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II. Laplace-Like Phase Angles to Facilitate Libration Searches in Multiplanetary N-Body Simulations

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Abstract: We describe a method of determining three-body and four-body Laplace-like phase angles with the potential to librate about a mean value in multiplanet extrasolar systems. Unlike in past searches of N-body results, this method relies on global mean-motion resonances (MMRs) and takes into consideration the location of the most massive planet which defines the 1:1 global MMR in each (sub)system. We compile lists of potentially librating phase angles and prevalent MMRs in 35 real multibody systems, and we discuss their properties in conjunction with recent investigations of librations discovered in sophisticated N-body simulations. We hope that our results will facilitate systematic libration searches in dynamical models of compact systems with three or more orbiting bodies.

Keywords: exoplanet dynamics; orbital resonances; tidal interaction

1. Motivation

Discovery of librating kinematic phase angles in N-body dynamical multiplanet simulations is based on analyses of local mean-motion resonances (MMRs) between adjacent planets, the classical Laplace resonance (LR), and some guesswork of experimental higher-order variations of the LR [1–11]. Kinematic searches in vast N-body output data would have been much more difficult, were it not for these few guiding principles.

Be that as it may, most libration searches in N-body models have not been systematic, and some unwarranted modifications/omissions of potentially librating angles could not thus be avoided. On many occasions, common factors were dropped from three-body phases, causing distortions to libration centers and confusion between comparisons (e.g., Refs. [2,3,10,12–15]); whereas in other cases, the angles were kept intact either by design [8] or because they did not have any common factors to be dropped (e.g., Refs. [5,6,9]).

In the few studies of four-body MMR chains, one of the two phases either was not detected (e.g., Kepler-223 [3]), or it was discovered and was given a geometric rather than a dynamical interpretation (e.g., the four Galilean satellites [16]). To remedy the situation, we have also developed four-body Laplace-like phase angles in a systematic way, and we describe their properties and some typical examples in an Appendix to the paper.

A case in point, of interest in this work, is the remarkable discovery of Goździewski et al. [2] (decidedly confirmed by Jontof-Hutter et al. [7]) that a peculiar combination α_0 of mean longitudes λ_n librate about 45° with an amplitude of ±10° in Kepler-60, viz.

$$\alpha_0 = \lambda_1 - 2\lambda_2 + \lambda_3 \,, \tag{1}$$

where subscripts $n = 1, 2, 3$ describe the orbits of planets b-c-d in order from the star. This result was widely misinterpreted: angle α_0 does not correspond to a triple MMR of any kind or order, thus 45° is not the center of libration in Kepler-60 ([17], hereafter Paper I). We determined that the actual mean libration phase angle is $4\alpha_0$, and it is thus centered at 180° (as listed in Tables 1 and 2 below; see also the librations near ∼180° in the general-purpose models of Siegel and Fabrycky [8], hereafter SF21); a

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familiar center that relates to past studies of the classical Galilean LR [18] in which the Laplace phase angle

$$\varphi_L \equiv \lambda_1 - 3\lambda_2 + 2\lambda_3\,, \tag{2}$$

also librates about 180° with an amplitude of $0.06423° \pm 0.00224°$ [19,20].

We describe in Section 2 a method of determining resonant Laplace-like angles in N-body models of planetary systems that contain three planets; and we extend the method to four-body systems in the Appendix, where we also revisit the global MMR chains of HD 110067 and Kepler-223. Then, we demonstrate our three-body results in Section 3, where we examine 35 observed multibody (sub)systems, including a Galilean chain featuring Callisto and our Earth's local chain with the inferior planets; and we collect our determinations of Laplace-like phases in Tables 1 and 2. In the end, we discuss and summarize our most distinct conclusions in Section 4. We hope that the tabulated results, as well as the many cross-comparisons described in the notes to the tables, will prove helpful to research teams building long-term dynamical simulations of multiplanetary exosystems.

Table 1. Observed Global MMRs, Local Resonant Pairs, and their Laplace-like Phase Angles.

Global MMRs (1)	Local MMR Pairs (2)	Pair Order [a] (3)	Phase [b] Angle φ (4)	Multibody System (5)
1 : 3/2 : 2	3:2 & 4:3	1	$2\varphi_L$ (**)	HD 110067 (most recently found LR multiple; Paper I)
2/3 : 1 : 4/3	3:2 & 4:3	1	$2\varphi_L$ (**)	Kepler-223, TOI-178
1/2 : 3/4 : 1	3:2 & 4:3	1	$2\varphi_L$ (**)	Kepler-90, TOI-1136, TRAPPIST-1
2/3 : 1 : 3/2	3:2 & 3:2	1	$\varphi_L + \alpha_0$ (*) (**)	Kepler-11, K2-138, HD 110067, TRAPPIST-1 (TR1)
2/3 : 1 : 5/3	3:2 & 5:3	M	$2\varphi_L + \alpha_0 + (\lambda_1 - \lambda_2)$ (‡)	HD 23472 (discussed in Section 3.3(d) in relation to Kepler-444)
2/3 : 1 : 2/1	3:2 & 2:1	1	$2\alpha_0$	TOI-1136,[c] Kepler-20 ($\frac{1}{9}$:$\frac{1}{6}$:$\frac{1}{3}$); no librations (∦)
2/3 : 1 : 5/1	3:2 & 5:1	M	$2\varphi_L + \alpha_0 + 5(\lambda_1 - \lambda_2)$	HD 34445
3/4 : 1 : 3/2	4:3 & 3:2	1	$3\alpha_0$ (**)	TOI-1136, HIP 41378,[d] Kepler-223,[e] HD 23472,[e] TRAPPIST-1
3/4 : 1 : 5/3	4:3 & 5:3	M	$2\varphi_L + \alpha_0 + 3(\lambda_1 - \lambda_2)$	Kepler-90
(3/4 : 1 : 5/4)	(4:3 & 5:4)	(1)	$(2\varphi_L + \alpha_0)$ (**)	(Not observed; discussed in Section 3.3(d) and in Table 2)
1/3 : 1 : 4/3	3:1 & 4:3	M	$\varphi_L + 6(\lambda_3 - \lambda_2)$	HD 10180
1/3 : 1 : 3/2	3:1 & 3:2	M	$\varphi_L + 4(\lambda_3 - \lambda_2)$ (***)	Kepler-80

