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A Novel TOPSIS Framework for Multi-Criteria Decision Making with Random Hypergraphs: Enhancing Decision Processes

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Abstract: In today's complex decision-making landscape, multi-criteria decision-making (MCDM) frameworks play a crucial role in addressing conflicting criteria. This paper introduces a novel framework that combines the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) with random hypergraphs to enhance decision processes. Traditional MCDM methods often face challenges due to uncertainty and interdependencies among criteria. Our approach leverages random hypergraphs to better capture the relationships between criteria, offering a refined representation of decision problems. We delve into the theoretical foundations of this framework, detailing its algorithmic implementation and methodologies for evaluating alternatives under uncertainty. Performance comparisons illustrate the advantages of the proposed TOPSIS framework, emphasizing how random hypergraphs enrich TOPSIS's analytical capabilities. This research advances the theoretical understanding of MCDM frameworks while providing practical insights for practitioners seeking robust solutions in complex and uncertain decision-making environments.

Keywords: random hypergraph; TOPSIS; MCDM

MSC: 05C85, 05C38, 05C05, 05C90

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1. Introduction

One of the structured ways to making decisions involved criteria interaction is known as multi-criteria decision making (MCDM). In MCDM, the problem is framed as one involving multiple criteria, where data is essential for making an informed decision, and a balance needs to be found among competing goals. MCDM is specifically designed to handle decisions with multiple criteria. Techniques like the Technique for the Order of Preference by Similarity to Ideal Solution (TOPSIS) help decision-makers prioritize criteria and make more balanced decisions [4,17]. Not all criteria have the same level of importance in decision making. Some may be critical, while others are secondary and less significant. Decisions can be influenced by changes in the criteria or their weights. Sensitivity analysis helps us to understand the stability of decision outcomes under different weighting scenarios or criteria values and how robust a decision is if circumstances change. Weighting ensures that critical factors have more influence on the final decision than less significant ones. Ensuring a best approximate weighting based on the informed data can lead to robust decision making, and this could be achieved through a dynamic weighting system based on the data involved. Decision making often involves uncertainty and risk, where the outcomes cannot be predicted with certainty. Softness and / or fuzzy techniques incorporate uncertainty into the decision-making process, allowing more robust decisions even when the information is incomplete or uncertain [19–22]. However, none of these decision processes utilized dynamic weighting in the interaction of criteria. Instead, most authors rely on fixed weighting systems based on the expertise of the decision-makers. This approach often leads to uncertainty and confusion, making it difficult to achieve precise and robust decision-making. Rahman et al. [32] introduced an

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informed dynamic weighting system based on the data involved to address the MCDM problem using random hypergraphs. The authors first defined a Choquet integral operator over a random hypergraph and applied it to the MCDM problem.

Graphs and hypergraphs are ideal for representing complex, non-linear network systems. Networks often involving uncertainties can be modeled using random graphs and hypergraphs. The use of random graphs and hypergraphs [15,34] explicitly addresses these uncertainties. In 1959, Erdős introduced random graphs [7], which have since been extensively studied and applied in numerous network problems [14]. The foundation of random graph theory was further solidified with a series of papers [8–12] by Erdős and R'enyi, which have been widely utilized by researchers. While graphs represent interactions between pairs of objects, hypergraphs capture interactions within groups [3,15], where the groups are referred to as hyperedges, and the objects as vertices [3]. The random hypergraphs used to define the Choquet integral operator are a generalized version of the Erdős-R'enyi model of random hypergraphs [32]. The Choquet integral [28] plays a significant role in MCDM. Several key properties of the Choquet integral operator over random hypergraphs are discussed with relevant examples in [32].

TOPSIS is a multi-criteria decision-making technique that allows individuals and organizations to evaluate and rank alternatives based on multiple criteria or attributes [17]. Developed by Hwang and Yoon in the early 1980s, this method has since gained widespread popularity across various fields, including engineering, business, environmental management, healthcare, and social affinity [16,40]. TOPSIS facilitates systematic analysis of both qualitative and quantitative factors, supporting informed decision-making in social and community contexts [24]. In this article, the TOPSIS method is applied to solve multi-criteria decision-making problems within random hypergraphs. Our discussion covers the use of TOPSIS with dynamic weights in random hypergraphs, as well as its application with fixed weights, supported by relevant examples.

2. Preliminaries

This section presents the preliminaries that will be used in the subsequent sections. Every effort has been made to ensure that the article is self-contained.

2.1. Graph and Hypergraph

A graph is a mathematical structure used to represent pairwise relationships between objects. It consists of two main components: vertices and edges. Vertices are points or entities within the graph, representing various concepts such as people, cities, or abstract ideas. Edges are the connections between pairs of vertices, illustrating the relationships between the objects they represent. Graphs have widespread applications across fields like computer science, social network analysis, operations research, biology, and more. They provide a flexible and intuitive framework for representing and analyzing relationships between objects or entities.

