equations (51), (54), (65), and (66).

As shown in many literatures [13–18], it is well-known that Equations (51), (54), and (66) are reduced to the single master equation for a single variable. We trace this procedure.

Equation (66) is an initial value constraint for the variables $(X_{(e)}, \tilde{F})$, while Equations (51) and (61) are evolution equations. Equation (65) directly yields that the variable $Y_{(e)}$ is determined by the solution $(X_{(e)}, \tilde{F})$ to Equations (66), (51) and (61), if $l \neq 0$. If the initial value constraint (66) is reduced to the equation of a variable $\Phi_{(e)}$ and $\tilde{F}$, we may expect that $\Phi_{(e)}$ linearly depends on $fX_{(e)}$, $\tilde{F}$, and $rf\partial_r\tilde{F}$. To show this, we introduce the variable $\Phi_{(e)}$ as

$$\alpha\Phi_{(e)} := fX_{(e)} + \beta\tilde{F} + \gamma rf\partial_r\tilde{F}, \quad (67)$$

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where α, β, and γ may depend on r. Substituting Equation (67) into Equation (66), we obtain

$$\begin{aligned} 0 \;=\;& -4rf\alpha'\Phi_{(e)} - 2[l(l+1)+2f]\alpha\Phi_{(e)} - 4rf\alpha\partial_r\Phi_{(e)} + 4\left(\gamma - \frac{1}{2}\right) rf\partial_r[rf\partial_r\tilde{F}] \\ & + \left[4\beta + 4rf\gamma' + 2\left\{l(l+1)+2f\right\}\gamma - (5f-1)\right] rf\partial_r\tilde{F} \\ & + \left[4rf\beta' + 2\left\{l(l+1)+2f\right\}\beta + (l-1)(l+2)f\right]\tilde{F} \\ & -32\pi r^2\left[S_{(\mathbb{F})tt} + \partial_t S_{(ec)t} + f^2\partial_r S_{(ec)r} + \frac{4}{r}f^2 S_{(ec)r} - \frac{f}{2}S_{(F)}\right]. \end{aligned} \tag{68}$$

Here, we choose

$$\gamma = \frac{1}{2} \tag{69}$$

to eliminate the term of the second derivative of $\tilde{F}$. Owing to this choice, we obtain

$$\begin{aligned} 0 \;=\;& -4rf\alpha'\Phi_{(e)} - 2[l(l+1)+2f]\alpha\Phi_{(e)} - 4rf\alpha\partial_r\Phi_{(e)} + [4\beta + \Lambda]rf\partial_r\tilde{F} \\ & + \left[4rf\beta' + 2\left\{l(l+1)+2f\right\}\beta + (l-1)(l+2)f\right]\tilde{F} \\ & -32\pi r^2\left[S_{(\mathbb{F})tt} + \partial_t S_{(ec)t} + f^2\partial_r S_{(ec)r} + \frac{4}{r}f^2 S_{(ec)r} - \frac{f}{2}S_{(F)}\right]. \end{aligned} \tag{70}$$

Here, we choose β as

$$\beta = -\frac{1}{4}\Lambda := -\frac{1}{4}[(l-1)(l+2) + 3(1-f)], \quad \Lambda := (l-1)(l+2) + 3(1-f) \tag{71}$$

to eliminate the term of the first derivative of $\tilde{F}$. Due to this choice, we obtain

$$\begin{aligned} l(l+1)\Lambda\tilde{F} \;=\;& -8rf\partial_r\left(\alpha\Phi_{(e)}\right) - 4[l(l+1)+2f]\alpha\Phi_{(e)} \\ & -64\pi r^2\left[S_{(\mathbb{F})tt} + \partial_t S_{(ec)t} + f^2\partial_r S_{(ec)r} + \frac{4}{r}f^2 S_{(ec)r} - \frac{f}{2}S_{(F)}\right]. \end{aligned} \tag{72}$$

This equation yields that the variable $\tilde{F}$ is determined by the single variable $\Phi_{(e)}$ and the source terms if $l \neq 0$ and if the coefficient α is determined.

At this moment, the variable $\Phi_{(e)}$ is determined up to its normalization α as

$$\alpha\Phi_{(e)} := fX_{(e)} - \frac{1}{4}\Lambda\tilde{F} + \frac{1}{2}rf\partial_r\tilde{F}. \tag{73}$$

Eliminating $X_{(e)}$ in Equation (51) through Equation (73), we obtain

$$-\partial_t^2\tilde{F} + f\partial_r(f\partial_r\tilde{F}) + \frac{1}{r^2}3(1-f)f\tilde{F} + \frac{4}{r^2}f\alpha\Phi_{(e)} = 16\pi f S_{(F)}. \tag{74}$$

Similarly, eliminating $X_{(e)}$ in Equation (54) through Equations (72)–(74), we obtain

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$$\begin{aligned}
0 \;=\; & -\alpha\partial_t^2\Phi_{(e)} + \alpha f\partial_r\left[f\partial_r\Phi_{(e)}\right] + 2\alpha\left[\frac{\alpha'}{\alpha} + \frac{1}{r} + \frac{1}{r\Lambda}3(1-f)\right]f^2\partial_r\Phi_{(e)} \\
& + \left[\alpha'' f + \alpha'\frac{1}{r}(1+f) + \alpha'\frac{1}{r\Lambda}3(1-f)2f + \alpha\frac{3(1-f)\left\{l(l+1)+2f\right\}}{r^2\Lambda}\right. \\
& \quad \left. -\alpha\frac{(l-1)(l+2)+1+f}{r^2}\right]f\Phi_{(e)} \\
& + 16\pi f\frac{\Lambda+3(1-f)}{\Lambda}\left[S_{(\mathbb{F})tt} + \partial_t S_{(ec)t} + f^2\partial_r S_{(ec)r} + \frac{4f^2}{r}S_{(ec)r} - \frac{f}{2}S_{(F)}\right] \\
& - 16\pi f S_{(\mathbb{F})tt} - 32\pi\frac{3f-1}{r}f^2 S_{(ec)r} - 16\pi(-\frac{1}{4}\Lambda)fS_{(F)} - 16\pi r\frac{1}{2}f\partial_r\left[fS_{(F)}\right].
\end{aligned} \tag{75}$$

We determine α so that the terms proportional to $\partial_r\Phi_{(e)}$ vanish. Then, we obtain the equation for α as

$$\frac{\alpha'}{\alpha} + \frac{1}{r} + \frac{1}{r\Lambda}3(1-f) = 0. \tag{76}$$

From this equation, we obtain

$$\frac{1}{\alpha} = \frac{Cr}{\Lambda}, \tag{77}$$

where C is a constant of integration. In this paper, we choose $C = 1$. Then, we obtain

$$\Phi_{(e)} := \frac{1}{\alpha}\left[fX_{(e)} - \frac{1}{4}\Lambda\tilde{F} + \frac{1}{2}rf\partial_r\tilde{F}\right] = \frac{r}{\Lambda}\left[fX_{(e)} - \frac{1}{4}\Lambda\tilde{F} + \frac{1}{2}rf\partial_r\tilde{F}\right]. \tag{78}$$

This is the Moncrief variable.

From Equation (77), we obtain

$$\alpha' = -\frac{1}{r}\alpha - \frac{1}{\Lambda}3\frac{1-f}{r}\alpha, \quad \alpha'' = +\frac{2}{r^2}\alpha + \frac{12(1-f)}{\Lambda r^2}\alpha. \tag{79}$$

Then, using

$$\mu := (l-1)(l+2), \quad \Lambda = \mu + 3(1-f), \tag{80}$$

Equation (76) is given by

$$\begin{aligned}
& -\frac{1}{f}\partial_t^2\Phi_{(e)} + \partial_r\left[f\partial_r\Phi_{(e)}\right] - \frac{1}{r^2\Lambda^2}\left[\mu^2[(\mu+2)+3(1-f)] + 9(1-f)^2(\mu+1-f)\right]\Phi_{(e)} \\
= \; & 16\pi\frac{r}{\Lambda}\left[-\partial_t S_{(ec)t} - f^2\partial_r S_{(ec)r} + 2f\frac{f-1}{r}S_{(ec)r} + \frac{r}{2}f\partial_r S_{(F)} + \frac{1}{2}S_{(F)} - \frac{1}{4}\Lambda S_{(F)}\right. \\
& \left. + \frac{3(1-f)}{\Lambda}\left[-S_{(\mathbb{F})tt} - \partial_t S_{(ec)t} - f^2\partial_r S_{(ec)r} - \frac{4}{r}f^2 S_{(ec)r} + \frac{f}{2}S_{(F)}\right]\right].
\end{aligned} \tag{81}$$

This is the Zerilli equation for the Moncrief variable (78).

Here, we summarize the equations for even-mode perturbations. We derive the definition of the Moncrief variable as Equation (78), i.e.,

$$\Phi_{(e)} := \frac{r}{\Lambda}\left[fX_{(e)} - \frac{1}{4}\Lambda\tilde{F} + \frac{1}{2}rf\partial_r\tilde{F}\right], \tag{82}$$

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where Λ is defined by

$$\Lambda = \mu + 3(1-f), \quad \mu := (l-1)(l+2). \tag{83}$$

This definition of the variable $\Phi_{(e)}$ implies that if we obtain the variables $\Phi_{(e)}$ and $\tilde{F}$ are determined, the component of $X_{(e)}$ of the metric perturbation is determined through the equation

$$fX_{(e)} = \frac{1}{r}\Lambda\Phi_{(e)} + \frac{1}{4}\Lambda\tilde{F} - \frac{1}{2}rf\partial_r\tilde{F}. \tag{84}$$

As the initial value constraint for the variable $\tilde{F}$ and $Y_{(e)}$, we have Equations (72) and (65) as

$$l(l+1)\Lambda\tilde{F} = -8f\Lambda\partial_r\Phi_{(e)} + \frac{4}{r}\left[6f(1-f) - l(l+1)\Lambda\right]\Phi_{(e)} - 64\pi r^2 S_{(\Lambda\tilde{F})}, \tag{85}$$

$$l(l+1)Y_{(e)} = r\partial_t\left(2X_{(e)} + r\partial_r\tilde{F}\right) + \frac{3f-1}{2f}r\partial_t\tilde{F} + 16\pi r^2 S_{(Y_{(e)})}, \tag{86}$$

where the source term $S_{(\Lambda\tilde{F})}$ and $S_{(Y_{(e)})}$ are given by

$$S_{(\Lambda\tilde{F})} := S_{(\mathbb{F})tt} + \partial_t S_{(ec)t} + f^2\partial_r S_{(ec)r} + \frac{4}{r}f^2 S_{(ec)r} - \frac{f}{2}S_{(F)} \tag{87}$$

$$= \tilde{T}_{tt} + rf^2\partial_r\tilde{T}_{(e2)} + 2f(f+1)\tilde{T}_{(e2)} + \frac{1}{2}f(l-1)(l+2)\tilde{T}_{(e2)}, \tag{88}$$

and

$$S_{(Y_{(e)})} := S_{(\mathbb{F})tr} + \partial_t S_{(ec)r} - \frac{1-f}{rf}S_{(ec)t} + \partial_r S_{(ec)t} \tag{89}$$

$$= \tilde{T}_{tr} + r\partial_t\tilde{T}_{(e2)}. \tag{90}$$

Equation (85) implies that the variable $\tilde{F}$ of the metric perturbation is determined if the variable $\Phi_{(e)}$ and source term $S_{(\Lambda\tilde{F})}$ are specified. Equation (86) implies that the component $Y_{(e)}$ of the metric perturbation is determined if the variables $X_{(e)}$, $\tilde{F}$, and the source term $S_{(Y_{(e)})}$ are specified.