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Table 1. *Cont.*

Global MMRs (1)	Local MMR Pairs (2)	Pair Order [a] (3)	Phase [b] Angle φ (4)	Multibody System (5)
$1/2:1:3/2$	2:1 & 3:2	1	$\varphi_L + (\lambda_3 - \lambda_2)$ (*) (**)	Kepler-32, Kepler-82 ($\frac{1}{3}$:$\frac{2}{3}$:1), Umbriel-Titania-Oberon (Uranus) (U)
$1/2:1:5/3$	2:1 & 5:3	M	$2\varphi_L + (\lambda_3 - \lambda_2)$	HD 40307
$1/2:1:2/1$	2:1 & 2:1	1	φ_L	GJ 876, HR 8799, HR 8832, Kepler-176 (Paper I)
$1/4:1/2:1$	2:1 & 2:1	1	φ_L	HIP 41378, Io-Europa-Ganymede (Galilean LR) (Paper I)
$1/4:5/8:1$	5:2 & 8:5	3	$[2\varphi_L + 4(\lambda_3 - \lambda_2)]_{MVE}$	Mercury-Venus-Earth (MVE) secondary MMR
$1/2:1:7/3$	2:1 & 7:3	M	$(3\varphi_L + \alpha_0)_{EGC}$	Europa-Ganymede-Callisto (EGC) MMR, no librations
$1/5:1/2:1$	5:2 & 2:1	M	$2\varphi_L + 2(\lambda_3 - \lambda_2)$	HD 40307
$2/5:1:3/2$	5:2 & 3:2	M	$2\varphi_L + 5(\lambda_3 - \lambda_2)$	PSR B1257+12 [f] (although $\frac{3}{8}$:1:$\frac{3}{2}$ is a strong alternative)
$3/5:1:8/5$	5:3 & 8:5	M	$8\varphi_L + (\lambda_1 - \lambda_2)$	Kepler-90 [g]
$3/5:1:2/1$	5:3 & 2:1	M	$\varphi_L + 2\alpha_0$	TOI-270 [h] (and Ariel-Umbriel-Titania $\frac{3}{10}$:$\frac{1}{2}$:1 MMR in Uranus) (U)
$3/5:1:9/4$	5:3 & 9:4	M	$9\varphi_L + 6(\lambda_1 - \lambda_2)$	HD 108236
$4/5:1:4/3$	5:4 & 4:3	1	$4\alpha_0$ (**)	Kepler-60
$5/8:1:7/4$	8:5 & 7:4	3	$3\varphi_L + \alpha_0 + (\lambda_1 - \lambda_2)$ (‡)	TOI-700
$5/8:1:8/5$	8:5 & 8:5	3	$[4\varphi_L + (\lambda_1 - \lambda_2)]_{VET}$	Venus-Earth-Toro (VET) secondary MMR [i]
$2/7:1:4/3$	7:2 & 4/3	M	$[2\varphi_L + 16(\lambda_3 - \lambda_2)]_{RTH}$	Rhea-Titan-Hyperion (RTH) (Saturn) (S) (Section 3.3(g) and Paper I)
$4/7:1:9/4$	7:4 & 9/4	M	$13\varphi_L + \alpha_0 + 6(\lambda_1 - \lambda_2)$	Kepler-20

NOTES:
[a] M: Mixed order of the pair [25]; absence of order-2 MMRs is notable.
[b] Laplace phase $\varphi_L = \lambda_1 - 3\lambda_2 + 2\lambda_3$ [18], angle $\alpha_0 = \lambda_1 - 2\lambda_2 + \lambda_3$ [2].
[c] In TOI-1136, the innermost chain $\frac{1}{6}$:$\frac{1}{4}$:$\frac{1}{2}$ is recast to $\frac{2}{3}$:1:2.
[d] In HIP 41378, the outermost chain $\frac{9}{2}$:6:9 is recast to $\frac{3}{4}$:1:$\frac{3}{2}$.
[e] In Kepler-223, the innermost chain $\frac{1}{2}$:$\frac{2}{3}$:1 is recast to $\frac{3}{4}$:1:$\frac{3}{2}$. Similarly for the middle chain $\frac{1}{2}$:$\frac{2}{3}$:1 in HD 23472.
[f] Pulsar planets b-c-d [26–28]. For $(\varpi_c - \varpi_d)$ apsidal librations (about 180° with amplitude ±45°) and the $\leq 60°$ lock on the nodal line of planet b relative to those of planets c and d, see Ref. [29]. The phase angle φ' that involves the difference $(\varpi_c - \varpi_d)$ is not covered in this or any previous work; it has the form $\varphi' = \varphi + 3(\varpi_c - \varpi_d)$.
[g] In Kepler-90, the outermost chain $\frac{3}{8}$:$\frac{5}{8}$:1 is recast to $\frac{3}{5}$:1:$\frac{8}{5}$. But the planets (f-g-h) are too far apart to possibly develop librations.
[h] Known stable non-LR resonance [30–32] of type II [2], analyzed in depth in Paper I; the same phase angle is also found in the first-order $\frac{3}{4}$:1:$\frac{4}{3}$ MMR chain (not yet observed—librating about 180°; SF21).
[i] Only the trailing 8:5 MMR librates (amplitude $\sim 32°$), although Venus leverages the orbital elements of asteroid 1685 Toro [33]. For the leading MMR, we did not adopt the commonly-quoted 8:13 ratio; we argue that 13 is not a small integer and that the order 5 is too high for such an MMR to be effectual. Besides 8:3 (Kepler-90, HD 110067), the only order-5 weak MMRs seen in exosystems are the 9:4 trailing MMRs in HD 108236 and Kepler-20. By the same token, order-4 MMRs either do not exist (e.g., $(p+4)$:p with p odd and $p \geq 5$) or do not librate because the two bodies are too far apart (e.g., 7:3 of Callisto with Ganymede and 5:1 of Iapetus with Titan).
(*) These angles were found to librate about $\varphi = 180°$ [5,25] in numerical models not specific for the subsystems quoted in the last column. This implies that the MMRs are of type I or 'double resonances' [2].
(**) These angles were found to librate (mostly about $\varphi = 180°$) in type-I first-order MMR models (SF21).
(***) Many MMR triples were found to be librating in Kepler-80 [34].
(TR1) In TRAPPIST-1, the middle chain $\frac{1}{3}$:$\frac{1}{2}$:$\frac{3}{4}$ is recast to $\frac{2}{3}$:1:$\frac{3}{2}$. Another two inner chains, also found to be librating [13–15], are listed in Table 2 below.
(‡) An identity relation: $2\varphi_L + \alpha_0 + (\lambda_1 - \lambda_2) = \varphi_L + 3\alpha_0$ (see also Section 3.3(e) for more identities).
(♯) No librations were seen in 30 type-I first-order (3:2 & 2:1) MMR models (SF21).
(U) Refs. [35,36].
(S) Ref. [37].

2. Method

For local resonant pairs of any order, $P_{n+1}{:}P_n = (p+q){:}p$, where the orbital periods $P_{n+1} > P_n$, index $p = r_1, r_2$ and index $q = s_1, s_2$, respectively, two of the commensurability relations between the mean longitudes λ_n (e.g., Ref. [21]) are

$$\begin{array}{rcl} \theta_1 & = & (r_1+s_1)\lambda_2 - r_1\lambda_1 - s_1\varpi_2 \\ \theta_2 & = & (r_2+s_2)\lambda_3 - r_2\lambda_2 - s_2\varpi_2 \end{array}, \quad (3)$$

where ϖ_2 is the longitude of the pericenter of the middle planet orbiting at radial location $n = 2$.

Angle ϖ_2 can be introduced in both equations because the $n = 2$ orbit is the only one that participates in both local MMRs. This does not imply that the local angles θ_1 and θ_2 describe genuine resonances even when they may be locked individually in librations. This point was first discovered in experiments [2,22], and then it was proven formally by Lari and Saillenfest [23] whose Hamiltonian formalism [24] showed that only LRs are true MMRs in three-body systems; because, unlike the individual two-body MMRs, their phase-space trajectories are bounded (thus isolated and protected) by separatrices. It is for these reasons that we have adopted the distinction of multibody MMRs into double/local and pure/global [2] and we do not require librating phases in the definition of MMR chains of three or more bodies (see also Paper I).

By eliminating ϖ_2 between equations (3), we obtain the global phase angle φ of the triple, viz.

$$\varphi = A\lambda_1 - (A+B)\lambda_2 + B\lambda_3\,, \quad (4)$$

where $A = r_1 s_2$ and $B = r_2 s_1 + s_1 s_2$ (for $s_1 \neq s_2$), or $A = r_1$ and $B = r_2 + s$ (for $s_1 = s_2 \equiv s$). In the special case of a first-order resonant pair [25], then $s = 1$ leading to $A = r_1$ and $B = r_2 + 1$ (see Paper I).