A hypergraph generalizes the graph structure by allowing edges to connect any number of vertices. In a hypergraph, a hyperedge may connect one, two, or more vertices. Formally, let $V = \{v_1, v_2, v_3, ..., v_n\}$ be a finite set of vertices, and let $E = \{e_1, e_2, e_3, ..., e_m\}$ be a family of subsets of V. The pair $H = (V, E)$ is called a hypergraph, where V is the vertex set and E is the edge set. The order of the hypergraph, denoted $n(H)$, is the cardinality of V, that is, $|V| = n$. The elements $v_1, v_2, v_3, ..., v_n$ are referred to as nodes or vertices, and the subsets $e_1, e_2, e_3, ..., e_m$ are called hyperedges or edges. The size of the hypergraph is the cardinality of E, i.e., $|E| = m$. A hypergraph with no edges and no vertices, where $V = \phi$ and $E = \phi$, is called an empty hypergraph. Edges within a hypergraph can have different relationships. Some edges may be included within others, while edges that coincide are called multiple edges. A hypergraph with no included edges is termed a simple hypergraph.

For a vertex $v \in V$, let $E(v)$ denote the set of edges containing v. The number $|E(v)|$ is the degree of the vertex v, while the number $|E_i|$ is the degree of the edge E_i. A hypergraph is called k-regular if

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all vertices have the same degree $k \geq 0$, and it is called r-uniform if all edges have the same degree $r \geq 0$.

Now, let's recall the generalized version of the standard Erdős - Rényi and Edgar Gilbert models for r-uniform random hypergraphs on n vertices, as described by Rahman et al. [32]. Consider a random experiment with a finite number of outcomes, where the sample space corresponds to the vertex set. Through inherent connections or links, certain vertices are grouped together to form hyperedges. Since vertices represent outcomes of a random experiment, the subsets of the vertex set (hyperedges) can also be viewed as events. The combination of the vertex set and these grouped vertices forms a random hypergraph, where each hyperedge can be assigned a weight representing the probability of the corresponding event.

This random hypergraph generalizes the binomial r-uniform random hypergraph $G^r(n, p)$, where $p \in (0, 1)$. Here, $\binom{n}{r}$ possible hyperedges are included, and each hyperedge is present with probability p. Importantly, the hyperedges are mutually independent events. For further details, refer to Rahman et al. [32].

2.2. Multi-Criteria Decision Making and TOPSIS

Multi-criteria decision-making (MCDM) is a field of study that addresses decision-making problems involving multiple criteria or objectives. The goal is to evaluate and select the best alternative from a set of options, considering various factors.

According to Roszkowska [33], the primary steps in the MCDM process are:

1. Establish system evaluation criteria that link system capabilities to overall goals.
2. Develop alternative systems or approaches for achieving those goals (generating alternatives).
3. Evaluate the alternatives based on the established criteria.
4. Apply a normative multiple criteria analysis method.
5. Select the optimal (preferred) alternative.
6. If the final solution is not satisfactory, gather new information and repeat the process in another iteration of multiple criteria optimization.

One of the widely used MCDM methods is TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution). TOPSIS helps in selecting the best alternative from a set of options by considering multiple criteria or attributes. The core principle of TOPSIS is to choose the alternative that has the shortest distance from the ideal solution and the greatest distance from the negative ideal solution [17]. It is a popular, straightforward, and intuitive method that provides a systematic approach to decision-making, especially when multiple criteria need to be considered simultaneously.

TOPSIS has been applied across a wide range of fields. In business [35], it has been used for tasks such as supplier selection, project prioritization, and performance evaluation. In engineering and technology [1,5,18,23], it aids in optimal design selection, risk assessment, and quality control. In environmental management [2], TOPSIS is employed for site selection, pollution control strategy evaluation, and sustainable development assessments. In healthcare, the method has been used for hospital rankings, medical equipment selection, and evaluating healthcare systems. In agriculture, it has been applied to crop selection, land-use planning, and agricultural technology evaluation. Educational contexts [29] also benefit from TOPSIS, where it is used for school rankings, faculty performance evaluation, and curriculum development.

These examples illustrate the versatility of TOPSIS across various domains, demonstrating its broad applicability.

2.3. Choquet Integral Operator over Random Hypergraph

Let's recall the Choquet integral operator over a random hypergraph, as introduced by Rahman et al. [32]. Let $V = \{v_1, v_2, v_3, ..., v_n\}$ be the vertex set of a simple random hypergraph H. A function $f : V \to [0, \infty)$ is referred to as a score function. Let $E = \{e_1, e_2, e_3, ..., e_k\}$ represent the set of edges, and let $w_1, w_2, w_3, ..., w_k$ be the corresponding weights of the groups $e_1, e_2, e_3, ..., e_k$. These weights

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are assumed to satisfy the equation $\sum_{i=1}^{k} w_i = 1$. If the assigned weights do not satisfy this condition, appropriate adjustments can be made through proper formulation.

