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Article

3-Heisenberg-Robertson-Schrodinger Uncertainty Principle

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Abstract: Let $\mathcal{X}$ be a 3-product space. Let $A : \mathcal{D}(A) \subseteq \mathcal{X} \to \mathcal{X}$, $B : \mathcal{D}(B) \subseteq \mathcal{X} \to \mathcal{X}$ and $C : \mathcal{D}(C) \subseteq \mathcal{X} \to \mathcal{X}$ be possibly unbounded 3-self-adjoint operators. Then for all $x \in \mathcal{D}(ABC) \cap \mathcal{D}(ACB) \cap \mathcal{D}(BAC) \cap \mathcal{D}(BCA) \cap \mathcal{D}(CAB) \cap \mathcal{D}(CBA)$ with $\langle x, x, x \rangle = 1$, we show that (1) $\quad \Delta_x(3, A)\Delta_x(3, B)\Delta_x(3, C) \geq |\langle (ABC - aBC - bAC - cAB)x, x, x \rangle + 2abc|$, where $\Delta_x(3, A) := \|Ax - \langle Ax, x, x\rangle x\|$, $\quad a := \langle Ax, x, x\rangle$, $\quad b := \langle Bx, x, x\rangle$, $\quad c := \langle Cx, x, x\rangle$. We call Inequality (1) as 3-Heisenberg-Robertson-Schrodinger uncertainty principle. Classical Heisenberg-Robertson-Schrodinger uncertainty principle (by Schrodinger in 1930) considers two operators whereas Inequality (1) considers three operators.

Keywords: Uncertainty Principle; Banach space

MSC: 46C50; 46B99

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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and A be a possibly unbounded self-adjoint linear operator defined on domain $\mathcal{D}(A) \subseteq \mathcal{H}$. For $h \in \mathcal{D}(A)$ with $\|h\| = 1$, define the **uncertainty** (also known as variance) of A at the point h as

$$\Delta_h(A) := \|Ah - \langle Ah, h\rangle h\| = \sqrt{\|Ah\|^2 - \langle Ah, h\rangle^2}.$$

In 1929, Robertson [1] derived the following mathematical form of the uncertainty principle (term due to Condon [2]) of Heisenberg derived in 1927 [3]. Recall that, for two linear operators $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \to \mathcal{H}$, we define $[A, B] := AB - BA$ and $\{A, B\} := AB + BA$.

Theorem 1. *[1,3–7] (**Heisenberg-Robertson Uncertainty Principle**) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \to \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have*

$$\frac{1}{2}\left(\Delta_h(A)^2 + \Delta_h(B)^2\right) \geq \frac{1}{4}(\Delta_h(A) + \Delta_h(B))^2 \geq \Delta_h(A)\Delta_h(B) \geq \frac{1}{2}|\langle [A, B]h, h\rangle|. \tag{1}$$

In 1930, Schrodinger improved Inequality (1) [8].

Theorem 2. *[8] (**Heisenberg-Robertson-Schrodinger Uncertainty Principle**) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \to \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have*

$$\Delta_h(A)\Delta_h(B) \geq |\langle Ah, Bh\rangle - \langle Ah, h\rangle\langle Bh, h\rangle| = \frac{\sqrt{|\langle [A, B]h, h\rangle|^2 + |\langle \{A, B\}h, h\rangle - 2\langle Ah, h\rangle\langle Bh, h\rangle|^2}}{2}.$$

Theorem 2 leads to the following question.

Question 3. *What is the version of Theorem 2 for three operators?*

In this note, we answer Question 3 by deriving an uncertainty principle for three operators acting on classes of Banach spaces, using trilinear forms. We note that there are uncertainty principles derived for three operators on Hilbert spaces, but ours differ from them [9–13].

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2. 3-Heisenberg-Robertson-Schrodinger Uncertainty Principle

The Heisenberg-Robertson-Schrodinger uncertainty principle requires the inner product to handle two operators; for three operators, we need a 3-product defined as follows.