Equation (4) also represents the zeroth-order form of the 'generalized resonance equation' (e.g., equation (15) in Ref. [5], with the additional order of the 'three-planet MMR' set to zero). It constitutes the main Laplace-like MMR relation that we used to compile Table 1. We discuss the properties of the tabulated phase angles in Section 3 and in the detailed remarks appended to Table 2 below.

3. Applications

3.1. Table 1

Table 1 shows potentially librating resonances found in 35 (extra)solar subsystems. The list includes established LRs (phases φ_L); the recently discovered Laplace-like MMRs (phases $2\varphi_L$ and the additional phases marked by asterisks); and the Earth's secondary MMRs (not involving Jupiter) in our inner solar system (denoted by MVE and VET).

The observed MMRs are listed in column 1; they are used to obtain the corresponding local MMR pairs listed in column 2. The overall orders of the MMR pairs according to the classification scheme of Celletti et al. [25] are listed next in column 3, followed by the derived phase angles φ and the names of the (sub)systems in which these MMRs were found (columns 4 and 5, respectively).

The notes to Table 1 include many cross-comparisons with a number of numerical N-body models that have tracked the corresponding MMRs for librations [2,5,8,25].

3.2. Table 2

Table 2 is distilled from Table 1; it highlights librating and circulating Laplace-like angles of first-order global MMRs, as these were determined in the extensive numerical study of SF21. A number of remarks appended to Table 2 include detailed comparisons between these cases and an analysis of particular angular terms that are likely to signal libration or circulation of the overall phase angles. The overall phase angles are written without simplifying common factors, so when two distinct MMRs are

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subject to the same Laplace-like phase, our identification is precise (see items (d) and (f) in Section 3.3 below).

The commonly-appearing angular phase terms are summarized in Section 3.3(e) below. Furthermore, overlapping LRs and Laplace-like MMRs in first-order four-body (sub)systems are explored in the Appendix.

3.3. General Properties

Considering the phase angles and their host systems listed in Tables 1 and 2, we deduce the following general properties and characteristics:

(a) As noted in Section 1, angle $\alpha_0 = 45°$ is not the center of libration in any real system, but some even multiples may be librating (based on the results of SF21); as in TOI-270 ($\varphi = \varphi_L + 2\alpha_0$; Paper I) and Kepler-60 ($\varphi = 4\alpha_0$; Paper I and Ref. [2]); but not in TOI-1136 ($\varphi = 2\alpha_0$; 30 circulating models were presented in SF21). Furthermore, TRAPPIST-1 can be used to determine whether some odd multiples of α_0, such as $\varphi = 3\alpha_0$, may be resonating too (the same configuration appears also in TOI-1136, HIP 41378, and Kepler-223). For this angle, as well as for $\varphi = \varphi_L + \alpha_0$ (Kepler-11, K2-138, HD 110067, TRAPPIST-1 d-e-f MMR recast), the results obtained by SF21 (rows 7, 9 in Table 2) are affirmative.

(b) The $1{:}\frac{3}{2}{:}2$ global MMR of HD 110067 [38] was recently shown to be a Laplace-like resonance with a phase angle of $\varphi = 2\varphi_L$ (Paper I). The factor of 2 might point to librations about 0°, although this is clearly not confirmed by the models of SF21 showing libration centers close to 180° (162° and 198°). Thus, new N-body models tailored to the particular configuration of planet masses in HD 110067 are needed to explore this issue further. For the time being, it is interesting to note that this Laplace angle φ_L (without the factor of 2), albeit embedded in a Laplace-like MMR, would seem to librate near ∼90° according to the SF21 models of the (3:2 & 4:3) MMR pair (row 3 in Table 2); this is in contrast to the classical cases with libration centers of $\varphi_L = 0°$ or 180° [19,40].

(c) Several exosystems with 6 or more planets (Kepler-90, HD 110067, TOI-1136, HIP 41378, and TRAPPIST-1) show additional three-body and four-body MMR chains that should be investigated again for locked librations of type II ('pure' three-body MMRs) according to the classification scheme established by Goździewski et al. [2] (also used in the paired MMR study of Charalambous et al. [5]). The two overlapping MMR triples found in HD 110067 [38] are discussed in the notes to Table 2 (item 4) and in more depth in Appendix A.

(d) We included in Table 1 the global MMR $\frac{3}{4}{:}1{:}\frac{5}{4}$ that was not found in any real system. On the other hand, SF21 found that the associated local MMR pair (4:3 & 5:4) has a libration center of 180° (row 11 in Table 2). We think that this MMR chain is suppressed in real systems in favor of a nearly superimposed chain: although unusual, the *geometric MMR chain* $\left(\frac{4}{5}\right)^2{:}\frac{4}{5}{:}1$ likely appears in Kepler-444 [41,42]. The corresponding local MMR pair (5:4 & 5:4) also librates about 180° in the 30 N-body models of SF21. The librating phase is $\varphi = 2\varphi_L + \alpha_0 + (\lambda_1 - \lambda_2)$, the same as that of the mixed-order MMR pair (3:2 & 5:3) in HD 23472 (Table 1). Thus, we expect a librating phase to be found in HD 23472 as well (planets f-b-c [43]).

We also expect that relatively few such degeneracies will be found in phases of three-body systems because quite a few prerequisites are needed to establish an identical phase angle. For the case at hand (5:4 & 5:4 and 3:2 & 5:3), one has to adopt the conditions $\{r_1 = r_2, s_1 = s_2 = s = 1\}$ and $\{R_2 = R_1 + 1, S_1 = 1\}$, respectively, to derive $r_1 = 4$, $R_1 = 2$, and $S_2 = 2$ from an underdetermined system of 2 Diophantine equations with 8 unknowns that is finally reduced to the 2×3 Diophantine system $R_1 + S_2 = r_1 = R_1 S_2$.

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Table 2. Librating and Circulating Phase Angles in First-order MMRs Based on the Findings of SF21.