Since the hypergraph is simple, neither $e_i \subseteq e_j$ nor $e_j \subseteq e_i$ holds.

A measure $\rho : 2^V \to [0,1]$, where 2^V represents the power set of V can be defined as follows:

$$\rho(A) = \begin{cases} 0, \text{ if there does exists any } e_i \text{ such that } e_i \subset A \\ \sum_{i=1}^{l} w_i^{'}, \text{ if } e_i^{'} \subset A, \end{cases} \tag{1}$$

where $w_i^{'}, i = 1, 2, ..., l$ are levels of $e_i^{'}, i = 1, 2, ..., l$ with $e_i^{'} \in E$ such that $e_i^{'} \subset A$. Then ρ is a capacity on V satisfies normalization.

Suppose, $v_i\,(i = 1, 2, \ldots, n)$'s are reordered as $v_{\sigma(1)} \preccurlyeq v_{\sigma(2)} \preccurlyeq \ldots \preccurlyeq v_{\sigma(n)}$ according to the preference order $f\left(v_{\sigma(1)}\right) \leq f\left(v_{\sigma(2)}\right) \leq ... \leq f\left(v_{\sigma(n)}\right)$ of the score function f. Here $\sigma : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ is a permutation on $\{1, 2, ..., n\}$.

The Choquet integral of the alternatives v_i (where $i = 1, 2, \ldots, n$) with respect to the measure ρ and score function f over the random hypergraph $H = (V, E)$ is defined as follows:

$$\mathrm{Ch}_{\rho,f}^{H}(v_1, v_2, \ldots, v_n) \text{ or } \mathrm{Ch}_{f}^{H}(v_1, v_2, \ldots, v_n) = \sum_{i=1}^{n}\left(f\left(v_{\sigma(i)}\right) - f\left(v_{\sigma(i-1)}\right)\right)\rho\left(A_{\sigma(i)}\right), \tag{2}$$

where, $f\left(v_{\sigma(0)}\right) = 0$ and $A_{\sigma(i)} = \{v_{\sigma(i)}, v_{\sigma(i+1)}, ..., v_{\sigma(n)}\}$.

Using the above operator, Rahman et al. [32] introduced an MCDM algorithm. In the subsequent sections, we extend the Choquet integral-based MCDM algorithm over random hypergraphs to model a TOPSIS framework for solving multi-criteria decision-making problems. A comparative study is conducted between the conventional TOPSIS method, which considers a set of alternatives with multiple criteria, and the TOPSIS approach using a random hypergraph introduced in this article.

3. TOPSIS Method over Random Hypergraph for Fixed Weights

To accommodate criteria interactions within the TOPSIS method, we propose a generalized TOPSIS approach that incorporates these interactions. In this approach, the set of criteria and their interaction groups are represented by a random hypergraph $H = (V, E)$, where $V = \{c_1, c_2, \ldots, c_m\}$ and $E = \{e_1, e_2, \ldots, e_k\}$. Further details on criteria interactions over a random hypergraph can be found in [32]. It is important to note that the weights of the hyperedges satisfy the condition $\sum_{i=1}^{k} w_i = 1$. The method proceeds through the following steps.

1. Express the assessment information of the alternatives $A_i\ (i = 1, 2, \ldots, n)$ corresponding to the criteria $C_j\ (j = 1, 2, \ldots, m)$ into the decision matrix $D = \left[x_{ij}\right]_{n\times m}$, where $x_{ij}\ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ indicates the partial evaluation of the i^{th} alternative A_i with respect to the j^{th} criteria $C_j\ (j = 1, 2, \ldots, m)$.
2. Criteria interacted decision matrix
$$Q = \left[y_{ij}\right]_{n\times k},$$
where $y_{ij} = \sum_{t=1}^{|e_t|} x_{it}$ if $c_t \in e_t\ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, k)$ indicates the partial evaluation of the i^{th} alternative A_i with respect to the j^{th} interacted group of criteria $e_j\ (j = 1, 2, \ldots, k)$.
3. Normalize the criteria interacted decision matrix
$$r_{ij} = \frac{y_{ij}}{\sqrt{\sum_{i=1}^{n} y_{ij}^2}},$$

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where y_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, k)$.