Definition 1. *Let $\mathcal{X}$ be a real Banach space with norm $\|\cdot\|$. A map $\langle\cdot,\cdot,\cdot\rangle : \mathcal{X}\times\mathcal{X}\times\mathcal{X}\to\mathbb{R}$ is said to be a* ***3-product*** *if following conditions hold.*

(i) $\langle x,y,z\rangle = \langle\sigma(x),\sigma(y),\sigma(z)\rangle$ *for all* $x,y,z\in\mathcal{X}$, *for all bijections* $\sigma : \{x,y,z\}\to\{x,y,z\}$.
(ii) $\langle\alpha x,y,z\rangle = \alpha\langle x,y,z\rangle$ *for all* $x,y,z\in\mathcal{X}$, *for all* $\alpha\in\mathbb{R}$.
(iii) $\langle x+w,y,z\rangle = \langle x,y,z\rangle + \langle w,y,z\rangle$ *for all* $x,y,z,w\in\mathcal{X}$.
(iv) $|\langle x,y,z\rangle| \leq \|x\|\|y\|z\|$ *for all* $x,y,z\in\mathcal{X}$.

In this case, we say that $\mathcal{X}$ is a ***3-product space****.*

Following is the standard example we keep in mind.

Example 1. *Let (Ω,μ) be a measure space and $\mathcal{L}^3(\Omega,\mu)$ be the standard real Lebesgue space. Generalized Holder's inequality says that*

$$\int_\Omega |f_1(x)f_2(x)f_3(x)|\,d\mu(x) \leq \left(\int_\Omega |f_1(x)|^3\,d\mu(x)\right)^{\frac{1}{3}}\left(\int_\Omega |f_2(x)|^3\,d\mu(x)\right)^{\frac{1}{3}}\left(\int_\Omega |f_3(x)|^3\,d\mu(x)\right)^{\frac{1}{3}} < \infty, \quad \forall f_1,f_2,f_3\in\mathcal{L}^3(\Omega,\mu).$$

Therefore $\mathcal{L}^3(\Omega,\mu)$ is a 3-product space equipped with 3-product

$$\langle f_1,f_1,f_3\rangle := \int_\Omega f_1(x)f_2(x)f_3(x)\,d\mu(x), \quad \forall f_1,f_2,f_3\in\mathcal{L}^3(\Omega,\mu).$$

We next introduce the notion of self-adjointness for operators on 3-product spaces.

Definition 2. *Let $\mathcal{X}$ be a 3-product space. A possibly unbounded linear operator $A:\mathcal{D}(\mathcal{X})\subseteq\mathcal{X}\to\mathcal{X}$ is said to be* ***3-self-adjoint*** *if*

$$\langle Ax,y,z\rangle = \langle x,Ay,z\rangle = \langle x,y,Az\rangle, \quad \forall x,y,z\in\mathcal{D}(\mathcal{X}).$$

Example 2. *Consider $\mathbb{R}^n$ with the 3-product*

$$\left\langle (x_j)_{j=1}^n,(y_j)_{j=1}^n,(z_j)_{j=1}^n\right\rangle := \sum_{j=1}^n x_jy_jz_j, \quad \forall (x_j)_{j=1}^n,(y_j)_{j=1}^n,(z_j)_{j=1}^n\in\mathbb{R}^n.$$

Let $a_1,\dots,a_n$ be any real numbers. Define

$$A:\mathbb{R}^n\ni(x_j)_{j=1}^n\mapsto(a_jx_j)_{j=1}^n\in\mathbb{R}^n.$$

Then A is 3-self-adjoint.

Example 3. *Consider $\ell^3(\mathbb{N})$ (as a real sequence space) with the 3-product*

$$\langle\{x_n\}_{n=1}^\infty,\{y_n\}_{n=1}^\infty,\{z_n\}_{n=1}^\infty\rangle := \sum_{n=1}^\infty x_ny_nz_n, \quad \forall\{x_n\}_{n=1}^\infty,\{y_n\}_{n=1}^\infty,\{z_n\}_{n=1}^\infty\in\ell^3(\mathbb{N}).$$

Let $\{x_n\}_{n=1}^\infty$ be a bounded real sequence. Define

$$A:\ell^3(\mathbb{N})\ni\{x_n\}_{n=1}^\infty\mapsto\{a_nx_n\}_{n=1}^\infty\in\ell^3(\mathbb{N}).$$

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Then A is 3-self-adjoint.