Row Index	Global MMRs (1)	Local MMR Pairs (2)	Phase Angle φ (3)	Equivalent Form of φ (4)	Libration Centers (°) (5)	Multibody System (6)
1	$1/2:1:2/1$	2:1 & 2:1	φ_L		0	GJ 876, HR 8799, HR 8832, Kepler-176 (Paper I)
2	$1/4:1/2:1$	2:1 & 2:1	φ_L		180	HIP 41378, Io-Europa-Ganymede
3	$1:3/2:2/1$	3:2 & 4:3	$2\varphi_L$		180 ± 18	HD 110067
4	$2/3:1:4/3$	3:2 & 4:3	$2\varphi_L$		180 ± 18	Kepler-223, TOI-178
5	$1/2:3/4:1$	3:2 & 4:3	$2\varphi_L$		180 ± 18	Kepler-90, TOI-1136, TRAPPIST-1
6	$2/3:1:2/1$	3:2 & 2:1	$\varphi_L+\Lambda_{12}$	$=2\alpha_0$	None	TOI-1136, Kepler-20 ($\frac{1}{9}$:$\frac{1}{6}$:$\frac{1}{3}$); no librations
7	$3/4:1:3/2$	4:3 & 3:2	$\varphi_L+\Lambda_{12}+\alpha_0$	$=3\alpha_0$	180 ± 31, 180	TOI-1136, HIP 41378, Kepler-223, HD 23472, TRAPPIST-1
8	$4/5:1:4/3$	5:4 & 4:3	$\varphi_L+\Lambda_{12}+2\alpha_0$	$=4\alpha_0$	$180\pm\left\{\begin{smallmatrix}14\\5\end{smallmatrix}\right.$, 180 (*)	Kepler-60
9	$2/3:1:3/2$	3:2 & 3:2	$2\varphi_L-\Lambda_{32}$	$=\varphi_L+\alpha_0$	180 (**)	Kepler-11, K2-138, HD 110067, TRAPPIST-1
10	$1/2:1:3/2$	2:1 & 3:2	$\varphi_L+\Lambda_{32}$	$=2\varphi_L-\alpha_0$	180 (**)	Kepler-32, Kepler-82, Umbriel-Titania-Oberon
11	$3/4:1:5/4$	4:3 & 5:4	$3\varphi_L-\Lambda_{32}$	$=2\varphi_L+\alpha_0$	180	Not yet observed
12	$3/4:1:2/1$	4:3 & 2:1	$\varphi_L+2\Lambda_{12}$		None	Not yet observed, no librations
13	$4/5:1:2/1$	5:4 & 2:1	$\varphi_L+3\Lambda_{12}$		None	Not yet observed, no librations
14	$1:5/3:5/2$	5:3 & 3:2	$3\varphi_L$		150 (***)	TRAPPIST-1 ($\frac{1}{5}$:$\frac{1}{3}$:$\frac{1}{2}$), Kepler-11 ($\frac{2}{5}$:$\frac{2}{3}$:1), HD 40307
15	$5/8:1:5/3$	8:5 & 5:3	$5(2\varphi_L-\Lambda_{32})$	$=5(\varphi_L+\alpha_0)$	80 (***)	TRAPPIST-1 ($\frac{1}{8}$:$\frac{1}{5}$:$\frac{1}{3}$)

NOTES:
(*) In Refs. [2,7].
(**) Also in Refs. [5,25].
(***) In Refs. [13,15], although the angles φ (with centers at 50° and 160°) were reduced by factors of 3 and 5, respectively.
THREE-BODY DEFINITIONS AND RELATIONS:
$\alpha_0\equiv\lambda_1-2\lambda_2+\lambda_3$, $\varphi_L\equiv\lambda_1-3\lambda_2+2\lambda_3$, $\Lambda_{12}\equiv\lambda_1-\lambda_2$, $\Lambda_{32}\equiv\lambda_3-\lambda_2$ and $\alpha_0=\Lambda_{12}+\Lambda_{32}$, $\varphi_L=\Lambda_{12}+2\Lambda_{32}=\alpha_0+\Lambda_{32}$.
REMARKS AND COMPARISONS:
(1) Inspecting the Laplace-like angles in column 3 (rows 6-13): (1a) Three angles depend on Λ_{12} but not on α_0 (rows 6, 12, 13); all three circulate. (1b) Two angles depend on both Λ_{12} and α_0 (rows 7, 8); both librate. (1c) Three angles depend on Λ_{32} but not on α_0 (rows 9-11); all three librate. (1d) In general, in the absence of α_0, angle Λ_{32} signals libration of φ (rows 9-11), whereas Λ_{12} signals circulation of φ (rows 6, 12, 13); but α_0 terms added to Λ_{12} induce librations (rows 7, 8).
(2) Comparing the forms of φ in columns 3 and 4 (rows 6-11): (2a) If a φ_L term is present in both forms, then angle α_0 is as good a libration tracer as Λ_{32} (rows 9-11). (2b) If an α_0 term is present in both forms, then φ librates, otherwise $\varphi=2\alpha_0=\varphi_L+\Lambda_{12}$ circulates (rows 6-8); equivalently, when α_0 terms are added to circulating $\varphi_L+\Lambda_{12}$ (or more α_0 terms are added to $2\alpha_0$), they induce libration (rows 7, 8).
(3) Turning to the local MMR pairs in column 2: (3a) The classical LR pair (2:1 & 2:1) shows librations of $\varphi\equiv\varphi_L$ (rows 1, 2). (3b) The pairs with trailing MMR 2:1 and leading MMR other than 2:1 (rows 6, 12, 13) all show circulations; evidently, the trailing 2:1 MMR is responsible for this outcome. (3c) No such circulating primary MMRs have been observed in (extra)solar subsystems (rows 12, 13), and the ($\frac{1}{6}$:$\frac{1}{4}$:$\frac{1}{2}$) chain of TOI-1136 in row 6 is only a secondary triple that does not include the most massive planet (see note *c* below Table 1); therefore, no such primary MMRs (with a trailing 2:1 MMR) have been observed in (extra)solar subsystems at all. This is compounding evidence that the primary global 2:1 MMR must be vacant, except in LRs and Laplace-like multiples, such as those listed in rows 1 and 3.

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(4) Focusing on the two MMR chains observed in HD 110067 (rows 3, 9): Planets c-d-e-f show an extended sequence of local first-order MMRs of the form (3:2 & 3:2 & 4:3) in which planet d is the most massive body ($M_d = 8.52 M_\oplus$ [38]). Both triples should librate according to the models of SF21. For this extended sequence of planets (indexed by 1-4), we find two corresponding phase angles, $\varphi_1 = 2\lambda_1 - 7\lambda_2 + 9\lambda_3 - 4\lambda_4$ and $\varphi_2 = 2\lambda_1 - 3\lambda_2 - 3\lambda_3 + 4\lambda_4$ (see Appendix A), that ought to be checked for librations individually. Their sum and difference also point to librations: $\varphi_1 + \varphi_2 = 2(\varphi_L + \alpha_0)_{123}$ for planets 1-3, and $\varphi_2 - \varphi_1 = 4(\varphi_L)_{234}$ for planets 2-4. These angles are $2\times$ the librating angles listed in rows 9 and 3, respectively. The particular $2\times$ scaling is universal in four-body MMR chains; we derive it also for the planets b-c-d-e in Kepler-223 (with global MMR $\frac{1}{2}$:$\frac{2}{3}$:1:$\frac{4}{3}$; rows 7 and 4, respectively) in Appendix B.3 from the observations and modeling of Mills et al. [3].

(5) Focusing on the three secondary MMR chains (not involving planet g) of mostly mixed order observed in TRAPPIST-1 (rows 9, 14, 15): All secondary MMRs librate, just as the primary MMRs (e-f-g, f-g-h) listed in rows 5 and 7, respectively. This is because all consecutive local pairs are individually locked in resonance (including the innermost 8:5 & 5:3 pair) [12,13,39].

(e) A widespread characteristic of the angles in Table 1 is the succinct appearance of the differences of adjacent mean longitudes $\Lambda_{12} \equiv +(\lambda_1 - \lambda_2)$ and $\Lambda_{32} \equiv +(\lambda_3 - \lambda_2)$, so that we can write

$$\begin{array}{rcl} \Lambda_{12} & = & -\varphi_L + 2\alpha_0 \\ \Lambda_{32} & = & \varphi_L - \alpha_0 \end{array}, \qquad (5)$$

and

$$\begin{array}{rcl} \Lambda_{12} + \Lambda_{32} & = & \alpha_0 \\ \Lambda_{12} + 2\Lambda_{32} & = & \varphi_L \end{array}. \qquad (6)$$

The appearance of only Λ_{12} or Λ_{32} along with φ_L in a phase angle φ has some notable consequences going forward:

(i) The corresponding local MMR (1-2 or 3-2, leading or trailing) may determine whether the overall phase will librate or not. In cases of librations (e.g., Kepler-32, Kepler-82 in row 10 of Table 2), the other local MMR determines only the type (I or II) of the global MMR [2].

