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Relativistic Adjusted Bohr Radius

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Abstract: The Bohr radius was derived using the non-relativistic de Broglie approximation, not the relativistic de Broglie equation. In this work, we will derive a relativistic Bohr radius from both the relativistic de Broglie approximation and relativistic kinetic energy. The Bohr model of the hydrogen atom has since been replaced by modern quantum mechanics and the Schrödinger equation. It is well known that the Schrödinger equation confirms the first Bohr radius as the most probable radius, but the Schrödinger equation itself is non-relativistic. The Dirac equation can likely be used, but there seem to be very few publications on the relativistically corrected Bohr radius. This is something we will discuss.

Keywords: Bohr radius; relativistic adjustments; Schrödinger equation; Dirac equation; Hydrogen atom

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1. The Bohr Radius

Bohr [1] published his atomic model in 1913. The Bohr [1] radius is given by:

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{e^2 m_e} = \frac{\hbar}{m_e c\alpha} = \frac{\bar{\lambda}_e}{\alpha} \approx 5.29 \times 10^{-11}\ \text{m} \tag{1}$$

where $\epsilon_0 = \frac{1}{\mu_0 c^2} = \frac{1}{4\pi \times 10^{-7} c^2}$ is the permittivity of free space, α is the fine structure constant, m_e is the electron mass, e is the elementary charge, $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant, and $\bar{\lambda}_e$ is the reduced Compton wavelength of the electron.

The reduced Bohr radius, which takes into account the reduced mass in the hydrogen atom, is given by:

$$a_0^* = \frac{m_e}{\mu} a_0 = \frac{\bar{\lambda}_e}{\alpha}\left(1 + \frac{\bar{\lambda}_P}{\bar{\lambda}_e}\right) = \frac{\bar{\lambda}_e + \bar{\lambda}_P}{\alpha} \tag{2}$$

where $\mu = \frac{m_e m_P}{m_e + m_P}$ is the reduced mass. According to NIST CODATA (2019), the Bohr radius is $a_0 = 5.29177210544 \pm 0.00000000082 \times 10^{-11}$ m. The uncertainty arises both from the uncertainty in the reduced Compton [2] wavelength of the electron, which can be determined by Compton scattering, and the uncertainty in the fine structure constant. There are multiple ways to derive the Bohr radius; here, we will show some of these methods before moving on to make relativistic corrections. One such method is the following:

$$\begin{aligned}
k_e \frac{e^2}{r^2} &= ma = m_e \frac{v^2}{r} \\
\frac{e^2}{4\pi\epsilon_0 r^2} &= m_e a = m_e \frac{v^2}{r} \\
\frac{\hbar c\alpha}{r^2} &= m_e a = m_e \frac{v^2}{r} \\
r &= \frac{\hbar c\alpha}{m_e c^2 \alpha^2} \\
r &= \frac{\hbar}{m_e c\alpha} \\
r &= \frac{\hbar}{\frac{\hbar}{\bar{\lambda}_e}\frac{1}{c} c\alpha} \\
r &= \frac{\bar{\lambda}_e}{\alpha} \approx 5.29 \times 10^{-11}\ m
\end{aligned} \tag{3}$$

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Another approach is to to do like Bohr and relay on the de Broglie wavelength to do the derivation. Assume that the standing circumference of the electron around the proton must be an integer multiple, n, of the de Broglie wavelength $\lambda_b = \frac{h}{mv}$. This gives:

$$2\pi r = n\frac{h}{mv} \tag{4}$$

alternatively this can be seen as the angular momentum must be quantized:

$$mrv = n\frac{h}{2\pi} \tag{5}$$

which is basically the same formula written in a different way. Solved for v this gives:

$$v = n\frac{h}{m2\pi r} \tag{6}$$

Next, we take advantage of the fact that the centripetal force must equal the Coulomb [3] force. This gives:

$$\frac{m_e v^2}{r} = k_e\frac{ee}{r^2} \tag{7}$$

Next, we substitute the velocity from equation (6) into the equation above and solve for r. This gives:

$$\begin{aligned} \frac{m_e n^2 \frac{h^2}{m_e^2 4\pi^2 r^2}}{r} &= k_e\frac{ee}{r^2} \\ n^2\frac{h^2}{m_e 4\pi^2 r} &= \hbar c\alpha \\ r = n^2\frac{\hbar}{m_e c\alpha} &= n\frac{\bar{\lambda}_e}{\alpha} \end{aligned} \tag{8}$$

which is, once again, equal to the Bohr radius.