4. Construct the weightage matrix

$$v_{ij} = w_j y_{ij}, \text{ where } (i = 1, 2, \ldots, n; j = 1, 2, \ldots, k).$$

5. Determine the positive ideal (A^+) and negative ideal (A^-). Positive ideal can be defined as $A^+ = \{V_1^+, V_2^+, \ldots, V_k^+\}$ where

$$V_j^+ = \begin{cases} \max v_{ij}, & \text{if } j \text{ is an attribute of benefit.} \\ \min v_{ij}, & \text{if } j \text{ is attribute of cost.} \end{cases}$$

Also negative ideal is defined as
$A^- = \{V_1^-, V_2^-, \ldots, V_k^-\}$

$$V_j^- = \begin{cases} \min v_{ij}, & \text{if } j \text{ is attribute of benefit.} \\ \max v_{ij}, & \text{if } j \text{ is attribute of cost.} \end{cases}$$

6. calculation separation measure

The distance between the alternative with the positive ideal solution is defined as $S_i^+ = \sqrt{\sum_{j=1} (v_{ij} - V_j^+)^2}$

The distance between the alternative with the negative ideal solution is defined as $S_i^- = \sqrt{\sum_{j=1} (v_{ij} - V_j^-)^2}$

7. Relative closeness from ideal solution or preference value for each alternatives Pi is given as $P_i = \frac{S_i^-}{S_i^+ + S_i^-}$

If $A_i = A_i^+$, then $P_i = 1$ and if $A_i = A_i^-$, then $P_i = 0$.
Greater P_i value indicates that the preferred alternative A_i

Remark 1. *The TOPSIS method over a random hypergraph differs from the conventional TOPSIS method only in how it handles interactions among criteria and the weighting of criteria. All other computational steps remain almost identical to the original TOPSIS method. By doing so, the process also takes into account the impact of criteria interactions when determining the best alternatives.*

4. Numerical Example Based on Fixed Weights TOPSIS Method over Random Hypergraph

We consider the same MCDM problem discussed in [32] to facilitate a comparative study. The number of alternatives in the problem is 10, namely, $V = v_1, v_2, \ldots, v_{10}$, and there are 4 criteria denoted as $C_j; (j = 1, 2, 3, 4)$. For our convenience, we recall the decision matrix, which is given as follows:

Step 1. Decision matrix

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Table 1. The assessment decision matrix D.

Criteria/Alternative	C_1	C_2	C_3	C_4
v_1	7.45	8.51	7.37	9.32
v_2	7.72	8.47	8.53	8.43
v_3	8.23	8.13	7.48	8.32
v_4	7.31	7.63	6.85	7.41
v_5	7.21	6.24	6.61	7.28
v_6	9.14	8.72	9.11	9.51
v_7	8.23	7.15	9.74	9.33
v_8	8.43	9.34	9.32	9.48
v_9	8.45	7.81	9.23	8.68
v_{10}	9.78	8.91	8.64	9.34

***Step* 2.** The hyperedges of the random hypergraph are $e_1 = \{c_1, c_2, c_4\}$, $e_2 = \{c_1, c_3\}$, $e_3 = \{c_3, c_4\}$. Based on these hyperedges, we construct the interacted decision matrix Q as follows.

Table 2. The interacted decision matrix Q.

Hyperedge/Alternative	e_1	e_2	e_3
v_1	25.28	14.82	16.69
v_2	24.62	16.25	16.96
v_3	24.68	15.71	15.8
v_4	22.35	14.16	14.26
v_5	20.73	13.82	13.89
v_6	27.37	18.25	18.62
v_7	24.71	17.97	19.07
v_8	27.25	17.75	18.8
v_9	24.94	17.68	17.91
v_{10}	28.03	18.42	17.98

***Step* 3.** Construct the normalize matrix

Table 3. The normalize matrix.

Hyperedge/Alternative	e_1	e_2	e_3
v_1	0.318661782	0.282867613	0.308863646
v_2	0.310342289	0.310161857	0.313860242
v_3	0.311098607	0.29985494	0.292393386
v_4	0.281728276	0.27027027	0.263894284
v_5	0.261307703	0.26378073	0.257047097
v_6	0.345006842	0.348335624	0.344580054
v_7	0.311476765	0.342991296	0.352907713
v_8	0.343494207	0.338792182	0.347911118
v_9	0.314375983	0.3374561	0.331440857
v_{10}	0.353326335	0.351580394	0.332736271

***Step* 4.** Construct the weightage matrix

Weightage to the criteria. i.e. $w_1 = 0.44, w_2 = 0.27, w_3 = 0.29$.

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Table 4. The weightage matrix.