(ii) Based on the experimental results of SF21, the first-order pairs in rows 6, 9-13 of Table 2 seem to indicate that Λ_{12} signals circulation, whereas Λ_{32} signals libration of the overall phase φ. But an α_0 term joining in with Λ_{12} signals instead libration of φ (rows 7, 8 in Table 2 and also item (d) above).

(iii) This scenario also signifies that the same global MMR may or may not be librating in individual systems, as the incorporated Λ-difference may or may not be locked in local resonance. The location of the most massive planet (i.e., the 1:1 orbit) in the MMR chain, as well as perturbations from neighboring planets outside of the resonance, may play an important role in determining the outcome of early dynamical evolution.

(iv) The angles φ of five mostly *mixed-order* MMRs at the bottom half of Table 1 (HD 40307, PSR B1257+12, Kepler-90, VET, RTH) combine even multiples of φ_L with either Λ_{32} or Λ_{12}. If the φ_L terms could possibly be set aside for a moment, then the librations of angles φ would be determined solely by the single local Λ-pair present in the phase angle. If the presumption in item (ii) holds for mixed/high-order MMRs as well, then the (5:3 & 8:5) MMR in Kepler-90 f-g-h and the (8:5 & 8:5) VET MMR that depend on Λ_{12} should show circulating phases. Some local MMRs of mixed order (but not quite those listed in Table 1) have been studied numerically in the recent past (see, e.g., Refs. [5,6,25,34,40]).

(v) An analysis of the local first-order MMRs studied by SF21 is summarized in the notes below Table 2. The questions raised therein by cross-comparisons between cases will need to be addressed by numerical N-body models tailored to the particular exosystems listed in column 6 of Table 2.

(f) In the Galilean satellite subsystem, the EGC resonant angle does not librate (Table 1). This result was demonstrated to us by a reviewer, and it is also implicit in the synthetic orbital elements obtained by Lainey et al. [22] using L1 ephemerides. Thus, Callisto is not locked on to the main Galilean LR. The corresponding circulating phase angle is $\varphi_{EGC} = 4\lambda_2 - 11\lambda_3 + 7\lambda_4$, where subscripts $2, 3, 4$ correspond to E, G, C, respectively. This angle is equivalent to that of the global first-order MMR

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$\frac{4}{5}$:1:$\frac{7}{6}$ (built from the local pair of 5:4 & 7:6 MMRs[1]). This global resonance has not been found in any of the 50 systems that we have studied. Specifically, the leading 4:5 MMR may be an important component of the $4\alpha_0$ MMR in Kepler-60 (see row 8 in Table 2 and Paper I), but a trailing 7:6 MMR has never been seen in any real system (see also item (g) below and Appendix B.2).

Despite being out of phase, the oscillations of the orbit of Callisto do however have implications for spin-orbit coupling in the Galilean moons and the rotational librations of their spin axes which, in turn, are needed in studies of the oceanic content beneath their icy surfaces [46–48].

(g) Trailing MMRs, such as 7:6, 6:5, and 5:4, do not occur in the observed (exo)systems (items (d) and (f) above); they are described by those period ratios $(p+1){:}p$ with $p \geq 4$. On the other hand, MMRs with $p \leq 3$ are quite common in Table 1. In this category, a well-known example is the 4:3 RTH MMR involving the tiny pumice-moon Hyperion [49] orbiting right next to giant moon Titan (occupying the 1:1 MMR in the $\frac{2}{7}$:1:$\frac{4}{3}$ global chain), the second most massive moon in our solar system; their semimajor axes scale as $a_4/a_3 = 1.21$ [37]. The nearest scale among the $p \geq 4$ (unrealized) semimajor axes is, by Kepler's third law, $a_5/a_4 = (5/4)^{2/3} = 1.16$ for $p = 4$. Thus, in general, trailing $p \geq 4$ orbits do not occur as they are not permitted to maintain stability so close to the most massive (1:1) body.

In the case of Saturn's Titan (where masses and semimajor axes are available), we confirm the unrealized ($p \geq 4$) orbits and the tolerated ($p < 3$) orbits, but the approximation fails marginally for the intermediate case of Hyperion's 4:3 MMR ($p = 3$): Titan's Landau wavelength is $\lambda_T = 0.26a_3$, where a_3 is the semimajor axis of the orbit of Titan (Paper I and Ref. [50]); then, the 'screening' (or Debye) belt $(D_T)_\pm$ of tidal dominance of Titan extends to radii

$$(D_T)_\pm = a_3 \pm \lambda_T = (1 \pm 0.26)\,a_3,$$

i.e., a little past Hyperion's orbit ($a_4 = 1.21a_3$; a relative mismatch of 4%); although clearly shorter than the (vacant) orbits of the 3:2 and 2:1 trailing MMRs (with semimajor axes of $a_2 = 1.31a_3$ and $a_1 = 1.59a_3$, respectively). Thus, a moon orbiting at the 3:2 global MMR of Titan would not have been a surprise (and the vacant 2:1 MMR is not a surprise), unlike the startling 4:3 orbit of tiny Hyperion.

4. Conclusions

With help from the listings and the notes compiled in the tables, we summarize briefly our most distinct conclusions. Row numbers refer to the rows of Table 2:

1. We found no global MMRs of the LR form 1:2:4 with the most massive planet occupying radial position $n = 1$, where $n = 1, 2, 3$ (rows 1, 2, and Paper I). It is conceivable that such LRs may not at all form in multibody (sub)systems.

2. Laplace-like MMRs whose phase angles φ are integer multiples of the classical Laplace phase φ_L (equation (2)) are apparently locked in librations centered in the neighborhood of $\varphi = 180°$, never near 0° (rows 3-5, 14). The same property seems to hold for linear combinations of φ_L and the angle α_0 defined in equation (1) (rows 9-11, and also row 15 but with its libration center located at $\varphi = 80°$).

3. With the most massive planet defining the location of the 1:1 MMR, the 2:1 global MMR appears to always be empty, except in LRs $\frac{1}{2}$:1:2 (row 1) and in Laplace-like MMRs $1\frac{3}{2}$:2 (row 3). The inner MMR chain listed for TOI-1136 in row 6 is not global; its global form is $\frac{1}{6}$:$\frac{1}{4}$:$\frac{1}{2}$ (note *c* in Table 1). The unobserved global 1:2 MMR chains listed in rows 12, 13 also prop up this argument.

4. Unless specialized initial conditions are used (SF21), the overall phase angle $\varphi = 2\alpha_0 = \varphi_L + \Lambda_{12}$ of the paired MMRs (3:2 & 2:1) does not librate (row 6), but some higher multiples of α_0 do (rows 7, 8). Of the α_0 multiples, phase $2\alpha_0$ is the only one that can be written in terms of φ_L and only

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Λ_{12} (but note also the phases of Kepler-90, HD 108236, and VET at the bottom of Table 1 having the same property).

5. In general, the linear combinations of $+\varphi_L$ and $\pm\Lambda_{32}$ likely signal librations (rows 9-11, 15), whereas the opposite is likely for the linear combinations of $+\varphi_L$ and $+\Lambda_{12}$ (rows 6, 12-13). The former case is equivalent to combinations of $+\varphi_L$ and $\pm\alpha_0$ because of the relation $\Lambda_{32} = \varphi_L - \alpha_0$ (equation (5)). The latter case is equivalent to combinations of $-\varphi_L$ and $+\alpha_0$ because of the relation $\Lambda_{12} = 2\alpha_0 - \varphi_L$ (equation (5) as well). (We note that in rows 12, 13 of Table 2 column 4 was not filled for clarity.)