Thus, apart from the source terms, the component $\tilde{F}$ of the metric perturbation is determined through Equation (85) if the Moncrief variable $\Phi_{(e)}$ is specified. The component $X_{(e)}$ of the metric perturbation is determined through Equation (84) if the variables $\Phi_{(e)}$ and $\tilde{F}$ are specified. Finally, the component $Y_{(e)}$ of the metric perturbation is determined through Equation (86) if the variables $\tilde{F}$ and $X_{(e)}$ are specified. Namely, the components $X_{(e)}$, $Y_{(e)}$, and $\tilde{F}$ of the metric perturbation are determined by the Moncrief variable $\Phi_{(e)}$. The Moncrief variable $\Phi_{(e)}$ is determined by the master equation

$$-\frac{1}{f}\partial_t^2\Phi_{(e)} + \partial_r\left[f\partial_r\Phi_{(e)}\right] - V_{even}\Phi_{(e)} = 16\pi\frac{r}{\Lambda}S_{(\Phi_{(e)})}, \tag{91}$$

where the potential function V_{even} is defined by

$$V_{even} := \frac{1}{r^2\Lambda^2}\left[\mu^2[(\mu+2)+3(1-f)] + (3(1-f))^2\left(+\mu+(1-f)\right)\right]$$

$$= \frac{1}{r^2\Lambda^2}\left[\Lambda^3 - 2(2-3f)\Lambda^2 + 6(1-3f)(1-f)\Lambda + 18f(1-f)^2\right], \tag{92}$$

and the source term in Equation (91) is given by

$$\begin{aligned} S_{(\Phi_{(e)})} \;:=\;& -\partial_t S_{(ec)t} - f^2\partial_r S_{(ec)r} + 2f\frac{f-1}{r}S_{(ec)r} + \frac{r}{2}f\partial_r S_{(F)} + \frac{1}{2}S_{(F)} - \frac{1}{4}\Lambda S_{(F)} \\ & + \frac{3(1-f)}{\Lambda}\left[-S_{(\mathbb{F})tt} - \partial_t S_{(ec)t} - f^2\partial_r S_{(ec)r} - \frac{4}{r}f^2 S_{(ec)r} + \frac{f}{2}S_{(F)}\right] \qquad (93) \\ =\;& \frac{1}{2}\left(\frac{\Lambda}{2f}-1\right)\tilde{T}_{tt} + \frac{1}{2}\left((2-f)-\frac{1}{2}\Lambda\right)f\tilde{T}_{rr} - \frac{1}{2}r\partial_r\tilde{T}_{tt} + \frac{1}{2}f^2 r\partial_r\tilde{T}_{rr} \\ & -\frac{f}{2}\tilde{T}_{(e0)} - l(l+1)f\tilde{T}_{(e1)r} \\ & +\frac{1}{2}r^2\partial_t^2\tilde{T}_{(e2)} - \frac{1}{2}f^2r^2\partial_r^2\tilde{T}_{(e2)} - \frac{1}{2}3(1+f)rf\partial_r\tilde{T}_{(e2)} \\ & -\frac{1}{2}(7-3f)f\tilde{T}_{(e2)} + \frac{1}{4}(l(l+1)-1-f)(l(l+1)+2)\tilde{T}_{(e2)} \\ & -\frac{3(1-f)}{\Lambda}\left[\tilde{T}_{tt} + rf^2\partial_r\tilde{T}_{(e2)} + \frac{1}{2}(1+7f)f\tilde{T}_{(e2)}\right]. \qquad (94) \end{aligned}$$

To solve the master equation (91) we have to impose appropriate boundary conditions and solve as the Cauchy problem. In the book [19], it is shown that the Zerilli equation (91) without the source term, i.e., $S_{(\Phi_{(e)})} = 0$, can be transformed to the Regge-Wheeler equation. This transformation is called the Chandrasekhar transformation. Since the Regge-Wheeler equation can be solved by MST (Mano Suzuki Takasugi) formulation [53–55], we may say that the solution to the Zerilli equation (91) without the source term is obtained through MST formulation.

Finally, we note that the solutions $\Phi_{(e)}$ and $\tilde{F}$ satisfy the equation (74), as the consistency of the linearized Einstein equation. Here, the source term $S_{(F)}$ is explicitly given by Equation (53). Here, we check this consistency of the initial value constraint (85) and the evolution equation (74). From Equations (74) and (91), we obtain

$$\begin{aligned} 0 \;=\;& r^2\Lambda\partial_t^2 S_{(\Lambda\tilde{F})} - \left[(5-3f)\Lambda + 3(1-f)(1+f) + 18\frac{1}{\Lambda}f(1-f)^2\right] fS_{(\Lambda\tilde{F})} \\ & -2\left[3(1-f)+2\Lambda\right] f^2 r\partial_r S_{(\Lambda\tilde{F})} - \Lambda r^2 f\partial_r\left[f\partial_r S_{(\Lambda\tilde{F})}\right] \\ & +\frac{1}{4}\left[(1-3f)-\Lambda\right]\Lambda^2 fS_{(F)} \\ & -2rf^2\Lambda\partial_r S_{(\Phi_{(e)})} - \left[\Lambda + (1+3f)\right]\Lambda fS_{(\Phi_{(e)})}. \end{aligned} \qquad (95)$$

This is an identity of the source terms. We have confirmed Equation (96) is an identity due to the definitions (88)–(53) and the continuity equations (58)–(60) of the perturbative energy-momentum tensor. This means that the evolution equation (74) is trivial when $l \neq 0$. Thus, we have confirmed that the above strategy for $l \neq 0$ modes are consistent.

Of course, this strategy is valid only when $l \neq 0$. In the $l = 0$ case, we have to consider the different strategy to obtain the variable $X_{(e)}$, $Y_{(e)}$, and $\tilde{F}$. This will be discussed Section 4.

Before going to the discussion on the strategy to solve $l = 0$ mode Einstein equation, we comment on the original equation derived by Zerilli [13,14] for $l \geq 2$. We consider the original time derivative of the Moncrief master variable (82) as

$$\partial_t\Phi_{(e)} \;=\; \frac{r}{\Lambda}\left[f\partial_t X_{(e)} - \frac{1}{4}\Lambda\partial_t\tilde{F} + \frac{1}{2}rf\partial_t\partial_r\tilde{F}\right]. \qquad (96)$$

On the other hand, Equation (86) is given by

$$\partial_t X_{(e)} \;=\; \frac{l(l+1)}{2r}Y_{(e)} - \frac{r}{2}\partial_t\partial_r\tilde{F} - \frac{3f-1}{4f}\partial_t\tilde{F} - 8\pi r S_{(Y_{(e)})}. \qquad (97)$$

Substituting Equation (97) into Equation (96), for $l \neq 0$ modes, we obtain

$$\frac{1}{l(l+1)}\partial_t\Phi_{(e)} = \frac{1}{2\Lambda}\left[fY_{(e)} - \frac{r}{2}\partial_t\tilde{F}\right] - 8\pi r^2 f\frac{1}{l(l+1)\Lambda}S_{(Y_{(e)})}. \tag{98}$$

Here, if we define the variable $\Psi_{(e)}$ by

$$\Psi_{(e)} := \frac{1}{2\Lambda}\left[fY_{(e)} - \frac{r}{2}\partial_t\tilde{F}\right] \tag{99}$$

$$= \frac{1}{l(l+1)}\partial_t\Phi_{(e)} + 8\pi r^2 f\frac{1}{l(l+1)\Lambda}S_{(Y_{(e)})}, \tag{100}$$

the variable $\Psi_{(e)}$ corresponds to original Zerilli's master variable. Roughly speaking, the variable $\Psi_{(e)}$ corresponds to the time-derivative of the variable $\Phi_{(e)}$ with additional source terms from the matter fields. Therefore, it is trivial $\Psi_{(e)}$ also satisfies the Zerilli equation with different source terms. In other words, the Zerilli equation for $\Psi_{(e)}$ is derived by the time derivative of the Zerilli equation for $\Phi_{(e)}$. This means that the solution to the Zerilli equation for $\Psi_{(e)}$ may include an additional arbitrary function of r as an "integration constants." This "integration constants" do not included in the solution $\Phi_{(e)}$ for the Zerilli equation (91). In this sense, the restriction of the initial value of Equation (91) for $\Phi_{(e)}$ is stronger than that of Equation (91) for $\Psi_{(e)}$.

4. $l = 0$ Mode Perturbations on the Schwarzschild Background

Here, we consider the $l = 0$ mode perturbations based on the perturbation equations for the even-mode on Schwarzschild background which are summarized in Section 3. Since Proposal 1 enable us to carry out the mode-by-mode analyses including $l = 0, 1$ modes, all equations in Section 3 except for Equations (98) and (100) are valid even in $l = 0$ mode. However, the strategy to solve these equations is different from that $l \neq 0$ modes, because Equations (85) and (86) do not directly give the components $(\tilde{F}, Y_{(e)})$ of the metric perturbation for $l = 0$ mode.