2. Relativistic Bohr Radius

It is well known that the Schrödinger [4] equation confirms the first Bohr radius as the most probable radius at which to find the electron, but this does not mean that the Bohr radius does not require relativistic adjustments, as the Schrödinger equation is non-relativistic. On the other hand, the Dirac [5] equation is relativistic, and we will return to it at the end of this section. Bohr also relied on the Rydberg [6] formula, which was published in 1890 and is clearly non-relativistic. However, the Rydberg formula has recently been relativistically adjusted [7], and the relativistic version has, for example, been used in high-energy physics by the W-7 Max Planck research group [8,9] in connection with their stellarator experiments. The alternative is to use quantum electron dynamics, but for large atoms in high energy physics the calculations can then be very complicated and computer intensive and more basic relativistic adjustments is therefore likely used by for example the W-7 Max Planck research in part of their research work.

The de Broglie [10] wavelength formula: $\lambda_b \approx \frac{h}{mv}$, is in reality only an approximation valid when $v \ll c$ (and is not valid when $v = 0$). De Broglie [11] also presented a relativistic extension of his formula:

$$\lambda_b \approx \frac{h}{mv\gamma} \tag{9}$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ is the Lorentz factor. The formula $\lambda_b \approx \frac{h}{mv}$ is simply the first term in the Taylor expansion of this expression and is therefore clearly non-relativistic..

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If we now follow the same approach as in the section above to derive the Bohr radius, but include the relativistic de Broglie formula instead of the non relativistic, we end up with the following equation for what we will call the relativistic modified Bohr radius:

$$a_{0,r} = \frac{\bar{\lambda}_e n(\sqrt{1-4\alpha^2}+1)}{2\alpha} \tag{10}$$

For $n = 1$ we get:

$$a_{0,r} = \frac{\bar{\lambda}_e(\sqrt{1-4\alpha^2}+1)}{2\alpha} \approx 5.29149 \times 10^{-11}$$

which is slightly shorter than the classical Bohr radius due to a relativistic correction. The relativistic correction itself is equal to:

$$\begin{aligned} a_0 - a_{0,r} &= \frac{\bar{\lambda}_e}{\alpha} - \frac{\bar{\lambda}_e(\sqrt{1-4\alpha^2}+1)}{2\alpha} \\ a_0 - a_{0,r} &= \frac{2\bar{\lambda}_e}{2\alpha} - \frac{\bar{\lambda}_e(\sqrt{1-4\alpha^2}+1)}{2\alpha} \\ a_0 - a_{0,r} &= \frac{\bar{\lambda}_e(1-\sqrt{1-4\alpha^2})}{2\alpha} \approx 2.81809 \times 10^{-15}\ m \end{aligned} \tag{11}$$

That is, the Bohr radius for the Hydrogen atom is likely off by approximately 2.81809×10^{-15} meters, which is about only off by about 0.0053%, but still this is approximately 13.39 times the reduced Compton wavelength of the proton and an error equal to approximately 3.35 times the proton charge radius, so it could clearly be of important if one really want hig precession when trying to study the quantum world.

Our equation (10) should likely be confirmable or at least studied in relation to predictions from the Dirac equation. When examining the relativistically adjusted Bohr radius from the Dirac equation's perspective. Even if the Hydrogen atom has been extensively studied based on also the Dirac equation it somewhat surprisingly appears that very limited work has been done so far, despite some efforts [12]. Therefore, we encourage more researchers to investigate this further.

3. Conclusion

We have pointed out that the Bohr radius, $a_0 = \frac{4\pi\epsilon_0\hbar^2}{e^2 m_e} = \frac{\hbar}{m_e c\alpha} = \frac{\bar{\lambda}_e}{\alpha}$, is based on the non-relativistic form of the de Broglie wavelength. If, instead, we derive it using the relativistic de Broglie formula that de Broglie also provided, we get: $a_{0,r} = \frac{\bar{\lambda}_e(\sqrt{1-4\alpha^2}+1)}{2\alpha}$. This suggests that the Bohr radius is off by approximately: $a_0 - a_{0,r} = \frac{\bar{\lambda}_e(1-\sqrt{1-4\alpha^2})}{2\alpha} \approx 2.81809 \times 10^{-15}\ m$. This could potentially impact multiple interpretations and calculations in physics, given how frequently the Bohr radius is used.

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