6. We found no first-order local MMRs of the form $(p+1){:}p$ with $p \geq 4$, just outside the principal 1:1 orbits (Section 3.3(g)). On the other hand, outer MMRs with $p \leq 3$ are quite common (we count ~ 20 of them in Table 1 or in Table 2). It seems plausible that the most massive body in each (sub)system does not allow other bodies of any mass to settle into such nearby ($p \geq 4$) orbits.

7. *Geometric MMR sequences*.—Classical LRs form geometric sequences with a common ratio of $r = 2$. We have found more geometric MMRs with smaller common ratios, as listed below. As in Paper I, the most massive planet in each system is denoted by bb and radial positions (n) are shown in parentheses for bb planets, as well as for the most massive planet in each geometric sequence:

$r = \frac{3}{2}$:

- $\left(\frac{2}{3}\right)^2{:}\frac{2}{3}{:}1$ HD 23472 e-f-b(4), $bb = $ b(4) [43];
- $\left(\frac{2}{3}\right)^2{:}\frac{2}{3}{:}1{:}\frac{3}{2}$ HD 110067 b-c-d(3)-e, $bb = $ d(3) [38];
- $\frac{2}{3}{:}1{:}\frac{3}{2}$ K2-138 d-e(4)-f, $bb = $ e(4) [9];
- $\frac{1}{3}{:}\frac{1}{2}{:}\frac{3}{4}$ TRAPPIST-1 d-e-f(5), $bb = $ g(6) [12];

$r = \frac{4}{3}$:

- $\frac{3}{2}{:}2{:}\frac{8}{3}$ HD 110067 e-f-g(6), $bb = $ d(3) [38];

$r = \frac{5}{4}$:

- $\left(\frac{4}{5}\right)^2{:}\frac{4}{5}{:}1$ Kepler-444 d-e-f(5), $bb = $ f(5) [41].

The overall chain of HD 110067 is of vital importance (see Appendix A and Figure A1 below). Not only does it show two adjoined geometric sequences coalescing in the middle of its exact Laplace-like resonance ($1{:}\frac{3}{2}{:}2$), but it also shows very small deviations of the orbits from the corresponding precise MMRs [38].

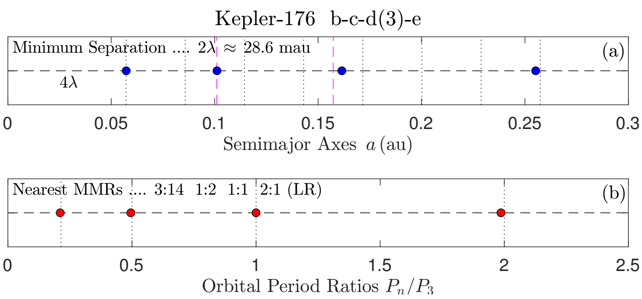

Figure 1. Kepler-176: (a) Distribution of semimajor axes of planets (filled blue circles) and nearest radial tidal potential minima (vertical dotted lines separated by a distance of 2λ, where $\lambda \approx 14.3$ mau is the wavelength of the tidal field). Equations (10), (12) of Paper I predict that $\lambda \simeq 17.9$ mau for planet d and $\lambda \simeq 8$ mau for planet b. The vertical dashed lines split the 2λ intervals into equal segments. (b) Distribution of orbital period ratios of planets (filled red circles) and nearest global MMRs (vertical dotted lines). The orbits deviate from the nearby exact MMRs by $\leq 1.6\%$. Planet d ($n = 3$) is the most massive within the LR ($M_d = 4.64 M_\oplus$ [51]); it is taken at the 1:1 MMR for illustration purposes.

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8. *Overlapping LRs.*—Connected overlapping LRs do not exist in four-body (sub)systems (Appendix B.2). The four planets in overlapping LRs (2:1 & 2:1 & 2:1) do not seem to face an issue of proximity, so their absence must be due to another dynamical cause. The cases of Callisto (7:3) and HR 8799 b (9:4) that were not allowed to join the LRs of their respective (2:1 & 2:1) inner bodies are discussed in Appendix B.2. But a far more sensational example is Kepler-176 [51] in which the innermost ($n = 1$) and most massive planet b ($M_b = 9.2M_\oplus$) in the 1:1 orbit did not join the LR $\frac{14}{3}\left(\frac{1}{2}:1:2\right)$ of planets c-d-e. (This however may be due to the observation made in item 1 above.) Thus, the global 2:1 MMR remains vacant in Kepler-176 as well (the adjacent MMR to the 1:1 ($n = 1$) orbit turns out to be the familiar 7:3 MMR). The planetary system of Kepler-176 is shown in Figure 1, where we reset the 1:1 MMR to the second most massive planet d to highlight this unique LR among the 73 (sub)systems that we have studied so far.

9. Finally, we summarize the types of high-order global MMRs that appear in actual multibody systems (Table 1):

(a) We call principal resonant orbits those in the MMR set $\left\{\frac{1}{k}:1 \cup 1:1 \cup k:1\right\}$ with $k \geq 2$. The principal MMRs are available for orbiting bodies to settle in safely, although the 2:1 MMR appears to be vacant (except in LRs and Laplace-like chains). High-order MMRs of this type provide shelter to orbiting bodies, especially to small distant planets, dwarf planets, and minor moons. For instance, Mercury and Pluto occupy the 1:50 and 21:1 MMRs of Jupiter, respectively; and Pan and Kiviuq occupy the 1:28 and 28:1 MMRs of Titan, respectively, in the Saturnian system.
(b) Besides principal MMRs, very few rational outer MMRs $(p + q):p$ and their reciprocal inner MMRs are common in multibody (sub)systems, and even fewer MMRs with $p, q > 1$ appear scarcely. Our survey of 73 exosystems and solar subsystems is mapped out in Table 3. The scarce MMRs ($p \geq 4$, $q \geq 2$) are enclosed in parentheses. An intriguing feature in this chart is that the MMRs commonly appear in reciprocal pairs. The three exceptional cases (4:5, 2:7, 8:3) are also interesting in their own right.
(c) Extending Table 3 out to $p = 7$ and $q = 7$, we find only the 9:2 MMR in exosystems. Surprisingly, this MMR is common in systems that contain complete LRs (HIP 41378 d, HR 8799 b) or uncompleted (would-be) LRs (Kepler-48 d, Kepler-332 d). In the former case, two overlapping LRs are avoided (see also item 8 above); in the latter case, classical LRs are not assembled, and the global 2:1 MMR (that would have formed the LR) remains vacant.