Before showing the strategy to solve even-mode Einstein equations for $l = 0$ mode, we note that

$$\hat{D}_p k_{(\hat{\Delta})} = 0 = \hat{D}_p\hat{D}_q k_{(\hat{\Delta})} \tag{101}$$

if we impose the regularity $\delta = 0$ to the harmonic function $k_{(\hat{\Delta})}$. In this case, the only remaining components of the linearized energy-momentum tensor is

$$\mathscr{T}_{ab} = \tilde{T}_{AB}k_{(\hat{\Delta})}(dx^A)_a(dx^B)_b + \frac{1}{2}r^2\gamma_{pq}\tilde{T}_{(e0)}k_{(\hat{\Delta})}(dx^p)_a(dx^q)_b. \tag{102}$$

Therefore, we can safely regard that

$$\tilde{T}_{(e2)} = 0, \quad \tilde{T}_{(e1)A} = 0. \tag{103}$$

Owing to Equation (103), the trace of the perturbation $\tilde{F}_{AB}$ is determined by the Einstein equation (39), i.e.,

$$\tilde{F}_D{}^D = 0. \tag{104}$$

In the case of $l = 0$ mode, Λ defined by Equation (83) is given by

$$\Lambda = 1 - 3f. \tag{105}$$

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Then, the Moncrief master variable $\Phi_{(e)}$ is given by Equation (82), i.e.,

$$\Phi_{(e)} := \frac{r}{1-3f}\left[fX_{(e)} - \frac{1}{4}(1-3f)\tilde{F} + \frac{1}{2}rf\partial_r\tilde{F}\right]. \tag{106}$$

This is equivalent to Equation (84) with $l = 0$ as

$$fX_{(e)} = \frac{1-3f}{r}\Phi_{(e)} + \frac{1-3f}{4}\tilde{F} - \frac{1}{2}rf\partial_r\tilde{F}. \tag{107}$$

As in the case of $l \neq 0$ mode, this equation yields the component $X_{(e)}$ of the metric perturbation is determined by $(\tilde{F}, \Phi_{(e)})$.

The crucial difference between the $l = 0$ mode and $l \neq 0$ modes is Equations (85) and (86). In the $l = 0$ case, these equations yield

$$\partial_r\left[(1-3f)\Phi_{(e)}\right] = -\frac{8\pi r^2}{f}\tilde{T}_{tt}, \tag{108}$$

$$\partial_t\left[(1-3f)\Phi_{(e)}\right] = -8\pi r^2 f\tilde{T}_{tr}, \tag{109}$$

where we used Equation (107) to derive Equation (109).

The components of the divergence of the energy momentum tensor are summarized as

$$\partial_t\tilde{T}_{tt} - f^2\partial_r\tilde{T}_{rt} - \frac{(1+f)f}{r}\tilde{T}_{rt} = 0, \tag{110}$$

$$\partial_t\tilde{T}_{tr} - \frac{1-f}{2rf}\tilde{T}_{tt} - f^2\partial_r\tilde{T}_{rr} - \frac{(3+f)f}{2r}\tilde{T}_{rr} = 0, \tag{111}$$

$$\tilde{T}_{(e0)} = 0. \tag{112}$$

Here, we check the integrability condition of Equations (108) and (109). Differentiating Equation (108) with respect to t and differentiating Equation (109) with respect to r, we obtain the integrability condition of Equations (108) and (109) follows

$$\begin{aligned} 0 &= \partial_t\left(-8\pi\frac{r^2}{f}\tilde{T}_{tt}\right) - \partial_r\left(-8\pi r^2 f\tilde{T}_{tr}\right) \\ &= -8\pi r^2\frac{1}{f}\left[\partial_t\tilde{T}_{tt} - f^2\partial_r\tilde{T}_{tr} - \frac{(1+f)f}{r}\tilde{T}_{tr}\right]. \end{aligned} \tag{113}$$

This coincides with the component (110) of the continuity equation of the matter field. Thus, Equations (108) and (109) are integrable and there exist the solution $\Phi_{(e)} = \Phi_{(e)}[T_{tt}, T_{tr}]$ to these equations.

In the case of $l = 0$ mode, the evolution equation (91) has the same form, but the potential V_{even} defined by Equation (92) with $l = 0$ is given by

$$V_{even} = \frac{3(1-f)(1+3f^2)}{r^2(1-3f)^2} \tag{114}$$

and the source term in Equation (94) is given by

$$S_{(\Phi_{(e)})} = -\tilde{T}_{tt} - \frac{r}{2}\partial_r\tilde{T}_{tt} + \frac{r}{2}\partial_t\tilde{T}_{tr} - \frac{3(1-f)}{1-3f}\tilde{T}_{tt}. \tag{115}$$

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Through Equations (108) and (109), we obtain

$$-\frac{1}{f}\partial_t^2\Phi_{(e)} + \partial_r(f\partial_r\Phi_{(e)}) - V_{even}\Phi_{(e)}$$
$$= \frac{16\pi r}{1-3f}\left[-\tilde{T}_{tt} - \frac{r}{2}\partial_r\tilde{T}_{tt} + \frac{r}{2}\partial_t\tilde{T}_{tr} - \frac{3(1-f)}{1-3f}\tilde{T}_{tt}\right]. \tag{116}$$

This coincides with the master equation (91) with $l = 0$. Thus, the master equation (91) does not give us any information other than that of Equations (108) and (109).

As in the case of $l \neq 0$ modes, the metric component $X_{(e)}$ is determined by the variables $(\tilde{F}, \Phi_{(e)})$ as seen in Equation (107). Although $\tilde{F}$ is determined by Equation (85) in the $l \neq 0$ case, this is impossible in the $l = 0$ case. Therefore, we have to consider Equation (74) for the variable $\tilde{F}$ which is trivial in the $l \neq 0$ case

$$-\frac{1}{f}\partial_t^2\tilde{F} + \partial_r(f\partial_r\tilde{F}) + \frac{1}{r^2}3(1-f)\tilde{F} + \frac{4}{r^3}(1-3f)\Phi_{(e)} = 16\pi\left(-\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right). \tag{117}$$

This equation has the same form of the inhomogeneous version of the Regge-Wheeler equation with $l = 0$, while the original Regge-Wheeler equation is valid only for the $l \geq 2$ modes. If we solve this equation (117), we can determine the variable $\tilde{F}$ which depends on the variable $\Phi_{(e)}$ and the matter fields $\tilde{T}_{tt}$ and $\tilde{T}_{rr}$. Then, through this solution $\tilde{F} = \tilde{F}[\Phi_{(e)}, \bar{T}_{tt}, \bar{T}_{rr}]$ and the solution to Equations (108) and (109), we can obtain the variable $X_{(e)}$ through Equation (107) as a solution to the linearized Einstein equation for the $l = 0$ mode.

The remaining component to be obtained is the component $Y_{(e)}$ of the metric perturbation. To obtain the variable $Y_{(e)}$, we remind the original initial value constraints (48) and (49). In the $l = 0$ mode case, the source term $S_{(ec)t}$ and $S_{(ec)r}$ are given by $S_{(ec)t} = S_{(ec)r} = 0$ from Equations (50) and (103). Then, the initial value constraints (48) and (49) are given by

$$\partial_r(fY_{(e)}) = \frac{1}{2}\partial_t\tilde{F} - \partial_t(X_{(e)}), \tag{118}$$
$$\partial_t(fY_{(e)}) = -f\partial_r(fX_{(e)}) - \frac{1}{2}f^2\partial_r\tilde{F}. \tag{119}$$

We may regard that Equations (118) and (119) are equations to obtain the variable $Y_{(e)}$. Actually, the integrability of these equations is guaranteed by Equations (107), (108), (109), (111), and (117). Then, we can obtain the component $Y_{(e)}$ of the metric perturbation by the direct integration of Equations (118) and (119).

We may carry out the above strategy to obtain the $l = 0$ mode solution to the linearized Einstein equations, but it is instructive to consider the vacuum case where all components of the linearized energy-momentum tensor ${}^{(1)}\mathscr{T}_{ab}$ vanishes before the derivation of the non-vacuum case.

4.1. $l = 0$ Mode Vacuum Case

Here, we consider the vacuum case of the above equations for $l = 0$ mode perturbations. First, we consider Equations (108) and (109) with the vacuum condition:

$$\partial_r\left[(1-3f)\Phi_{(e)}\right] = 0, \tag{120}$$
$$\partial_t\left[(1-3f)\Phi_{(e)}\right] = 0. \tag{121}$$

These equations are easily integrated as

$$(1-3f)\Phi_{(e)} = -2M_1, \quad M_1 \in \mathbb{R}. \tag{122}$$

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Furthermore, the variable $\tilde{F}$ is determined by Equation (117) with vacuum condition:

$$-\frac{1}{f}\partial_t^2\tilde{F} + \partial_r(f\partial_r\tilde{F}) + \frac{1}{r^2}3(1-f)\tilde{F} - \frac{8M_1}{r^3} = 0. \tag{123}$$

From Equations (107) and (122), we obtain the component $X_{(e)}$ of the metric perturbation as follows:

$$fX_{(e)} = -\frac{2M_1}{r} + \frac{1-3f}{4}\tilde{F} - \frac{1}{2}rf\partial_r\tilde{F}. \tag{124}$$

Moreover, the components $Y_{(e)}$ is obtain the direct integration of Equations (118) and (119), because the integrability is already guaranteed. Substituting Equation (124) into Equations (118) and (119), we obtain

$$f\partial_r(fY_{(e)}) = -\frac{1}{4}(1-5f)\partial_t\tilde{F} + \frac{1}{2}rf\partial_r\partial_t\tilde{F}, \tag{125}$$

$$\partial_t(fY_{(e)}) = \frac{2M_1 f}{r^2} - \frac{3}{4r}f(1-f)\tilde{F} - \frac{1}{4}f(1-3f)\partial_r\tilde{F} + \frac{1}{2}r\partial_t^2\tilde{F}, \tag{126}$$

where we used Equation (123).