Example 4. *We continue from Example 1. Let $\phi \in \mathcal{L}^\infty(\Omega, \mu)$. Define*

$$A : \mathcal{L}^3(\Omega, \mu) \ni f \mapsto Af \in \mathcal{L}^3(\Omega, \mu); \quad Af : \Omega \ni \alpha \mapsto (Af)(\alpha) := \phi(\alpha)f(\alpha) \in \mathbb{R}.$$

Then A is 3-self-adjoint.

Let $A : \mathcal{D}(\mathcal{X}) \subseteq \mathcal{X} \to \mathcal{X}$ be a 3-self-adjoint operator. For $x \in \mathcal{D}(A)$ with $\langle x, x, x\rangle = 1$, define the **3-uncertainty** of A at the point x as

$$\Delta_x(3, A) := \|Ax - \langle Ax, x, x\rangle x\|.$$

Theorem 4. *(**3-Heisenberg-Robertson-Schrodinger Uncertainty Principle**) Let $\mathcal{X}$ be a 3-product space. Let $A : \mathcal{D}(A) \subseteq \mathcal{X} \to \mathcal{X}$, $B : \mathcal{D}(B) \subseteq \mathcal{X} \to \mathcal{X}$ and $C : \mathcal{D}(C) \subseteq \mathcal{X} \to \mathcal{X}$ be possibly unbounded 3-self-adjoint operators. Then for all*

$$x \in \mathcal{D}(ABC) \cap \mathcal{D}(ACB) \cap \mathcal{D}(BAC) \cap \mathcal{D}(BCA) \cap \mathcal{D}(CAB) \cap \mathcal{D}(CBA)$$

with $\langle x, x, x\rangle = 1$, we have

$$\begin{aligned}&\frac{1}{27}(\Delta_x(3, A) + \Delta_x(3, B) + \Delta_x(3, C))^3 \geq \Delta_x(3, A)\Delta_x(3, B)\Delta_x(3, C) \geq \\ &|\langle(ABC - \langle Ax, x, x\rangle BC - \langle Bx, x, x\rangle AC - \langle Cx, x, x\rangle AB)x, x, x\rangle + 2\langle Ax, x, x\rangle\langle Bx, x, x\rangle\langle Cx, x, x\rangle|.\end{aligned}$$

Proof. First inequality follows from AM-GM inequality for three positive reals. Given

$$x \in \mathcal{D}(ABC) \cap \mathcal{D}(ACB) \cap \mathcal{D}(BAC) \cap \mathcal{D}(BCA) \cap \mathcal{D}(CAB) \cap \mathcal{D}(CBA)$$

with $\langle x, x, x\rangle = 1$, set

$$a := \langle Ax, x, x\rangle, \quad b := \langle Bx, x, x\rangle, \quad c := \langle Cx, x, x\rangle.$$

Then

$$\begin{aligned}&\Delta_x(3, A)\Delta_x(3, B)\Delta_x(3, C) \geq |\langle Ax - ax, Bx - bx, Cx - cx\rangle| \\ &= |\langle ABCx, x, x\rangle - \langle(aBC + bAC + cAB)x, x, x\rangle + \langle(abC + bcA + caB)x, x, x\rangle - abc| \\ &= |\langle ABCx, x, x\rangle - \langle(aBC + bAC + cAB)x, x, x\rangle + 2abc| \\ &= |\langle(ABC - \langle Ax, x, x\rangle BC - \langle Bx, x, x\rangle AC - \langle Cx, x, x\rangle AB)x, x, x\rangle + 2\langle Ax, x, x\rangle\langle Bx, x, x\rangle\langle Cx, x, x\rangle|.\end{aligned}$$

□

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