Hyperedge/Alternative	e_1	e_2	e_3
v_1	0.140211184	0.076374256	0.089570457
v_2	0.136550607	0.083743701	0.09101947
v_3	0.136883387	0.080960834	0.084794082
v_4	0.123960442	0.072972973	0.076529342
v_5	0.114975389	0.071220797	0.074543658
v_6	0.151803011	0.094050618	0.099928216
v_7	0.137049777	0.09260765	0.102343237
v_8	0.151137451	0.091473889	0.100894224
v_9	0.138325432	0.091113147	0.096117849
v_{10}	0.155463587	0.094926706	0.096493519

Step 5. Positive ideal and negative ideal Positive ideal $A^+ = \{V_1^+, V_2^+, \dots, V_l^+\}$ where

$$V_j^+ = \begin{cases} \max v_{ij}, & \text{if } j \text{ is attribute of benefit.} \\ \min v_{ij}, & \text{if } j \text{ is attribute of cost.} \end{cases}$$

Negative ideal

$A^- = \{V_1^-, V_2^-, \dots, V_l^-\}$

$$V_j^- = \begin{cases} \min v_{ij}, & \text{if } j \text{ is attribute of benefit.} \\ \max v_{ij}, & \text{if } j \text{ is attribute of cost.} \end{cases}$$

V_j^+={0.155463587, 0.094926706, 0.102343237}

V_j^-={0.114975389, 0.071220797, 0.074543658}

Step 6. S_i^+, S_i^+, and P_i can be calculated as

$S_i^+ = \sqrt{\sum_{j=1} (v_{ij} - V_j^+)^2}$

$S_i^- = \sqrt{\sum_{j=1} (v_{ij} - V_j^-)^2}$

$P_i = \frac{S_i^-}{S_i^+ + S_i^-}$

Table 5. Table for S_i^+ S_i^- P_i *Score*.

Distance/edege	S_i^+	S_i^-	P_i	*Score*
v_1	0.000739991	0.000889215	0.545796814	7
v_2	0.000611005	0.000893766	0.593954857	6
v_3	0.000848258	0.000679906	0.444917035	8
v_4	0.002140798	$8.77492E-05$	0.039375042	9
v_5	0.002974115	$3.09713E-13$	$1.04136E-10$	10
v_6	$2.00021E-05$	0.002521852	0.992130913	1
v_7	0.000344463	0.001717482	0.832942694	4
v_8	$3.2742E-05$	0.002412237	0.986608492	3
v_9	0.000347028	0.001406371	0.802082492	5
v_{10}	$3.42164E-05$	0.002683068	0.987407855	2

Step 7. End

5. TOPSIS Method over Random Hypergraph with Dynamic Weights

The working principle of this method is almost same as the previous method with exception that to incorporate the criteria interactions and their weights, the dynamic weights for each individual

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shall be taken instead of fixed weight. We note that we shall take dynamic weights computation technique that adopted in Pisand algorithm [32]. We consider a random hypergraph $H = (V, E)$, where $V = \{c_1, c_2, \ldots, c_m\}$ (the set of criteria) and $E = \{e_1, e_2, \ldots, e_k\}$ (interacted groups). The process of the method involves the following steps.

1. Express the assessment information of the alternatives A_i $(i = 1, 2, \ldots, n)$ corresponding to the criteria C_j $(j = 1, 2, \ldots, m)$ into the decision matrix $D = \left[x_{ij}\right]_{n\times m}$, where x_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ indicates the partial evaluation of the i^{th} alternative A_i with respect to the j^{th} criteria C_j $(j = 1, 2, \ldots, m)$.
2. Criteria interacted decision matrix
$$Q = \left[y_{ij}\right]_{n\times k},$$
where $y_{ij} = \sum_{t=1}^{|e_t|} x_{it}$ if $c_t \in e_t$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, k)$ indicates the partial evaluation of the i^{th} alternative A_i with respect to the j^{th} interacted group of criteria e_j $(j = 1, 2, \ldots, k)$.
3. Normalize the criteria interacted decision matrix
$$r_{ij} = \frac{y_{ij}}{\sqrt{\sum_{i=1}^{n} y_{ij}^2}},$$
where y_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, k)$.
4. Calculate the dynamic weights of the hyperedges for each individual.
   - Construct the interaction matrix B as follows:
   $$B = \left[b_{ij}\right]_{n\times m},$$
   where $b_{ij} = v_{ij} l_j$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ and l_j $(j = 1, 2, \ldots, m)$ are the interaction levels of C_j $(j = 1, 2, \ldots, m)$, respectively.
   - Estimation of the weightage for the hyperedges e_k $(k = 1, 2, \ldots, l)$ for each alternative v_i $(i = 1, 2, \ldots, n)$.
   The procedure represents the partial evaluation for each alternative v_i $(i = 1, 2, \ldots, n)$ corresponding to the interrelated interactions of the criteria C_k $(1 \le k \le l)$.
   Define
   $$e_{il} = \{b_{ij} \,| C_j \in e_l \ (i = 1, 2, \ldots, n;\ 1 \le l \le k)\}.$$
   Construct the set E_i, where $E_i = \{e_{i1}, e_{i2}, \ldots, e_{ik}\}$ $(i = 1, 2, \ldots, n)$. Then for each $i (i = 1, 2, \ldots, n)$, the set E_i represents a copy of the edge set E of the random hypergraph H.
   Since the hypergraph H is simple, for any two distinct groups e_{il} and $e_{il'}$, neither $e_{il} \subset e_{il'}$ nor $e_{il'} \subset e_{il}$ for all $l \neq l'$. The degree assignment w_{il} $(l = 1, 2, \ldots, k)$ corresponding to each alternative i, $(i = 1, 2, \ldots, n)$ for hyperedges e_l $(l = 1, 2, \ldots, k)$ is determined by
   $$w_{il} = \frac{\sum_{b_{ij} \in e_{il}} b_{ij}}{\sum_{j=1}^{m} b_{ij} d_j} \ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m),$$
   where d_j is the degree of C_j $(j = 1, 2, \ldots, m)$ in H. Then
   $$\sum_{l=1}^{k} w_{il} = 1.$$
5. Construct the weightage matrix