Here, we assume the existence of the solution to Equation (123) and we denote this solution by

$$\tilde{F} =: \partial_t \Upsilon, \tag{127}$$

for our convention. Substituting Equation (127) into Equation (123) and integrating by t, we obtain

$$-\frac{1}{f}\partial_t^2\Upsilon + \partial_r(f\partial_r\Upsilon) + \frac{1}{r^2}3(1-f)\Upsilon - \frac{8M_1}{r^3}t + \zeta(r) = 0. \tag{128}$$

where $\zeta(r)$ is an arbitrary function of r. Using Equation (127) and integrating by t, Equation (126) yields

$$fY_{(e)} = \frac{2M_1 f}{r^2}t - \frac{3}{4r}f(1-f)\Upsilon - \frac{1}{4}f(1-3f)\partial_r\Upsilon + \frac{1}{2}r\partial_t^2\Upsilon + \Xi(r), \tag{129}$$

where $\Xi(r)$ is an arbitrary function of r. Substituting Equation (129) into Equation (125) and using Equation (128), we obtain

$$\zeta(r) = -\frac{4}{1-3f}\partial_r\Xi(r). \tag{130}$$

In summary, we have obtained the components of $X_{(e)}$, $Y_{(e)}$, and $\tilde{F}$ of the metric perturbations as follows:

$$fX_{(e)} = -\frac{2M_1}{r} + \frac{1-3f}{4}\partial_t\Upsilon - \frac{1}{2}rf\partial_r\partial_t\Upsilon, \tag{131}$$

$$fY_{(e)} = \frac{2M_1 f}{r^2}t - \frac{3}{4r}f(1-f)\Upsilon - \frac{1}{4}f(1-3f)\partial_r\Upsilon + \frac{1}{2}r\partial_t^2\Upsilon + \Xi(r), \tag{132}$$

and

$$\tilde{F} = \partial_t\Upsilon, \quad -\frac{1}{f}\partial_t^2\Upsilon + \partial_r(f\partial_r\Upsilon) + \frac{1}{r^2}3(1-f)\Upsilon - \frac{8M_1}{r^3}t - \frac{4}{1-3f}\partial_r\Xi(r) = 0, \tag{133}$$

and $\Xi(r)$ is an arbitrary function of r.

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Here, we consider the covariant form $\mathscr{F}_{ab}$ of the $l = 0$ mode metric perturbation. According to Proposal 1, we impose the regularity on S^2 to the harmonic function $k_{(\Delta)}$ so that

$$k_{(\Delta)} = 1. \tag{134}$$

Since $\tilde{F}_D{}^D = 0$ by Equation (104) for $l = 0$ mode perturbations, the gauge-invariant metric perturbation $\mathscr{F}_{ab}$ for the $l = 0$ mode is given by

$$\begin{aligned}\mathscr{F}_{ab} &= \tilde{F}_{AB}(dx^A)_a(dx^B)_b + \frac{1}{2}\gamma_{pq}r^2\tilde{F}(dx^p)_a(dx^q)_b \\ &= -(fX_{(e)})\left\{(dt)_a(dt)_b + f^{-2}(dr)_a(dr)_b\right\} + 2(fY_{(e)})f^{-1}(dt)_{(A}(dr)_{B)} \\ &\quad +\frac{1}{2}\gamma_{pq}r^2\tilde{F}(dx^p)_a(dx^q)_b. \end{aligned} \tag{135}$$

As in the case of the $l = 1$ odd-mode perturbation in Part I paper [30], the solutions (131)–(133) may include the terms in the form of $\pounds_V g_{ab}$ for a vector field V^a. To find the term $\pounds_V g_{ab}$, we consider the generator V_a whose components are given by

$$V_a = V_t(t,r)(dt)_a + V_r(r,t)(dr)_a. \tag{136}$$

Then, the nonvanishing components of $\pounds_V g_{ab}$ are given by

$$\pounds_V g_{tt} = 2\partial_t V_t - ff'V_r, \tag{137}$$
$$\pounds_V g_{tr} = \partial_t V_r + \partial_r V_t - \frac{f'}{f}V_t, \tag{138}$$
$$\pounds_V g_{rr} = 2\partial_r V_r + \frac{f'}{f}V_r, \tag{139}$$
$$\pounds_V g_{\theta\theta} = 2rfV_r, \tag{140}$$
$$\pounds_V g_{\phi\phi} = 2rf\sin^2\theta V_r. \tag{141}$$

From Equations (135), (140), and (141), we choose

$$V_r = \frac{1}{4f}r\tilde{F} = \frac{1}{4f}r\partial_t \Upsilon, \quad \pounds_V g_{\theta\theta} = \frac{1}{\sin^2\theta}\pounds_V g_{\phi\phi} = \frac{1}{2}r^2\tilde{F} = \frac{1}{2}r^2\partial_t \Upsilon. \tag{142}$$

Substituting Equation (142) into Equations (137)–(139), we obtain

$$\pounds_V g_{tt} = 2\partial_t V_t - \frac{1}{4}(1-f)\partial_t \Upsilon, \tag{143}$$
$$\pounds_V g_{tr} = \frac{1}{4f}r\partial_t^2 \Upsilon + \partial_r V_t - \frac{1}{fr}(1-f)V_t, \tag{144}$$
$$\pounds_V g_{rr} = -\frac{1}{4f^2}(1-3f)\partial_t \Upsilon + \frac{1}{2f}r\partial_r\partial_t \Upsilon. \tag{145}$$

To identify the degree of freedom which expressed as $\pounds_V g_{ab}$ in $X_{(e)}$, we choose

$$\partial_t V_t = \frac{1}{4}f\partial_t \Upsilon + \frac{1}{4}rf\partial_r\partial_t \Upsilon \tag{146}$$

so that

$$\pounds_V g_{tt} = -\frac{1}{4}(1-3f)\partial_t \Upsilon + \frac{1}{2}rf\partial_r\partial_t \Upsilon. \tag{147}$$

Then, we obtain

$$V_t = \frac{1}{4} f \Upsilon + \frac{1}{4} r f \partial_r \Upsilon + \gamma(r), \tag{148}$$

where $\gamma(r)$ is an arbitrary function of r. Substituting Equation (148) into Equation (144) and using the equation (133) for Υ, we obtain

$$\begin{aligned} \pounds_V g_{tr} &= \frac{2M_1}{r^2} t + \frac{r}{2f} \partial_t^2 \Upsilon - \frac{1}{4}(1-3f)\partial_r \Upsilon - \frac{3}{4r}(1-f)\Upsilon \\ &\quad + \frac{r}{1-3f} \partial_r \Xi(r) + \partial_r \gamma(r) - \frac{1}{fr}(1-f)\gamma(r). \end{aligned} \tag{149}$$

From the solutions (131), (132), and (133), and the expression (135) of the gauge-invariant part of the metric perturbation, and the components (142), (145), (147), and (149) of $\pounds_V g_{ab}$, we obtain

$$\begin{aligned} \mathscr{F}_{ab} &= \frac{2M_1}{r} \left((dt)_a (dt)_b + f^{-2} (dr)_a (dr)_b \right) + \pounds_V g_{ab} \\ &\quad + 2\left(\frac{1}{f} \Xi(r) - \frac{r}{1-3f} \partial_r \Xi(r) - \partial_r \gamma(r) + \frac{1}{fr}(1-f)\gamma(r) \right) (dt)_{(a} (dr)_{b)}. \end{aligned} \tag{150}$$

As a choice of the generator V_a, we choose the arbitrary function $\gamma(r)$ in V_a such that

$$\gamma(r) = -\frac{r}{(1-3f)} \Xi(r) + f \int dr \frac{2}{f(1-3f)^2} \Xi(r). \tag{151}$$

Then, we obtain

$$\mathscr{F}_{ab} = \frac{2M_1}{r} \left((dt)_a (dt)_b + f^{-2} (dr)_a (dr)_b \right) + \pounds_V g_{ab}, \tag{152}$$

where

$$V_a = \left(\frac{f}{4} \Upsilon + \frac{rf}{4} \partial_r \Upsilon - \frac{r}{1-3f} \Xi(r) + f \int dr \frac{2}{f(1-3f)^2} \Xi(r) \right) (dt)_a + \frac{r}{4f} \partial_t \Upsilon (dr)_a. \tag{153}$$

The function $\Upsilon(t,r)$ is the solution to the second equation (133).

The solution (152) is the $O(\epsilon)$ mass parameter perturbation $M + \epsilon M_1$ of the Schwarzschild spacetime apart from the term the Lie derivative of the background metric g_{ab}. Since $l = 0$ mode is the spherically symmetric perturbations, the solution (152) is the realization of the linearized gauge-invariant version of Birkhoff's theorem [56]. We also note that the vector field V_a is also gauge-invariant in the sense of the second kind. Here, we have to emphasize that the generator (153) with the second equation in Equation (133) is necessary if we include the perturbative Schwarzschild mass parameter M_1 as the solution to the linearized Einstein equation in our framework. This can be seen from the second equation in Equation (133). This equation indicates that $M_1 = 0$ if we choose $\Upsilon = 0$ for arbitrary time t.