Table 3. Common global outer MMRs of the form $(p + q):p$ and their reciprocal inner MMRs. Six scarcely-appearing MMRs with $p \geq 4$ and orders $q \geq 2$ are enclosed in parentheses.

		q				
		1	2	3	4	5
p	1	1:2, 2:1	1:3, 3:1	1:4, 4:1	1:5, 5:1	1:6, 6:1
	2	2:3, 3:2	—	2:5, 5:2	—	2:7
	3	3:4, 4:3	3:5, 5:3	—	3:7, 7:3	8:3
	4	4:5	—	(4:7, 7:4)	—	(4:9, 9:4)
	5	—	(5:7)	(5:8)	—	—

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Abbreviations

The following abbreviations are used in this manuscript:

EGC	Europa-Ganymede-Callisto
LR	Laplace Resonance
MMR	Mean-Motion Resonance
MVE	Mercury-Venus-Earth
RTH	Rhea-Titan-Hyperion
SF21	Siegel and Fabrycky (2021), Ref. [8]
VET	Venus-Earth-Toro

Appendix A. Four-Planet Laplace-Like Phase Angles in HD 110067

We derive the phase angles relating the mean longitudes of four adjacent orbiting bodies in the global MMR chain of HD 110067 ($\frac{2}{3}$:1:$\frac{3}{2}$:2; see remark (4) in Table 2 and Figure A1). We use subscripts $n = 1$-4 for planets c-d-e-f, respectively, in increasing order from the star (not counting in the process planet b or g).

For planets c-d-e, the local MMR pair is 3:2 and 3:2, thus $r_1 = r_2 = 2$ and $s_1 = s_2 = 1$ in equation (3) give

$$\begin{aligned} \theta_1 &= 3\lambda_2 - 2\lambda_1 - \varpi_2 \\ \theta_2 &= 3\lambda_3 - 2\lambda_2 - \varpi_2 \end{aligned} , \tag{A1}$$

where ϖ_2 is the longitude of the pericenter of planet d orbiting at radial location $n = 2$.

For planets d-e-f, the local MMR pair is 3:2 and 4:3, thus $r_1 = 2$, $r_2 = 3$, $s_1 = s_2 = 1$, and $\varpi_2 \to \varpi_3$ in equation (3) give

$$\begin{aligned} \theta_3 &= 3\lambda_3 - 2\lambda_2 - \varpi_3 \\ \theta_4 &= 4\lambda_4 - 3\lambda_3 - \varpi_3 \end{aligned} , \tag{A2}$$

where ϖ_3 is the longitude of the pericenter of planet e orbiting at radial location $n = 3$.

One phase angle, φ_1, is obtained from the linear combination $(\theta_2 + \theta_3) - (\theta_1 + \theta_4)$ in which ϖ_2 and ϖ_3 are eliminated, viz.

$$\varphi_1 = 2\lambda_1 - 7\lambda_2 + 9\lambda_3 - 4\lambda_4 \,. \tag{A3}$$

Another phase angle, φ_2, is obtained from the linear combination $(\theta_2 + \theta_4) - (\theta_1 + \theta_3)$ in which ϖ_2 and ϖ_3 are again eliminated, viz.

$$\varphi_2 = 2\lambda_1 - 3\lambda_2 - 3\lambda_3 + 4\lambda_4 \,. \tag{A4}$$

A striking property is detected in these four-planet angles: Their sum and difference describe overlapping triple MMR chains. In particular,

$$\begin{aligned} \varphi_2 + \varphi_1 &= 2(2\lambda_1 - 5\lambda_2 + 3\lambda_3) \\ &= 2(\varphi_{\rm L} + \alpha_0)_{123} \end{aligned} , \tag{A5}$$

and

$$\begin{aligned} \varphi_2 - \varphi_1 &= 4(\lambda_2 - 3\lambda_3 + 2\lambda_4) \\ &= 4(\varphi_{\rm L})_{234} \end{aligned} . \tag{A6}$$

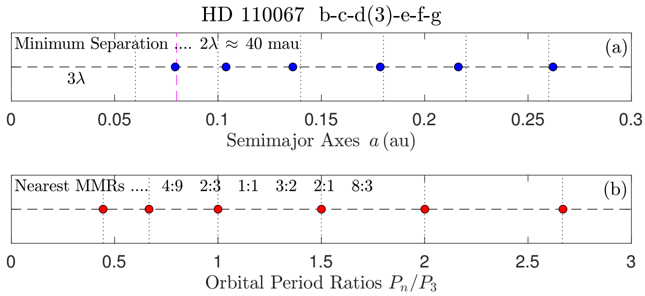

Figure A1. HD 110067: (a) Distribution of semimajor axes of planets (filled blue circles) and nearest radial tidal potential minima (vertical dotted lines separated by a distance of 2λ, where $\lambda \approx 20$ mau is the wavelength of the tidal field). Equations (10) and (12) of Paper I predict that $\lambda \simeq 19$ mau for planets d and e. The vertical dashed line splits the 2λ interval into equal segments. (b) Distribution of orbital period ratios of planets (filled red circles) and nearest global MMRs (vertical dotted lines). The orbits deviate from the nearby exact MMRs by $\leq 0.093\%$. Planet d ($n = 3$) is the most massive ($M_{\rm d} = 8.52 M_{\oplus}$ [38]).

The sum and difference equations show that librations in four-body systems are connected to Laplace and Laplace-like librations in their overlapping triple MMR chains. Thus, four-body librating chains clearly resemble the shorter type-I double MMRs of Goździewski et al. [2], and there are no type-II pure MMRs in chains of four or more bodies. This conclusion makes classical LRs and Laplace-like resonances appear more prominent than previously thought.

The conclusions obtained from equations (A5) and (A6) are valid for any overlapping triple MMRs, not only for those in HD 110067. We show this in Appendix B.1 below, where we derive the generalized equations for a chain of three consecutive local MMRs, all of the first order.

Appendix B. Overlapping First-order Four-body MMRs

In a chain of four orbiting bodies indexed by 1-4 radially outward, we consider three consecutive local MMRs of the first order with period ratios $(r_j + 1){:}r_j$, where $j = 1, 2, 3$.

Appendix B.1. Four-body Phase Angles

For the leading pair ($j = 1, 2$), equation (3) with $s_j = 1$ gives

$$\begin{array}{rcl} \theta_1 & = & (r_1 + 1)\lambda_2 - r_1\lambda_1 - \varpi_2 \\ \theta_2 & = & (r_2 + 1)\lambda_3 - r_2\lambda_2 - \varpi_2 \end{array} , \tag{B1}$$

where ϖ_2 is the longitude of the pericenter of body 2.

Similarly, for the trailing pair ($j = 2, 3$) and for $s_j = 1$ and $\varpi_2 \to \varpi_3$, we obtain

$$\begin{array}{rcl} \theta_3 & = & (r_2 + 1)\lambda_3 - r_2\lambda_2 - \varpi_3 \\ \theta_4 & = & (r_3 + 1)\lambda_4 - r_3\lambda_3 - \varpi_3 \end{array} , \tag{B2}$$

where ϖ_3 is the longitude of the pericenter of body 3.

One phase angle, φ_1, is obtained from the linear combination $(\theta_2 + \theta_3) - (\theta_1 + \theta_4)$, viz.

$$\varphi_1 = r_1\lambda_1 - (r_1 + 2r_2 + 1)\lambda_2 + (r_3 + 2r_2 + 2)\lambda_3 - (r_3 + 1)\lambda_4 \,. \tag{B3}$$

Another phase angle, φ_2, is obtained from the linear combination $(\theta_2 + \theta_4) - (\theta_1 + \theta_3)$, viz.

$$\varphi_2 = r_1\lambda_1 - (r_1 + 1)\lambda_2 - r_3\lambda_3 + (r_3 + 1)\lambda_4 \,. \tag{B4}$$

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The sum and difference of φ_2 and φ_1 are then given by the equations

$$\varphi_2 + \varphi_1 = 2[r_1\lambda_1 - (r_1 + r_2 + 1)\lambda_2 + (r_2 + 1)\lambda_3], \tag{B5}$$

and

$$\varphi_2 - \varphi_1 = 2[r_2\lambda_2 - (r_2 + r_3 + 1)\lambda_3 + (r_3 + 1)\lambda_4]. \tag{B6}$$

Equations (A5) and (A6) for planets c-d-e-f in HD11067 are recovered from these equations for $r_1 = r_2 = 2$ and $r_3 = 3$.