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$$v_{ij} = w_{ij} r_{ij}, \text{ where } (i = 1,2,\ldots,n; j = 1,2,\ldots,k).$$

6. Determine the positive ideal (A^+) and negative ideal (A^-). Positive ideal can be defined as $A^+ = \{V_1^+, V_2^+, \ldots, V_k^+\}$ where

$$V_j^+ = \begin{cases} \max v_{ij}, & \text{if } j \text{ is an attribute of benefit.} \\ \min v_{ij}, & \text{if } j \text{ is attribute of cost.} \end{cases}$$

The negative ideal is also defined as
$A^- = \{V_1^-, V_2^-, \ldots, V_k^-\}$

$$V_j^- = \begin{cases} \min v_{ij}, & \text{if } j \text{ is an attribute of benefit.} \\ \max v_{ij}, & \text{if } j \text{ is attribute of cost.} \end{cases}$$

7. Calculation of separation measure.
The distance between the alternative and the positive ideal solution is defined as $S_i^+ = \sqrt{\sum_{j=1} (v_{ij} - V_j^+)^2}$.
The distance between the alternative and the negative ideal solution is defined as $S_i^- = \sqrt{\sum_{j=1} (v_{ij} - V_j^-)^2}$.
8. Relative closeness from ideal solution or preference value for each alternatives Pi is given as
$P_i = \frac{S_i^-}{S_i^+ + S_i^-}$
If $A_i = A_i^+$, then $P_i = 1$ and if $A_i = A_i^-$, then $P_i = 0$. The higher value of P_i indicates the preferred alternative A_i.

6. Numerical Example Based on Dynamic Weights TOPSIS Method over Random Hypergraph

In order to conduct a comparative study, we revisit the issue outlined above. The problem involves 10 alternatives, denoted as $V = \{v_1, v_2, \ldots, v_{10}\}$, and encompasses 4 criteria, represented as C_j where j ranges from 1 to 4. To streamline our analysis, we reiterate the steps whose computation tables align with those utilized in previous methods.

Step 1. For the assessment information of the alternatives, we refer to Table 1.
Step 2. For the interacted decision matrix of criteria, we refer to Table 2.
Step 3. For normalization of the interacted decision matrix of criteria, we refer to Table 3.
Step 4. For the dynamic weights, we construct the Table 6.

Table 6. The dynamic weights matrix w_{ij}

Criteria/Alternative	e_1	e_2	e_3
v_1	0.454170958	0.254573096	0.291255946
v_2	0.437387628	0.274405948	0.288206424
v_3	0.449199187	0.274507859	0.276292954
v_4	0.449903714	0.27394773	0.276148556
v_5	0.439931153	0.279231211	0.280837636
v_6	0.438048865	0.277758456	0.284192679
v_7	0.416072968	0.281980306	0.301946726
v_8	0.438605757	0.271454978	0.289939266
v_9	0.425990271	0.284873755	0.289135974
v_{10}	0.445773766	0.28091392	0.273312314

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***Step* 5.** Construct the weightage matrix v_{ij} (see Table 7).

Table 7. The weightage matrix v_{ij}

Criteria/Alternative	e_1	e_2	e_3
v_1	0.144726927	0.072010484	0.089958374
v_2	0.135739878	0.085110258	0.090456538
v_3	0.139745241	0.082312538	0.080786232
v_4	0.126750598	0.074039927	0.072874025
v_5	0.114957399	0.073655813	0.072188499
v_6	0.151129856	0.096753165	0.097927129
v_7	0.129597062	0.096716791	0.106559329
v_8	0.150658537	0.091966824	0.100873094
v_9	0.13392111	0.096132386	0.095831475
v_{10}	0.157503611	0.098763827	0.09094092

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***Step* 6.** Computed positive ideal and negative ideal.

V_j^+={0.157503611, 0.099993819, 0.106559329}

V_j^-={0.114957399, 0.07197597, 0.072188499}

For the ranking table, we refer to Table 8.