4.2. $l = 0$ Mode Non-Vacuum Case

Inspecting the above vacuum case, we apply the method of variational constants. In Equation (122), the Schwarzschild mass parameter perturbation M_1 is an integration constant. Then, we choose the function $m_1(t,r)$ so that

$$m_1(t,r) := -\frac{1}{2}(1-3f)\Phi_{(e)}. \tag{154}$$

The integrability of Equations (108) and (109) was already confirmed in Equation (113). Then, we obtain

$$m_1(t,r) = 4\pi \int dr \frac{r^2}{f} \tilde{T}_{tt} + M_1 = 4\pi \int dt r^2 f \tilde{T}_{tr} + M_1. \tag{155}$$

Equation (107) yields the component $X_{(e)}$ of the metric perturbation as follows:

$$f X_{(e)} = -\frac{2m_1(t,r)}{r} + \frac{1-3f}{4}\tilde{F} - \frac{1}{2} r f \partial_r \tilde{F}. \tag{156}$$

As discussed in above, the variable $\tilde{F}$ is determined by Equation (117). As in the vacuum case in Section 4.1, we introduce the function Υ such that

$$\tilde{F} =: \partial_t \Upsilon, \tag{157}$$

$$\begin{aligned} & -\frac{1}{f}\partial_t^2 \Upsilon + \partial_r(f\partial_r \Upsilon) + \frac{3(1-f)}{r^2}\Upsilon - \frac{8}{r^3}\int dt m_1(t,r) + \zeta(r) \\ = {} & 16\pi \int dt \left(-\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr} \right), \end{aligned} \tag{158}$$

where $\zeta(r)$ is an arbitrary function of r. Through the variable Υ and Equation (156), Equation (119) is integrated as follows:

$$\begin{aligned} f Y_{(e)} = {} & \frac{2f}{r^2}\int dt m_1(t,r) + 8\pi r f^2 \int dt \tilde{T}_{rr} \\ & -\frac{3f(1-f)}{4r}\Upsilon - \frac{f(1-3f)}{4}\partial_r \Upsilon + \frac{r}{2}\partial_t^2 \Upsilon + \Xi(r), \end{aligned} \tag{159}$$

where $\Xi(r)$ is an different arbitrary function of r from $\zeta(r)$. Substituting Equation (159) into Equation (118), and using Equations (155), (156), (158), and the component (111) of the continuity equation, we obtain

$$\zeta(r) = -\frac{4}{1-3f}\partial_r \Xi(r) \tag{160}$$

as expected from the vacuum case in Section 4.1.

In summary, we have obtained the solution to the components of the metric perturbations $X_{(e)}$, $Y_{(e)}$, and $\tilde{F}$ as follows:

$$f X_{(e)} = -\frac{2m_1(t,r)}{r} + \frac{1-3f}{4}\partial_t \Upsilon - \frac{1}{2} r f \partial_r \partial_t \Upsilon, \tag{161}$$

$$\begin{aligned} f Y_{(e)} = {} & \frac{2f}{r^2}\int dt m_1(t,r) + 8\pi r f^2 \int dt \tilde{T}_{rr} \\ & -\frac{3f(1-f)}{4r}\Upsilon - \frac{f(1-3f)}{4}\partial_r \Upsilon + \frac{r}{2}\partial_t^2 \Upsilon + \Xi(r), \end{aligned} \tag{162}$$

$$\tilde{F} =: \partial_t \Upsilon, \tag{163}$$

$$\begin{aligned} & \partial_t^2 \Upsilon - f\partial_r(f\partial_r \Upsilon) - \frac{3f(1-f)}{r^2}\Upsilon + \frac{8f}{r^3}\int dt m_1(t,r) + \frac{4f\partial_r \Xi(r)}{1-3f} \\ = {} & 16\pi \int dt \left(\tilde{T}_{tt} - f^2 \tilde{T}_{rr} \right). \end{aligned} \tag{164}$$

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Here, we consider the covariant form of the above $l = 0$ mode non-vacuum solutions. As in the vacuum case in Section 4.1, we show the expression (135) of the above non-vacuum solution

$$\begin{aligned}\mathscr{F}_{ab} &= -(fX_{(e)})\left[(dt)_a(dt)_b + f^{-2}(dr)_a(dr)_b\right] + 2(fY_{(e)})f^{-1}(dt)_{(A}(dr)_{B)} \\ &\quad +\frac{1}{2}\gamma_{pq}r^2\partial_t\Upsilon(dx^p)_a(dx^q)_b. \end{aligned} \tag{165}$$

The components of $\mathscr{F}_{ab}$ are given by

$$\mathscr{F}_{tt} = \frac{2m_1(t,r)}{r} - \frac{1-3f}{4}\partial_t\Upsilon + \frac{1}{2}rf\partial_r\partial_t\Upsilon, \tag{166}$$

$$\begin{aligned}\mathscr{F}_{tr} &= \frac{2}{r^2}\int dt m_1(t,r) + 8\pi rf\int dt\tilde{T}_{rr} \\ &\quad -\frac{3(1-f)}{4r}\Upsilon - \frac{(1-3f)}{4}\partial_r\Upsilon + \frac{r}{2f}\partial_t^2\Upsilon + \frac{1}{f}\Xi(r),\end{aligned} \tag{167}$$

$$\mathscr{F}_{rr} = \frac{2m_1(t,r)}{rf^2} - \frac{1-3f}{4f^2}\partial_t\Upsilon + \frac{r}{2f}\partial_r\partial_t\Upsilon, \tag{168}$$

$$\mathscr{F}_{\theta\theta} = \frac{r^2}{2}\partial_t\Upsilon = \frac{1}{\sin^2\theta}\mathscr{F}_{\phi\phi}. \tag{169}$$

As in the vacuum case, we consider the term in the form $\pounds_V g_{ab}$ with the generator

$$V_a = V_t(t,r)(dt)_a + V_r(r,t)(dr)_a. \tag{170}$$

Then, we obtain Equations (137)–(141). Comparing Equations (140), (141), and (169), we choose V_r so that

$$V_r = \frac{1}{4f}r\partial_t\Upsilon, \quad \pounds_V g_{\theta\theta} = \frac{1}{\sin^2\theta}\pounds_V g_{\phi\phi} = \frac{1}{2}r^2\partial_t\Upsilon, \tag{171}$$

and we have

$$\mathscr{F}_{\theta\theta} = \pounds_V g_{\theta\theta}, \quad \mathscr{F}_{\phi\phi} = \pounds_V g_{\phi\phi}. \tag{172}$$

Substituting the choice (171) into Equation (139) and compare with Equation (168), we obtain

$$\pounds_V g_{rr} = -\frac{1-3f}{4f^2}\partial_t\Upsilon + \frac{1}{2f}r\partial_r\partial_t\Upsilon, \quad \mathscr{F}_{rr} = \frac{2m_1(t,r)}{rf^2} + \pounds_V g_{rr}. \tag{173}$$

Substituting the choice V_r in Equation (171) into Equation (137) and comparing with Equation (166), we choose

$$V_t = \frac{1}{4}f\Upsilon + \frac{1}{4}rf\partial_r\Upsilon + \gamma(r), \tag{174}$$

and obtain

$$\pounds_V g_{tt} = -\frac{1-3f}{4}\partial_t\Upsilon + \frac{1}{2}rf\partial_r\partial_t\Upsilon, \quad \mathscr{F}_{tt} = \frac{2m_1(t,r)}{r} + \pounds_V g_{tt}. \tag{175}$$

Finally, from Equation (138) with the choice (174) of V_t and the choice (171) of V_r, we obtain

$$\pounds_V g_{tr} = \frac{1}{4f}r\partial_t^2\Upsilon - \frac{1-3f}{4}\partial_r\Upsilon + \frac{1}{4}r\partial_r(f\partial_r\Upsilon) + \partial_r\gamma(r) - \frac{1-f}{fr}\gamma(r). \tag{176}$$

Furthermore, using Equation (164), we have

$$\begin{aligned} \pounds_V g_{tr} &= \frac{2}{r^2}\int dt m_1(t,r) - 4\pi\frac{r}{f}\int dt\tilde{T}_{tt} + 4\pi r f\int dt\tilde{T}_{rr} \\ &+\frac{1}{2f}r\partial_t^2\Upsilon - \frac{1-3f}{4}\partial_r\Upsilon - \frac{3(1-f)}{4r}\Upsilon \\ &+\partial_r\gamma(r) - \frac{1-f}{fr}\gamma(r) + \frac{r\partial_r\Xi(r)}{1-3f}. \end{aligned} \tag{177}$$

Through Equation (167), we obtain

$$\begin{aligned} \mathscr{F}_{tr} &= 4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right) + \pounds_V g_{tr} \\ &+f\left(\frac{2}{f(1-3f)^2}\Xi(r) - \partial_r\left(\frac{r}{f(1-3f)}\Xi(r)\right) - \partial_r(\frac{1}{f}\gamma(r))\right). \end{aligned} \tag{178}$$

The same choice of $\gamma(r)$ in the generator V_a as Equation (151) yields

$$\mathscr{F}_{tr} = 4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right) + \pounds_V g_{tr}. \tag{179}$$

Thus, we have obtained

$$\begin{aligned} \mathscr{F}_{ab} &= \frac{2}{r}\left(M_1 + 4\pi\int dr\frac{r^2}{f}T_{tt}\right)\left((dt)_a(dt)_a + \frac{1}{f^2}(dr)_a(dr)_a\right) \\ &+2\left[4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right)\right](dt)_{(a}(dr)_{b)} + \pounds_V g_{ab}, \end{aligned} \tag{180}$$

where

$$V_a = \left(\frac{f}{4}\Upsilon + \frac{rf}{4}\partial_r\Upsilon - \frac{r\Xi(r)}{(1-3f)} + f\int dr\frac{2\Xi(r)}{f(1-3f)^2}\right)(dt)_a + \frac{1}{4f}r\partial_t\Upsilon(dr)_a. \tag{181}$$

The variable Υ must satisfy Equation (164). We also note that the expression of $\mathscr{F}_{ab}$ is not unique, since we may choose different vector field V_a. We can also choose the time component V_t of the vector field V_a so that $\mathscr{F}_{tr} = \pounds_V g_{tr}$. In this case, the additional terms appear in the component $\mathscr{F}_{tt}$.

We also note that the term $\pounds_V g_{ab}$ in Equation (180) is gauge-invariant of the second kind. Furthermore, unlike the vacuum case, the variable Υ in this term includes information of the matter field through Equation (164). In this sense, the term $\pounds_V g_{ab}$ in Equation (180) is physical.

5. $l = 1$ Mode Non-Vacuum Perturbations on the Schwarzschild Background

In this section, we consider the $l = 1$ mode perturbations based through the variables defined in Sections 2 and 3. Even in the case of $l = 1$ mode, the gauge-invariant variables given by Equations (33)–(35) are valid. Since the mode-by-mode analyses are possible in our formulation, we can consider $l = 1$ modes, separately. For the $l = 1$ even-mode perturbations, the component $\mathscr{F}_{Ap}$ of the gauge-invariant part of the metric perturbation vanishes and the other components are given by

$$\mathscr{F}_{AB} := \sum_{m=-1}^{1}\tilde{F}_{AB}k_{(\hat{\Delta+2})m}, \quad \mathscr{F}_{pq} := \frac{1}{2}\gamma_{pq}r^2\sum_{m=-1}^{1}\tilde{F}k_{(\hat{\Delta}+2)m}. \tag{182}$$

We can also separate the trace part $\tilde{F}_D{}^D$ and the traceless part $\tilde{\mathbb{F}}_{AB}$ for the metric perturbation $\tilde{F}_{AB}$ as Equation (41). We also consider the components of the traceless part $\mathbb{F}_{AB}$ as Equation (47).