For $r_3 + 1 = 2r_2$ in the trailing MMR, equation (B6) then produces an exact Laplace multiple $\varphi_2 - \varphi_1 = 2r_2(\varphi_{\rm L})_{234}$, as in equation (A6) above. Similarly, equation (B5) produces an exact Laplace multiple $\varphi_2 + \varphi_1 = 2r_1(\varphi_{\rm L})_{123}$ for $r_2 + 1 = 2r_1$. In the event that both the sum and the difference of the four-body angles are Laplace multiples, then both conditions hold simultaneously, and they are described by the conjunction

$$4r_1 \;=\; 2r_2 + 2 \;=\; r_3 + 3\,, \tag{B7}$$

for integer values of $r_j \geq 1$ ($j = $ 1-3) and for r_2, r_3 both odd.

Appendix B.2. The Classical Double LR with $r_j = 1$

The case with all $r_j = 1$ corresponds to a chain of two overlapping classical LRs with global MMR of type 1:2:4:8, a layout that has not so far been observed (see below) and that was not permitted to occur in the Galilean satellites of Jupiter. At this point, we find it plausible that the double Laplace-like MMRs described by equation (B7) may not exist at all, and one reason is the appearance of unconventional MMRs (6:5, 8:7, 10:9, 14:13, $\cdots$ for $r_1 \geq 2$ in equation (B7)) that have also not been observed in multibody systems (see also items (f) and (g) in Section 3.3).

On the other hand, librating double LRs ($r_j = 1$) have been generated in some N-body experiments of four-body configurations tailored after the four Galilean satellites [16,47] and the four planets of HR 8799 [52]. These MMRs result from specialized initial conditions and do not invalidate our presumption. In particular, the Galilean double LR is formed in some N-body models in which the classical LR migrates outward without disruption until it captures Callisto as well [16,47].

Another migration scenario unfolds in the N-body models of HR 8799 e-d-c-b, where the planets seek stability in or near a double LR; nevertheless, the best-fit strictly periodic MMR chain $\frac{1}{2}$:1:2:$\frac{9}{2}$ turns out to display only one classical LR (planets e-d-c; Refs. [52,53], with the orbital periods calculated from Kepler's third law).

The similarity of the observed displaced trailing MMRs of HR 8799 b (9:4) and Callisto (7:3) is compelling. Along with the dispacement of Kepler-176 b (Figure 1 and item 8 in Section 4), it certainly seems that a fully-formed LR is powerful enough to prevent a fourth planet of any mass from joining in to the LR MMR chain.

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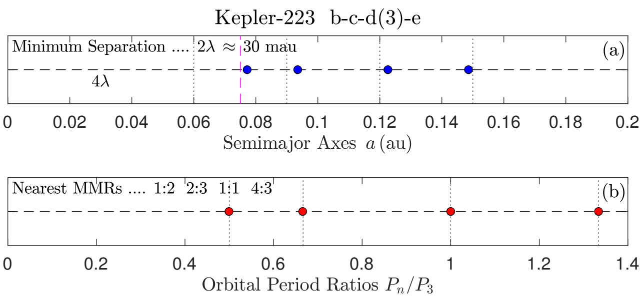

Figure B1. Kepler-223: (a) Distribution of semimajor axes of planets (filled blue circles) and nearest radial tidal potential minima (vertical dotted lines separated by a distance of 2λ, where λ ≈ 15 mau is the wavelength of the tidal field). Equations (10) and (12) of Paper I predict that λ ≃ 15 mau for planets d and e. The vertical dashed line splits the 2λ interval into equal segments. (b) Distribution of orbital period ratios of planets (filled red circles) and nearest global MMRs (vertical dotted lines). The orbits deviate from the nearby exact MMRs by $\leq 0.14\%$. Planet d ($n = 3$) is the most massive ($M_d = 8.0 M_\oplus$ [3]).

Appendix B.3. Application to Kepler-223

Kepler-223 [3,10] is an exquisite resonant four-planet system with a global MMR chain $\frac{1}{2}$:$\frac{2}{3}$:1:$\frac{4}{3}$ (rows 7, 4 in Table 2 and note e in Table 1) that was mentioned briefly in Figure 1 of Paper I. The four-planet system of Kepler-223 b-c-d-e competes with the recently discovered six-planet HD 110067 system (Appendix A) and Kepler-60 b-c-d [2] for resonance proximity, and Kepler-223 places a close third. The pristine resonant system of Kepler-223 is shown in Figure B1.

The local MMRs are 4:3, 3:2, and 4:3, all of order 1. The paired MMRs are also of the first order in the classification scheme of Celletti et al. [25]. The phase angle of the leading pair is $\varphi = 3(\alpha_0)_{123}$ (row 7 in Table 2), whereas the phase angle of the trailing pair is $\varphi = 2(\varphi_L)_{234}$ (row 3 in Table 2). The N-body models of Mills et al. [3] produced libration centers and amplitudes for reduced versions of the two overlapping triple chains; transformed to our φ-space, these are $175.5° \pm 25.5°$ and $122° \pm 28°$, respectively.

Now, equations (B3) and (B4) for $r_1 = 3, r_2 = 2, r_3 = 3$ produce two four-planet phase angles, viz.

$$\begin{array}{l} \varphi_1 = 3\lambda_1 - 8\lambda_2 + 9\lambda_3 - 4\lambda_4 \\ \varphi_2 = 3\lambda_1 - 4\lambda_2 - 3\lambda_3 + 4\lambda_4 \end{array}, \tag{B8}$$

and their sum and difference are

$$\begin{array}{l} \varphi_2 + \varphi_1 = 6(\alpha_0)_{123} \\ \varphi_2 - \varphi_1 = 4(\varphi_L)_{234} \end{array}; \tag{B9}$$

that is, twice as large as the phases φ of the paired MMRs given above (this is a universal scaling for all four-body phase angles irrespective of librations). According to the N-body results [3], the libration centers and amplitudes of $\varphi_2 \pm \varphi_1$ are $351° \pm 51°$ and $244° \pm 56°$, respectively.

Mills et al. [3] also monitored the four-planet phase φ_2 (equation (B8)); they measured librations about $306° \pm 11°$ in agreement with the theoretical expectation of $297.5° \pm 38°$ (derived from the librations of φ angles of the paired MMRs given above). Although they did not seem to be aware of the four-body phase φ_1 (equation (B8)), we have sufficient information from their simulations to determine the librations of φ_1 as well. We recast the second of equations (B9) in the form

$$\varphi_1 = \varphi_2 - 4(\varphi_L)_{234}, \tag{B10}$$

and we use the libration results from the simulations [3], i.e., $306° \pm 11°$ for φ_2 and $61° \pm 14°$ for $(\varphi_L)_{234}$. We determine thus librations of φ_1 about $62° \pm 57°$.

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Repeating the steps but using the first of equations (B9), we find a similar result, librations of φ_1 about $45^\circ \pm 52^\circ$. Actual N-body models should show a φ_1 libration center near these values (45°-62°). But as Mills et al. [3] also remarked, the four-planet libration amplitudes are expected to be significantly smaller than the φ_1 amplitudes of $\sim 50^\circ$ determined above, owing to the strong coupling of the mean longitudes in equations (B8).

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