Table 8. S_i^+ S_i^- P_i *Score*.

Distance/edege	S_i^+	S_i^-	P_i	*Score*
v_1	0.001280303	0.001235767	0.491149661	6
v_2	0.001077535	0.00099295	0.479573549	7
v_3	0.001449611	0.000827512	0.36340245	8
v_4	0.002969536	0.000149246	0.047854057	9
v_5	0.003938637	$6.80852E-06$	0.001725664	10
v_6	0.000196396	0.002670691	0.931499956	2
v_7	0.00087541	0.002110528	0.706822601	4
v_8	0.00018419	0.002581181	0.933394221	1
v_9	0.000818269	0.001583742	0.659340064	5
v_{10}	0.000343108	0.002954994	0.895968049	3

***Step* 7.** End

7. Numerical Example Based on TOPSIS Method

We consider the same MCDM problem as in the previous example to facilitate a comparative study. The number of alternatives in the problem is 10, namely, $V = v_1, v_2, \ldots, v_{10}$, and there are 4 criteria denoted as $C_j; (j = 1, 2, 3, 4)$. To solve this problem by TOPSIS method, we follow the following steps.

***Step* 1.** Construct the decision matrix D (the same as in the previous example).

Table 9. The assessment decision matrix D

Criteria/Alternative	C_1	C_2	C_3	C_4
v_1	7.45	8.51	7.37	9.32
v_2	7.72	8.47	8.53	8.43
v_3	8.23	8.13	7.48	8.32
v_4	7.31	7.63	6.85	7.41
v_5	7.21	6.24	6.61	7.28
v_6	9.14	8.72	9.11	9.51
v_7	8.23	7.15	9.74	9.33
v_8	8.43	9.34	9.32	9.48
v_9	8.45	7.81	9.23	8.68
v_{10}	9.78	8.91	8.64	9.34

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***Step* 2.** Normalised the decision matrix.

We normalised the decision matrix by the relation

$r_{ij} = \frac{y_{ij}}{\sqrt{\sum_{j=1}^{m} y_{ij}^2}}$, where y_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, l)$

Table 10. The normalize matrix.

Criteria/Alternative	C_1	C_2	C_3	C_4
v_1	0.28619107	0.330702352	0.278917226	0.336966692
v_2	0.296563095	0.329147934	0.322817359	0.304788543
v_3	0.316154698	0.315935384	0.28308017	0.300811468
v_4	0.280812982	0.296505164	0.259237856	0.267910214
v_5	0.276971492	0.242489151	0.25015507	0.263210034
v_6	0.351112265	0.338863045	0.344767426	0.343836185
v_7	0.316154698	0.277852152	0.36860974	0.337328245
v_8	0.32383768	0.362956518	0.352714864	0.342751528
v_9	0.324605978	0.303500043	0.349308819	0.313827349
v_{10}	0.375697807	0.346246528	0.326980303	0.337689797

***Step* 3.** Weightage matrix

The same weight has been given to the criteria. i.e. $w_i = 0.25$, $(i = 1, 2, \ldots, 4)$.

Table 11. The weightage matrix.

Criteria/Alternative	C_1	C_2	C_3	C_4
v_1	0.071547767	0.082675588	0.069729306	0.084241673
v_2	0.074140774	0.082286984	0.08070434	0.076197136
v_3	0.079038675	0.078983846	0.070770042	0.075202867
v_4	0.070203246	0.074126291	0.064809464	0.066977553
v_5	0.069242873	0.060622288	0.062538767	0.065802509
v_6	0.087778066	0.084715761	0.086191856	0.085959046
v_7	0.079038675	0.069463038	0.092152435	0.084332061
v_8	0.08095942	0.090739129	0.088178716	0.085687882
v_9	0.081151495	0.075875011	0.087327205	0.078456837
v_{10}	0.093924452	0.086561632	0.081745076	0.084422449

***Step* 4.** Positive ideal and negative ideal Positive ideal $A^+ = \{V_1^+, V_2^+, \ldots, V_l^+\}$ where

$$V_j^+ = \begin{cases} \max v_{ij}, & \text{if } j \text{ is a benefit attribute.} \\ \min v_{ij}, & \text{if } j \text{ is attribute of cost.} \end{cases}$$

Negative ideal

$A^- = \{V_1^-, V_2^-, \ldots, V_l^-\}$

$$V_j^- = \begin{cases} \min v_{ij}, & \text{if } j \text{ is an attribute of benefit.} \\ \max v_{ij}, & \text{if } j \text{ is attribute of cost.} \end{cases}$$

V_j^+={0.093924452, 0.090739129, 0.092152435, 0.085959046}

V_j^-={0.069242873, 0.060622288, 0.062538767, 0.065802509}

***Step* 5.** S_i^+, S_i^+, and P_i can be calculated as

$S_i^+ = \sqrt{\sum_{j=1} (v_{ij} - V_j^+)^2}$

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$$S_i^- = \sqrt{\sum_{j=1} (v_{ij} - V_j^-)^2}$$

$$P_i = \frac{S_i^-}{S_i^+ + S_i^-}$$

Table 12. S_i^+ S_i^- P_i *Score*.