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Following Proposal 1, we impose the regularity to the harmonic function $k_{(\hat{\Delta}+2)m}$. Then, we have

$$\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{\Delta}\right) k_{(\hat{\Delta}+2)m} = \epsilon_{r(p}\hat{D}_{q)}\hat{D}^r k_{(\hat{\Delta}+2)m} = 0. \tag{183}$$

In this case, the only remaining components of the linearized energy-momentum tensor ${}^{(1)}\mathscr{T}_{ab}$ are given by

$$\begin{aligned} {}^{(1)}\mathscr{T}_{ab} &= \sum_{m=-1}^{1} \tilde{T}_{AB} k_{(\hat{\Delta}+2)} (dx^A)_a (dx^B)_b \\ &+2r \sum_{m=-1}^{1} \left\{ \tilde{T}_{(e1)A}\hat{D}_p k_{(\hat{\Delta}+2)m} + \tilde{T}_{(o1)A}\epsilon_{pr}\hat{D}^r k_{(\hat{\Delta}+2)m} \right\} (dx^A)_{(a}(dx^p)_{b)} \\ &+\frac{1}{2} r^2 \gamma_{pq} \sum_{m=-1}^{1} \tilde{T}_{(e0)} k_{(\hat{\Delta}+2)m} (dx^p)_a (dx^q)_b. \end{aligned} \tag{184}$$

Therefore, for even-mode perturbations, we can safely regard that

$$\tilde{T}_{(e2)} = 0. \tag{185}$$

From Equations (39) and (185), the components $\tilde{F}_{AB}$ is traceless. Then, we may concentrate on the components $X_{(e)}$ and $Y_{(e)}$ defined by Equation (47) and the component $\tilde{F}$ as the metric perturbations. Furthermore, all arguments in Section 3 are valid even in the case of $l = 1$ modes. Therefore, we may use Equations (82)–(96) when we derive the $l = 1$ mode solutions to the linearized Einstein equations.

From the definition (83) of Λ, we obtain

$$\Lambda = 3(1 - f). \tag{186}$$

Then, the Moncrief variable $\Phi_{(e)}$ defined by Equation (82) is given by

$$\Phi_{(e)} := \frac{r}{3(1-f)} \left[f X_{(e)} - \frac{3(1-f)}{4}\tilde{F} + \frac{1}{2} r f \partial_r \tilde{F} \right]. \tag{187}$$

In other words, the components $X_{(e)}$ is given by

$$f X_{(e)} = \frac{3(1-f)}{r}\Phi_{(e)} + \frac{3(1-f)}{4}\tilde{F} - \frac{1}{2} r f \partial_r \tilde{F} \tag{188}$$

as a solution to the linearized Einstein equation, if the variables $\Phi_{(e)}$ and $\tilde{F}$ are given as solutions to the linearized Einstein equation. Furthermore, from Equations (85) and (86), we obtain

$$\tilde{F} = -4 f \partial_r \Phi_{(e)} - \frac{4(1-f)}{r}\Phi_{(e)} - \frac{32\pi r^2}{3(1-f)}\tilde{T}_{tt}, \tag{189}$$

$$\begin{aligned} f Y_{(e)} &= r f \partial_t \left(X_{(e)} + \frac{r}{2}\partial_r \tilde{F} \right) + \frac{3f-1}{4} r \partial_t \tilde{F} + 8\pi r^2 f \tilde{T}_{tr} \\ &= (1-f)\partial_t \Phi_{(e)} - 2 r f \partial_t \partial_r \Phi_{(e)} - \frac{16\pi r^3}{3(1-f)} \partial_t \tilde{T}_{tt} + 8\pi r^2 f \tilde{T}_{tr}, \end{aligned} \tag{190}$$

where we used Equation (188) and (189) in the derivation of Equation (190). Under the given the components $\tilde{T}_{tt}$ and $\tilde{T}_{tr}$ of the linearized energy-momentum tensor, Equations (189) and (190) yield

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that the component $\tilde{F}$ and $Y_{(e)}$ are determined by $\Phi_{(e)}$. Furthermore, substituting Equation (189) into Equation (188), we obtain

$$\begin{aligned} fX_{(e)} &= -\frac{f(1-f)}{r}\Phi_{(e)} - f(1-f)\partial_r\Phi_{(e)} + 2rf\partial_r(f\partial_r\Phi_{(e)}) \\ &\quad -8\pi r^2\tilde{T}_{tt} + \frac{16\pi r^2 f}{(1-f)}\tilde{T}_{tt} + \frac{16\pi r^3 f}{3(1-f)}\partial_r\tilde{T}_{tt}. \end{aligned} \tag{191}$$

This also yields that the component $X_{(e)}$ is determined by $\Phi_{(e)}$ under the given components of the linearized energy-momentum tensor. Thus, the components $X_{(e)}$, $Y_{(e)}$, and $\tilde{F}$ are determined by the single variable $\Phi_{(e)}$ apart from the contribution from the components of the linearized energy-momentum tensor.

The determination of the Moncrief variable $\Phi_{(e)}$ is accomplished by solving the master Equation (91):

$$-\frac{1}{f}\partial_t^2\Phi_{(e)} + \partial_r\left[f\partial_r\Phi_{(e)}\right] - \frac{1-f}{r^2}\Phi_{(e)} = 16\pi\frac{r}{\Lambda}S_{(\Phi_{(e)})}, \tag{192}$$

and the source term in Equation (91) is given by

$$S_{(\Phi_{(e)})} = \frac{1}{2}r\partial_t\tilde{T}_{tr} - \frac{1}{2}r\partial_r\tilde{T}_{tt} + \frac{1-4f}{2f}\tilde{T}_{tt} - \frac{f}{2}\tilde{T}_{rr} - f\tilde{T}_{(e1)r}. \tag{193}$$

The master variable $\Phi_{(e)}$ is determined through the master equation (192) with appropriate initial conditions.

Furthermore, we have to take into account of the perturbation of the divergence of the energy-momentum tensor, which are summarized as follows:

$$-\partial_t\tilde{T}_{tt} + f^2\partial_r\tilde{T}_{rt} + \frac{(1+f)f}{r}\tilde{T}_{rt} - \frac{2f}{r}\tilde{T}_{(e1)t} = 0, \tag{194}$$

$$-\partial_t\tilde{T}_{tr} + \frac{1-f}{2rf}\tilde{T}_{tt} + f^2\partial_r\tilde{T}_{rr} + \frac{(3+f)f}{2r}\tilde{T}_{rr} - \frac{2f}{r}\tilde{T}_{(e1)r} - \frac{f}{r}\tilde{T}_{(e0)} = 0, \tag{195}$$

$$-\partial_t\tilde{T}_{(e1)t} + f^2\partial_r\tilde{T}_{(e1)r} + \frac{(1+2f)f}{r}\tilde{T}_{(e1)r} + \frac{f}{2r}\tilde{T}_{(e0)} = 0. \tag{196}$$

The expression of (193) for the source term $S_{(\Phi_{(e)})}$ in Equation (193) was derived by using Equation (195).

5.1. $l = 1$ Mode Vacuum Case

As in the case of $l = 0$ modes, it is instructive to consider the vacuum case where all components of the linearized energy-momentum tensor vanish before the derivation of the non-vacuum case.

Here, we consider the covariant form $\mathscr{F}_{ab}$ of the $l = 1$-mode metric perturbation as follows:

$$\mathscr{F}_{ab} = \sum_{m=-1}^{1}\tilde{F}_{AB}k_{(\Delta+2)m}(dx^A)_a(dx^B)_b + \frac{1}{2}\sum_{m=-1}^{1}\gamma_{pq}r^2\tilde{F}k_{(\Delta+2)m}(dx^p)_a(dx^q)_b. \tag{197}$$

The harmonic function $k_{(\Delta+2)m}$ is explicitly given by Equations (31) and (32). If we impose the regularity on these harmonics by the choice $\delta = 0$, these harmonics are given by the spherical harmonics $Y_{l=1,m}$ with $l = 1$:

$$Y_{l=1,m=0} \propto \cos\theta, \quad Y_{l=1,m=1} \propto \sin\theta e^{i\phi}, \quad Y_{l=1,m=-1} \propto \sin\theta e^{-i\phi}. \tag{198}$$

Since the extension of our arguments to $m = \pm 1$ modes is straightforward, we concentrate only on the $m = 0$ modes.

For the $m=0$ mode, the gauge-invariant part $\mathscr{F}_{ab}$ of the metric perturbation is given by

$$\begin{aligned}\mathscr{F}_{ab} &= \left(fX_{(e)}\right)\cos\theta\left\{-(dt)_a(dt)_b - f^{-2}(dr)_a(dr)_b\right\} + \frac{2}{f}(fY_{(e)})\cos\theta(dt)_{(a}(dr)_{b)} \\ &\quad +\frac{1}{2}r^2\tilde{F}\cos\theta\left\{(d\theta)_a(d\theta)_b + \sin^2\theta(d\phi)_a(d\phi)_b\right\}. \end{aligned} \tag{199}$$

By choosing $\tilde{T}_{tt} = \tilde{T}_{tr} = 0$ in Equations (189), (190), and (191), we obtain the vacuum solutions $\tilde{F}$, $Y_{(e)}$, and $X_{(e)}$ of the metric perturbation as follows:

$$X_{(e)} = -\frac{1}{r}(1-f)\Phi_{(e)} - (1-f)\partial_r\Phi_{(e)} + 2r\partial_r\left[f\partial_r\Phi_{(e)}\right], \tag{200}$$

$$Y_{(e)} = \partial_t\left[+\frac{1}{f}(1-f)\Phi_{(e)} - 2r\partial_r\Phi_{(e)}\right], \tag{201}$$

$$\tilde{F} = -4f\partial_r\Phi_{(e)} - 4\frac{1-f}{r}\Phi_{(e)}. \tag{202}$$

Here, $\Phi_{(e)}$ is a solution to the equation

$$-\frac{1}{f}\partial_t^2\Phi_{(e)} + \partial_r\left[f\partial_r\Phi_{(e)}\right] - \frac{1-f}{r^2}\Phi_{(e)} = 0. \tag{203}$$

As in the case of $l=0$ mode, we consider the problem whether the solution (199) with Equations (200)–(202) is described by $\pounds_V g_{ab}$ for an appropriate vector field V_a, or not. From the symmetry of the above solution, we consider the case where the vector field V_a is given by

$$V_a = V_t(dt)_a + V_r(dr)_a + V_\theta(d\theta)_a, \quad \partial_\phi V_t = \partial_\phi V_r = \partial_\phi V_\theta = 0 \tag{204}$$

and calculate all components of $\pounds_V g_{ab}$. We note that all components of $\mathscr{F}_{ab}$ given by Equation (199) are proportional to $\cos\theta$. Therefore, if we may identify some components of $\mathscr{F}_{ab}$ with $\pounds_V g_{ab}$, the θ-dependence of the components in Equation (204) should be given by

$$V_a = v_t(t,r)\cos\theta(dt)_a + v_r\cos\theta(dr)_a + v_\theta\sin\theta(d\theta)_a. \tag{205}$$

Then, the non-trivial components of $\pounds_V g_{ab}$ are given by

$$\pounds_V g_{tt} = (2\partial_t v_t - ff'v_r)\cos\theta \neq 0, \tag{206}$$

$$\pounds_V g_{tr} = \left(\partial_t v_r + \partial_r v_t - \frac{f'}{f}v_t\right)\cos\theta \neq 0, \tag{207}$$

$$\pounds_V g_{t\theta} = (\partial_t v_\theta - v_t)\sin\theta = 0, \tag{208}$$

$$\pounds_V g_{rr} = 2f^{-1/2}\partial_r\left(f^{1/2}v_r\right)\cos\theta \neq 0, \tag{209}$$

$$\pounds_V g_{r\theta} = \left(r^2\partial_r\left(\frac{1}{r^2}v_\theta\right) - v_r\right)\sin\theta = 0, \tag{210}$$

$$\pounds_V g_{\theta\theta} = 2\left(v_\theta + rfv_r\right)\cos\theta \neq 0, \tag{211}$$

$$\pounds_V g_{\phi\phi} = 2\left(rfv_r + v_\theta\right)\sin^2\theta\cos\theta \neq 0. \tag{212}$$

From Equations (208) and (210), we obtain

$$r^2 v(t,r) := v_\theta, \quad v_t = \partial_t v_\theta = r^2\partial_t v, \quad v_r = r^2\partial_r\left(\frac{1}{r^2}v_\theta\right) = r^2\partial_r v, \tag{213}$$

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i.e.,

$$V_a = r^2 \partial_t v \cos\theta (dt)_a + r^2 \partial_r v \cos\theta (dr)_a + r^2 v \sin\theta (d\theta)_a. \tag{214}$$

Then, Equations (206)–(212) are summarized as

$$\pounds_V g_{tt} = r^2 \left(2\partial_t^2 v - \frac{f(1-f)}{r} \partial_r v \right) \cos\theta, \tag{215}$$

$$\pounds_V g_{tr} = \partial_t \left(2r^2 \partial_r v - \frac{1-3f}{f} r v \right) \cos\theta, \tag{216}$$

$$\pounds_V g_{rr} = 2f^{-1/2} \partial_r \left(f^{1/2} r^2 \partial_r v \right) \cos\theta, \tag{217}$$

$$\pounds_V g_{\theta\theta} = 2r^2 \left(r f \partial_r v + v \right) \cos\theta. \tag{218}$$

As the first trial, we consider the correspondence

$$\pounds_V g_{\theta\theta} = \mathscr{F}_{\theta\theta}, \tag{219}$$

i.e.,

$$r f \partial_r v + v = -f \partial_r \Phi_{(e)} - \frac{1-f}{r} \Phi_{(e)}. \tag{220}$$

As the second trial, we consider the correspondence

$$\pounds_V g_{rr} = \mathscr{F}_{rr}, \tag{221}$$

i.e.,

$$-\frac{1-5f}{f} r \partial_r v + \frac{2r^2}{f} \partial_r \left(f \partial_r v \right) = \frac{1-f}{rf} \Phi_{(e)} + \frac{1-f}{f} \partial_r \Phi_{(e)} - \frac{2r}{f} \partial_r \left[f \partial_r \Phi_{(e)} \right]. \tag{222}$$

From Equations (220) and (222), we obtain

$$v = -\frac{1}{r} \Phi_{(e)}. \tag{223}$$

Substituting Equation (223) into Equation (207), we obtain

$$\pounds_V g_{tr} = \partial_t \left(-2r \partial_r \Phi_{(e)} + \frac{1}{f} (1-f) \Phi_{(e)} \right) \cos\theta = \mathscr{F}_{tr}. \tag{224}$$

Furthermore, substituting Equation (223) into Equation (206), we obtain

$$\begin{aligned} \pounds_V g_{tt} &= \left(-2r \partial_t^2 \Phi_{(e)} - \frac{f(1-f)}{r} \Phi_{(e)} + f(1-f) \partial_r \Phi_{(e)} \right) \cos\theta \\ &= -f \left(-\frac{1}{r} (1-f) \Phi_{(e)} - (1-f) \partial_r \Phi_{(e)} + 2r \partial_r \left[f \partial_r \Phi_{(e)} \right] \right) \cos\theta \\ &= \mathscr{F}_{tt}, \end{aligned} \tag{225}$$

where we used Equation (203).

Then, we have shown that

$$\mathscr{F}_{ab} = \pounds_V g_{ab}, \tag{226}$$

where

$$V_a = -r\partial_t\Phi_{(e)}\cos\theta(dt)_a + \left(\Phi_{(e)} - r\partial_r\Phi_{(e)}\right)\cos\theta(dr)_a - r\Phi_{(e)}\sin\theta(d\theta)_a. \tag{227}$$

Thus, the vacuum solution of $l = 1$-mode perturbations described by the Lie derivative of the background metric through the master equation (203).

5.2. $l = 1$ Mode Non-Vacuum Case

Here, we consider the non-vacuum solution to the $l = 1$ even-mode linearized Einstein equations. In this non-vacuum case, we concentrate only on the $m = 0$ mode perturbations as in the vacuum case, because the extension to our arguments to $m = \pm 1$ modes is straightforward. The solution is given by the covariant form (199) as in the case of the vacuum case. The non-vacuum solutions for the variable $\tilde{F}$, $Y_{(e)}$, and $X_{(e)}$ are given by Equations (189), (190), and (191), respectively. The master variable $\Phi_{(e)}$ must satisfy the master equation (192) with the source term (193). We have to emphasize that the components of the linear perturbation of energy-momentum tensor satisfy the continuity equations (194)–(196). Then, the components of the gauge-invariant part $\mathscr{F}_{ab}$ for $l = 1$ even-mode non-vacuum perturbations are summarized as follows:

$$\begin{aligned}
\mathscr{F}_{tt} &= f\left[\frac{1}{r}(1-f)\Phi_{(e)} + (1-f)\partial_r\Phi_{(e)} - 2r\partial_r\left[f\partial_r\Phi_{(e)}\right]\right]\cos\theta \\
&\quad + \frac{8\pi r^2}{3(1-f)}\left[3(1-3f)\tilde{T}_{tt} - 2rf\partial_r\tilde{T}_{tt}\right]\cos\theta, & (228)\\
\mathscr{F}_{tr} &= r\partial_t\left[\frac{1-f}{rf}\Phi_{(e)} - 2\partial_r\Phi_{(e)} - \frac{16\pi r^2}{3f(1-f)}\tilde{T}_{tt}\right]\cos\theta + 8\pi r^2\tilde{T}_{tr}\cos\theta, & (229)\\
\mathscr{F}_{rr} &= \frac{1}{f}\left[\frac{1-f}{r}\Phi_{(e)} + (1-f)\partial_r\Phi_{(e)} - 2r\partial_r(f\partial_r\Phi_{(e)})\right]\cos\theta \\
&\quad + \frac{8\pi r^2}{3f^2(1-f)}\left[3(1-3f)\tilde{T}_{tt} - 2rf\partial_r\tilde{T}_{tt}\right]\cos\theta, & (230)\\
\mathscr{F}_{\theta\theta} &= -2r\left[rf\partial_r\Phi_{(e)} + (1-f)\Phi_{(e)} + \frac{8\pi r^3}{3(1-f)}\tilde{T}_{tt}\right]\cos\theta, & (231)\\
\mathscr{F}_{\phi\phi} &= -2r\left[rf\partial_r\Phi_{(e)} + (1-f)\Phi_{(e)} + \frac{8\pi r^3}{3(1-f)}\tilde{T}_{tt}\right]\sin^2\theta\cos\theta. & (232)
\end{aligned}$$

As seen in the vacuum case, if we choose the generator V_a as Equation (227), i.e.,

$$\begin{aligned}
V_a = V_{(vac)a} \quad :=& \quad -r\partial_t\Phi_{(e)}\cos\theta(dt)_a + \left(\Phi_{(e)} - r\partial_r\Phi_{(e)}\right)\cos\theta(dr)_a \\
& -r\Phi_{(e)}\sin\theta(d\theta)_a, & (233)
\end{aligned}$$

we obtain

$$\begin{aligned}
£_V g_{tt} &= f\left[-\frac{2r}{f}\partial_t^2\Phi_{(e)} + (1-f)\partial_r\Phi_{(e)} - \frac{1-f}{r}\Phi_{(e)}\right]\cos\theta, & (234)\\
£_V g_{tr} &= r\partial_t\left(\frac{1-f}{rf}\Phi_{(e)} - 2\partial_r\Phi_{(e)}\right)\cos\theta, & (235)\\
£_V g_{rr} &= \frac{1}{f}\left[\frac{1-f}{r}\Phi_{(e)} + (1-f)\partial_r\Phi_{(e)} - 2r\partial_r\left(f\partial_r\Phi_{(e)}\right)\right]\cos\theta, & (236)\\
£_V g_{\theta\theta} &= -2r\left[rf\partial_r\Phi_{(e)} + (1-f)\Phi_{(e)}\right]\cos\theta, & (237)\\
£_V g_{\phi\phi} &= -2r\left(rf\partial_r\Phi_{(e)} + (1-f)\Phi_{(e)}\right)\sin^2\theta\cos\theta, & (238)\\
£_V g_{t\theta} &= £_V g_{t\phi} = £_V g_{r\theta} = £_V g_{r\phi} = £_V g_{\theta\phi} = 0. & (239)
\end{aligned}$$