Distance/edege	S_i^+	S_i^-	P_i	*Score*
v_1	0.001071441	0.000883358	0.451891995	7
v_2	0.000689156	0.000931377	0.574735227	6
v_3	0.000932639	0.00058922	0.387171277	8
v_4	0.001946566	0.000189823	0.088852068	9
v_5	0.002799402	$3.94555E-13$	$1.40943E-10$	10
v_6	0.000109575	0.001889782	0.945194772	1
v_7	0.000676886	0.001394401	0.673204957	5
v_8	0.000183941	0.00209712	0.919361703	3
v_9	0.000463636	0.001149041	0.712505331	4
v_{10}	0.000128115	0.001997596	0.939730543	2

Step 6. End

8. Results and Discussions

As discussed in earlier, the formation of the individual sets E_i $(i = 1, 2, \ldots, 10)$ is contingent upon the levels of interaction w_{ik} $(k = 1, 2, \ldots, l < 4)$, which, in turn, is influenced by the criteria C_j $(j = 1, 2, 3, 4)$ and their interactions within the problem. Consequently, the outcomes derived from the dynamic weights TOPSIS method, as defined over the random hypergraph, reflect the combined effects of various interactions among individuals and criteria.

It's worth noting that in cases where the criteria are independent, meaning there is no interaction among them, the random hypergraph $H = (C, E)$ comprises only singleton subsets of C as its hyperedges (refer to Figure 1).

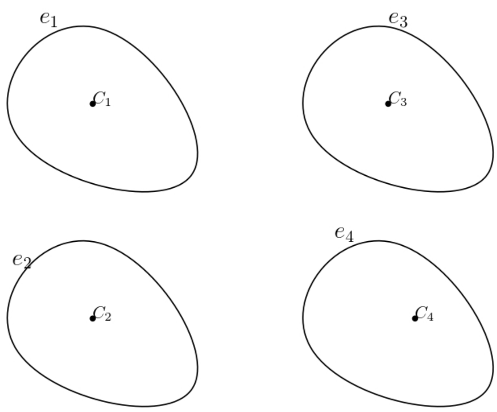

Figure 1. Random hypergraph when there is no interaction, i.e., criteria are independent.

Furthermore, the interaction level w_{ik} of each hyperedge $e_k = \{C_k\}$ (where $k = 1, 2, \ldots, l = m$) corresponding to each alternative i (where $i = 1, 2, \ldots, n$) is $1/m$. In such scenarios, both proposed TOPSIS methods reduce to the general TOPSIS method.

To offer a comprehensive comparison across various approaches, the overall evaluation of the alternatives v_i $(i = 1, 2, \ldots, 10)$ for each method is presented in Table 13.

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Table 13. Overall preference ordering of the alternatives obtained using TOPSIS method, TOPSIS using dynamic weights (DW) and TOPSIS using fixed weights(FW).

Ranking	Ranking by TOPSIS	Ranking by DW	Ranking by FW
r_1	v_6	v_8	v_6
r_2	v_{10}	v_6	v_{10}
r_3	v_8	v_{10}	v_8
r_4	v_9	v_7	v_7
r_5	v_7	v_9	v_9
r_6	v_2	v_1	v_2
r_7	v_1	v_2	v_1
r_8	v_3	v_3	v_3
r_9	v_4	v_4	v_4
r_{10}	v_5	v_5	v_5

It is evident from Table 13 that there is a noticeable variation in the ranking of alternatives between the proposed TOPSIS methods and the traditional TOPSIS method. This disparity arises from the distinct computational processes employed by each method. Furthermore, even within the two proposed methods, differences in alternative rankings are apparent, attributed to the use of dynamic weights versus fixed weights. However, the rankings produced by TOPSIS and the fixed weights method are nearly identical, except for ranks 4 and 5. Based on this comparison, we have made the following observations.

The dynamic weights method accounts for the cumulative effects of interactions among individuals and criteria, while the fixed weight method evaluates interactions solely among criteria. Despite these differences, the optimal (best and least preferable) alternatives identified through all three methods show proximity, validating the soundness and applicability of our proposed models. Additionally, with the dynamic weights method, there is no need to assign values to the interacted groups of criteria, as it generates weights based on interactions among individuals and criteria.

Given that the dynamic weights method incorporates the combined effects of interactions among individuals and criteria, it can be regarded as superior to the other two methods.

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