Through these formulae of the components $\pounds_V g_{ab}$ and Equations (228)–(232) for the components of $\mathscr{F}_{ab}$, we obtain

$$\mathscr{F}_{tt} = \pounds_V g_{tt} - \frac{16\pi r^2 f^2}{3(1-f)} \left[\frac{1+f}{2} \tilde{T}_{rr} + rf\partial_r \tilde{T}_{rr} - \tilde{T}_{(e0)} - 4\tilde{T}_{(e1)r} \right] \cos\theta, \quad (240)$$

$$\mathscr{F}_{tr} = \pounds_V g_{tr} - \frac{16\pi r^3}{3f(1-f)} \left[\partial_t \tilde{T}_{tt} - \frac{3f(1-f)}{2r} \tilde{T}_{tr} \right] \cos\theta, \quad (241)$$

$$\mathscr{F}_{rr} = \pounds_V g_{rr} - \frac{16\pi r^3}{3f(1-f)} \left[\partial_r \tilde{T}_{tt} - \frac{3(1-3f)}{2rf} \tilde{T}_{tt} \right] \cos\theta, \quad (242)$$

$$\mathscr{F}_{\theta\theta} = \pounds_V g_{\theta\theta} - \frac{16\pi r^4}{3(1-f)} \tilde{T}_{tt} \cos\theta, \quad (243)$$

$$\mathscr{F}_{\phi\phi} = \pounds_V g_{\phi\phi} - \frac{16\pi r^4}{3(1-f)} \tilde{T}_{tt} \sin^2\theta \cos\theta, \quad (244)$$

where we used Equation (192) with the source term (193) and the component (195) of the continuity equation in Equation (240). Equations (240)–(244) are summarized as

$$\begin{aligned} \mathscr{F}_{ab} = \pounds_V g_{ab} - \frac{16\pi r^2}{3(1-f)} \Bigg[& f^2 \left\{ \frac{1+f}{2} \tilde{T}_{rr} + rf\partial_r \tilde{T}_{rr} - \tilde{T}_{(e0)} - 4\tilde{T}_{(e1)r} \right\} (dt)_a (dt)_b \\ & + \frac{2r}{f} \left\{ \partial_t \tilde{T}_{tt} - \frac{3f(1-f)}{2r} \tilde{T}_{tr} \right\} (dt)_{(a} (dr)_{b)} \\ & + \frac{r}{f} \left\{ \partial_r \tilde{T}_{tt} - \frac{3(1-3f)}{2rf} \tilde{T}_{tt} \right\} (dr)_a (dr)_b \\ & + r^2 \tilde{T}_{tt} \gamma_{ab} \Bigg] \cos\theta. \end{aligned} \quad (245)$$

We note that there may be exist the term $\pounds_W g_{ab}$ in the right-hand side of Equations (245) in addition to the term $\pounds_V g_{ab}$ discussed above. Such term will depend on the equation of state of the matter field. This situation can be seen in the Part III paper [46]. Even if we consider such terms, we will not have a simple expression of the metric perturbation, in general. Therefore, we will not carry out such further considerations, here.

6. Summary and Discussion

In summary, after reviewing our general framework of the general-relativistic gauge-invariant perturbation theory and our strategy for the linear perturbations on the Schwarzschild background spacetime proposed in Refs. [29,30], we developed the component treatments of the even-mode linearized Einstein equations. Our proposal in Refs. [29,30] was on the gauge-invariant treatments of the $l = 0, 1$ mode perturbations on the Schwarzschild background spacetime. Since we used singular harmonic functions at once in our proposal, we have to confirm whether our proposal is physically reasonable, or not.

To confirm this, in the Part I paper [30], we carefully discussed the solutions to the Einstein equations for odd-mode perturbations. We obtain the Kerr parameter perturbations in the vacuum case, which is physically reasonable. In this paper, we carefully discussed the solutions to the even-mode perturbations. Due to Proposal 1, we can treat the $l = 0, 1$ mode perturbations through the equivalent manner to the $l \geq 2$-mode perturbations. For this reason, we derive the equations for even-mode perturbations without making distinction among $l \geq 0$ modes for even-mode perturbations.

To derive the even-mode perturbations, it is convenient to introduce the Moncrief variable. In this paper, we explain the introduction of the Moncrief variable through an initial value constraint (66) is regard as an equation for the component $\tilde{F}$ of the metric perturbation and the Moncrief variable $\Phi_{(e)}$. This consideration leads to the well-known definition of the Moncrief variable $\Phi_{(e)}$. Furthermore, from the evolution equation (54), we obtain the well-known master equation (91) for the Moncrief variable $\Phi_{(e)}$.

Moreover, we obtain the constraint equations (85) and (86) together with the definition (84) of the Moncrief variable. From their derivations, we have shown that these equations are valid not only for $l \geq 2$ but also for $l = 0, 1$ modes. We also checked the consistency of these equations, and we derived the identity of the source terms which are given by the components of the linear perturbation of the energy-momentum tensor. This identity is confirmed by the components of the linear perturbation of the energy-momentum tensor.

In this paper, we also carefully discussed the $l = 0, 1$ mode solutions to the linearized Einstein equations for even-mode perturbations to check that Proposal 1 is physically reasonable.

The $l = 0$-mode solutions are discussed in Section 4. After summarizing the linearized Einstein equations and the linearized continuity equations for generic matter field for $l = 0$ mode, we first considered the vacuum solution of the $l = 0$-mode perturbations following Proposal 1. Then, we showed that the additional mass parameter perturbation of the Schwarzschild spacetime is the only solution apart from the terms of the Lie derivative of the background metric g_{ab} in the vacuum case. This is the gauge-invariant realization of the linearized version of the Birkhoff theorem [56].

In the non-vacuum case, we use the method of the variational constant with the Schwarzschild mass constant parameter in vacuum case. Then, we obtained the general non-vacuum solution to the linearized Einstein equation for the $l = 0$ mode. As the result, we obtained the linearized metric (180). The solution (180) includes the additional mass parameter perturbation M_1 of the Schwarzschild mass and the integration of the energy density. Furthermore, in the solution (180), we have the $2(dt)_{(a}(dr)_{b)}$ term due to the integration of the components of the energy-momentum tensor. In the solution (180), we also have the term which have the form of the Lie derivative of the background metric g_{ab}. The off-diagonal term of $2(dt)_{(a}(dr)_{b)}$ can be eliminate by the replacement of the generator V_a of the term of the Lie derivative of the g_{ab}. However, if we eliminate the off-diagonal term of $2(dt)_{(a}(dr)_{b)}$ through the replacement of the generator V_a, we have additional term to the diagonal components of the linearized metric perturbation (180). Since these diagonal components have complicated forms, we do not carry out this displacement.

We also discussed the $l = 1$-mode perturbations in Section 5. In this paper, we concentrated only on the $m = 0$ mode, since the extension to $m = \pm 1$ modes are straightforward. The solution of the $l = 1$ mode is obtained through the similar strategy to the case of $l \geq 2$ modes that are discussed in Section 3. As in the case of $l = 0$-mode perturbations, we first discuss the vacuum solution for $l = 1$-mode perturbations. As the result, $l = 1$-mode vacuum metric perturbations are described by the Lie derivative of the background metric g_{ab} with an appropriate operator. On the other hand, in the non-vacuum $l = 1$-mode perturbations, the $l = 1$ mode metric perturbation have the contribution from the components of the energy-momentum tensor of the matter field in addition to the term of the Lie derivative of the background metric g_{ab} which is derived as the above vacuum solution.

As the odd-mode solutions in the Part I paper [30], we also have the terms of the Lie derivative of the background metric g_{ab} in the derived solutions in the $l = 0, 1$ even-mode solutions. We have to remind that our definition of gauge-invariant variables is not unique, and we may always add the term of the Lie derivative of the background metric g_{ab} with a gauge-invariant generator as emphasized in Section 2.1. In other words, we may have such terms in derived solutions at any time, and we may say that the appearance of such terms is a natural consequence due to the symmetry in the definition of gauge-invariant variables. Furthermore, since our formulation completely excludes the second kind gauge through Proposal 1, these terms of the Lie derivative should be regarded as the degree of freedom of the first kind gauge, i.e., the coordinate transformation of the physical spacetime $\mathscr{M}_\epsilon$ as emphasized in the Part I paper [30]. This discussion is the consequence of our distinction of the first- and second-kind of gauges and the complete exclusion of the gauge degree of freedom of the second kind as emphasized in the Part I paper [30].

We also note that the existence of the additional mass parameter perturbation M_1 requires the perturbations of $\tilde{F}$ due to the linearized Einstein equations. In this sense, the term described by the Lie derivative of the background spacetime is necessary. The solutions derived in this paper and the Part I paper [30] are local perturbative solutions. If we construct the global solution, we have to use the solutions obtained in this paper and in the Part I paper [30] as local solutions and have to match these local solutions. We expect that the term of the Lie derivative derived here will play important roles in this case.

Besides the term of the Lie derivative of the background metric g_{ab}, we have realized the Birkhoff theorem for $l = 0$ even-mode solutions and the Kerr parameter perturbations in $l = 1$ odd-mode solutions. These solutions are physically reasonable. This also implies that Proposal 1 is physically reasonable nevertheless we used singular mode functions at once to construct gauge-invariant variables and imposed the regular boundary condition on the functions on S^2 when we solve the linearized Einstein equations, while the conventional treatment through the decomposition by the spherical harmonics Y_{lm} corresponds to the imposition of the regular boundary condition from the starting point.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism

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